Unitarity of the modular tensor categories associated to unitary vertex operator algebras, I

BIN GUI

Department of Mathematics, Vanderbilt University

bin.gui@vanderbilt.edu

Abstract

This is the first part in a two-part series of papers constructing a unitary structure for the modular tensor category (MTC) associated to a unitary rational vertex operator algebra (VOA). Given a rational VOA, we know that its MTC is constructed using the (finite dimensional) vector spaces of intertwining operators of this VOA. Moreover, the tensor-categorical structures can be described by the monodromy behaviors of the intertwining operators. Thus, constructing a unitary structure for the MTC of a unitary rational VOA amounts to defining an inner product on each (finite dimensional) vector space of intertwining operators, and showing that the monodromy matrices of the intertwining operators (e.g. braiding matrices, fusion matrices) are unitary under these inner products.

In this paper, we develop necessary tools and techniques for constructing our unitary structures. This includes giving a systematic treatment of one of the most important functional analytic properties of the intertwining operators: the energy bounds condition. On the one side, we give some useful criteria for proving the energy bounds condition of intertwining operators. On the other side, we show that energy bounded intertwining operators can be smeared to give rise to (unbounded) closed operators. We prove that the (well-known) braid relations and adjoint relations for unsmeared intertwining operators have the corresponding smeared versions. We also give criteria on the strong commutativity between smeared intertwining operators and smeared vertex operators localized in disjoint open intervals of S^1 (the strong intertwining property). Besides investigating the energy bounds condition, we also study certain genus 0 geometric properties of intertwining operators. Most importantly, we prove the convergence of certain mixed products-iterations of intertwining operators. Many useful braid and fusion relations will also be discussed.

Contents

0 Introduction

Vertex operator algebras: unitarity and reflection positivity, intertwining operators, and modular tensor categories

Wightman axioms and algebraic quantum field theories (AQFTs) are two major ways to formulate quantum field theories (QFTs) in the rigorous language of mathematics. Roughly speaking, the main difference between these two approaches is that the first one focuses on field operators localized at points, whereas the latter one studies bounded or unbounded but (pre)closed field operators (as well as the von Neumann algebras they generate) localized on open subsets of the space-time. For 2d conformal field theories (CFTs), the AQFT approach goes under the name "conformal net". Many fruitful results have been achieved in this functional analytic approach. We refer the reader to [Kaw15] for a brief survey on this topic.

In many senses, the theory of vertex operator algebras (VOAs) can be regarded as the Wightman axiomatization of CFT. In fact, given a VOA we have a vertex operator Y, which associates to each state vector v and each point $z \in \mathbb{C}$ a field operator $Y(v, z)$ localized at z. However, one has to be careful when regarding VOAs as Wightman CFTs for the following reasons.

1. Wightman QFTs are defined on the Minkowski space-time, while VOAs actually correspond to CFTs in the Euclidean picture. It is well known that one can do Wick rotation to pass from Minkowskian QFTs to Euclidean ones. However, it is not true that any Euclidean QFT can arise from a Minkowskian one. One has to ensure that the Euclidean QFT satisfies *reflection positivity* [OS73]. A VOA satisfying reflection positivity is called **unitary** [DL14].

2. In most cases, a VOA does not give all field operators of a (closed-string) CFT. In fact, a VOA V^L is the chiral part of a CFT, consisting only of meromorphic fields, and the anti-chiral part corresponds to the complex conjugate of another VOA V^R. However, in a CFT there are field operators which are locally neither holomorphic nor anti-holomorphic. The typical way of studying these general field operators is through conformal blocks, or equivalently, through **intertwining operators**. An intertwining operator $\mathcal{Y}$ of a VOA V is a generalization of the vertex operator Y, which intertwines the actions of V on three V-modules (the charge space, the source space, and the target space), and which is locally holomorphic but globally multi-valued field.[1] Then a field operator $\Phi(z,\overline{z})$ should look like $\Phi(z,\overline{z}) = \sum_{\alpha,\beta} \mathcal{Y}_\alpha(w^\alpha, z)\overline{\mathcal{Y}_\beta(w^\beta, z)}$, where each $\mathcal{Y}_\alpha$ (resp. $\mathcal{Y}_\beta$) is an intertwining operator of V^L (resp. V^R), and w^α (resp. w^β) is a vector inside the source space of $\mathcal{Y}_\alpha$ (resp. $\mathcal{Y}_\beta$).[2] This means that intertwining operators are indeed the "chiral halves" of the full field operators.

Since full field operators satisfy commutativity (locality), associativity (existence of operator product expansions), and modular invariance,[3] one may expect that their chiral halves should also satisfy similar properties. But since intertwining operators are multi-valued functions, monodromy behaviors will appear when considering these properties. Thus, for intertwining operators, one should expect braiding and fusion instead of commutativity and associativity, and, rather than thinking of the modular invariance of (the trace of) one single intertwining operator, one should consider the modular invariance of the vector space of intertwining operators [MS88].[4] Hence one will have braid, fusion and modular (S and T) matrices, and, written in a coordinate-independent way, one has a **modular tensor category** (MTC) [MS89, MS90].[5]

3. The reason why we have commutativity, associativity, and modular invariance in CFTs is not quite obvious from the Wightman axioms. These properties have highly geometric nature, and can more easily be seen in the Euclidean picture, where the CFTs are defined, not only on the flat complex plane (or punctured Riemann spheres), but on

[1] See section 1.3 for the precise definition of intertwining operators.

[2] cf. [MS88]. Intertwining operators are called chiral vertex operators in that paper.

[3] cf. [MS88]. In [HK07, HK10] the reader can find the precise statement of these properties in the language of vertex operator algebras.

[4] See section 2 for the statement of braid and fusion relations. Modular invariance in its most general form can be found in [Hua05b].

[5] A mathematically rigorous and complete construction is due to Y.Z.Huang and J.Lepowsky. See [HL13] for a brief review of their theory.

any compact Riemann surface. Indeed, these properties are among the most important examples of the *sewing property*[6], which is clear from a (highly geometric) axiomatization of CFT not yet mentioned: G.Segal's definition of CFTs [Seg88].

Motivations

Thus, VOAs are deeply rooted in the geometric nature of CFT, but can be formulated without assuming unitarity (or reflection positivity). On the other hand, conformal nets, the Wightman axiomatization of CFT, are not so geometric but manifestly unitary. The goal of this paper (as well as the forthcoming second part of this series) is to develop a unitary theory for the MTCs of unitary rational VOAs. We explain some motivations behind this theory.

First, we have seen that MTCs are important for the construction of full CFTs. Having constructed MTCs from rational VOAs, one can use Frobenius algebras over MTCs to classify full rational CFTs [Kong06, Kong08] (the word "rational" means that the sum $\Phi = \sum \mathcal{Y}_\alpha \overline{\mathcal{Y}_\beta}$ mentioned earlier is always finite). However, in order to classify *unitary* full CFTs, i.e., full CFTs with *reflection positivity*, one needs the unitarity of these MTCs, and then one studies unitary Frobenius algebras (i.e., Q-systems) over these unitary MTCs. Besides full (closed-string) CFTs, the unitarity of MTCs is also necessary for studying the unitary extensions of unitary rational VOAs, and unitary open-string CFTs, just as MTCs are important for studying general VOA extensions [HKL15, CKM17] and general open-string CFTs [Kong08].

The second motivation is to prepare for the investigation of the relations between conformal nets and unitary VOAs. Just like unitary VOAs, conformal nets also describe the chiral parts of unitary CFTs, and one can construct MTCs from rational conformal nets [DHR71, FRS89, KLM01], which are automatically unitary. It is important to know whether the MTCs constructed from conformal nets and from unitary VOAs are equivalent. Clearly, if one can show the equivalence, then the unitarizability of the MTCs of conformal nets will imply that of the MTCs of unitary VOAs. It turns out, however, that in order to prove this equivalence, one has to first equip the MTCs of unitary VOAs with a unitary structure.

A glance at the theory

Now we briefly explain what we shall do in this series of papers in order to find a unitary structure on the MTCs. For simplicity, we assume that V is a unitary "rational"[7] VOA whose representations are always unitarizable. (For example, V can be a unitary

[6] Sewing property says that if a punctured Riemann surfaces M is obtained by attaching another two M_1 and M_2, then the correlation function on M can always be obtained by taking the composition of two correlation functions defined on M_1 and M_2 respectively. Note that intertwining operators are nothing but the chiral halves of the correlation functions on the Riemann sphere with three holes.

[7] The exact meaning of rationality in this paper will be made clear later. See conditions (0.4)-(0.6).

Virasoro VOA (minimal model), or a unitary affine VOA (WZW model).) If W_i, W_j, W_k are unitary representations of V, then a type $\binom{k}{i\ j}$ intertwining operator $\mathcal{Y}_\alpha$ linearly associates to each $w^{(i)} \in W_i$ a multivalued holomorphic operator-valued function

$$\mathcal{Y}_\alpha(w^{(i)}, z) : W_j \to \widehat{W}_k,$$

where $\widehat{W}_k$ is the algebraic completion of W_k (see section 1.2). Moreover, one requires that $\mathcal{Y}$ "intertwines" the actions of V on W_i, W_j, W_k (Jacobi identity), and that $\mathcal{Y}_\alpha$ is conformal covariant (translation property).[8] The V-modules W_i, W_j, W_k are called, respectively, the charge space, the source space, and the target space of $\mathcal{Y}_\alpha$. We denote by $\mathcal{V}\binom{k}{i\ j}$ the vector space of type $\binom{k}{i\ j}$ intertwining operators. Note that if we set $W_0 = V$, then the vertex operator Y is a type $\binom{0}{0\ 0}$ intertwining operator.

Now, for each equivalence class of irreducible unitary V-module, we choose a representing element to form a set $\{W_k : k \in \mathcal{E}\}$. With abuse of notation, we also let $\mathcal{E}$ denote this set. For any unitary V-modules W_i, W_j, their tensor product $W_i \boxtimes W_j$ is a V-module defined by

$$W_i \boxtimes W_j = \bigoplus_{k \in \mathcal{E}} \mathcal{V}\binom{k}{i\ j}^* \otimes W_k.$$

[9] By rationality of V, $\mathcal{E}$ is a finite set (i.e., there are only finitely many equivalence classes of irreducible V-modules), and $\mathcal{V}\binom{k}{i\ j}$, as well as its dual space $\mathcal{V}\binom{k}{i\ j}^*$, is finite-dimensional. Note that although $W_i \boxtimes W_j$ is unitarizable, we don't know how to choose a canonical unitary structure on $W_i \boxtimes W_j$, because we don't know how to choose a meaningful inner product on the vector space $\mathcal{V}\binom{k}{i\ j}^*$. But this is exactly the goal of our theory. In part II of this series, we will define a sesquilinear form Λ on $\mathcal{V}\binom{k}{i\ j}^*$ for each W_i, W_j and irreducible W_k. After choosing a basis of $\mathcal{V}\binom{k}{i\ j}$, Λ will be defined using certain fusion or braid matrix under this basis. The most difficult part of our theory is to prove that these sesquilinear forms (or equivalently, the corresponding fusion or braid matrices) are positive definite, i.e., they are inner products. Once this is proved, then it is not hard to show the unitarity of all braid and fusion matrices under any orthonormal basis with respect to this inner product, and hence the unitarity of the MTC.

Smeared intertwining operators

The non-degeneracy of Λ will follow from the rigidity of the MTC. So what we actually need to prove is the positivity of Λ. Although this problem is purely vertex-operator-algebraic, it seems very difficult to solve it using only VOA methods. We tackle this problem by investigating some analytic and algebraic properties of the **smeared intertwining operators** of V, so that many results in conformal nets (most importantly, the Haag duality) can be used in our theory. Here, for any type $\binom{k}{i\ j}$ intertwining operator $\mathcal{Y}_\alpha$, $w^{(i)} \in W_i$,

[8] Rigorous definition can be found in definition 1.11.

[9] This definition is due to Y.Z.Huang and J.Lepowsky, cf. [HL95a].

I an open interval in S^1, and $f \in C_c^\infty(I)$, the smeared intertwining operator is defined to be

$$\mathcal{Y}_\alpha(w^{(i)}, f) := \oint_{S^1} \mathcal{Y}_\alpha(w^{(i)}, z) f(z) \frac{dz}{2i\pi}.$$

This generalizes the smeared vertex operators considered in [CKLW15]. Similar to [CKLW15], we require that $\mathcal{Y}_\alpha(w^{(i)}, \cdot)$ satisfies the following **energy bounds condition**: there exist $M, r, t \geqslant 0$, such that for any open interval $I \in S^1$, $f \in C_c^\infty(I)$, $w^{(j)} \in W_j$,

$$\|\mathcal{Y}_\alpha(w^{(i)}, f) w^{(j)}\| \leqslant M|f|_t \|(1 + L_0)^r w^{(j)}\|,$$

where $|f|_t$ is the t-th order Sobolev norm of f. Then $\mathcal{Y}_\alpha(w^{(i)}, f)$ will be a (pre)closed unbounded operator mapping $\mathcal{H}_j \to \mathcal{H}_k$.

One of the main purposes of the present paper is to prove the algebraic and analytic properties of smeared intertwining operators that are necessary for showing the positivity of Λ. First we discuss **braiding of smeared intertwining operators**. As we mentioned above, braiding, fusion, and modular invariance are among the most important geometric properties of intertwining operators. However, only braid relation can be translated onto smeared intertwining operators. More specifically, if I, J are disjoint open intervals in S^1 with chosen continuous $\arg$ functions, and we have intertwining operators $\mathcal{Y}_\alpha, \mathcal{Y}_\beta, \mathcal{Y}_{\alpha'}, \mathcal{Y}_{\beta'}$ such that the braid relation

$$\mathcal{Y}_\beta(w^{(j)}, \zeta)\mathcal{Y}_\alpha(w^{(i)}, z) = \mathcal{Y}_{\alpha'}(w^{(i)}, z)\mathcal{Y}_{\beta'}(w^{(j)}, \zeta)$$

holds for any vectors $w^{(i)}, w^{(j)}$, and any $z \in I, \zeta \in J$, and if these four intertwining operators are energy bounded, then we will show that the corresponding braid relation for smeared intertwining operators

$$\mathcal{Y}_\beta(w^{(j)}, g)\mathcal{Y}_\alpha(w^{(i)}, f) = \mathcal{Y}_{\alpha'}(w^{(i)}, f)\mathcal{Y}_{\beta'}(w^{(j)}, g)$$

hold for any vectors $w^{(i)}, w^{(j)}$, and any $f \in C_c^\infty(I), g \in C_c^\infty(J)$. Note that these two braid relations are understood in different ways. The second one is a completely algebraic relation, where products of smeared intertwining operators just mean compositions. However, as compositions of (non-smeared) intertwining operators, the two sides of the first braid relation cannot be defined on the same region. Braiding of intertwining operators, unlike its smeared version, should be understood in the sense of analytic continuation.

Braid relations tell us what we shall get if we exchange the product of two smeared intertwining operators localized in disjoint open intervals. With the help of **adjoint relation**, we can obtain the result of exchanging the product of an intertwining operator with the adjoint of another one, say $\mathcal{Y}_\beta(w^{(j)}, g)\mathcal{Y}_\alpha(w^{(i)}, f)^\dagger$, which is also very important in our theory. Given a type $\binom{k}{i\ j}$ intertwining operator $\mathcal{Y}_\alpha$, one can define in a canonical way a type $\binom{j}{\bar{i}\ k}$ intertwining operator $\mathcal{Y}_{\alpha^*}$, called the **adjoint intertwining operator** of $\mathcal{Y}_\alpha$. (Here $W_{\bar{i}}$ is the contragredient module (the dual) of W_i.) For any eigenvector $w^{(i)} \in W_i$ of L_0 (with eigenvalue Δ) satisfying $L_1 w^{(i)} = 0$ (i.e., $w^{(i)}$ is a quasi-primary vector), $\mathcal{Y}_\alpha(w^{(i)}, z)^\dagger$ can be related to $\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, z)$ by the following very simple relation

$$\mathcal{Y}_\alpha(w^{(i)}, z)^\dagger = e^{-i\pi\Delta}\overline{z^{-2\Delta}}\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, \overline{z^{-1}}).$$

We shall prove a similar relation for smeared intertwining operators, so that the result of exchanging $\mathcal{Y}_\beta(w^{(j)}, g)\mathcal{Y}_\alpha(w^{(i)}, f)^\dagger$ will follow from the braiding of $\mathcal{Y}_\beta(w^{(j)}, g)\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, f)$

Braiding and adjoint relations are algebraic properties of smeared intertwining operators. To be able to use the powerful machinery of conformal nets, we need an analytic property of smeared intertwining operators: the **strong intertwining property**. It says that for any disjoint open intervals $I, J \in S^1$, and $f \in C_c^\infty(I), g \in C_c^\infty(J)$, the commuting relation

$$Y_k(v, g)\mathcal{Y}_\alpha(w^{(i)}, f) = \mathcal{Y}_\alpha(w^{(i)}, f)Y_j(v, g)$$

(as a special case of braiding) not only holds when acting on W_j, but also holds in a **strong** sense, which means that $\mathcal{Y}_\alpha(w^{(i)}, f)$, when extended to an unbounded operator on $\mathcal{H}_j \oplus \mathcal{H}_k$ mapping $\mathcal{H}_k$ to zero, commutes with the von Neumann algebra generated by $Y_j(v, g) \oplus Y_k(v, g)$.[10] The strong intertwining property could be understood as a generalization of the strong locality property (i.e., the strong commutativity of smeared vertex operators) discussed in [CKLW15].[11]

Generalized (smeared) intertwining operators

The above discussion is based on the assumption that the intertwining operators are energy-bounded. However, in practice it might be not easy to show the energy bounds condition for *all* intertwining operators of a given unitary rational VOA. Let us choose V to be the unitary level-l affine $\mathfrak{su}_n$ VOA for instance. Then the energy bounds condition for type $\binom{k}{i\ j}$ intertwining operators is established only when the charge space W_i is a direct sum of V-modules equivalent to $W_0 = V$ or $W_\square = L_{\mathfrak{su}_n}(\square, l)$ [Was98]. Here $W_\square$ corresponds to the irreducible level l integrable highest weight representation of the affine Lie algebra $\widehat{\mathfrak{su}_n}$ whose highest weight $\square$ is the one of the vector representation $\mathfrak{su}_n \curvearrowright \mathbb{C}^n$. So when W_i is a general V-module, it might not be helpful to consider smeared intertwining operators of type $\binom{k}{i\ j}$.

To overcome this difficulty, we consider **generalized intertwining operators** and their smeared versions. The key observation is that $W_\square$ is a generating object inside the tensor category of V. Let us assume, without loss of generality, that W_i is irreducible. Then from the well-known fusion rules of V, one can easily find $n = 1, 2, \ldots$ such that W_i is

[10] That the commutativity of two unbounded operators acting on a common invariant core does not imply the strong commutativity of these two operators is well known due to Nelson's counterexample [Nel59].

[11] A natural question is whether one can generalize the strong intertwining property one step further to the strong braiding between smeared intertwining operators. Strong braiding is very important for showing the equivalence between the fusion categories of a unitary VOA and the corresponding conformal net. But since it will not be used in our present theory, we leave the discussion of this interesting topic to future work.

equivalent to a V-submodule of $\underbrace{W_{\square}\boxtimes\cdots\boxtimes W_{\square}}_{n}$. It follows that there exist intertwining operators $\mathcal{Y}_{\sigma_2},\dots,\mathcal{Y}_{\sigma_n}$ with charge spaces equaling $W_{\square}$, such that the source space of $\mathcal{Y}_{\sigma_2}$ is $W_{\square}$, the target space of $\mathcal{Y}_{\sigma_n}$ is W_i, and for any $3\leqslant m\leqslant n$ the source space of $\mathcal{Y}_{\sigma_m}$ equals the target space of $\mathcal{Y}_{\sigma_{m-1}}$. (Any sequence of intertwining operators satisfying the last condition is called a **chain of intertwining operators**.) Now, for any type $\binom{k}{i\ j}$ intertwining operator $\mathcal{Y}_\alpha$, we define a **generalized intertwining operator** $\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}$ which linearly associates to any $w_1^{(\square)},\dots,w_n^{(\square)}\in W_{\square}$ a $\mathrm{Hom}(W_j,\widehat{W}_k)$-valued multi-valued holomorphic function $\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w_n^{(\square)},z_n;\dots;w_1^{(\square)},z_1)$ of the complex variables $z_1,\dots,z_n$ by setting

$$\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w_n^{(\square)},z_n;\dots;w_1^{(\square)},z_1)=\mathcal{Y}_\alpha\big(\mathcal{Y}_{\sigma_n}(w_n^{(\square)},z_n-z_1)\cdots\mathcal{Y}_{\sigma_2}(w_2^{(\square)},z_2-z_1)w_1^{(\square)},z_1\big). \quad (0.1)$$

Then for any mutually disjoint open intervals $I_1,\dots,I_n\subset S^1$ and $f_1\in C_c^\infty(I_1),\dots,f_n\in C_c^\infty(I_n)$, the corresponding **smeared generalized intertwining operator** is defined to be

$$\begin{aligned}&\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w_n^{(\square)},f_n;\dots;w_1^{(\square)},f_1)\\ =&\oint_{S^1}\cdots\oint_{S^1}\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w_n^{(\square)},z_n;\dots;w_1^{(\square)},z_1)f_n(z_n)\cdots f_1(z_1)\frac{dz_1}{2i\pi}\cdots\frac{dz_n}{2i\pi}.\end{aligned}$$

Thanks to fusion relations, there exist a chain of intertwining operators $\mathcal{Y}_{\alpha_1},\dots,\mathcal{Y}_{\alpha_n}$ with charge spaces equaling $W_{\square}$ (hence these intertwining operators are energy-bounded!), such that

$$\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w_n^{(\square)},z_n;\dots;w_1^{(\square)},z_1)=\mathcal{Y}_{\alpha_n}(w_n^{(\square)},z_n)\cdots\mathcal{Y}_{\alpha_1}(w_1^{(\square)},z_1). \quad (0.2)$$

So the smeared generalized intertwining operator will be a product of smeared intertwining operators. This shows that $\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w_n^{(\square)},f_n;\dots;w_1^{(\square)},f_1)$ has similar analytic properties as smeared intertwining operators: it is a (pre)closed unbounded operator mapping $\mathcal{H}_j\to\mathcal{H}_k$, and it satisfies the strong intertwining property.

Braiding and adjoint of smeared generalized intertwining operators are much harder to prove than those analytic properties. The difficulty is mainly on the unsmeared side: we want to determine the braid relation

$$\begin{aligned}&\mathcal{Y}_{\tau_m\cdots\tau_2,\beta}(\widetilde{w}_m^{(\square)},\zeta_m;\cdots\widetilde{w}_1^{(\square)},\zeta_1)\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w_n^{(\square)},z_n;\cdots w_1^{(\square)},z_1)\\ =&\mathcal{Y}_{\sigma_n\cdots\sigma_2,?}(w_n^{(\square)},z_n;\cdots w_1^{(\square)},z_1)\mathcal{Y}_{\tau_m\cdots\tau_2,?}(\widetilde{w}_m^{(\square)},\zeta_m;\cdots\widetilde{w}_1^{(\square)},\zeta_1)\end{aligned} \quad (0.3)$$

and the adjoint relation (when the vectors are quasi-primary)

$$\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w_n^{(\square)},z_n;\cdots w_1^{(\square)},z_1)^\dagger=(\cdots)\cdot\mathcal{Y}_{?\cdots?,\alpha^*}(\overline{w_n^{(\square)}},\overline{z_n^{-1}};\cdots\overline{w_1^{(\square)}},\overline{z_1^{-1}}).$$

These problems will be treated in part II of this series. However, certain preparatory results, including general fusion relations (that generalized intertwining operators can be written as the products of several intertwining operators), general braid relations (braiding of the products of more than two intertwining operators), and the well-definedness (convergence) of the products of generalized intertwining operators (the convergence of both sides of (0.3) for instance), will be proved in this paper.

Outline of this paper

Part I is organized as follows. In chapter 1 we review the basic definitions of (unitary) VOAs, their (unitary) representations, and intertwining operators. We define unitary representations of unitary VOAs, adjoint intertwining operators, creation and annihilation operators, and prove some basic properties.

Chapter 3 is devoted to the study of energy bounds condition and smeared intertwining operators. In section 3.1 we define the energy bounds condition for intertwining operators, and give some useful criteria. In section 3.2 we define, for energy-bounded intertwining operators, the corresponding smeared intertwining operators. We prove the braid relations, the adjoint relation, and the strong intertwining property of smeared intertwining operators. We also prove the rotation covariance of smeared intertwining operators[12], which will be used in part II to prove some density results.

The purpose of chapter 2 needs more explanations. One of the main goals of this chapter is to give a brief and self-contained introduction to Huang-Lepowsky's tensor product theory of rational VOAs based on the braid and fusion relations of intertwining operators. So, unlike chapter 1, before reading which we suggest that the reader has some basic knowledge on VOAs, this chapter does not require any previous knowledge on Huang-Lepowsky's theory. Moreover, the results that we shall cite but not prove again in our papers will be kept to a minimum. Such results include: (1) The absolute convergence of the products of intertwining operators (theorem 2.2). (2) The analytic continuation principle of (chiral) correlation functions due to the existence of holomorphic differential equations (theorem 2.4.) (3) The existence of the fusion relation for two intertwining operators. (Theorem 2.5 in the special case when $n = 2$. The proof of the general case in section A.3 relies on this special case.) (4) The rigidity of the braided tensor category of rational VOAs. (It will only be used in part II to prove the non-degeneracy of Λ.)

All the other results used in our theory are proved in chapter 2 or A. These results are either known in Huang-Lepowsky's theory explicitly or implicitly, or can be easily derived using the machinery they have developed. Such results include the description of a linear basis of the vector space of correlation functions (proposition 2.3), the braiding of two or more than two intertwining operators (theorem 2.8), the relation between the braiding of intertwining operators and the maps $B_\pm : \mathcal{V}\binom{k}{i\ j} \to \mathcal{V}\binom{k}{j\ i}$ introduced in section 1.3 (proposition 2.12), and the fusion and braiding of intertwining operators with vertex operators or creation operators (section 2.3). We give complete proofs of these results in this paper, since the language and notations used in Huang and Lepowsky's papers are very different from ours, and also because there are many analytic subtleties in the proofs of these results.[13] Readers with a background in functional analysis or con-

[12] In fact, the more general conformal covariance can be proved for smeared intertwining operators using the similar argument for proving the conformal covariance of smeared vertex operators (cf. [CKLW15] proposition 6.4).

[13] As an examples of these analytic subtleties, let us assume that we have a braid relation of intertwining operators looking like $AB = B'A'$. If we have another intertwining operator C, then the braid relation of three intertwining operators $CAB = CB'A'$ does *not* follow directly from "multiplying" both sides of

formal net might especially care about these subtleties.

What's new in chapter 2 is the convergence of the products of generalized intertwining operators (theorem 2.6). Another type of convergence property (corollary 2.7), which will be used in part II to prove the braid and adjoint relations of generalized (smeared) intertwining operators, is also given. The conditions on the complex variables under which the absolute convergence holds are especially important for our theory.

Acknowledgment

The author would like to thank Professor Vaughan Jones. This paper, as well as the second half of the series, cannot be finished without his constant support, guidance, and encouragement. The author was supported by NSF grant DMS-1362138.

Notations.

In this paper, we assume that V is a vertex operator algebra of CFT type. Except in chapter 1, we assume that V also satisfies the following conditions:

(1) V is isomorphic to V'. (0.4)

(2) Every $\mathbb{N}$-gradable weak V-module is completely reducible. (0.5)

(3) V is C_2-cofinite. (0.6)

(See [Hua05b] for the definitions of these terminologies.) The following notations are used throughout this paper.

A^{t}: the transpose of the linear operator A.
$A^{\dagger}$: the formal adjoint of the linear operator A.
A^*: the ajoint of the possibly unbounded linear operator A.
$\overline{A}$: the closure of the pre-closed linear operator A.
C_i: the antiunitary map $W_i \to W_{\bar{i}}$.
$\mathbb{C}^\times = \{z \in \mathbb{C} : z \neq 0\}$.
$\mathrm{Conf}_n(\mathbb{C}^\times)$: the n-th configuration space of $\mathbb{C}^\times$.
$\widetilde{\mathrm{Conf}}_n(\mathbb{C}^\times)$: the universal covering space of $\mathrm{Conf}_n(\mathbb{C}^\times)$.
$\mathscr{D}(A)$: the domain of the possibly unbounded operator A.
$đ\theta = \frac{e^{i\theta}}{2\pi} d\theta$.
$e_r(e^{i\theta}) = e^{ir\theta} \quad (-\pi < \theta < \pi)$.
$\mathcal{E}$: a complete list of mutually inequivalent irreducible V-modules.
$\mathcal{E}^{\mathrm{u}}$: the set of unitary V-modules in $\mathcal{E}$.
$\mathrm{Hom}_V(W_i, W_j)$: the vector space of V-module homomorphisms from W_i to W_j.

---

the original braid relation by C. As we have emphasized before, the braiding of intertwining operators is understood using analytic continuation, and not as the direct composition of operators. Therefore, the braiding of several intertwining operators does not follow from that of two intertwining operators through a direct algebraic argument.

$\mathcal{H}_i$: the norm completion of the vector space W_i.
$\mathcal{H}_i^r$: the vectors of $\mathcal{H}_i$ that are inside $\mathscr{D}((1+\overline{L_0})^r)$.
$\mathcal{H}_i^\infty = \bigcap_{r\geqslant 0}\mathcal{H}_i^r$.
I^c: the complement of the open interval I.
$I_1 \subset\subset I_2$: $I_1, I_2 \in \mathcal{J}$ and $\overline{I_1} \subset I_2$.
$\mathrm{id}_i = \mathrm{id}_{W_i}$: the identity operator of W_i.
$\mathcal{J}$: the set of (non-empty, non-dense) open intervals of S^1.
$\mathcal{J}(U)$: the set of open intervals of S^1 contained in the open set U.
P_s: the projection operator of W_i onto $W_i(s)$.
$\mathfrak{r}(t) : S^1 \to S^1$: $\mathfrak{r}(t)(e^{i\theta}) = e^{i(\theta+t)}$.
$\mathfrak{r}(t) : C^\infty(S^1) \to C^\infty(S^1)$: $\mathfrak{r}(t)h = h \circ \mathfrak{r}(-t)$.
$\mathrm{Rep}(V)$: the modular tensor category of the representations of V.
$\mathrm{Rep}^{\mathrm{u}}(V)$: the category of the unitary representations of V.
$\mathrm{Rep}^{\mathrm{u}}_{\mathcal{G}}(V)$: When $\mathcal{G}$ is additively closed, it is the subcategory of $\mathrm{Rep}^{\mathrm{u}}(V)$ whose objects are unitary V-modules in $\mathcal{G}$. When $\mathcal{G}$ is multiplicatively closed, then it is furthermore equipped with the structure of a ribbon tensor category.
$S^1 = \{z \in \mathbb{C} : |z| = 1\}$.
$\mathcal{V}\binom{k}{i\ j}$: the vector space of type $\binom{k}{i\ j}$ intertwining operators.
$W_0 = V$, the vacuum module of V.
W_i: a V-module.
$\widehat{W_i}$: the algebraic completion of W_i.
$W_{\bar{i}} \equiv W_i'$: the contragredient module of W_i.
$W_{ij} \equiv W_i \boxtimes W_j$: the tensor product of W_i, W_j.
$w^{(i)}$: a vector in W_i.
$\overline{w^{(i)}} = C_i w^{(i)}$.
x: a formal variable.
Y_i: the vertex operator of W_i.
$\mathcal{Y}_\alpha$: an intertwining operator of V.
$\mathcal{Y}_{\overline{\alpha}} \equiv \overline{\mathcal{Y}_\alpha}$: the conjugate intertwining operator of $\mathcal{Y}_\alpha$.
$\mathcal{Y}_{\alpha^*} \equiv \mathcal{Y}_\alpha^\dagger$: the adjoint intertwining operator of $\mathcal{Y}_\alpha$.
$\mathcal{Y}_{B_\pm\alpha} \equiv B_\pm\mathcal{Y}_\alpha$: the braided intertwining operators of $\mathcal{Y}_\alpha$.
$\mathcal{Y}_{C\alpha} \equiv C\mathcal{Y}_\alpha$: the contragredient intertwining operator of $\mathcal{Y}_\alpha$.
$\mathcal{Y}_{i0}^i$: the creation operator of W_i.
$\mathcal{Y}_{\bar{i}i}^0$: the annihilation operator of W_i.
Δ_i: the conformal weight of W_i.
Δ_w: the conformal weight (the energy) of the homogeneous vector w.
Θ_{ij}^k: a linear basis of $\mathcal{V}\binom{k}{i\ j}$.
$\Theta_{i*}^k = \coprod_{j\in\mathcal{E}}\Theta_{ij}^k, \Theta_{*j}^k = \coprod_{i\in\mathcal{E}}\Theta_{ij}^k, \Theta_{ij}^* = \coprod_{k\in\mathcal{E}}\Theta_{ij}^k$.
θ: the PCT operator of V, or a real variable.
ϑ_i: the twist of W_i.
ν: the conformal vector of V.
$\sigma_{i,j}$: the braid operator $\sigma_{i,j} : W_i \boxtimes W_j \to W_j \boxtimes W_i$.

Ω: the vacuum vector of V.

1 Intertwining operators of unitary vertex operator algebras (VOAs)

We refer the reader to [FHL93] for the general theory of VOAs, their representations, and intertwining operators. Other standard references on VOAs include [FB04, FLM89, Kac98, LL12]. Unitary VOAs were defined by Dong, Lin in [DL14]. Our approach in this article follows [CKLW15].

1.1 Unitary VOAs

Let x be a formal variable. For a complex vector space U, we set

$$U[[x]] = \Big\{ \sum_{n \in \mathbb{Z}_{\geqslant 0}} u_n x^n : u_n \in U \Big\}, \tag{1.1}$$

$$U((x)) = \Big\{ \sum_{n \in \mathbb{Z}} u_n x^n : u_n \in U, u_n = 0 \text{ for sufficiently small } n \Big\}, \tag{1.2}$$

$$U[[x^{\pm 1}]] = \Big\{ \sum_{n \in \mathbb{Z}} u_n x^n : u_n \in U \Big\}, \tag{1.3}$$

$$U\{x\} = \Big\{ \sum_{s \in \mathbb{R}} u_s x^s : u_s \in U \Big\}. \tag{1.4}$$

We define the formal derivative $\frac{d}{dx}$ to be

$$\frac{d}{dx}\Big(\sum_{n \in \mathbb{R}} u_n x^n \Big) = \sum_{n \in \mathbb{R}} n u_n x^{n-1}. \tag{1.5}$$

Let V be a complex vector space with grading $V = \bigoplus_{n \in \mathbb{Z}} V(n)$. Assume that $\dim V(n) < \infty$ for each $n \in \mathbb{Z}$, and $\dim V(n) = 0$ for n sufficiently small. We say that V is a **vertex operator algebra** (VOA), if the following conditions are satisfied:
(a) There is a linear map

$$V \to (\text{End } V)[[x^{\pm 1}]]$$
$$u \mapsto Y(u,x) = \sum_{n \in \mathbb{Z}} Y(u,n) x^{-n-1}$$
$$(\text{where } Y(u,n) \in \text{End } V),$$

such that for any $v \in V$, $Y(u,n)v = 0$ for n sufficiently large.
(b) (**Jacobi identity**) For any $u, v \in V$ and $m, n, h \in \mathbb{Z}$, we have

$$\sum_{l \in \mathbb{Z}_{\geqslant 0}} \binom{m}{l} Y(Y(u,n+l)v, m+h-l)$$

$$= \sum_{l\in\mathbb{Z}_{\geqslant 0}} (-1)^l \binom{n}{l} Y(u, m+n-l)Y(v, h+l) - \sum_{l\in\mathbb{Z}_{\geqslant 0}} (-1)^{l+n} \binom{n}{l} Y(v, n+h-l)Y(u, m+l). \tag{1.6}$$

(c) There exists a vector $\Omega \in V(0)$ (the **vacuum vector**) such that $Y(\Omega, x) = \mathrm{id}_V$.
(d) For any $v \in V$ and $n \in \mathbb{Z}_{\geqslant 0}$, we have $Y(v, n)\Omega = 0$, and $Y(v, -1)\Omega = v$. This condition is simply written as $\lim_{x\to 0} Y(v, x)\Omega = v$.
(e) There exists a vector $\nu \in V(2)$ (the **conformal vector**) such that the operators $L_n = Y(\nu, n+1)$ ($n \in \mathbb{Z}$) satisfy the Virasoro relation: $[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3-m)\delta_{m,-n}c$. Here the number $c \in \mathbb{C}$ is called the **central charge** of V.
(f) If $v \in V(n)$ then $L_0 v = nv$. n is called the **conformal weight** (or the **energy**) of v and will be denoted by Δ_v. L_0 is called the **energy operator**.
(g) (**Translation property**) $\frac{d}{dx}Y(v, x) = Y(L_{-1}v, x)$.

Convention 1.1. In this article, we always assume that V is a VOA of **CFT type**, i.e., $V(0) = \mathbb{C}\Omega$, and $\dim V(n) = 0$ when $n < 0$.

Given a (anti)linear bijective map $\phi : V \to V$, we say that ϕ is an **(antilinear) automorphism** of V if the following conditions are satisfied:

$$(a)\phi\Omega = \Omega, \quad \phi\nu = \nu. \tag{1.7}$$

$$(b)\text{For any } v \in V, \quad \phi Y(v, x) = Y(\phi v, x)\phi. \tag{1.8}$$

It is easy to deduce from these two conditions that $\phi L_n = L_n\phi$ (for any $n \in \mathbb{Z}$). In particular, since ϕ commutes with L_0, we have $\phi V(n) = V(n)$ for each $n \in \mathbb{Z}$.

Definition 1.2. Suppose that V is equipped with an inner product $\langle\cdot|\cdot\rangle$ (antilinear on the second variable) satisfying $\langle\Omega|\Omega\rangle = 1$. Then we call V a **unitary vertex operator algebra**, if there exists an antilinear automorphism θ, such that for any $v \in V$ we have

$$Y(v, x)^\dagger = Y(e^{xL_1}(-x^{-2})^{L_0}\theta v, x^{-1}), \tag{1.9}$$

where $\dagger$ is the formal adjoint operation. More precisely, this equation means that for any $v, v_1, v_2 \in V$ we have

$$\langle Y(v, x)v_1|v_2\rangle = \langle v_1|Y(e^{xL_1}(-x^{-2})^{L_0}\theta v, x^{-1})v_2\rangle. \tag{1.10}$$

Remark 1.3. Such θ, if exists, must be unique. Moreover, θ is anti-unitary (i.e. $\langle\theta v_1|\theta v_2\rangle = \langle v_2|v_1\rangle$ for any $v_1, v_2 \in V$), and $\theta^2 = id_V$ (i.e. θ is an involution). We call θ the **PCT operator** of V. (cf. [CKLW15] proposition 5.1.) In this article, θ denotes either the PCT operator of V, or a real variable. These two meanings will be used in different situations. So no confusion will arise.

We say that a vector $v \in V$ is **homogeneous** if $v \in V(n)$ for some $n \in \mathbb{Z}$. If moreover, $L_1 v = 0$, we say that v is **quasi-primary**. It is clear that the vacuum vector Ω is quasi-primary. By translation property, we have $L_{-1}\Omega = 0$. Therefore, $L_1\nu = L_1 Y(\nu, -1)\Omega = L_1 L_{-2}\Omega = [L_1, L_{-2}]\Omega = 3L_{-1}\Omega = 0$. We conclude that *the conformal vector is quasi-primary*.

Now suppose that V is unitary and $v \in V$ is quasi-primary, then equation (1.9) can be simplified to

$$Y(v,x)^\dagger = (-x^{-2})^{\Delta_v} Y(\theta v, x^{-1}). \tag{1.11}$$

If we take $v = \nu$, then we obtain

$$L_n^\dagger = L_{-n} \quad (n \in \mathbb{Z}). \tag{1.12}$$

In particular, we have $L_0^\dagger = L_0$. This shows that different energy subspaces are orthogonal, i.e., *the grading $V = \bigoplus_{n \geqslant 0} V(n)$ is orthogonal under the inner product $\langle \cdot | \cdot \rangle$*.

1.2 Unitary representations of unitary VOAs

Definition 1.4. Let W_i be a complex vector space with grading $W_i = \bigoplus_{s \in \mathbb{R}} W_i(s)$. Assume $\dim W_i(s) < \infty$ for each $s \in \mathbb{R}$, and $\dim W_i(s) = 0$ for s sufficiently small. We say that W_i is a **representation of** V (or V**-module**), if the following conditions are satisfied:
(a) There is a linear map

$$\begin{gathered} V \to (\text{End } W_i)[[x^{\pm 1}]] \\ v \mapsto Y_i(v,x) = \sum_{n \in \mathbb{Z}} Y_i(v,n) x^{-n-1} \\ (\text{where } Y(v,n) \in \text{End } W_i), \end{gathered}$$

such that for any $w^{(i)} \in W_i$, $Y_i(v,n)w^{(i)} = 0$ for n sufficiently large. Y_i is called the **vertex operator** of W_i.
(b) (**Jacobi identity**) For any $u, v \in V$ and $m, n, h \in \mathbb{Z}$, we have

$$\begin{aligned} &\sum_{l \in \mathbb{Z}_{\geqslant 0}} \binom{m}{l} Y_i(Y(u, n+l)v, m+h-l) \\ =& \sum_{l \in \mathbb{Z}_{\geqslant 0}} (-1)^l \binom{n}{l} Y_i(u, m+n-l) Y_i(v, h+l) - \sum_{l \in \mathbb{Z}_{\geqslant 0}} (-1)^{l+n} \binom{n}{l} Y_i(v, n+h-l) Y_i(u, m+l). \end{aligned} \tag{1.13}$$

(c) $Y_i(\Omega, x) = \mathrm{id}_{W_i}$.
(d) The operators $L_n = Y_i(\nu, n+1)$ $(n \in \mathbb{Z})$ satisfy the Virasoro relation: $[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m,-n}c$, where c is the central charge of V.
(e) If $w^{(i)} \in W_i(s)$ then $L_0 w^{(i)} = s w^{(i)}$. s is called the **conformal weight** (or the **energy**) of $w^{(i)}$ and will be denoted by $\Delta_{w^{(i)}}$, and L_0 is called the **energy operator**.
(f) (**Translation property**) $\frac{d}{dx} Y_i(v,x) = Y_i(L_{-1}v, x)$.

Clearly V itself is a representation of V. We call it the **vacuum module** of V. Modules of V are denoted by $W_i, W_j, W_k, \ldots$, or simply $i, j, k, \ldots$. The vacuum module is sometimes denoted by 0. We let $\mathrm{id}_i = \mathrm{id}_{W_i}$ and $\mathrm{id}_0 = \mathrm{id}_V$ be the identity operators on W_i and V respectively.

A V**-module homomorphism** is, by definition, a linear map $\phi : W_i \to W_j$, such that for any $v \in V$ we have $\phi Y_i(v, x) = Y_j(v, x)\phi$. It is clear that ϕ preserves the gradings of W_i, W_j, for ϕ intertwines the actions of L_0 on these spaces. The vector space of homomorphisms $W_i \to W_j$ is denoted by $\mathrm{Hom}_V(W_i, W_j)$.

Remark 1.5. If the V-module W_i has a subspace W that is invariant under the action of V, then the restricted action of V on W produces a submodule of W_i. In fact, the only non-trivial thing to check is that W inherits the grading of W_i. But this follows from the fact that L_0, when restriced to W, is diagonalizable on W. (In general, if a linear operator of a complex vector space is diagonalizable, then by polynomial interpolations, it must also be diagonalizable on any invariant subspace.)

From the remark above, we see that a module W_i is irreducible if and only if the vector space W_i has no V-invariant subspace. If W_i is irreducible, we call

$$\Delta_i = \inf\{s : \dim W_i(s) > 0\}$$

the **conformal weight** of W_i. It is easy to show that $W_i = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} W_i(n + \Delta_i)$.

We now review the definition of contragredient modules introduced in [FHL93]. Let again W_i be a V-module. First we note that the dual space W_i^* of W_i has the grading $W_i^* = \prod_{s\in\mathbb{R}} W_i(s)^*$. Here $W_i(s)^*$ is the dual space of the finite dimensional vector space $W(s)$, and if $s \neq t$, the evaluations of $W_i(s)^*$ on $W_i(t)$ are set to be zero. Now we consider the subspace $W_{\bar{i}} \equiv W_i' = \bigoplus_{s\in\mathbb{R}} W(s)^*$ of W^*. We define the action of V on $W_{\bar{i}}$ as follows:

$$Y_{\bar{i}}(v, x) = Y_i(e^{xL_1}(-x^{-2})^{L_0}v, x^{-1})^{\mathrm{t}} \tag{1.14}$$

where the superscript "t" stands for the transpose operation. In other words, for any $w^{(\bar{i})} \in W_{\bar{i}} \subset W_i^*$ and $w^{(i)} \in W_i$, we have

$$\langle Y_{\bar{i}}(v, x)w^{(\bar{i})}, w^{(i)}\rangle = \langle w^{(\bar{i})}, Y_i(e^{xL_1}(-x^{-2})^{L_0}v, x^{-1})w^{(i)}\rangle. \tag{1.15}$$

We refer the reader to [FHL93] section 5.2 for a proof that $(W_{\bar{i}}, Y_{\bar{i}})$ is a representation of V. This representation is called the **contragredient module** of W_i.

In general, for each V-module W_i, the vector space $\widehat{W_i} = \prod_{s\in\mathbb{R}} W_i(s)$ is called the **algebraic completion** of W_i. The action Y_i of V on W_i can be clearly extended onto $\widehat{W_i}$. It is clear that $\widehat{W_i}$ can be identified with $W_{\bar{i}}^*$.

Equation (1.14) can be written in terms of modes: if $v \in V$ is a quasi-primary vector with conformal weight Δ_v, then

$$Y_{\bar{i}}(v, n) = \sum_{m\in\mathbb{Z}_{\geqslant 0}} \frac{(-1)^{\Delta_v}}{m!} Y_i(L_1^m v, -n - m - 2 + 2\Delta_v)^{\mathrm{t}}. \tag{1.16}$$

In particular, by letting $v = \nu$, we obtain $L_n^{\mathrm{t}} = L_{-n}$. More precisely, if $w^{(i)} \in W_i, w^{(\bar{i})} \in W_{\bar{i}}$, we have $\langle L_n w^{(i)}, w^{(\bar{i})} \rangle = \langle w^{(i)}, L_{-n} w^{(\bar{i})} \rangle$.

The contragredient operation is an involution: W_i is the contragredient module of $W_{\bar{i}}$. In particular, we have

$$Y_i(v,x) = Y_{\bar{i}}(e^{xL_1}(-x^{-2})^{L_0} v, x^{-1})^{\mathrm{t}}. \tag{1.17}$$

Hence we identify i with $\bar{\bar{i}}$, the contragredient module of $\bar{i}$.

Now we turn to the definition of unitary VOA modules.

Definition 1.6. Suppose that V is unitary and W_i is a V-module equipped with an inner product $\langle \cdot | \cdot \rangle$. We call W_i **unitary** if for any $v \in V$ we have

$$Y_i(v,x)^\dagger = Y_i(e^{xL_1}(-x^{-2})^{L_0} \theta v, x^{-1}). \tag{1.18}$$

In the remaining part of this section, we assume that V is unitary. Let W_i be a unitary V-module. Then formula (1.18), with $v = \nu$, implies that the action of the Virasoro subalgebras $\{L_n\}$ on W_i satisfies $L_n^\dagger = L_{-n}$. In particular, L_0 is symmetric, and hence the decompsition $W_i = \bigoplus_{s \in \mathbb{R}} W_i(s)$ is orthogonal. If we let P_s be the projection operator of W_i onto $W_i(s)$ (this operator can be defined whether W_i is unitary or not), we have $P_s^\dagger = P_s$.

Proposition 1.7 (Positive energy)**.** *If W_i is unitary, then we have the grading $W_i = \bigoplus_{s \geqslant 0} W_i(s)$. In particular, if W_i is irreducible, then $\Delta_i \geqslant 0$.*

Proof. We choose an arbitrary non-zero homogeneous vector $w^{(i)} \in W_i$ and show that $\Delta_{w^{(i)}} \geqslant 0$. First, assume that $w^{(i)}$ is **quasi-primary** (i.e., $L_1 w^{(i)} = 0$). Then we have

$$2\Delta_{w^{(i)}} \langle w^{(i)} | w^{(i)} \rangle = 2\langle L_0 w^{(i)} | w^{(i)} \rangle = \langle [L_1, L_{-1}] w^{(i)} | w^{(i)} \rangle = \| L_{-1} w^{(i)} \|^2 \geqslant 0,$$

which implies that $\Delta_{w^{(i)}} \geqslant 0$. In general, we may find $m \in \mathbb{Z}_{\geqslant 0}$ such that $L_1^m w^{(i)} \neq 0$, and $L_1^{m+1} w^{(i)} = 0$. So $\Delta_{L_1^m w^{(i)}} \geqslant 0$, and hence $\Delta_{w^{(i)}} = \Delta_{L_1^m w^{(i)}} + m \geqslant 0$. □

Proposition 1.8. *If W_i is unitary, then its contragredient module $W_{\bar{i}}$ is unitarizable.*

Proof. Assume that W_i has inner product $\langle \cdot | \cdot \rangle$. Define an anti-linear bijective map $C_i : W_i \to W_{\bar{i}}$ such that $\langle C_i w_1^{(i)}, w_2^{(i)} \rangle = \langle w_2^{(i)} | w_1^{(i)} \rangle$ for any $w_1^{(i)}, w_2^{(i)} \in W$. We simply write $C_i w^{(i)} = \overline{w^{(i)}}$. Now we may define the inner product on $W_{\bar{i}}$ such that C_i becomes antiunitary.

For any $v \in V$, we show that $Y_{\bar{i}}(v,x)$ satisfies equation (1.18). Note that for any $A \in \mathrm{End}(W_i)$, if A has a transpose $A^{\mathrm{t}} \in \mathrm{End}(W_{\bar{i}})$, then A also has a formal adjoint $A^\dagger \in \mathrm{End}(W)$, and it satisfies $A^\dagger = C_i^{-1} A^{\mathrm{t}} C_i$. Thus we have

$$\begin{aligned} &Y_{\bar{i}}(v,x) = Y_i(e^{xL_1}(-x^{-2})^{L_0} v, x^{-1})^{\mathrm{t}} \\ =&C_i Y_i(e^{xL_1}(-x^{-2})^{L_0} v, x^{-1})^\dagger C_i^{-1} = C_i Y_i(\theta v, x) C_i^{-1}, \end{aligned} \tag{1.19}$$

which implies that $Y_{\bar{i}}$ satisfies (1.18). □

From now on, if W_i is a unitary V-module, we fix an inner product on $W_{\bar{i}}$ to be the one constructed in the proof of proposition 1.8. We view $W_{\bar{i}}$ as a unitary V-module under this inner product.

Note that if we let $v = \nu$, then (1.19) implies that $L_n C_i = C_i L_n$ $(n \in \mathbb{Z})$.

Since we use W_0 (or simply 0) to denote the vacuum module V, it is natural to let C_0 represent the conjugation map from V onto its contragredient module $W_{\bar{0}} \equiv V'$. By equation (1.19) (with $i = 0$) and (1.8), we have:

Corollary 1.9. *$C_0\theta : V \to V'$ is a unitary V-module isomorphism.*

Therefore, we identify the vacuum module V with its contragredient module V'. This fact can be simply written as $\bar{0} = 0$. The operators θ and C_0 are also identified. The evaluation map $V \otimes V' \to \mathbb{C}$ is equivalent to the symmetric bilinear form $V \otimes V \to \mathbb{C}$ defined by $\langle v_1, v_2 \rangle = \langle v_1 | \theta v_2 \rangle$, where $v_1, v_2 \in V$.

Recall that we also identify W_i with $W_{\bar{\bar{i}}}$. It is easy to see that the anti-unitary map $C_{\bar{i}} : W_{\bar{i}} \to W_i = W_{\bar{\bar{i}}}$ satisfies $C_{\bar{i}} = C_i^{-1}$.

We now give a criterion for unitary V-modules. First, we say that V is **generated by** a subset E if V is spanned by vectors of the form $Y(v_1, n_1) \cdots Y(v_k, n_m)\Omega$ where $v_1, v_2, \ldots, v_m \in E$ and $n_1, \ldots, n_m \in \mathbb{Z}$. By the Jacoby identity (1.13) (with $m = 0$), any vertex operator Y_i is determined by its values on E.

Now we have a useful criterion for unitarity of V-modules.

Proposition 1.10. *If V is unitary, W_i is a V-module equipped with an inner product $\langle \cdot | \cdot \rangle$, E is a generating subset of V, and equation (1.18) holds under the inner product $\langle \cdot | \cdot \rangle$ for any $v \in E$, then W_i is a unitary V-module.*

Proof. For any $v \in V$ we define $\widetilde{Y}_i(v, x) = Y_i(e^{xL_1}(-x^{-2})^{L_0}\theta v, x^{-1})^\dagger$. As in the proof of proposition 1.8, we have $\widetilde{Y}_i(v, x) = C_i^{-1} Y_{\bar{i}}(\theta v, x) C_i$. It follows that $\widetilde{Y}_i$ satisfies the Jacobi identity. Since Y_i also satisfies the Jacobi identity, and since $Y_i(v, x) = \widetilde{Y}_i(v, x)$ for any $v \in E$, we must have $Y_i(v, x) = \widetilde{Y}_i(v, x)$ for all $v \in V$, which proves that W_i is unitary. $\square$

1.3 Intertwining operators of unitary VOAs

Definition 1.11. Let W_i, W_j, W_k be V-modules. A type $\binom{W_k}{W_i W_j}$ (or type $\binom{k}{i\ j}$) **intertwining operator** $\mathcal{Y}_\alpha$ is a linear map

$$
\begin{gathered}
W_i \to (\mathrm{Hom}(W_j, W_k))\{x\}, \\
w^{(i)} \mapsto \mathcal{Y}_\alpha(w^{(i)}, x) = \sum_{s \in \mathbb{R}} \mathcal{Y}_\alpha(w^{(i)}, s) x^{-s-1} \\
(\text{where } \mathcal{Y}_\alpha(w^{(i)}, s) \in \mathrm{Hom}(W_j, W_k)),
\end{gathered}
$$

such that:

(a) For any $w^{(j)} \in W_j$, $\mathcal{Y}_\alpha(w^{(i)}, s) w^{(j)} = 0$ for s sufficiently large.

(b) (**Jacobi identity**) For any $u\in V, w^{(i)}\in W_i, m,n\in\mathbb{Z}, s\in\mathbb{R}$, we have

$$\begin{aligned}&\sum_{l\in\mathbb{Z}_{\geqslant 0}}\binom{m}{l}\mathcal{Y}_\alpha\big(Y_i(u,n+l)w^{(i)},m+s-l\big)\\ =&\sum_{l\in\mathbb{Z}_{\geqslant 0}}(-1)^l\binom{n}{l}Y_k(u,m+n-l)\mathcal{Y}_\alpha(w^{(i)},s+l)\\ &-\sum_{l\in\mathbb{Z}_{\geqslant 0}}(-1)^{l+n}\binom{n}{l}\mathcal{Y}_\alpha(w^{(i)},n+s-l)Y_j(u,m+l).\end{aligned}\tag{1.20}$$

(c) (**Translation property**) $\frac{d}{dx}\mathcal{Y}_\alpha(w^{(i)},x)=\mathcal{Y}_\alpha(L_{-1}w^{(i)},x)$.

Intertwining operators will be denoted by $\mathcal{Y}_\alpha,\mathcal{Y}_\beta,\mathcal{Y}_\gamma,\dots$, or just $\alpha,\beta,\gamma,\dots$.

Note that if we let $n=0$ and $m=0$ respectively, (1.20) becomes:

$$\sum_{l\geqslant 0}\binom{m}{l}\mathcal{Y}_\alpha(Y_i(u,l)w^{(i)},m+s-l)=Y_k(u,m)\mathcal{Y}_\alpha(w^{(i)},s)-\mathcal{Y}_\alpha(w^{(i)},s)Y_j(u,m),\tag{1.21}$$

$$\begin{aligned}&\mathcal{Y}_\alpha(Y_i(u,n)w^{(i)},s)\\ =&\sum_{l\geqslant 0}(-1)^l\binom{n}{l}Y_k(u,n-l)\mathcal{Y}_\alpha(w^{(i)},s+l)-\sum_{l\geqslant 0}(-1)^{l+n}\binom{n}{l}\mathcal{Y}_\alpha(w^{(i)},n+s-l)Y_j(u,l).\end{aligned}\tag{1.22}$$

In particular, if we let $u=\nu$ and $m=0,1$ respectively, the first equation implies that

$$[L_{-1},\mathcal{Y}_\alpha(w^{(i)},x)]=\mathcal{Y}_\alpha(L_{-1}w^{(i)},x)=\frac{d}{dx}\mathcal{Y}_\alpha(w^{(i)},x);\tag{1.23}$$

$$[L_0,\mathcal{Y}_\alpha(w^{(i)},x)]=\mathcal{Y}_\alpha(L_0w^{(i)},x)+\frac{d}{dx}\mathcal{Y}_\alpha(w^{(i)},x).\tag{1.24}$$

The second equation is equivalent to that

$$[L_0,\mathcal{Y}_\alpha(w^{(i)},s)]=(-s-1+\Delta_{w^{(i)}})\mathcal{Y}_\alpha(w^{(i)},s)\quad\text{if } w^{(i)} \text{ is homogeneous.}\tag{1.25}$$

Hence $\mathcal{Y}_\alpha(w^{(i)},s)$ raises the energy by $-s-1+\Delta_{w^{(i)}}$. Equation (1.25) implies the relation

$$z^{L_0}\mathcal{Y}_\alpha(w^{(i)},x)z^{-L_0}=\mathcal{Y}_\alpha(z^{L_0}w^{(i)},zx)\tag{1.26}$$

(cf. [FHL93] section 5.4), where z is either a non-zero complex number, or a formal variable which commutes with and is independent of x. In the former case, we need to assign to z an argument, i.e., a real number $\arg z$ such that $z=|z|e^{i\arg z}$. Then, for any $s\in\mathbb{R}$, we let $z^s=|z|^se^{is\arg z}$, i.e., we let the argument of z^s be $s\arg z$.

Convention 1.12. In this article, unless otherwise stated, we make the following assumptions:
(1) If $t\in\mathbb{R}$ then $\arg e^{it}=t$.
(2) If $z\in\mathbb{C}^\times$ with argument $\arg z$, then $\arg\overline{z}=-\arg z$. If $s\in\mathbb{R}$, then $\arg(z^s)=s\arg z$.
(3) If $z_1,z_2\in\mathbb{C}^\times$ with arguments $\arg z_1$ and $\arg z_2$ respectively, then $\arg(z_1z_2)=\arg z_1+\arg z_2$.

Definition 1.13. Let U be an open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}^\times$ be a continuous function. Suppose that $z_1, z_2 \in U$, and for any $t \in [0,1]$, $tz_1 + (1-t)z_2 \in U$. Then we say that the argument $\arg f(z_2)$ **is close to** $\arg f(z_1)$ **as** $z_2 \to z_1$, if there exists a (unique) continuous function $A : [0,1] \to \mathbb{R}$, such that $A(0) = \arg z_1$, $A(1) = \arg z_2$, and that for any $t \in [0,1]$, $A(t)$ is an argument of $f(tz_1 + (1-t)z_2)$.

Let $\mathcal{V}\binom{k}{i\,j}$ be the vector space of type $\binom{k}{i\,j}$ intertwining operators. If $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\,j}$, we say that W_i, W_j and W_k are the **charge space**, the **source space**, and the **target space** of $\mathcal{Y}_\alpha$ respectively. We say that $\mathcal{Y}_\alpha$ is **irreducible** if W_i, W_j, W_k are irreducible V-modules. If $\mathcal{Y}_\alpha$ is irreducible, then by (1.25), it is easy to see that $\mathcal{Y}(w^{(i)}, s) = 0$ except possibly when $s \in \Delta_i + \Delta_j - \Delta_k + \mathbb{Z}$. If V is unitary, and W_i, W_j, W_k are unitary V-modules, then we say that $\mathcal{Y}_\alpha$ is **unitary**.

We have several ways to construct new intertwining operators from old ones. First, for any $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\,j}$, we define its **contragredient intertwining operator** (cf. [FHL93]) $C\mathcal{Y}_\alpha \equiv \mathcal{Y}_{C\alpha} \in \mathcal{V}\binom{\overline{j}}{i\,\overline{k}}$ by letting

$$\mathcal{Y}_{C\alpha}(w^{(i)}, x) = \mathcal{Y}_\alpha(e^{xL_1}(e^{-i\pi}x^{-2})^{L_0}w^{(i)}, x^{-1})^{\mathrm{t}}, \quad w^{(i)} \in W_i. \tag{1.27}$$

In other words, if $w^{(j)} \in W_j$ and $w^{(\overline{k})} \in W_{\overline{k}}$, then

$$\langle \mathcal{Y}_{C\alpha}(w^{(i)}, x)w^{(\overline{k})}, w^{(j)}\rangle = \langle w^{(\overline{k})}, \mathcal{Y}_\alpha(e^{xL_1}(e^{-i\pi}x^{-2})^{L_0}w^{(i)}, x^{-1})w^{(j)}\rangle. \tag{1.28}$$

We also define, for each $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\,j}$, an intertwining operator $C^{-1}\mathcal{Y}_\alpha \equiv \mathcal{Y}_{C^{-1}\alpha} \in \mathcal{V}\binom{\overline{j}}{i\,\overline{k}}$ such that

$$\mathcal{Y}_{C^{-1}\alpha}(w^{(i)}, x) = \mathcal{Y}_\alpha(e^{xL_1}(e^{i\pi}x^{-2})^{L_0}w^{(i)}, x^{-1})^{\mathrm{t}}, \quad w^{(i)} \in W_i. \tag{1.29}$$

One can show that $C^{-1}C\alpha = CC^{-1}\alpha = \alpha$. (To prove this, we first show that $(xL_1)x_0^{L_0} = x_0^{L_0}(xx_0L_1)$ by checking this relation on any homogeneous vector. We then show that

$$e^{xL_1}x_0^{L_0} = x_0^{L_0}e^{xx_0L_1}, \tag{1.30}$$

where x, x_0 are independent commuting formal variables. Finally, we may use (1.30) to prove the desired relation.)

We now define, for any $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\,j}$, a pair of **braided intertwining operators** (cf. [FHL93]) $B_\pm\mathcal{Y}_\alpha \equiv \mathcal{Y}_{B_\pm\alpha} \in \mathcal{V}\binom{k}{j\,i}$ in the following way: If $w^{(i)} \in W_i, w^{(j)} \in W_j$, then

$$\mathcal{Y}_{B_+\alpha}(w^{(j)}, x)w^{(i)} = e^{xL_{-1}}\mathcal{Y}_\alpha(w^{(i)}, e^{i\pi}x)w^{(j)}, \tag{1.31}$$
$$\mathcal{Y}_{B_-\alpha}(w^{(j)}, x)w^{(i)} = e^{xL_{-1}}\mathcal{Y}_\alpha(w^{(i)}, e^{-i\pi}x)w^{(j)}. \tag{1.32}$$

It's easy to see that $B_\mp$ is the inverse operation of $B_\pm$. We refer the reader to [FHL93] chapter 5 for a proof that contragredient intertwining operators and braided intertwining operators satisfy the Jacobi identity.

In the remaining part of this section, we assume that V is unitary. Let W_i, W_j, W_k be unitary V-modules with conjugation maps $C_i : W_i \to W_{\bar{i}}, C_j : W_j \to W_{\bar{j}}, C_k : W_k \to W_{\bar{k}}$ respectively. Given $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$, we define its **conjugate intertwining operator** $\overline{\mathcal{Y}_\alpha} \equiv \mathcal{Y}_{\overline{\alpha}} \in \mathcal{V}\binom{\bar{k}}{\bar{i}\ \bar{j}}$ by setting

$$\mathcal{Y}_{\overline{\alpha}}(\overline{w^{(i)}}, x) = C_k \mathcal{Y}_\alpha(w^{(i)}, x) C_j^{-1}, \quad w^{(i)} \in W_i. \tag{1.33}$$

It is clear that $\mathcal{Y}_{\overline{\alpha}}$ satisfies the Jacobi identity.

For any $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$, it is easy to check that

$$\mathcal{Y}_{\overline{B_\pm \alpha}} = \mathcal{Y}_{B_\mp \overline{\alpha}}, \qquad \mathcal{Y}_{\overline{C^{\pm 1}\alpha}} = \mathcal{Y}_{C^{\mp 1}\overline{\alpha}}.$$

We define $\mathcal{Y}_\alpha^\dagger \equiv \mathcal{Y}_{\alpha^*} = \mathcal{Y}_{\overline{C\alpha}} \in \mathcal{V}\binom{j}{\bar{i}\ k}$ and call it the **adjoint intertwining operator** of $\mathcal{Y}_\alpha$. One can easily check, for any $w^{(i)} \in W_i$, that

$$\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, x) = \mathcal{Y}_\alpha(e^{xL_1}(e^{-i\pi}x^{-2})^{L_0} w^{(i)}, x^{-1})^\dagger. \tag{1.34}$$

where the symbol $\dagger$ on the right hand side means the formal adjoint. In other words, for any $w^{(j)} \in W_j, j, w^{(k)} \in W_k$, we have

$$\langle \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, x) w^{(k)} | w^{(j)} \rangle = \langle w^{(k)} | \mathcal{Y}_\alpha(e^{xL_1}(e^{-i\pi}x^{-2})^{L_0} w^{(i)}, x^{-1}) w^{(j)} \rangle. \tag{1.35}$$

If $w^{(i)}$ is homogeneous, we can write (1.34) in terms of modes:

$$\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, s) = \sum_{m \in \mathbb{Z}_{\geqslant 0}} \frac{e^{i\pi\Delta_{w^{(i)}}}}{m!} \mathcal{Y}(L_1^m w^{(i)}, -s - m - 2 + 2\Delta_{w^{(i)}})^\dagger \tag{1.36}$$

for all $s \in \mathbb{R}$.

It is also obvious that the adjoint operation is an involution, i.e., $\mathcal{Y}_{\alpha^{**}} = \mathcal{Y}_\alpha$. Hence $* : \mathcal{V}\binom{k}{i\ j} \to \mathcal{V}\binom{j}{\bar{i}\ k}$ is an antiunitary map.

We define the cardinal number N_{ij}^k to be the dimension of the vector space $\mathcal{V}\binom{k}{i\ j}$. N_{ij}^k is called a **fusion rule** of V. The above constructions of intertwining operators imply the following:

$$N_{ij}^k = N_{i\bar{k}}^{\bar{j}} = N_{ji}^k = N_{\bar{i}\,\bar{j}}^{\bar{k}} = N_{\bar{i}k}^j. \tag{1.37}$$

We now construct several intertwining operators related to a given V-module W_i. First, note that $Y_i \in \mathcal{V}\binom{i}{0\ i}$. It is obvious that $B_+ Y_i = B_- Y_i \in \mathcal{V}\binom{i}{i\ 0}$. We define $\mathcal{Y}_{i0}^i = B_\pm Y_i$ and call it the **creation operator** of W_i. Using the definition of $B_\pm$, we have, for any $w^{(i)} \in W_i, v \in V$,

$$\mathcal{Y}_{i0}^i(w^{(i)}, x)v = e^{xL_{-1}} Y_i(v, -x) w^{(i)}. \tag{1.38}$$

In particular, we have

$$\mathcal{Y}^i_{i0}(w^{(i)},x)\Omega = e^{xL_{-1}}w^{(i)}. \tag{1.39}$$

We define $\mathcal{Y}^0_{i\bar{i}} := C^{-1}\mathcal{Y}^i_{i0} = C^{-1}B_\pm Y_i \in \mathcal{V}\binom{0}{i\,\bar{i}}$. Thus for any $w^{(i)}_1 \in W_i$ and $w^{(\bar{i})}_2 \in W_{\bar{i}}$, we may use (1.39) and (1.30) to compute that

$$\begin{aligned}
\langle \mathcal{Y}^0_{i\bar{i}}(w^{(i)}_1,x)w^{(\bar{i})}_2,\Omega\rangle =&\langle w^{(\bar{i})}_2, \mathcal{Y}^i_{i0}(e^{xL_1}(e^{i\pi}x^{-2})^{L_0}w^{(i)}_1,x^{-1})\Omega\rangle\\
=&\langle w^{(\bar{i})}_2, e^{x^{-1}L_{-1}}e^{xL_1}(e^{i\pi}x^{-2})^{L_0}w^{(i)}_1\rangle\\
=&\langle e^{x^{-1}L_1}w^{(\bar{i})}_2, e^{xL_1}(e^{i\pi}x^{-2})^{L_0}w^{(i)}_1\rangle\\
=&\langle e^{x^{-1}L_1}w^{(\bar{i})}_2, (e^{i\pi}x^{-2})^{L_0}e^{-x^{-1}L_1}w^{(i)}_1\rangle.
\end{aligned} \tag{1.40}$$

Note that by (1.14), $Y_{\bar{i}} = C^{\pm 1}Y_i \in \mathcal{V}\binom{\bar{i}}{0\,\bar{i}}$. $\mathcal{Y}^0_{\bar{i}i} = C^{-1}B_\pm Y_{\bar{i}}$ is called the **annihilation operator** of W_i.

Define $\vartheta_i \in \mathrm{End}_V(W_i)$ by setting $\vartheta_i = e^{2i\pi L_0}$. That ϑ_i is a V-module homomorphism follows from (1.26). ϑ_i is called the **twist** of W_i. Then the intertwining operators $\mathcal{Y}^0_{i\bar{i}}$ and $\mathcal{Y}^0_{\bar{i}i}$ can be related in the following way:

Proposition 1.14.

$$\mathcal{Y}^0_{i\bar{i}}(w^{(i)},x) = (B_+\mathcal{Y}^0_{\bar{i}i})(\vartheta_i w^{(i)},x) = (B_-\mathcal{Y}^0_{\bar{i}i})(\vartheta_i^{-1}w^{(i)},x), \tag{1.41}$$

$$\mathcal{Y}^0_{i\bar{i}}(w^{(i)},x) = (B_+\mathcal{Y}^0_{\bar{i}i})(w^{(i)},x)\vartheta_i = (B_-\mathcal{Y}^0_{\bar{i}i})(w^{(i)},x)\vartheta_i^{-1}. \tag{1.42}$$

Proof. Using equations (1.30), (1.40), and that $L_1\Omega = 0$, we see that for any $w^{(i)}_1 \in W_i, w^{(\bar{i})}_2 \in W_{\bar{i}}$,

$$\begin{aligned}
&\langle (B_\pm\mathcal{Y}^0_{\bar{i}i})(\vartheta_i^{\pm 1}w^{(i)}_1,x)w^{(\bar{i})}_2,\Omega\rangle\\
=&\langle \mathcal{Y}^0_{\bar{i}i}(w^{(\bar{i})}_2,e^{\pm i\pi}x)e^{\pm 2i\pi L_0}w^{(i)}_1,\Omega\rangle\\
=&\langle e^{-x^{-1}L_1}e^{\pm 2i\pi L_0}w^{(i)}_1,(e^{i\pi\mp 2i\pi}x^{-2})^{L_0}e^{x^{-1}L_1}w^{(\bar{i})}_2\rangle\\
=&\langle e^{\pm 2i\pi L_0}e^{-x^{-1}L_1}w^{(i)}_1,(e^{i\pi\mp 2i\pi}x^{-2})^{L_0}e^{x^{-1}L_1}w^{(\bar{i})}_2\rangle\\
=&\langle (e^{i\pi}x^{-2})^{L_0}e^{-x^{-1}L_1}w^{(i)}_1,e^{x^{-1}L_1}w^{(\bar{i})}_2\rangle\\
=&\langle \mathcal{Y}^0_{i\bar{i}}(w^{(i)}_1,x)w^{(\bar{i})}_2,\Omega\rangle.
\end{aligned} \tag{1.43}$$

Since V is of CFT type and isomorphic to V' as a V-module, V is a simple VOA, i.e., V is an irreducible V-module (cf., for example, [CKLW15] proposition 4.6-(iv)). Hence Ω is a cyclic vector in V. By (1.21), we have $\langle (B_\pm\mathcal{Y}^0_{\bar{i}i})(\vartheta_i^{\pm 1}w^{(i)}_1,x)w^{(\bar{i})}_2,v\rangle = \langle \mathcal{Y}^0_{i\bar{i}}(w^{(i)}_1,x)w^{(\bar{i})}_2,v\rangle$ for any $v \in V$, which proves (1.41). (1.42) can be proved in a similar way. $\square$

When W_i is unitary, we also have

$$\mathcal{Y}^0_{\bar{i}i} = (\mathcal{Y}^i_{i0})^\dagger. \tag{1.44}$$

Indeed, by (1.19), $Y_{\bar{i}} = \overline{Y_i}$. Hence

$$\mathcal{Y}^0_{\bar{i}i} = C^{-1}B_\pm\overline{Y_i} = \overline{CB_\mp Y_i} = (B_\mp Y_i)^\dagger = (\mathcal{Y}^i_{i0})^\dagger.$$

2 Braiding and fusion of intertwining operators

Starting from this chapter, we assume that V satisfies conditions (0.4), (0.5), and (0.6). Recall that, by corollary 1.9, a unitary VOA automatically satisfies condition (0.4).

By [Hua05a] theorem 3.5, *the fusion rules of V are finite numbers, and there are only finitely many equivalence classes of irreducible V-modules.* Let us choose, for each equivalence class $[W_k]$ of irreducible V-module, a representing element W_k, and let these modules form a finite set $\{W_k : k \in \mathcal{E}\}$. (With abuse of notations, we also let $\mathcal{E}$ denote this finite set.) In other words, $\mathcal{E}$ is a complete list of mutually inequivalent irreducible V-modules. We also require that V is inside $\mathcal{E}$. If, moreover, V is unitary, then for any unitarizable W_k ($k \in \mathcal{E}$), we fix a unitary structure on W_k. The unitary structure on V is the standard one. We let $\mathcal{E}^{\mathrm{u}}$ be the set of all unitary V-modules in $\mathcal{E}$.

Let W_i, W_j, W_k be V-modules. Then Θ^k_{ij} will always denote (the index set of) a basis $\{\mathcal{Y}_\alpha : \alpha \in \Theta^k_{ij}\}$ of the vector space $\mathcal{V}\binom{k}{i\,j}$. If bases of the vector spaces of intertwining operators are chosen, then for any W_i, W_k, we set $\Theta^k_{i*} = \coprod_{j\in\mathcal{E}} \Theta^k_{ij}$. The notations $\Theta^k_{*j}, \Theta^*_{ij}$ are understood in a similar way.

2.1 Genus 0 correlation functions

In this section, we review the construction of genus 0 correlation functions from intertwining operators. We first give a complex analytic point of view of intertwining operators. Let $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\,j}$. For any $w^{(i)} \in W_i, w^{(j)} \in W_j, w^{(\overline{k})} \in W_{\overline{k}}$,

$$\langle \mathcal{Y}_\alpha(w^{(i)}, z)w^{(j)}, w^{(\overline{k})}\rangle = \langle \mathcal{Y}_\alpha(w^{(i)}, x)w^{(j)}, w^{(\overline{k})}\rangle\big|_{x=z} = \sum_{s\in\mathbb{R}} \langle \mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}, w^{(\overline{k})}\rangle z^{-s-1} \tag{2.1}$$

is a finite sum of powers of z. (Indeed, if all the vectors are homogeneous then, by (1.25), the coefficient before each z^{-s-1} is zero, except when $s = \Delta_{w^{(i)}} + \Delta_{w^{(j)}} - \Delta_{w^{(k)}} - 1$.) Since the powers of z are not necessarily integers, (2.1) is a *multivalued holomorphic function* defined for $z \in \mathbb{C}^\times = \mathbb{C}\backslash\{0\}$: the exact value of (2.1) depends not only on z, but also on $\arg z$. We can also regard $\mathcal{Y}_\alpha$ as a multivalued $(W_i \otimes W_j \otimes W_{\overline{k}})^*$-valued holomorphic function on $\mathbb{C}^\times$. Note that by proposition A.1, the transition from the formal series viewpoint to the complex analytic one is faithful.

Convention 2.1. At this point, the notations $\mathcal{Y}_\alpha(w^{(i)}, x)$, $\mathcal{Y}_\alpha(w^{(i)}, z)$, and $\mathcal{Y}_\alpha(w^{(i)}, s)$ seem confusing. We clarify their meanings as follows.

Unless otherwise stated, $\mathcal{Y}_\alpha(w^{(i)}, x)$ is a formal series of the formal variable x. If $z \neq 0$ is a complex number, or if z is a complex variable (possibly taking real values), $\mathcal{Y}_\alpha(w^{(i)}, z)$ is defined by (2.1). If s is a real number, $\mathcal{Y}_\alpha(w^{(i)}, s)$ is a mode of $\mathcal{Y}_\alpha(w^{(i)}, x)$, i.e., the coefficient before x^{-s-1} in $\mathcal{Y}_\alpha(w^{(i)}, x)$.

Intertwining operators are also called 3-point (correlation) functions. In [Hua05a], Y. Z. Huang constructed general n-point functions by taking the products of intertwining operators. His approach can be sketched as follows:

For any $n = 1, 2, 3, \ldots$, we define the **configuration space** $\mathrm{Conf}_n(\mathbb{C}^\times)$ to be the complex sub-manifold of $(\mathbb{C}^\times)^n$ whose points are $(z_1, z_2, \ldots, z_n) \in \mathrm{Conf}_n(\mathbb{C}^\times)$ satisfying that $z_m \neq z_l$ whenever $1 \leqslant m < l \leqslant n$. We let $\widetilde{\mathrm{Conf}}_n(\mathbb{C}^\times)$ be the universal covering space of $\mathrm{Conf}_n(\mathbb{C}^\times)$.

Let $\mathcal{Y}_{\alpha_1}, \mathcal{Y}_{\alpha_2}, \ldots, \mathcal{Y}_{\alpha_n}$ be intertwining operators V. We say that they form a **chain of intertwining operators**, if for each $1 \leqslant m \leqslant n-1$, the target space of $\mathcal{Y}_{\alpha_m}$ equals the source space of $\mathcal{Y}_{\alpha_{m+1}}$. The following theorem was proved by Huang.

Theorem 2.2 (cf. [Hua05a] theorem 3.5). *Suppose that $\mathcal{Y}_{\alpha_1}, \ldots, \mathcal{Y}_{\alpha_n}$ form a chain of intertwining operators. For each $1 \leqslant m \leqslant n$, we let W_{i_m} be the charge space of $\mathcal{Y}_{\alpha_m}$. We let W_{i_0} be the source space of $\mathcal{Y}_{\alpha_1}$, and let W_k be the target space of $\mathcal{Y}_{\alpha_n}$. Then for any $w^{(i_0)} \in W_{i_0}, w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_n)} \in W_{i_n}, w^{(\overline{k})} \in W_{\overline{k}}$, and $z_1, z_2, \ldots, z_n \in \mathbb{C}$ such that $0 < |z_1| < |z_2| < \cdots < |z_n|$, the expression*

$$\langle \mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1}) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1)w^{(i_0)}, w^{(\overline{k})}\rangle \tag{2.2}$$

***converges absolutely**, which means that the series*

$$\sum_{s_1, s_2, \ldots, s_{n-1} \in \mathbb{R}} \big|\langle \mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)P_{s_{n-1}}\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1})P_{s_{n-2}} \\ \cdots P_{s_1}\mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1)w^{(i_0)}, w^{(\overline{k})}\rangle\big| \tag{2.3}$$

converges, where each P_{s_m} ($1 \leqslant m \leqslant n-1$) is the projection of the target space of $\mathcal{Y}_{\alpha_m}$ onto its weight-s_m component.

Note that (2.2) also **converges absolutely and locally uniformly**, which means that there exists a neighborhood $U \subset \mathrm{Conf}_n(\mathbb{C}^\times)$ of $(z_1, z_2, \ldots, z_n)$, and a finite number $M > 0$, such that for any $(\zeta_1, \zeta_2, \ldots, \zeta_n) \in U$, (2.3) is bounded by M if we replace each $z_1, z_2, \ldots$ with $\zeta_1, \zeta_2, \ldots$ in that expression.

To see this, we assume, without loss of generality, that all the vectors in (2.2) are homogeneous, and that all the intertwining operators are irreducible. Consider a new set of coordinates $\omega_1, \omega_2, \ldots, \omega_n$ such that $z_m = \omega_m\omega_{m+1}\cdots\omega_n$ ($1 \leqslant m \leqslant n$). Then the condition that $0 < |z_1| < |z_2| < \cdots < |z_n|$ is equivalent to that $0 < |\omega_1| < 1, \ldots, 0 < |\omega_{n-1}| < 1, 0 < |\omega_n|$. By (1.26), expression (2.2) as a formal series also equals

$$\begin{aligned}
&\Big\langle \mathcal{Y}_{\alpha_n}\big(w^{(i_n)}, \omega_n\big)\mathcal{Y}_{\alpha_{n-1}}\big(w^{(i_{n-1})}, \omega_{n-1}\omega_n\big) \cdots \mathcal{Y}_{\alpha_1}\big(w^{(i_1)}, \omega_1\omega_2\cdots\omega_n\big)w^{(i_0)}, w^{(\overline{k})}\Big\rangle \\
=&\Big\langle \omega_n^{L_0}\mathcal{Y}_{\alpha_n}\big(\omega_n^{-L_0}w^{(i_n)}, 1\big)\omega_{n-1}^{L_0}\mathcal{Y}_{\alpha_{n-1}}\big((\omega_{n-1}\omega_n)^{-L_0}w^{(i_{n-1})}, 1\big) \cdots \\
&\cdot \omega_1^{L_0}\mathcal{Y}_{\alpha_1}\big((\omega_1\omega_2\cdots\omega_n)^{-L_0}w^{(i_1)}, 1\big)(\omega_1\omega_2\cdots\omega_n)^{-L_0}w^{(i_0)}, w^{(\overline{k})}\Big\rangle \\
=&\Big\langle \omega_n^{L_0}\mathcal{Y}_{\alpha_n}\big(w^{(i_n)}, 1\big)\omega_{n-1}^{L_0}\mathcal{Y}_{\alpha_{n-1}}\big(w^{(i_{n-1})}, 1\big) \cdots \omega_1^{L_0}\mathcal{Y}_{\alpha_1}\big(w^{(i_1)}, 1\big)w^{(i_0)}, w^{(\overline{k})}\Big\rangle \\
&\cdot \prod_{1 \leqslant m \leqslant n} \omega_m^{-\left(\Delta_{w^{(i_0)}} + \Delta_{w^{(i_1)}} + \cdots + \Delta_{w^{(i_m)}}\right)},
\end{aligned} \tag{2.4}$$

where $\mathcal{Y}_{\alpha_m}(w^{(i_m)}, 1) = \mathcal{Y}_{\alpha_m}(w^{(i_m)}, x)\big|_{x=1}$. Since the target space of each $\mathcal{Y}_{\alpha_m}$ is irreducible, (2.4) is a **quasi power series of** $\omega_1, \ldots, \omega_n$ (i.e., a power series of $\omega_1, \ldots, \omega_n$ multiplied by a monomial $\omega_1^{s_1}\cdots\omega_n^{s_n}$, where $s_1,\ldots,s_n\in\mathbb{C}$), and the convergence of (2.3) is equivalent to the absolute convergence of the quasi power series (2.4). Therefore, pointwise absolute convergence implies locally uniform absolute convergence.

We see that (2.2) is a multi-valued holomorphic function defined when $0<|z_1|<\cdots<|z_n|$. We let φ be the multi-valued $(W_{i_0}\otimes W_{i_1}\otimes\cdots\otimes W_{i_n}\otimes W_{\overline{k}})^*$-valued holomorphic function on $\{0<|z_1|<\cdots<|z_n|\}$ defined by (2.2). φ is called an $(n+2)$**-point (correlation) function**[14] of V, and is denoted by $\mathcal{Y}_{\alpha_n}\mathcal{Y}_{\alpha_{n-1}}\cdots\mathcal{Y}_{\alpha_1}$. We define $\mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_0}$ to be the vector space of $(W_{i_0}\otimes W_{i_1}\otimes\cdots\otimes W_{i_n}\otimes W_{\overline{k}})^*$-valued $n+2$-point functions of V. The following proposition can be used to find a basis of $\mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_0}$.

Proposition 2.3. *Define a linear map* Φ :

$$\bigoplus_{j_1,\ldots,j_{n-1}\in\mathcal{E}}\left(\mathcal{V}\binom{k}{i_n\ j_{n-1}}\otimes\mathcal{V}\binom{j_{n-1}}{i_{n-1}\ j_{n-2}}\otimes\mathcal{V}\binom{j_{n-2}}{i_{n-2}\ j_{n-3}}\otimes\cdots\otimes\mathcal{V}\binom{j_1}{i_1\ i_0}\right)$$
$$\to\mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_0},$$
$$\mathcal{Y}_{\alpha_n}\otimes\mathcal{Y}_{\alpha_{n-1}}\otimes\mathcal{Y}_{\alpha_{n-2}}\otimes\cdots\otimes\mathcal{Y}_{\alpha_1}\mapsto\mathcal{Y}_{\alpha_n}\mathcal{Y}_{\alpha_{n-1}}\mathcal{Y}_{\alpha_{n-2}}\cdots\mathcal{Y}_{\alpha_1}.$$

Then Φ *is an isomorphism.*

Therefore, if elements in $\{\mathcal{Y}_{\alpha_1}\},\ldots,\{\mathcal{Y}_{\alpha_n}\}$ are linearly independent respectively, then the correlation functions $\{\mathcal{Y}_{\alpha_n}\mathcal{Y}_{\alpha_{n-1}}\cdots\mathcal{Y}_{\alpha_1}\}$ are also linearly independent. The proof of this proposition is postponed to section A.2.

It was also shown in [Hua05a] that correlations functions satisfy a system of linear differential equations, the coefficients of which are holomorphic functions defined on $\mathrm{Conf}_n(\mathbb{C}^\times)$. More precisely, we have the following:

Theorem 2.4 (cf. [Hua05a], especially theorem 1.6)**.** *For any* $w^{(i_0)}\in W_{i_0},w^{(i_1)}\in W_{i_1},\ldots,w^{(i_n)}\in W_{i_n},w^{(\overline{k})}\in W_{\overline{k}}$*, there exist* $h_1,\ldots,h_n\in\mathbb{Z}_{\geqslant0}$*, and single-valued holomorphic functions* $a_{1,m}(z_1,\ldots,z_n),a_{2,m}(z_1,\ldots,z_n),\ldots,a_{h_m,m}(z_1,\ldots,z_n)$ *on* $\mathrm{Conf}_n(\mathbb{C}^\times)$*, such that for any* $(W_{i_0}\otimes W_{i_1}\otimes\cdots\otimes W_{i_n}\otimes W_{\overline{k}})^*$*-valued* $(n+2)$*-point correlation function* φ *defined on* $\{0<|z_1|<\cdots<|z_n|\}$*, the function* $\varphi(w^{(i_0)},w^{(i_1)},\ldots,w^{(i_n)},w^{(\overline{k})};z_1,z_2,\ldots,z_n)$ *of* $(z_1,\ldots,z_n)$ *satisfies the following system of differential equations:*

$$\frac{\partial^{h_m}\varphi}{\partial z_m^{h_m}}+a_{1,m}\frac{\partial^{h_m-1}\varphi}{\partial z_m^{h_m-1}}+a_{2,m}\frac{\partial^{h_m-2}\varphi}{\partial z_m^{h_m-2}}+\cdots+a_{h_m,m}\varphi=0\qquad(m=1,\ldots,n).\tag{2.5}$$

Hence, due to elementary ODE theory, φ can be analytically continued to a multivalued holomorphic function on $\mathrm{Conf}_n(\mathbb{C}^\times)$ (or equivalently, a single-valued holomorphic function on $\widetilde{\mathrm{Conf}}_n(\mathbb{C}^\times)$), which satisfies system (2.5) globally.

[14] So far our definition of genus 0 correlation functions is local. We will give a global definition at the end of next section.

Note that *(global) correlation functions are determined by their values at any fixed point in* $\widetilde{\mathrm{Conf}}_n(\mathbb{C}^\times)$. Indeed, since φ satisfies (2.5), the function φ is determined by the values of $\{\frac{\partial^l}{\partial z_m^l}\varphi : 1 \leqslant m \leqslant n, 0 \leqslant l \leqslant h_m - 1\}$ at any fixed point. On the other hand, by translation property and the locally uniform absolute convergence of (2.2), we have

$$\begin{aligned}&\frac{\partial}{\partial z_m}\varphi(w^{(i_0)}, w^{(i_1)}, \dots, w^{(i_n)}, w^{(\overline{k})}; z_1, z_2, \dots, z_n)\\ =&\varphi(w^{(i_0)}, w^{(i_1)}, \dots, L_{-1}w^{(i_m)}, \dots, w^{(i_n)}, w^{(\overline{k})}; z_1, z_2, \dots, z_n).\end{aligned} \tag{2.6}$$

Hence φ is determined by its value at a point.

2.2 General braiding and fusion relations for intertwining operators

The braid and the fusion relations for two intertwining operators were proved by Huang and Lepowsky in [HL95a, HL95b, HL95c, Hua95, Hua05a]. In this section, we generalize these relations to more than two intertwining operators. We also state some useful convergence theorems. The proofs are technical, so we leave them to section A.3.

General fusion relations and convergence properties

Theorem 2.5 (Fusion of a chain of intertwining operators)**.** *Let $\mathcal{Y}_{\sigma_2}, \mathcal{Y}_{\sigma_3}, \dots, \mathcal{Y}_{\sigma_n}$ be a chain of intertwining operators of V with charge spaces $W_{i_2}, W_{i_3}, \dots, W_{i_n}$ respectively. Let $\mathcal{Y}_\gamma$ be another intertwining operator of V, whose charge space is the same as the target space of $\mathcal{Y}_{\sigma_n}$. Let W_{i_0} be the source space of $\mathcal{Y}_\gamma$, W_{i_1} be the source space of $\mathcal{Y}_{\sigma_2}$, and W_k be the target space of $\mathcal{Y}_\gamma$. Then for any $w^{(i_0)} \in W_{i_0}, w^{(i_1)} \in W_{i_1}, \dots, w^{(i_n)} \in W_{i_n}, w^{(\overline{k})} \in W_{\overline{k}}$, and any $(z_1, z_2, \dots, z_n) \in \mathrm{Conf}_n(\mathbb{C}^\times)$ satisfying*

$$0 < |z_2 - z_1| < |z_3 - z_1| < \dots < |z_n - z_1| < |z_1|, \tag{2.7}$$

the expression

$$\begin{aligned}&\big\langle \mathcal{Y}_\gamma\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1)\mathcal{Y}_{\sigma_{n-1}}(w^{(i_{n-1})}, z_{n-1} - z_1)\\ &\qquad \cdots \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)w^{(i_0)}, w^{(\overline{k})}\big\rangle\end{aligned} \tag{2.8}$$

converges absolutely and locally uniformly*, which means that there exists a neighborhood $U \subset \mathrm{Conf}_n(\mathbb{C}^\times)$ of $(z_1, z_2, \dots, z_n)$, and a finite number $M > 0$, such that for any $(z_1, z_2, \dots, z_n) \in U$,*

$$\begin{aligned}\sum_{s_2,\dots,s_n \in \mathbb{R}} &\big|\langle \mathcal{Y}_\gamma(P_{s_n}\mathcal{Y}_{\sigma_n}(w^{(i_n)}, \zeta_n - \zeta_1)P_{s_{n-1}}\mathcal{Y}_{\sigma_{n-1}}(w^{(i_{n-1})}, \zeta_{n-1} - \zeta_1)\\ &\cdots P_{s_2}\mathcal{Y}_{\sigma_2}(w^{(i_2)}, \zeta_2 - \zeta_1)w^{(i_1)}, \zeta_1)w^{(i_0)}, w^{(\overline{k})}\rangle\big| < M.\end{aligned} \tag{2.9}$$

Moreover, if $(z_1, z_2, \dots, z_n)$ satisfies (2.7) and

$$0 < |z_1| < |z_2| < \cdots < |z_n|, \tag{2.10}$$

then (2.8) as a $(W_{i_0} \otimes W_{i_1} \otimes \cdots \otimes W_{i_n} \otimes W_{\overline{k}})^*$-valued holomorphic function defined near $(z_1, \dots, z_n)$ is an element in $\mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_0}$, and any element in $\mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_0}$ defined near $(z_1, \dots, z_n)$ can be written as (2.8).

The following convergence theorem for products of generalized intertwining operators is necessary for our theory. (See the discussion in the introduction.)

Theorem 2.6. *Let m be a positive integer. For each $a = 1, \dots, m$, we choose a positive integer n_a. Let $W_{i^1}, \dots, W_{i^m}$ be V-modules, and let $\mathcal{Y}_{\alpha^1}, \dots, \mathcal{Y}_{\alpha^m}$ be a chain of intertwining operators with charge spaces $W_{i^1}, \dots, W_{i^m}$ respectively. We let W_i be the source space of $\mathcal{Y}_{\alpha^1}$, and let W_k be the target space of $\mathcal{Y}_{\alpha^m}$. For each $a = 1, \dots, m$ we choose a chain of intertwining operators $\mathcal{Y}_{\alpha_2^a}, \dots, \mathcal{Y}_{\alpha_{n_a}^a}$ with charge spaces $W_{i_2^a}, \dots, W_{i_{n_a}^a}$ respectively. We let $W_{i_1^a}$ be the source space of $\mathcal{Y}_{\alpha_2^a}$, and assume that the target space of $\mathcal{Y}_{\alpha_{n_a}^a}$ is W_{i^a}.*

For any $a = 1, \dots, m$ and $b = 1, \dots, n_a$, we choose a non-zero complex number z_b^a. Choose $w_b^a \in W_{i_b^a}$. We also choose vectors $w^i \in W_i, w^{\overline{k}} \in W_{\overline{k}}$. Suppose that the complex numbers $\{z_b^a\}$ satisfy the following conditions:
(1) For each $a = 1, \dots, m$, $0 < |z_2^a - z_1^a| < |z_3^a - z_1^a| < \cdots < |z_{n_a}^a - z_1^a| < |z_1^a|$;
(2) For each $a = 1, \dots, m-1$, $|z_1^a| + |z_{n_a}^a - z_1^a| < |z_1^{a+1}| - |z_{n_{a+1}}^{a+1} - z_1^{a+1}|$,
then the expression

$$\begin{aligned}
&\Big\langle \Big[\prod_{m \geqslant a \geqslant 1} \mathcal{Y}_{\alpha^a} \Big(\Big(\prod_{n_a \geqslant b \geqslant 2} \mathcal{Y}_{\alpha_b^a}(w_b^a, z_b^a - z_1^a) \Big) w_1^a, z_1^a \Big) \Big] w^i, w^{\overline{k}} \Big\rangle \\
\equiv &\Big\langle \mathcal{Y}_{\alpha^m} \big(\mathcal{Y}_{\alpha_{n_m}^m}(w_{n_m}^m, z_{n_m}^m - z_1^m) \cdots \mathcal{Y}_{\alpha_2^m}(w_2^m, z_2^m - z_1^m) w_1^m, z_1^m \big) \\
&\vdots \\
&\cdot \mathcal{Y}_{\alpha^1} \big(\mathcal{Y}_{\alpha_{n_1}^1}(w_{n_1}^1, z_{n_1}^1 - z_1^1) \cdots \mathcal{Y}_{\alpha_2^1}(w_2^1, z_2^1 - z_1^1) w_1^1, z_1^1 \big) w^i, w^{\overline{k}} \Big\rangle
\end{aligned} \tag{2.11}$$

***converges absolutely and locally uniformly**, i.e., there exists $M > 0$ and a neighborhood U_b^a of each z_b^a, such that for any $\zeta_b^a \in U_b^a$ $(1 \leqslant a \leqslant m, 1 \leqslant b \leqslant n_a)$ we have:*

$$\sum_{s_1^a, s_b^a \in \mathbb{R}} \Big| \Big\langle \Big[\prod_{m \geqslant a \geqslant 1} P_{s_1^a} \mathcal{Y}_{\alpha^a} \Big(\Big(\prod_{n_a \geqslant b \geqslant 2} P_{s_b^a} \mathcal{Y}_{\alpha_b^a}(w_b^a, \zeta_b^a - \zeta_1^a) \Big) w_1^a, \zeta_1^a \Big) \Big] w^i, w^{\overline{k}} \Big\rangle \Big| < M. \tag{2.12}$$

Assume, moreover, that $\{z_b^a : 1 \leqslant a \leqslant m, 1 \leqslant b \leqslant n_a\}$ satisfies the following condition:
(3) For any $1 \leqslant a, a' \leqslant m, 1 \leqslant b \leqslant n_a, 1 \leqslant b' \leqslant n_{a'}$, the inequality $0 < |z_b^a| < |z_{b'}^{a'}|$ holds when $a < a'$, or $a = a'$ and $b < b'$.
Then (2.11) defined near $\{z_b^a : 1 \leqslant a \leqslant m, 1 \leqslant b \leqslant n_a\}$ is an element in $\mathcal{V}\binom{k}{i_{n_m}^m\ \cdots\ i_1^m\ \cdots\cdots\ i_{n_1}^1\ \cdots\ i_1^1\ i}$.

We need another type of convergence property. The notion of absolute and locally uniform convergence is understood as usual.

Corollary 2.7. *Let $\mathcal{Y}_{\sigma_2}, \mathcal{Y}_{\sigma_3}, \dots, \mathcal{Y}_{\sigma_m}$ be a chain of intertwining operators of V with charge spaces $W_{i_2}, W_{i_3}, \dots, W_{i_m}$ respectively. Let W_{i_1} be the source space of $\mathcal{Y}_{\sigma_2}$ and W_i be the target space of $\mathcal{Y}_{\sigma_m}$. Similarly we let $\mathcal{Y}_{\rho_2}, \mathcal{Y}_{\rho_3}, \dots, \mathcal{Y}_{\rho_m}$ be a chain of intertwining operators of V with charge spaces $W_{j_2}, W_{j_3}, \dots, W_{j_n}$ respectively. Let W_{j_1} be the source space of $\mathcal{Y}_{\rho_2}$ and W_j be the target space of $\mathcal{Y}_{\rho_n}$. Moreover we choose V-modules $W_{k_1}, W_{k_2}, W_{k_3}$, a type $\binom{k_1}{i\ j}$ intertwining operator $\mathcal{Y}_\alpha$ and a type $\binom{k_2}{k_1\ k_0}$ intertwining operator $\mathcal{Y}_\beta$. Choose $w^{(i_1)} \in W_{i_1}, w^{(i_2)} \in W_{i_2}, \dots, w^{(i_m)} \in W_{i_m}, w^{(j_1)} \in W_{j_1}, w^{(j_2)} \in W_{j_2}, \dots, w^{(j_m)} \in W_{j_m}, w^{(k_0)} \in W_{i_0}, w^{(\overline{k_2})} \in W_{\overline{k_2}}$. Then for any non-zero complex numbers $z_1, z_2, \dots, z_m, \zeta_1, \zeta_2, \dots, \zeta_n$, satisfying $0 < |\zeta_2 - \zeta_1| < |\zeta_3 - \zeta_1| < \dots < |\zeta_n - \zeta_1| < |z_1 - \zeta_1| - |z_m - z_1|$ and $0 < |z_2 - z_1| < |z_3 - z_1| < \dots < |z_m - z_1| < |z_1 - \zeta_1| < |\zeta_1| - |z_m - z_1|$, the expression*

$$\begin{aligned}\bigg\langle \mathcal{Y}_\beta\bigg(\mathcal{Y}_\alpha\Big(\mathcal{Y}_{\sigma_m}(w^{(i_m)}, z_m - z_1)\cdots\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1 - \zeta_1\Big)\\ \cdot\mathcal{Y}_{\rho_n}(w^{(j_n)}, \zeta_n - \zeta_1)\cdots\mathcal{Y}_{\rho_2}(w^{(j_2)}, \zeta_2 - \zeta_1)w^{(j_1)}, \zeta_1\bigg)w^{(k_0)}, w^{(\overline{k_2})}\bigg\rangle\end{aligned} \tag{2.13}$$

exists and converges absolutely and locally uniformly.

General braid relations

Let $z_1, z_2, \dots, z_n$ be distinct complex values in $\mathbb{C}^\times$. Assume that $0 < |z_1| = |z_2| = \dots = |z_n|$, and choose arguments $\arg z_1, \arg z_2, \dots, \arg z_n$. We define the expression

$$\langle \mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1})\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1)w^{(i_0)}, w^{(\overline{k})}\rangle \tag{2.14}$$

in the following way: Choose $0 < r_1 < r_2 < \dots < r_n$. Then the expression

$$\langle \mathcal{Y}_{\alpha_n}(w^{(i_n)}, r_n z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, r_{n-1} z_{n-1})\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, r_1 z_1)w^{(i_0)}, w^{(\overline{k})}\rangle \tag{2.15}$$

converges absolutely. We define (2.14) to be the limit of (2.15) as $r_1, r_2, \dots, r_n \to 1$. The existence of this limit is guaranteed by theorem 2.4.

Let S_n be the symmetric group of degree n, and choose any $\varsigma \in S_n$. The general braid relations can be stated in the following way:

Theorem 2.8 (Braiding of intertwining operators)**.** *Choose distinct $z_1, \dots, z_n \in \mathbb{C}^\times$ satisfying $0 < |z_1| = \dots = |z_n|$. Let $\mathcal{Y}_{\alpha_{\varsigma(1)}}, \mathcal{Y}_{\alpha_{\varsigma(2)}}, \dots, \mathcal{Y}_{\alpha_{\varsigma(n)}}$ be a chain of intertwining operators of V. For each $1 \leqslant m \leqslant n$, we let W_{i_m} be the charge space of $\mathcal{Y}_{\alpha_m}$. Let W_{i_0} be the source space of $\mathcal{Y}_{\alpha_{\varsigma(1)}}$, and let W_k be the target space of $\mathcal{Y}_{\alpha_{\varsigma(n)}}$. Then there exists a chain of intertwining operators $\mathcal{Y}_{\beta_1}, \mathcal{Y}_{\beta_2}, \dots, \mathcal{Y}_{\beta_n}$ with charge spaces $W_{i_1}, W_{i_2}, \dots, W_{i_n}$ respectively, such that the source space of $\mathcal{Y}_{\beta_1}$ is W_{i_0}, that the target space of $\mathcal{Y}_{\beta_n}$ is W_k, and that for any $w^{(i_0)} \in W_{i_0}, w^{(i_1)} \in W_{i_1}, \dots w^{(i_n)} \in W_{i_n}, w^{(\overline{k})} \in W_{\overline{k}}$, the following braid relation holds:*

$$\langle \mathcal{Y}_{\alpha_{\varsigma(n)}}(w^{(i_{\varsigma(n)})}, z_{\varsigma(n)})\cdots\mathcal{Y}_{\alpha_{\varsigma(1)}}(w^{(i_{\varsigma(1)})}, z_{\varsigma(1)})w^{(i_0)}, w^{(\overline{k})}\rangle$$

$$=\langle \mathcal{Y}_{\beta_n}(w^{(i_n)}, z_n)\mathcal{Y}_{\beta_{n-1}}(w^{(i_{n-1})}, z_{n-1})\cdots \mathcal{Y}_{\beta_1}(w^{(i_1)}, z_1)w^{(i_0)}, w^{(\overline{k})}\rangle. \tag{2.16}$$

We usually omit the vectors $w^{(i_0)}, w^{(\overline{k})}$, and write the above equation as

$$\mathcal{Y}_{\alpha_{\varsigma(n)}}(w^{(i_{\varsigma(n)})}, z_{\varsigma(n)})\cdots \mathcal{Y}_{\alpha_{\varsigma(1)}}(w^{(i_{\varsigma(1)})}, z_{\varsigma(1)}) = \mathcal{Y}_{\beta_n}(w^{(i_n)}, z_n)\cdots \mathcal{Y}_{\beta_1}(w^{(i_1)}, z_1). \tag{2.17}$$

When $n = 2$, the proof of braid relations is based on the following well-known property. For the reader's convenience, we include a proof in section A.3.

Proposition 2.9. *Let $\mathcal{Y}_\gamma, \mathcal{Y}_\delta$ be intertwining operators of V, and assume $\mathcal{Y}_\gamma \in \mathcal{V}\binom{k}{i\ j}$. Choose $z_i, z_j \in \mathbb{C}^\times$ satisfying $0 < |z_j - z_i| < |z_i|, |z_j|$. Choose $\arg(z_j - z_i)$, and let $\arg z_j$ be close to $\arg z_i$ as $z_j \to z_i$. Then for any $w^{(i)} \in W_i, w^{(j)} \in W_j$,*

$$\mathcal{Y}_\delta\big(\mathcal{Y}_{B_\pm\gamma}(w^{(j)}, z_j - z_i)w^{(i)}, z_i\big) = \mathcal{Y}_\delta\big(\mathcal{Y}_\gamma(w^{(i)}, e^{\pm i\pi}(z_j - z_i))w^{(j)}, z_j\big). \tag{2.18}$$

Remark 2.10. The braid relation (2.17) is unchanged if we scale the norm of the complex variables $z_1, z_2, \ldots, z_n$, or rotate each variable without meeting the others, and change its $\arg$ value continuously. The braid relation might change, however, if $z_1, z_2, \ldots, z_n$ are fixed, but their arguments are changed by 2π multiplied by some integers.

The proof of theorem 2.8 (see section A.3) implies the following:

Proposition 2.11. *Let $\mathcal{Y}_{\gamma_1}, \ldots, \mathcal{Y}_{\gamma_m}, \mathcal{Y}_{\alpha_{\varsigma(1)}}, \ldots, \mathcal{Y}_{\alpha_{\varsigma(n)}}, \mathcal{Y}_{\delta_1}, \ldots, \mathcal{Y}_{\delta_l}$ be a chain of intertwining operator of V with charge spaces $W_{i'_1}, \ldots, W_{i'_m}, W_{i_{\varsigma(1)}}, \ldots, W_{i_{\varsigma(n)}}, W_{i''_1}, \ldots, W_{i''_l}$ respectively. Let W_{j_1} be the source space of $\mathcal{Y}_{\gamma_1}$ and W_{j_2} be the target space of $\mathcal{Y}_{\delta_l}$. Let $z_1, \ldots, z_n, z'_1, \ldots, z'_m, z''_1, \ldots, z''_l$ be distinct complex numbers in S^1 with fixed arguments. Choose vectors $w^{(j_1)} \in W_{j_1}, w^{(i'_1)} \in W_{i'_1}, \ldots, w^{(i'_m)} \in W_{i'_m}, w^{(i''_1)} \in W_{i''_1}, \ldots, w^{(i''_l)} \in W_{i''_l}, w^{(\overline{j_2})} \in W_{\overline{j_2}}$. Let*

$$\begin{aligned}\mathcal{X}_1 &= \mathcal{Y}_{\gamma_m}(w^{(i'_m)}, z'_m)\cdots \mathcal{Y}_{\gamma_1}(w^{(i'_1)}, z'_1),\\ \mathcal{X}_2 &= \mathcal{Y}_{\delta_l}(w^{(i''_l)}, z''_l)\cdots \mathcal{Y}_{\delta_1}(w^{(i''_1)}, z''_1).\end{aligned}$$

Suppose that the braid relation (2.16) holds for all $w^{(i_0)} \in W_{i_0}, w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_n)} \in W_{i_n}, w^{(\overline{k})} \in W_{\overline{k}}$. Then we also have the braid relation

$$\begin{aligned}&\langle \mathcal{X}_2\mathcal{Y}_{\alpha_{\varsigma(n)}}(w^{(i_{\varsigma(n)})}, z_{\varsigma(n)})\cdots \mathcal{Y}_{\alpha_{\varsigma(1)}}(w^{(i_{\varsigma(1)})}, z_{\varsigma(1)})\mathcal{X}_1 w^{(j_1)}, w^{(\overline{j_2})}\rangle\\ =&\langle \mathcal{X}_2\mathcal{Y}_{\beta_n}(w^{(i_n)}, z_n)\cdots \mathcal{Y}_{\beta_1}(w^{(i_1)}, z_1)\mathcal{X}_1 w^{(j_1)}, w^{(\overline{j_2})}\rangle.\end{aligned} \tag{2.19}$$

The braiding operators $B_\pm$ and the braid relations of intertwining operators are related in the following way:

Proposition 2.12. *Let $z_i, z_j \in S^1$ and $\arg z_j < \arg z_i < \arg z_j + \pi/3$. Let $\arg(z_i - z_j)$ be close to $\arg z_i$ as $z_j \to 0$, and let $\arg(z_j - z_i)$ be close to $\arg z_j$ as $z_i \to 0$.*

Let $\mathcal{Y}_\beta, \mathcal{Y}_\alpha$ be a chain of intertwining operators with charge spaces W_j, W_i respectively, and let $\mathcal{Y}_{\alpha'}, \mathcal{Y}_{\beta'}$ be a chain of intertwining operators with charge spaces W_i, W_j respectively. Assume that the source spaces of $\mathcal{Y}_\beta$ and $\mathcal{Y}_{\alpha'}$ are W_{k_1}, and that the target spaces of $\mathcal{Y}_\alpha$ and $\mathcal{Y}_{\beta'}$ are W_{k_2}.

If there exist a V-module W_k, and $\mathcal{Y}_\gamma \in \binom{k}{i\ j}, \mathcal{Y}_\delta \in \binom{k_2}{k\ k_1}$, such that for any $w^{(i)} \in W_i, w^{(j)} \in W_j$, we have the fusion relations:

$$\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}_\beta(w^{(j)}, z_j) = \mathcal{Y}_\delta(\mathcal{Y}_\gamma(w^{(i)}, z_i - z_j)w^{(j)}, z_j), \tag{2.20}$$

$$\mathcal{Y}_{\beta'}(w^{(j)}, z_j)\mathcal{Y}_{\alpha'}(w^{(i)}, z_i) = \mathcal{Y}_\delta(\mathcal{Y}_{B_+\gamma}(w^{(j)}, z_j - z_i)w^{(i)}, z_i). \tag{2.21}$$

Then the following braid relation holds:

$$\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}_\beta(w^{(j)}, z_j) = \mathcal{Y}_{\beta'}(w^{(j)}, z_j)\mathcal{Y}_{\alpha'}(w^{(i)}, z_i). \tag{2.22}$$

Proof. Clearly we have $\arg(z_i - z_j) = \arg(z_j - z_i) + \pi$. So equation (2.22) follows directly from proposition 2.9. □

Using braid relations, we can give a global description of correlation functions. Consider the covering map $\pi_n : \widetilde{\mathrm{Conf}}_n(\mathbb{C}^\times) \to \mathrm{Conf}_n(\mathbb{C}^\times)$. Choose $\varsigma \in S_n$, let $U_\varsigma = \{(z_1, \ldots, z_n) : 0 < |z_{\varsigma(1)}| < |z_{\varsigma(2)}| < \cdots < |z_{\varsigma(n)}|\}$, and choose a connected component $\widetilde{U}_\varsigma$ of $\pi_n^{-1}(U_\varsigma)$. Then a $(W_{i_0} \otimes W_{i_{\varsigma(1)}} \otimes \cdots \otimes W_{i_{\varsigma(n)}} \otimes W_{\overline{k}})^*$-valued correlation function defined when $(z_{\varsigma(1)}, \ldots, z_{\varsigma(n)}) \in U_\varsigma$ by the left hand side of equation (2.16) can be lifted through $\pi_n : \widetilde{U}_\varsigma \to U_\varsigma$ and analytically continued to a (single-valued) holomorphic function φ on $\widetilde{\mathrm{Conf}}_n(\mathbb{C}^\times)$. We define the vector space $\mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_0}$ of $(W_{i_0} \otimes W_{i_1} \otimes \cdots \otimes W_{i_n} \otimes W_{\overline{k}})^*$**-valued (genus 0) correlation function** to be the vector space of holomorphic functions on $\widetilde{\mathrm{Conf}}_n(\mathbb{C}^\times)$ of the form φ. This definition does not depend on the choice of ς and $\widetilde{U}_\varsigma$: If $\varsigma' \in S_n$ and $\widetilde{U}'_{\varsigma'}$ is a connected component of $\pi_n^{-1}(U_{\varsigma'})$, then by theorem 2.8, for any $\varphi \in \mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_0}$ defined on $\widetilde{\mathrm{Conf}}_n(\mathbb{C}^\times)$, it is not hard to find a $(W_{i_0} \otimes W_{i_{\varsigma'(1)}} \otimes \cdots \otimes W_{i_{\varsigma'(n)}} \otimes W_{\overline{k}})^*$-valued correlation function defined when $(z_{\varsigma'(1)}, \ldots, z_{\varsigma'(n)}) \in U_{\varsigma'}$ which can be lifted through $\pi_n : \widetilde{U}'_{\varsigma'} \to U_{\varsigma'}$ and analytically continued to the function φ.

2.3 Braiding and fusion with vertex operators and creation operators

In this section, we prove some useful braid and fusion relations. These relations are not only important for constructing a braided tensor category of representations of V, but also necessary for studying generalized intertwining operators.

Braiding and fusion with vertex operators

The Jacobi identity (1.20) can be interpreted in terms of braid and fusion relations:

Proposition 2.13. *Let $\mathcal{Y}_\alpha$ be a type $\binom{k}{i\ j}$ intertwining operator of V. Choose $z, \zeta \in \mathbb{C}^\times$ satisfying $0 < |z - \zeta| < |z| = |\zeta|$. Choose an argument $\arg z$. Then for any $u \in V, w^{(i)} \in W_i$, we have*

$$Y_k(u, \zeta)\mathcal{Y}_\alpha(w^{(i)}, z) = \mathcal{Y}_\alpha(w^{(i)}, z)Y_j(u, \zeta) = \mathcal{Y}_\alpha\big(Y_i(u, \zeta - z)w^{(i)}, z\big). \tag{2.23}$$

Proof. The above braid and fusion relations are equivalent to the following statement: for any $w^{(j)} \in W_j, w^{(\overline{k})} \in W_{\overline{k}}$, and for any $z \in \mathbb{C}^\times$, the functions of ζ:

$$\langle \mathcal{Y}_\alpha(w^{(i)}, z) Y_j(u, \zeta) w^{(j)}, w^{(\overline{k})} \rangle, \tag{2.24}$$

$$\langle \mathcal{Y}_\alpha\big(Y_i(u, \zeta - z) w^{(i)}, z\big) w^{(j)}, w^{(\overline{k})} \rangle, \tag{2.25}$$

$$\langle Y_k(u, \zeta) \mathcal{Y}_\alpha(w^{(i)}, z) w^{(j)}, w^{(\overline{k})} \rangle \tag{2.26}$$

defined respectively near 0, near z, and near ∞ can be analytically continued to the same (single-valued) meromorphic function on $\mathbb{P}^1$ whose poles are inside $\{0, z, \infty\}$. This is equivalent to that for any $f(\zeta, z) \in \mathbb{C}[\zeta^{\pm 1}, (\zeta - z)^{-1}]$,

$$\begin{aligned} &\mathrm{Res}_{\zeta=0}\big(\langle \mathcal{Y}_\alpha(w^{(i)}, z) Y_j(u, \zeta) w^{(j)}, w^{(\overline{k})} \rangle \cdot f(\zeta, z) d\zeta\big) \\ +&\mathrm{Res}_{\zeta=z}\big(\langle \mathcal{Y}_\alpha\big(Y_i(u, \zeta - z) w^{(i)}, z\big) w^{(j)}, w^{(\overline{k})} \rangle \cdot f(\zeta, z) d\zeta\big) \\ +&\mathrm{Res}_{\zeta=\infty}\big(\langle Y_k(u, \zeta) \mathcal{Y}_\alpha(w^{(i)}, z) w^{(j)}, w^{(\overline{k})} \rangle \cdot f(\zeta, z) d\zeta\big) = 0. \end{aligned} \tag{2.27}$$

[15] Note that $\mathbb{C}[\zeta^{\pm 1}, (\zeta - z)^{-1}]$ is spanned by functions of the form $\zeta^m (\zeta - z)^n$, where $m, n \in \mathbb{Z}$. If we let $f(\zeta, z) = \zeta^m (\zeta - z)^n$, then one can compute that the left hand side of (2.27) becomes $\sum_{s \in \mathbb{R}} c_s z^{-s-1}$, where each coefficient c_s is the difference between the left hand side and the right hand side of the Jacobi identity (1.20). This shows that (2.27) is equivalent to the Jacobi identity (1.20). □

The above intertwining property can be generalized to any correlation function.

Proposition 2.14. [16] *Let $z_0 = 0$, choose $(z_1, z_2, \dots, z_n) \in \mathrm{Conf}_n(\mathbb{C}^\times)$, and choose a correlation function $\varphi \in \mathcal{V}\binom{k}{i_n\, i_{n-1} \cdots i_1\, i_0}$ defined near $(z_1, z_2, \dots, z_n)$. Then for any $u \in V, w^{(i_0)} \in W_{i_0}, w^{(i_1)} \in W_{i_1}, \dots, w^{(i_n)} \in W_{i_n}, w^{(\overline{k})} \in W_{\overline{k}}$, and any $0 \leqslant m \leqslant n$, the following formal series in $\mathbb{C}((\zeta - z_m))$:*

$$\begin{aligned} &\psi_{i_m}(\zeta, z_1, z_2, \dots, z_n) \\ =&\varphi(w^{(i_0)}, \dots, w^{(i_{m-1})}, Y_{i_m}(u, \zeta - z_m) w^{(i_m)}, w^{(i_{m+1})}, \dots, w^{(i_n)}, w^{(\overline{k})}; z_1, z_2, \dots, z_n), \end{aligned} \tag{2.28}$$

and the following formal series in $\mathbb{C}((\zeta^{-1}))$:

$$\psi_k(\zeta, z_1, z_2, \dots, z_n)$$

[15] Here we use the following well-known Mittag-Leffler type theorem: Let M be a compact Riemann surface, $p_1, \dots, p_n$ are distinct points on M, $\zeta_1, \dots, \zeta_n$ are local coordinates at $p_1, \dots, p_n$ respectively. For any $j = 1, \dots, n$ we associate a locally defined formal meromorphic function $\varphi_j \in \mathbb{C}((\zeta_j))$ near p_j. Then the following two conditions are equivalent. (a) There exists a global meromorphic function φ on M with no poles outside $p_1, \dots, p_n$, such that the series expansion of φ near each p_j is φ_j. (b) For any global meromorphic 1-form ω on M with no poles outside $p_1, \dots, p_n$, the relation $\sum_j \mathrm{Res}_{p_j} \varphi_i \omega = 0$ holds. (a)⇒(b) is obvious from residue theorem. (b)⇒(a) is not hard to prove using Serre duality. See [Ueno08] theorem 1.22, or [Muk10] theorem 1.

[16] One can use proposition 2.14 and the translation property to define correlation functions (parallel sections of conformal blocks). cf. [FB04] chapter 10.

$$=\varphi(w^{(i_0)}, w^{(i_1)}, w^{(i_2)}, \dots, w^{(i_n)}, Y_k(u,\zeta)^{\mathrm{t}} w^{(\overline{k})}; z_1, z_2, \dots, z_n) \tag{2.29}$$

are expansions of the same (single-valued) holomorphic function on $\mathbb{P}\backslash\{z_0, z_1, z_2, \dots, z_n, \infty\}$ near the poles $\zeta = z_m$ ($0 \leqslant m \leqslant n$) and $\zeta = \infty$ respectively.

Proof. When $0 < |z_1| < |z_2| < \cdots < |z_n|$, we can prove this property easily using proposition 2.11, proposition 2.13, and theorem 2.6. Note that this property is equivalent to that for any $f(\zeta, z_1, \dots, z_n) \in \mathbb{C}[\zeta^{\pm 1}, (\zeta - z_1)^{-1}, \dots, (\zeta - z_n)^{-1}]$,

$$\begin{aligned}&\sum_{0 \leqslant m \leqslant n} \mathrm{Res}_{\zeta = z_m}\big(\psi_{i_m}(\zeta, z_1, \dots, z_n) f(\zeta, z_1, \dots, z_n) d\zeta\big)\\ =& -\mathrm{Res}_{\zeta=\infty}\big(\psi_k(\zeta, z_1, \dots, z_n) f(\zeta, z_1, \dots, z_n) d\zeta\big).\end{aligned} \tag{2.30}$$

If we write down the above equations explicitly, we will find that condition (2.30) is equivalent to a set of linear equations of φ, the coefficients of which are $\mathrm{End}\big((W_{i_0} \otimes W_{i_1} \otimes \cdots \otimes W_{i_n} \otimes W_{\overline{k}})^*\big)$-valued single-valued holomorphic functions on $\mathrm{Conf}_n(\mathbb{C}^\times)$. Since φ satisfies these equations locally, it must satisfy them globally. Therefore φ satisfies the desired property at any point in $\mathrm{Conf}_n(\mathbb{C}^\times)$. □

As an application of this intertwining property, we prove a very useful uniqueness property for correlation functions.

Corollary 2.15. *Fix $(z_1, z_2, \dots, z_n) \in \mathrm{Conf}_n(\mathbb{C}^\times)$. Let $\varphi \in \mathcal{V}\binom{\overline{i_{n+1}}}{i_n\ i_{n-1}\ \cdots\ i_1\ i_0}$ be a correlation function defined near $(z_1, z_2, \dots, z_n)$. Choose $l \in \{0, 1, 2, \dots, n+1\}$. For any $m \in \{0, 1, 2, \dots, n+1\}$ such that $m \neq l$, we assume that W_{i_m} is irreducible, and choose a nonzero vector $w_0^{(i_m)} \in W_{i_m}$. Suppose that for any $w^{(i_l)} \in W_{i_l}$,*

$$\varphi(w_0^{(i_0)}, \dots, w_0^{(i_{l-1})}, w^{(i_l)}, w_0^{(i_{l+1})}, \dots, w_0^{(i_{n+1})}; z_1, z_2, \dots, z_n) = 0, \tag{2.31}$$

then $\varphi = 0$.

Proof. We assume that $l \leqslant n$. The case that $l = n+1$ can be proved in a similar way. Suppose that (2.31) holds. Then for any $u \in V$, the formal series in $\mathbb{C}((\zeta - z_l))$:

$$\varphi(w_0^{(i_0)}, \dots, w_0^{(i_{l-1})}, Y_{i_l}(u, \zeta - z_l) w^{(i_l)}, w_0^{(i_{l+1})}, \dots, w_0^{(i_{n+1})}; z_1, z_2, \dots, z_n) \tag{2.32}$$

equals zero. By proposition 2.14, (2.32) is the expansion of a global holomorphic function (which must be zero) on $\mathbb{P}\backslash\{z_0, z_1, \dots, z_n, \infty\}$, and when ζ is near $z_0 = 0$, this function becomes

$$\varphi(Y_{i_0}(u,\zeta) w_0^{(i_0)}, w_0^{(i_1)} \dots, w_0^{(i_{l-1})}, w^{(i_l)}, w_0^{(i_{l+1})}, \dots, w_0^{(i_{n+1})}; z_1, z_2, \dots, z_n), \tag{2.33}$$

which is zero. Therefore, for each mode $Y_{i_1}(u, s)$ ($s \in \mathbb{Z}$), we have

$$\varphi(Y_{i_0}(u,s) w_0^{(i_0)}, w_0^{(i_1)} \dots, w_0^{(i_{l-1})}, w^{(i_l)}, w_0^{(i_{l+1})}, \dots, w_0^{(i_{n+1})}; z_1, z_2, \dots, z_n) = 0. \tag{2.34}$$

Since W_{i_0} is irreducible, for any $w^{(i_0)} \in W_0$ we have

$$\varphi(w^{(i_0)}, w_0^{(i_1)} \ldots, w_0^{(i_{l-1})}, w^{(i_l)}, w_0^{(i_{l+1})}, \ldots, w_0^{(i_{n+1})}; z_1, z_2, \ldots, z_n) = 0. \tag{2.35}$$

If we repeat this argument several times, we see that for any $w^{(i_0)} \in W_{i_0}, w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_{n+1})} \in W_{i_{n+1}}$,

$$\varphi(w^{(i_0)}, w^{(i_1)}, \ldots, w^{(i_{n+1})}; z_1, \ldots, z_n) = 0. \tag{2.36}$$

Hence φ equals zero at $(z_1, \ldots, z_n)$. By theorem 2.4 and the translation property, the value of φ equals zero at any point. □

Braiding and fusion with creation operators

Lemma 2.16. *Let $\mathcal{Y}_\alpha$ be a type $\binom{k}{i\,j}$ intertwining operator. Then for any $w^{(i)} \in W_i, w^{(j)} \in W_j, w^{(\overline{k})} \in W_{\overline{k}}$, $z \in \mathbb{C}^\times$ and $z_0 \in \mathbb{C}$:*
(1) If $0 \leqslant |z_0| < |z|$, and $\arg(z - z_0)$ is close to $\arg z$ as $z_0 \to 0$, then

$$\sum_{s\in\mathbb{R}} \langle w^{(\overline{k})}, \mathcal{Y}_\alpha(w^{(i)}, z) P_s e^{z_0 L_{-1}} w^{(j)} \rangle \tag{2.37}$$

converges absolutely and equals

$$\langle w^{(\overline{k})}, e^{z_0 L_{-1}} \mathcal{Y}_\alpha(w^{(i)}, z - z_0) w^{(j)} \rangle. \tag{2.38}$$

We simply write

$$e^{z_0 L_{-1}} \mathcal{Y}_\alpha(w^{(i)}, z - z_0) = \mathcal{Y}_\alpha(w^{(i)}, z) e^{z_0 L_{-1}}. \tag{2.39}$$

(2) If $0 \leqslant |z_0| < |z|^{-1}$ and $\arg(1 - z z_0)$ is close to $\arg 1 = 0$ as $z_0 \to 0$, then

$$\sum_{s\in\mathbb{R}} \langle w^{(\overline{k})}, e^{z_0 L_1} P_s \mathcal{Y}_\alpha(w^{(i)}, z) w^{(j)} \rangle \tag{2.40}$$

converges absolutely and equals

$$\Big\langle w^{(\overline{k})}, \mathcal{Y}_\alpha\big(e^{z_0(1 - z z_0)L_1}(1 - z z_0)^{-2L_0} w^{(i)}, z/(1 - z z_0)\big) e^{z_0 L_1} w^{(j)} \Big\rangle. \tag{2.41}$$

We simply write

$$e^{z_0 L_1} \mathcal{Y}_\alpha(w^{(i)}, z) = \mathcal{Y}_\alpha\big(e^{z_0(1 - z z_0)L_1}(1 - z z_0)^{-2L_0} w^{(i)}, z/(1 - z z_0)\big) e^{z_0 L_1}. \tag{2.42}$$

Proof. Assume without loss of generality that all the vectors are homogeneous.

(1) Let x, x_0, x_1 be commuting independent formal variables. Note first of all that (2.39) holds in the formal sense:

$$\langle w^{(\overline{k})}, e^{x_0 L_{-1}} \mathcal{Y}_\alpha(w^{(i)}, x - x_0) w^{(j)} \rangle = \langle w^{(\overline{k})}, \mathcal{Y}_\alpha(w^{(i)}, x) e^{x_0 L_{-1}} w^{(j)} \rangle, \tag{2.43}$$

where

$$\mathcal{Y}_\alpha(w^{(i)}, x - x_0) = \sum_{s\in\mathbb{R}} \sum_{r\in\mathbb{Z}_{\geqslant 0}} \mathcal{Y}_\alpha(w^{(i)}, s)\binom{-s-1}{r} x^{-s-1-r}(-x_0)^r.$$

Equation (2.43) can be proved using the relation $[L_{-1}, \mathcal{Y}_\alpha(w^{(i)}, x)] = \frac{d}{dx}\mathcal{Y}_\alpha(w^{(i)}, x)$. (See [FHL93] section 5.4 for more details.) Write

$$\langle w^{(\overline{k})}, e^{x_0 L_{-1}} \mathcal{Y}_\alpha(w^{(i)}, x_1) w^{(j)}\rangle = \sum_{m\in\mathbb{Z}_{\geqslant 0}} c_m x_0^m x_1^{d-m} \tag{2.44}$$

where $d \in \mathbb{R}$ and $c_m \in \mathbb{C}$. Clearly $c_m = 0$ for all but finitely many m. Then the left hand side of (2.43) equals

$$\sum_{m,l\in\mathbb{Z}_{\geqslant 0}} c_m x_0^m \cdot \binom{d-m}{l} x^{d-m-l}(-x_0)^l.$$

We now substitute z and z_0 for x and x_0 in equation (2.43). For any z_0 satisfying $0 \leqslant |z_0| < |z|$, let $\arg(z - z_0)$ be close to $\arg z$ as $z_0 \to 0$. Then

$$\begin{aligned}
&\langle w^{(\overline{k})}, e^{z_0 L_{-1}} \mathcal{Y}_\alpha(w^{(i)}, z - z_0) w^{(j)}\rangle \\
=&\langle w^{(\overline{k})}, e^{x_0 L_{-1}} \mathcal{Y}_\alpha(w^{(i)}, x_1) w^{(j)}\rangle\big|_{x_0=z_0, x_1=z-z_0} \\
=&\sum_{m\in\mathbb{Z}_{\geqslant 0}} c_m z_0^m (z - z_0)^{d-m} \\
=&\sum_{m,l\in\mathbb{Z}_{\geqslant 0}} c_m z_0^m \cdot \binom{d-m}{l} z^{d-m-l}(-z_0)^l,
\end{aligned} \tag{2.45}$$

which converges absolutely and equals

$$\begin{aligned}
&\langle w^{(\overline{k})}, \mathcal{Y}_\alpha(w^{(i)}, x) e^{x_0 L_{-1}} w^{(j)}\rangle\big|_{x=z, x_0=z_0} \\
=&\langle w^{(\overline{k})}, \mathcal{Y}_\alpha(w^{(i)}, z) e^{z_0 L_{-1}} w^{(j)}\rangle.
\end{aligned} \tag{2.46}$$

This proves part (1).

(2) Since $\alpha = C^{-1}C\alpha$, we have

$$\begin{aligned}
&\sum_{s\in\mathbb{R}} \langle w^{(\overline{k})}, e^{z_0 L_1} P_s \mathcal{Y}_\alpha(w^{(i)}, z) w^{(j)}\rangle \\
=&\sum_{s\in\mathbb{R}} \langle P_s e^{z_0 L_1} w^{(\overline{k})}, \mathcal{Y}_{C^{-1}C\alpha}(w^{(i)}, z) w^{(j)}\rangle \\
=&\sum_{s\in\mathbb{R}} \Big\langle \mathcal{Y}_{C\alpha}\big(e^{zL_1}(e^{i\pi} z^{-2})^{L_0} w^{(i)}, z^{-1}\big) P_s e^{z_0 L_{-1}} w^{(\overline{k})}, w^{(j)}\Big\rangle,
\end{aligned}$$

which, according to part (1), converges absolutely and equals

$$\left\langle e^{z_0L_{-1}}\mathcal{Y}_{C\alpha}\left(e^{zL_1}(e^{i\pi}z^{-2})^{L_0}w^{(i)},z^{-1}-z_0\right)w^{(\bar{k})},w^{(j)}\right\rangle, \tag{2.47}$$

where $\arg(z^{-1}-z_0)$ is close to $\arg(z^{-1})=-\arg z$ as $z_0\to 0$. This is equivalent to saying that $\arg(1-zz_0)$ is close to 0 as $z_0\to 0$.

By the definition of $C\alpha$, (2.47) equals

$$\begin{aligned}&\left\langle \mathcal{Y}_{C\alpha}\left(e^{zL_1}(e^{i\pi}z^{-2})^{L_0}w^{(i)},z^{-1}-z_0\right)w^{(\bar{k})},e^{z_0L_1}w^{(j)}\right\rangle\\ =&\Big\langle w^{(\bar{k})},\mathcal{Y}_{\alpha}\Big(e^{(z^{-1}-z_0)L_1}(e^{-i\pi}(z^{-1}-z_0)^{-2})^{L_0}\\ &\cdot e^{zL_1}(e^{i\pi}z^{-2})^{L_0}w^{(i)},(z^{-1}-z_0)^{-1}\Big)e^{z_0L_1}w^{(j)}\Big\rangle.\end{aligned} \tag{2.48}$$

Note that (1.30) also holds when $x\in\mathbb{C},x_0\in\mathbb{C}^\times$. Therefore, by applying relation (1.30), expression (2.48) equals (2.41). This finishes the proof of part (2). $\square$

Proposition 2.17. *Let $z_1,\ldots,z_n\in\mathbb{C}^\times$ satisfy $|z_1|<|z_2|<\cdots<|z_n|$ and $|z_2-z_1|<\cdots<|z_n-z_1|<|z_1|$. Choose arguments $\arg z_1,\arg z_2,\ldots,\arg z_n$. For each $2\leqslant m\leqslant n$, we let $\arg(z_m-z_1)$ be close to $\arg z_m$ as $z_1\to 0$. Let $\mathcal{Y}_{\sigma_2},\ldots,\mathcal{Y}_{\sigma_n}$ be a chain of intertwining operators of V with charge spaces $W_{i_2},\ldots,W_{i_n}$ respectively. Let W_{i_1} be the source space of $\mathcal{Y}_{\sigma_2}$, and let W_i be the target space of $\mathcal{Y}_{\sigma_n}$. Then for any $w^{(i_1)}\in W_{i_1},w^{(i_2)}\in W_{i_2},\ldots,w^{(i_n)}\in W_{i_n}$, we have the fusion relation*

$$\begin{aligned}&\mathcal{Y}^{i}_{i0}\left(\mathcal{Y}_{\sigma_n}(w^{(i_n)},z_n-z_1)\cdots\mathcal{Y}_{\sigma_2}(w^{(i_2)},z_2-z_1)w^{(i_1)},z_1\right)\\ =&\mathcal{Y}_{\sigma_n}(w^{(i_n)},z_n)\cdots\mathcal{Y}_{\sigma_2}(w^{(i_2)},z_2)\mathcal{Y}^{i_1}_{i_10}(w^{(i_1)},z_1).\end{aligned} \tag{2.49}$$

Proof. We assume that $z_1,z_2,\ldots,z_n$ are on the same ray emitting from the origin (e.g. on $\mathbb{R}_{>0}$). (We don't assume, however, that these complex values have the same argument.) Then for each $2\leqslant m\leqslant n$, these complex numbers satisfy

$$|z_1|+|z_m-z_1|<|z_{m+1}|. \tag{2.50}$$

If (2.49) is proved at these points, then by theorem 2.4 and analytic continuation, (2.49) holds in general.

Choose any $w^{(\bar{i})}\in W_{\bar{i}}$. Using equations (1.39) and (2.39) several times, we have

$$\begin{aligned}&\langle\mathcal{Y}_{\sigma_n}(w^{(i_n)},z_n)\cdots\mathcal{Y}_{\sigma_3}(w^{(i_3)},z_3)\mathcal{Y}_{\sigma_2}(w^{(i_2)},z_2)\mathcal{Y}^{i_1}_{i_10}(w^{(i_1)},z_1)\Omega,w^{(\bar{i})}\rangle\\ =&\langle\mathcal{Y}_{\sigma_n}(w^{(i_n)},z_n)\cdots\mathcal{Y}_{\sigma_3}(w^{(i_3)},z_3)\mathcal{Y}_{\sigma_2}(w^{(i_2)},z_2)e^{z_1L_{-1}}w^{(i_1)},w^{(\bar{i})}\rangle\\ =&\langle\mathcal{Y}_{\sigma_n}(w^{(i_n)},z_n)\cdots\mathcal{Y}_{\sigma_3}(w^{(i_3)},z_3)e^{z_1L_{-1}}\mathcal{Y}_{\sigma_2}(w^{(i_2)},z_2-z_1)w^{(i_1)},w^{(\bar{i})}\rangle\\ =&\langle\mathcal{Y}_{\sigma_n}(w^{(i_n)},z_n)\cdots e^{z_1L_{-1}}\mathcal{Y}_{\sigma_3}(w^{(i_3)},z_3-z_1)\mathcal{Y}_{\sigma_2}(w^{(i_2)},z_2-z_1)w^{(i_1)},w^{(\bar{i})}\rangle\\ &\vdots\\ =&\langle e^{z_1L_{-1}}\mathcal{Y}_{\sigma_n}(w^{(i_n)},z_n-z_1)\cdots\mathcal{Y}_{\sigma_3}(w^{(i_3)},z_3-z_1)\mathcal{Y}_{\sigma_2}(w^{(i_2)},z_2-z_1)w^{(i_1)},w^{(\bar{i})}\rangle\end{aligned}$$

$$=\left\langle \mathcal{Y}^i_{i0}\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1)\cdots \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)\Omega, w^{(\bar{i})}\right\rangle. \tag{2.51}$$

Note that in order to make the above argument valid, we have to check that the expression in each step converges absolutely. To see this, we choose any $m = 1, \dots, n$, and let W_{j_m} be the target space of $\mathcal{Y}_{\sigma_m}$. Then

$$\begin{aligned}
&\langle \mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n)\cdots e^{z_1 L_{-1}}\mathcal{Y}_{\sigma_m}(w^{(i_m)}, z_m - z_1)\cdots \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, w^{(\bar{i})}\rangle\\
=&\sum_{s_1,\dots,s_{n-1}\in\mathbb{R}} \langle \mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n)P_{s_{n-1}}\cdots P_{s_1}e^{z_1 L_{-1}}P_{s_m}\mathcal{Y}_{\sigma_m}(w^{(i_m)}, z_m - z_1)P_{s_{m-1}}\\
&\qquad\cdots P_{s_2}\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, w^{(\bar{i})}\rangle\\
=&\sum_{s_1,\dots,s_{n-1}\in\mathbb{R}} \big\langle \mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n)P_{s_{n-1}}\cdots P_{s_1}\mathcal{Y}^{j_m}_{j_m 0}\big(P_{s_m}\mathcal{Y}_{\sigma_m}(w^{(i_m)}, z_m - z_1)P_{s_{m-1}}\\
&\qquad\cdots P_{s_2}\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)\Omega, w^{(\bar{i})}\big\rangle,
\end{aligned} \tag{2.52}$$

which, by (2.50) and theorem 2.6, converges absolutely. Therefore, equation (2.49) holds when both sides act on the vacuum vector Ω. By (the proof of) corollary 2.15, equation (2.49) holds when acting on any vector $v \in V$. □

Corollary 2.18. *Let $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$. Let $z_i, z_j \in S^1$ with arguments satisfying $\arg z_j < \arg z_i < \arg z_j + 2\pi$. Then for any $w^{(i)} \in W_i$ and $w^{(j)} \in W_j$, we have the braid relation*

$$\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}^j_{j0}(w^{(j)}, z_j) = \mathcal{Y}_{B_+\alpha}(w^{(j)}, z_j)\mathcal{Y}^i_{i0}(w^{(i)}, z_i). \tag{2.53}$$

Proof. By analytic continuation, we may assume, without loss of generality, that $0 < |z_i - z_j| < 1$. Let $\arg(z_i - z_j)$ be close to $\arg z_i$ as $z_j \to 0$, and let $\arg(z_j - z_i)$ be close to $\arg z_j$ as $z_i \to 0$. Then by propositions 2.17 and 2.12,

$$\begin{aligned}
&\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}^j_{j0}(w^{(j)}, z_j)\\
=&\mathcal{Y}^k_{k0}\big(\mathcal{Y}_\alpha(w^{(i)}, z_i - z_j)w^{(j)}, z_j\big)\\
=&\mathcal{Y}^k_{k0}\big(\mathcal{Y}_{B_+\alpha}(w^{(j)}, z_j - z_i)w^{(i)}, z_i\big)\\
=&\mathcal{Y}_{B_+\alpha}(w^{(j)}, z_j)\mathcal{Y}^i_{i0}(w^{(i)}, z_i).
\end{aligned}$$

□

2.4 The ribbon categories associated to VOAs

We refer the reader to [Tur16] for the general theory of ribbon categories and modular tensor categories. See also [BK01, EGNO04]. In this section, we review the construction of the ribbon category $\mathrm{Rep}(V)$ for V by Huang and Lepowspky. (cf. [HL94] and [Hua08b].) As an additive categoy, $\mathrm{Rep}(V)$ is the representation category of V: Objects of $\mathrm{Rep}(V)$ are V-modules, and the vector space of morphisms from W_i to W_j is $\mathrm{Hom}_V(W_i, W_j)$. We now equip with $\mathrm{Rep}(V)$ a structure of a ribbon category.

The **tensor product** of two V-modules W_i, W_j is defined to be

$$\begin{aligned} W_{ij} \equiv W_i \boxtimes W_j &= \bigoplus_{k\in\mathcal{E}} \mathcal{V}\binom{k}{i\ j}^* \otimes W_k, \\ Y_{ij}(v,x) &= \bigoplus_{k\in\mathcal{E}} \mathrm{id}\otimes Y_k(v,x) \qquad (v\in V), \end{aligned} \tag{2.54}$$

where $\mathcal{V}\binom{k}{i\ j}^*$ is the dual space of $\mathcal{V}\binom{k}{i\ j}$. (Recall our notations at the beginning of this chapter.) Thus for any $k\in\mathcal{E}$, we can define an isomorphism

$$\mathcal{V}\binom{k}{i\ j} \to \mathrm{Hom}_V(W_{ij}, W_k), \quad \mathcal{Y} \mapsto R_{\mathcal{Y}},$$

such that if $\check{\mathcal{Y}} \in \mathcal{V}\binom{k}{i\ j}^*$ and $w^{(k)}\in W_k$, then

$$R_{\mathcal{Y}}(\check{\mathcal{Y}}\otimes w^{(k)}) = \langle \check{\mathcal{Y}}, \mathcal{Y}\rangle w^{(k)}. \tag{2.55}$$

$R_{\mathcal{Y}}$ is called the homomorphism represented by $\mathcal{Y}$.

The tensor product of two morphisms are defined as follows: If $F \in \mathrm{Hom}_V(W_{i_1}, W_{i_2}), G\in \mathrm{Hom}_V(W_{j_1}, W_{j_2})$, then for each $k\in\mathcal{E}$ we have a linear map $(F\otimes G)^{\mathrm{t}} : \mathcal{V}\binom{k}{i_2\ j_2} \to \mathcal{V}\binom{k}{i_1\ j_1}$, such that if $\mathcal{Y}\in\mathcal{V}\binom{k}{i_2\ j_2}$, then $(F\otimes G)^{\mathrm{t}}\mathcal{Y} \in \mathcal{V}\binom{k}{i_1\ j_1}$, and for any $w^{(i_1)}\in W_{i_1}, w^{(j_1)}\in W_{j_1}$,

$$\big((F\otimes G)^{\mathrm{t}}\mathcal{Y}\big)(w^{(i_1)},x)w^{(j_1)} = \mathcal{Y}(Fw^{(i_1)},x)Gw^{(j_1)}. \tag{2.56}$$

Then $F\otimes G : \mathcal{V}\binom{k}{i_1\ j_1}^* \to \mathcal{V}\binom{k}{i_2\ j_2}^*$ is defined to be the transpose of $(F\otimes G)^{\mathrm{t}}$, and can be extended to a homomorphism

$$F\otimes G = \bigoplus_{k\in\mathcal{E}} (F\otimes G)\otimes \mathrm{id}_k : W_{i_1}\boxtimes W_{j_1} \to W_{i_2}\boxtimes W_{j_2}.$$

Hence we've define the tensor product $F\otimes G$ of F and G.

Let $W_0 = V$ be the unit object of $\mathrm{Rep}(V)$. The functorial isomorphisms $\lambda_i : W_0\boxtimes W_i \to W_i$ and $\rho_j : W_i\boxtimes W_0 \to W_i$ are defined as follows: If $i\in\mathcal{E}$, then λ_i is represented by the intertwining operator Y_i, and ρ_i is represented by $\mathcal{Y}^i_{i0}$. In general, λ_i (resp. ρ_i) is defined to be the unique isomorphism satisfying that for any $k\in\mathcal{E}$ and any $R\in\mathrm{Hom}_V(W_i, W_k)$, $R\lambda_i = \lambda_k(\mathrm{id}_0\otimes R)$ (resp. $R\rho_i = \rho_k(R\otimes\mathrm{id}_0)$).

We now define the associator. First of all, to simplify our notations, we assume the following:

Convention 2.19. Let $W_i, W_j, W_k, W_{i'}, W_{j'}, W_{k'}$ be V-modules. Let $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k'}{i'\ j'}$. If either $W_i \neq W_{i'}, W_j\neq W_{j'}$, or $W_k\neq W_{k'}$, then for any $w^{(i)}\in W_i, w^{(j)}\in W_j, w^{(\overline{k})}\in W_{\overline{k}}, z\in\mathbb{C}^\times$, we let

$$\langle \mathcal{Y}_\alpha(w^{(i)},z)w^{(j)}, w^{(\overline{k})}\rangle = 0.$$

Therefore, $\mathcal{Y}_\beta(w^{(i)},z_2)\mathcal{Y}_\alpha(w^{(j)},z_1) = 0$ if the target space of $\mathcal{Y}_\alpha$ does not equal the source space of $\mathcal{Y}_\beta$; $\mathcal{Y}_\gamma\big(\mathcal{Y}_\delta(w^{(i)},z_1-z_2)w^{(j)},z_2\big) = 0$ if the target space of $\mathcal{Y}_\delta$ does not equal the charge space of $\mathcal{Y}_\gamma$.

Given three V-modules W_i, W_j, W_k, we have

$$(W_i \boxtimes W_j) \boxtimes W_k = \bigoplus_{s,t \in \mathcal{E}} \mathcal{V}\binom{t}{s\,k}^* \otimes \mathcal{V}\binom{s}{i\,j}^* \otimes W_t, \tag{2.57}$$

$$W_i \boxtimes (W_j \boxtimes W_k) = \bigoplus_{r,t \in \mathcal{E}} \mathcal{V}\binom{t}{i\,r}^* \otimes \mathcal{V}\binom{r}{j\,k}^* \otimes W_t. \tag{2.58}$$

Choose basis $\Theta^t_{sk}, \Theta^s_{ij}, \Theta^t_{ir}, \Theta^r_{jk}$ of these spaces of intertwining operators. Choose $z_i, z_j \in \mathbb{C}^\times$ satisfying $0 < |z_i - z_j| < |z_j| < |z_i|$. Choose $\arg z_i$. Let $\arg z_j$ be close to $\arg z_i$ as $z_i - z_j \to 0$, and let $\arg(z_i - z_j)$ be close to $\arg z_i$ as $z_j \to 0$. For any $t \in \mathcal{E}, \alpha \in \Theta^t_{i*}, \beta \in \Theta^*_{jk}$, there exist complex numbers $F^{\beta'\alpha'}_{\alpha\beta}$ independent of the choice of z_i, z_j, such that for any $w^{(i)} \in W_i, w^{(j)} \in W_j$, we have the fusion relation

$$\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}_\beta(w^{(j)}, z_j) = \sum_{\alpha' \in \Theta^*_{ij}, \beta' \in \Theta^t_{*k}} F^{\beta'\alpha'}_{\alpha\beta} \mathcal{Y}_{\beta'}(\mathcal{Y}_{\alpha'}(w^{(i)}, z_i - z_j)w^{(j)}, z_j). \tag{2.59}$$

If the source space of $\mathcal{Y}_\alpha$ does not equal the target space of $\mathcal{Y}_\beta$, or if the target space of $\mathcal{Y}_{\alpha'}$ does not equal the charge space of $\mathcal{Y}_{\beta'}$, we set $F^{\beta'\alpha'}_{\alpha\beta} = 0$. Then, by the proof of proposition 2.3, the numbers $F^{\beta'\alpha'}_{\alpha\beta}$ are uniquely determined by the basis chosen. The matrix $\{F^{\beta'\alpha'}_{\alpha\beta}\}^{\alpha' \in \Theta^*_{ij}, \beta' \in \Theta^t_{*k}}_{\alpha \in \Theta^t_{i*}, \beta \in \Theta^*_{jk}}$ is called a **fusion matrix**. Define an isomorphism

$$\begin{aligned} A^{\mathrm{t}} : \bigoplus_{r \in \mathcal{E}} \mathcal{V}\binom{t}{i\,r} \otimes \mathcal{V}\binom{r}{j\,k} &\to \bigoplus_{s \in \mathcal{E}} \mathcal{V}\binom{t}{s\,k} \otimes \mathcal{V}\binom{s}{i\,j}, \\ \mathcal{Y}_\alpha \otimes \mathcal{Y}_\beta &\mapsto \sum_{\alpha' \in \Theta^*_{ij}, \beta' \in \Theta^t_{*k}} F^{\beta'\alpha'}_{\alpha\beta} \mathcal{Y}_{\beta'} \otimes \mathcal{Y}_{\alpha'}. \end{aligned} \tag{2.60}$$

Clearly A^{t} is independent of the basis chosen. Define

$$A : \bigoplus_{s \in \mathcal{E}} \mathcal{V}\binom{t}{s\,k}^* \otimes \mathcal{V}\binom{s}{i\,j}^* \to \bigoplus_{r \in \mathcal{E}} \mathcal{V}\binom{t}{i\,r}^* \otimes \mathcal{V}\binom{r}{j\,k}^* \tag{2.61}$$

to be the transpose of A^{t}, and extend it to

$$A = \sum_{t \in \mathcal{E}} A \otimes \mathrm{id}_t : (W_i \boxtimes W_j) \boxtimes W_k \to W_i \boxtimes (W_j \boxtimes W_k), \tag{2.62}$$

which is an **associator** of $\mathrm{Rep}(V)$. One can prove the pentagon axiom using theorem 2.6 and corollary 2.7, and prove the triangle axiom using propositions 2.13 and 2.9.

Recall the linear isomorphisms

$$B_\pm : \mathcal{V}\binom{k}{j\,i} \to \mathcal{V}\binom{k}{i\,j}, \quad \mathcal{Y} \mapsto B_\pm \mathcal{Y}.$$

We let $\sigma_{i,j} : \mathcal{V}\binom{k}{i\ j}^* \to \mathcal{V}\binom{k}{j\ i}^*$ be the transpose of B_+ and extend it to a morphism

$$\sigma_{i,j} = \sum_{t\in\mathcal{E}} \sigma_{i,j} \otimes \mathrm{id}_t : W_i \boxtimes W_j \to W_j \boxtimes W_i. \tag{2.63}$$

This gives the **braid operator**. The hexagon axiom can be proved using propositions 2.11, 2.12, and theorem 2.6.

For each object i, the twist is just the operator $\vartheta_i = \vartheta_{W_i}$ defined in section 1.3.

With these structural maps, Huang proved in [Hua05b, Hua08a, Hua08b] that $\mathrm{Rep}(V)$ is rigid and in fact a modular tensor category. From his proof, it is clear that $\bar{i}$ is the right dual of i: there exist homomorphisms $\mathrm{coev}_i : V \to W_i \boxtimes W_{\bar{i}}$ and $\mathrm{ev}_i : W_{\bar{i}} \boxtimes W_i \to V$ satisfying

$$(\mathrm{id}_i \otimes \mathrm{ev}_i) \circ (\mathrm{coev}_i \otimes \mathrm{id}_i) = \mathrm{id}_i, \tag{2.64}$$
$$(\mathrm{ev}_i \otimes \mathrm{id}_{\bar{i}}) \circ (\mathrm{id}_{\bar{i}} \otimes \mathrm{coev}_i) = \mathrm{id}_{\bar{i}}. \tag{2.65}$$

Since $i = \bar{\bar{i}}$, $\bar{i}$ is also the left dual of i.

Now assume that V is unitary. The additive category $\mathrm{Rep}^{\mathrm{u}}(V)$ is defined to be the representation category of unitary V-modules. We show that $\mathrm{Rep}^{\mathrm{u}}(V)$ is a C^*-category. First, we need the following easy consequence of Schur's lemma.

Lemma 2.20. *Choose for each $k \in \mathcal{E}^{\mathrm{u}}$ a number $n_k \in \mathbb{Z}_{\geqslant 0}$. Define the unitary V-module*

$$W = \bigoplus_{k\in\mathcal{E}^{\mathrm{u}}}^{\perp} W_k \otimes \mathbb{C}^{n_k} = \bigoplus_{k\in\mathcal{E}^{\mathrm{u}}}^{\perp} \underbrace{W_k \oplus^{\perp} W_k \oplus^{\perp} \cdots \oplus^{\perp} W_k}_{n_k}.$$

Then we have

$$\mathrm{End}_V(W) = \bigoplus_{k\in\mathcal{E}^{\mathrm{u}}} \mathrm{id}_k \otimes \mathrm{End}(\mathbb{C}^{n_k}). \tag{2.66}$$

Theorem 2.21. $\mathrm{Rep}^{\mathrm{u}}(V)$ *is a C^*-category, i.e., $\mathrm{Rep}^{\mathrm{u}}(V)$ is equipped with an involutive antilinear contravariant endofunctor $*$ which is the identity on objects; The positivity condition is satisfied: If W_i, W_j are unitary and $F \in \mathrm{Hom}_V(W_i, W_j)$, then there exists $R \in \mathrm{End}_V(W_i)$ such that $F^*F = R^*R$; The hom-spaces $\mathrm{Hom}_V(W_i, W_j)$ are normed spaces and the norms satisfy*

$$\|GF\| \leqslant \|G\|\|F\|, \quad \|F^*F\| = \|F\|^2 \tag{2.67}$$

for all $F \in \mathrm{Hom}(i,j), G \in \mathrm{Hom}(j,k)$.

Proof. For any $F \in \mathrm{Hom}_V(W_i, W_j)$, we let F^* be the formal adjoint of F , i.e. the unique homomorphism $F^* \in \mathrm{Hom}_V(W_j, W_i)$ satisfying $\langle Fw^{(i)}|w^{(j)}\rangle = \langle w^{(i)}|F^*w^{(j)}\rangle$ for all $w^{(i)} \in W_i, w^{(j)} \in W_j$. The existence of F^* follows from lemma 2.20 applied to $W \simeq W_i \oplus^{\perp} W_j$. Let $\|F\|$ be the operator norm of F, i.e., $\|F\| = \sup_{w^{(i)}\in W_i\setminus\{0\}}(\|Fw^{(i)}\|/\|w^{(i)}\|)$. Using lemma 2.20, it is easy to check that $\mathrm{Rep}^{\mathrm{u}}(V)$ satisfies all the conditions to be a C^*-category. □

It is not clear whether unitarizable V-modules are closed under tensor product. So it may not be a good idea to define a structure of a ribbon category on $\mathrm{Rep}^{\mathrm{u}}(V)$. We consider instead certain subcategories. Let $\mathcal{G}$ be a collection of unitary V-modules. We say that $\mathcal{G}$ is **additively closed**, if the following conditions are satisfied:
(1) If $i \in \mathcal{G}$ and W_j is isomorphic to a submodule of W_i, then $j \in \mathcal{G}$.
(2) If $i_1, i_2, \dots, i_n \in \mathcal{G}$, then $W_{i_1} \oplus^\perp W_{i_2} \oplus^\perp \cdots \oplus^\perp W_{i_n} \in \mathcal{G}$.
If $\mathcal{G}$ is additively closed, we define the additive category $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{G}}(V)$ to be the subcategory of $\mathrm{Rep}^{\mathrm{u}}(V)$ whose objects are elements in $\mathcal{G}$.

We say that $\mathcal{G}$ is **multiplicatively closed**, if $\mathcal{G}$ is additively closed, and the following conditions are satisfied:
(a) $0 \in \mathcal{G}$.
(b) If $i \in \mathcal{G}$, then $\bar{i} \in \mathcal{G}$.
(c) If $i, j \in \mathcal{G}$, then $W_{ij} = W_i \boxtimes W_j$ is unitarizable, and any unitarization of W_{ij} is inside $\mathcal{G}$.

Suppose that $\mathcal{G}$ is multiplicatively closed. A **unitary structure** on $\mathcal{G}$ assigns to each triplet $(i, j, k) \in \mathcal{G} \times \mathcal{G} \times \mathcal{E}$ an inner product on $\mathcal{V}{k \choose i\ j}^*$. For each unitary structure on $\mathcal{G}$, we define $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{G}}(V)$ to be a ribbon category in the following way: If $i, j \in \mathcal{G}$, then W_{ij} as a V-module is defined, as before, to be $\bigoplus_{k\in\mathcal{E}} \mathcal{V}{k \choose i\ j}^* \otimes W_k$. Since $\mathcal{G}$ is multiplicatively closed, each W_k in $\mathcal{E}$ satisfying $N^k_{ij} > 0$ must be equipped with a unitary structure. Hence the inner products on all $\mathcal{V}{k \choose i\ j}^*$'s give rise to a unitary structure on W_{ij}. W_{ij} now becomes a unitary V-module. The other functors and structural maps are defined in the same way as we did for $\mathrm{Rep}(V)$. Clearly $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{G}}(V)$ is a ribbon fusion category and is equivalent to a ribbon fusion subcategory of $\mathrm{Rep}(V)$.

Our main goal in this two-part series is to define a unitary structure on $\mathcal{G}$, under which $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{G}}(V)$ becomes a unitary ribbon fusion category. More explicitly, we want to show (cf. [Gal12]) that for any $i_1, i_2, j_1, j_2 \in \mathcal{G}$ and any $F \in \mathrm{Hom}_V(W_{i_1}, W_{i_2}), G \in \mathrm{Hom}_V(W_{j_1}, W_{j_2})$,

$$(F \otimes G)^* = F^* \otimes G^*; \tag{2.68}$$

that the associators, the operators λ_i, ρ_i ($i \in \mathcal{G}$), and the braid operators of $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{G}}(V)$ are unitary; and that for each $i \in \mathcal{G}$, ϑ_i is unitary, and ev_i and coev_i can be chosen in such a way that the following equations hold:

$$(\mathrm{coev}_i)^* = \mathrm{ev}_i \circ \sigma_{i,\bar{i}} \circ (\vartheta_i \otimes \mathrm{id}_{\bar{i}}), \tag{2.69}$$

$$(\mathrm{ev}_i)^* = (\mathrm{id}_{\bar{i}} \otimes \vartheta_i^{-1}) \circ \sigma^{-1}_{\bar{i},i} \circ \mathrm{coev}_i. \tag{2.70}$$

3 Analytic aspects of vertex operator algebras

3.1 Intertwining operators with energy bounds

The energy bounds conditions for vertex operators are important when one tries to construct conformal nets/loop groups representations from unitary VOAs/infinite dimensional Lie algebras. This can be seen, for instance, in [GW84], [BS90], and [CKLW15]. In this section, we generalize this notion to intertwining operators of VOAs.

We assume in this chapter that V is unitary. If W_i is a unitary V-module, we let the Hilbert space $\mathcal{H}_i$ be the norm completion of W_i, and view W_i as a norm-dense subspace of $\mathcal{H}_i$. It is clear that the unbounded operator L_0 on $\mathcal{H}_i$ (with domain W_i) is essentially self-adjoint, and its closure $\overline{L_0}$ is positive.

Now for any $r \in \mathbb{R}$, we let $\mathcal{H}_i^r$ be the domain $\mathscr{D}((1+\overline{L_0})^r)$ of $(1+\overline{L_0})^r$. If $\xi \in \mathcal{H}_i^r$, we define the r-th order Sobolev norm to be

$$\|\xi\|_r = \|(1+\overline{L_0})^r \xi\|.$$

Note that the 0-th Sobolev norm is just the vector norm. We let

$$\mathcal{H}_i^\infty = \bigcap_{r \geqslant 0} \mathcal{H}_i^r.$$

Clearly $\mathcal{H}_i^\infty$ contains W_i. Vectors inside $\mathcal{H}_i^\infty$ are said to be **smooth**.

Definition 3.1. Let W_i, W_j, W_k be unitary V-modules, $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\,j}$, and $w^{(i)} \in W_i$ be homogeneous. Choose $r \geqslant 0$. We say that $\mathcal{Y}_\alpha(w^{(i)}, x)$ satisfies **r-th order energy bounds**, if there exist $M \geqslant 0, t \geqslant 0$, such that for any $s \in \mathbb{R}$ and $w^{(j)} \in W_j$,

$$\|\mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}\| \leqslant M(1+|s|)^t \|w^{(j)}\|_r. \tag{3.1}$$

Here $\mathcal{Y}_\alpha(w^{(i)}, s)$ is the s-th mode of the intertwining operator $\mathcal{Y}_\alpha(w^{(i)}, x)$. It is clear that if $r_1 \leqslant r_2$ and $\mathcal{Y}_\alpha(w^{(i)}, x)$ satisfies r_1-th order energy bounds, then $\mathcal{Y}_\alpha(w^{(i)}, x)$ also satisfies r_2-th order energy bounds.

1-st order energy bounds are called **linear energy bounds**. We say that $\mathcal{Y}_\alpha(w^{(i)}, x)$ is **energy-bounded** if it satisfies energy bounds of some positive order. If for every $w^{(i)} \in W_i$, $\mathcal{Y}_\alpha(w^{(i)}, x)$ is energy-bounded, we say that $\mathcal{Y}_\alpha$ is **energy-bounded**. A unitary V-module W_i is called **energy-bounded** if Y_i is energy-bounded. The unitary VOA V is called **energy-bounded** if the vacuum module $V = W_0$ is energy-bounded. We now prove some useful properties concerning energy-boundedness.

Proposition 3.2. *If $w^{(i)} \in W_i$ is homogeneous and inequality (3.1) holds, then for any $p \in \mathbb{R}$, there exists $M_p > 0$ such that for any $w^{(j)} \in W_j$,*

$$\|\mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}\|_p \leqslant M_p(1+|s|)^{|p|+t} \|w^{(j)}\|_{p+r}. \tag{3.2}$$

Proof. (cf. [TL04] chapter II proposition 1.2.1) We want to show that

$$\|\mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}\|_p^2 \leqslant M_p^2(1+|s|)^{2(|p|+t)} \|w^{(j)}\|_{p+r}^2. \tag{3.3}$$

Since

$$\|\mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}\|_p^2 = \sum_{q \in \mathbb{R}} \|P_{q-s-1+\Delta_{w^{(i)}}} \mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}\|_p^2 = \sum_{q \in \mathbb{R}} \|\mathcal{Y}_\alpha(w^{(i)}, s)P_q w^{(j)}\|_p^2,$$

$$\|w^{(j)}\|_{p+r}^2 = \sum_{q\in\mathbb{R}} \|P_q w^{(j)}\|_{p+r}^2,$$

it suffices to assume that $w^{(j)}$ is homogeneous. We also assume that $\mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)} \neq 0$. Then by proposition 1.7, $\Delta_{w^{(i)}} + \Delta_{w^{(j)}} - 1 - s \geqslant 0$.

By (3.1) we have

$$\|\mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}\|^2 \leqslant M^2(1+|s|)^{2t}(1+\Delta_{w^{(j)}})^{2r}\|w^{(j)}\|^2. \tag{3.4}$$

Hence

$$\begin{aligned}
&\|\mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}\|_p^2\\
=&(\Delta_{w^{(i)}} + \Delta_{w^{(j)}} - s)^{2p}\|\mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}\|^2\\
\leqslant&(\Delta_{w^{(i)}} + \Delta_{w^{(j)}} - s)^{2p}M^2(1+|s|)^{2t}(1+\Delta_{w^{(j)}})^{2r}\|w^{(j)}\|^2\\
=&M^2\Big(\frac{\Delta_{w^{(i)}} + \Delta_{w^{(j)}} - s}{1+\Delta_{w^{(j)}}}\Big)^{2p}(1+|s|)^{2t}(1+\Delta_{w^{(j)}})^{2(p+r)}\|w^{(j)}\|^2\\
=&M^2\Big(\frac{\Delta_{w^{(i)}} - s + \Delta_{w^{(j)}}}{1+\Delta_{w^{(j)}}}\Big)^{2p}(1+|s|)^{2t}\|w^{(j)}\|_{p+r}^2.
\end{aligned} \tag{3.5}$$

If $p \geqslant 0$ then

$$\begin{aligned}
&\Big(\frac{\Delta_{w^{(i)}} - s + \Delta_{w^{(j)}}}{1+\Delta_{w^{(j)}}}\Big)^{2p}\\
\leqslant&\Big(\frac{1+\Delta_{w^{(i)}} + |s| + \Delta_{w^{(j)}}}{1+\Delta_{w^{(j)}}}\Big)^{2p}\\
\leqslant&(1+\Delta_{w^{(i)}} + |s|)^{2p} \leqslant (1+\Delta_{w^{(i)}})^{2p}(1+|s|)^{2p}.
\end{aligned} \tag{3.6}$$

If $p < 0$ and $1 \leqslant \Delta_{w^{(i)}} - s$, then

$$\begin{aligned}
&\Big(\frac{\Delta_{w^{(i)}} - s + \Delta_{w^{(j)}}}{1+\Delta_{w^{(j)}}}\Big)^{2p}\\
=&\Big(\frac{1+\Delta_{w^{(j)}}}{\Delta_{w^{(i)}} - s + \Delta_{w^{(j)}}}\Big)^{2|p|} \leqslant 1.
\end{aligned} \tag{3.7}$$

If $p < 0$ and $1 \geqslant \Delta_{w^{(i)}} - s$, then since $\Delta_{w^{(i)}} - s + \Delta_{w^{(j)}} \geqslant 1$,

$$\begin{aligned}
&\Big(\frac{1+\Delta_{w^{(j)}}}{\Delta_{w^{(i)}} - s + \Delta_{w^{(j)}}}\Big)^{2|p|}\\
=&\Big(1+\frac{1+s-\Delta_{w^{(i)}}}{\Delta_{w^{(i)}} - s + \Delta_{w^{(j)}}}\Big)^{2|p|}\\
\leqslant&(2-\Delta_{w^{(i)}} + s)^{2|p|}\\
\leqslant&(2+2\Delta_{w^{(i)}} + 2|s|)^{2|p|}\\
\leqslant&2^{2|p|}(1+\Delta_{w^{(i)}})^{2|p|}(1+|s|)^{2|p|}.
\end{aligned} \tag{3.8}$$

Therefore, if we let $M_p = 2^{|p|}(1+\Delta_{w^{(i)}})^{|p|}$, then (3.3) is always true. $\square$

The next property is obvious.

Proposition 3.3. *If $\mathcal{Y} \in \mathcal{V}\binom{k}{i\,j}$ is unitary, $w^{(i)} \in W_i$ is homogeneous, and $\mathcal{Y}_\alpha(w^{(i)},x)$ satisfies r-th order energy bounds, then $\mathcal{Y}_{\overline{\alpha}}(\overline{w^{(i)}},x)$ satisfies r-th order energy bounds.*

Proposition 3.4. *Suppose that $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\,j}$ is unitary, $w^{(i)} \in W_i$ is homogeneous, $r \geqslant 0$, and for any $m \in \mathbb{Z}_{\geqslant 0}$, $\mathcal{Y}_\alpha(L_1^m w^{(i)},x)$ satisfies r-th order energy bounds. Then $\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}},x)$ and $\mathcal{Y}_{C^{\pm 1}\alpha}(w^{(i)},x)$ satisfy r-th order energy bounds.*

Proof. First we note that $L_1^m w^{(i)} = 0$ for m large enough. Now suppose that (3.1) holds for all m if we replace $w^{(i)}$ by $L_1^m w^{(i)}$. Then by (1.36), for any $w^{(j)} \in W_j$, $w^{(k)} \in W_k$ and $s \in \mathbb{R}$,

$$\begin{aligned}
&\Big|\Big\langle \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}},s)w^{(j)}\Big|w^{(k)}\Big\rangle\Big|\\
\leqslant&\sum_{m\geqslant 0}\frac{1}{m!}\Big|\Big\langle w^{(j)}\Big|\mathcal{Y}_\alpha(L_1^m w^{(i)},-s-k-2+2\Delta_{w^{(i)}})w^{(k)}\Big\rangle\Big|\\
=&\sum_{m\geqslant 0}\frac{1}{m!}\Big|\Big\langle (1+L_0)^r w^{(j)}\Big|(1+L_0)^{-r}\mathcal{Y}_\alpha(L_1^m w^{(i)},-s-k-2+2\Delta_{w^{(i)}})w^{(k)}\Big\rangle\Big|\\
\leqslant&\sum_{m\geqslant 0}\frac{1}{m!}\big\|w^{(j)}\big\|_r\big\|\mathcal{Y}_\alpha(L_1^m w^{(i)},-s-k-2+2\Delta_{w^{(i)}})w^{(k)}\big\|_{-r}.
\end{aligned}$$

By proposition 3.2, we can find positive numbers C_1, C_2 independent of $w^{(j)}, w^{(k)}$, such that

$$\begin{aligned}
&\big\|\mathcal{Y}_\alpha(L_1^m w^{(i)},-s-m-2+2\Delta_{w^{(i)}})w^{(k)}\big\|_{-r}\\
\leqslant&C_1\big(1+|s+m+2-2\Delta_{w^{(i)}}|\big)^{r+t}\big\|w^{(k)}\big\|\\
\leqslant&C_2\big(1+|s|\big)^{r+t}\big\|w^{(k)}\big\|.
\end{aligned}$$

Thus there exists $C_3 > 0$ independent of $w^{(j)}, w^{(k)}$, such that

$$\Big|\Big\langle \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}},s)w^{(j)}\Big|w^{(k)}\Big\rangle\Big| \leqslant C_3\big(1+|s|\big)^{r+t}\big\|w^{(j)}\big\|_r\big\|w^{(k)}\big\|.$$

This proves that

$$\big\|\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}},s)w^{(j)}\big\| \leqslant C_3\big(1+|s|\big)^{r+t}\big\|w^{(j)}\big\|_r. \tag{3.9}$$

Therefore $\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}},x)$ satisfies r-th order energy bounds. Since $C\alpha = \overline{\alpha^*}$ and $\mathcal{Y}_{C^{-1}\alpha}(w^{(i)},x) = e^{2i\pi\Delta_{w^{(i)}}}\mathcal{Y}_{C\alpha}(w^{(i)},x)$, by proposition 3.3, $\mathcal{Y}_{C^{\pm1}\alpha}(w^{(i)},x)$ also satisfy r-th order energy bounds. □

Proposition 3.5. *Let W_i, W_j, W_k be unitary V-modules, $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\,j}$, and choose homogeneous vectors $w^{(i)} \in W_i, u \in V$. Suppose that $\mathcal{Y}_\alpha(w^{(i)},x), Y_j(u,x), Y_k(u,x)$ are energy-bounded. Then for any $n \in \mathbb{Z}$, $\mathcal{Y}_\alpha\big(Y_i(u,n)w^{(i)},x\big)$ is energy-bounded.*

Proof. By Jacobi Identity, for any $s \in \mathbb{R}$ we have

$$\begin{aligned}
&\mathcal{Y}_\alpha(Y_i(u,n)w^{(i)},s)\\
=&\sum_{l\in\mathbb{Z}_{\geqslant 0}}(-1)^l\binom{n}{l}Y_k(u,n-l)\mathcal{Y}_\alpha(w^{(i)},s+l)-\sum_{l\in\mathbb{Z}_{\geqslant 0}}(-1)^{l+n}\binom{n}{l}\mathcal{Y}_\alpha(w^{(i)},n+s-l)Y_j(u,l).
\end{aligned} \tag{3.10}$$

It can be shown by induction on $|n|$ that

$$\limsup_{l\to\infty}\left|\binom{n}{l}\right|l^{-|n|}<+\infty.$$

Choose any homogeneous vector $w^{(j)} \in W_j$ with energy $\Delta_{w^{(j)}}$. Then by energy-boundedness of $\mathcal{Y}_\alpha(w^{(i)},x), Y_j(u,x), Y_k(u,x)$ and proposition 3.2, there exist positive constants $C_1, C_2, \dots, C_8$ and $r_1, t_1, r_2, t_2, r_3, t_3$ independent of $w^{(j)}$ and s, such that

$$\begin{aligned}
&\Big\|\sum_{l\geqslant 0}(-1)^{l+n}\binom{n}{l}\mathcal{Y}_\alpha(w^{(i)},n+s-l)Y_j(u,l)w^{(j)}\Big\|\\
\leqslant&\sum_{l\geqslant 0}C_1l^{|n|}\big\|\mathcal{Y}_\alpha(w^{(i)},n+s-l)Y_j(u,l)w^{(j)}\big\|\\
\leqslant&\sum_{l\geqslant 0}C_2l^{|n|}\big(1+|n+s-l|\big)^{t_1}\big\|Y_j(u,l)w^{(j)}\big\|_{r_1}\\
\leqslant&\sum_{0\leqslant l\leqslant\Delta_u+\Delta_{w^{(j)}}-1}C_3l^{|n|}(1+|s|)^{t_1}(1+l)^{t_1}\cdot(1+l)^{r_1+t_2}\|w^{(j)}\|_{r_1+r_2}\\
\leqslant&\sum_{0\leqslant l\leqslant\Delta_u+\Delta_{w^{(j)}}-1}C_3(1+|s|)^{t_1}(1+l)^{|n|+t_1+r_1+t_2}\|w^{(j)}\|_{r_1+r_2}\\
\leqslant&C_4(1+|s|)^{t_1}(1+\Delta_{w^{(j)}})^{1+|n|+t_1+r_1+t_2}\|w^{(j)}\|_{r_1+r_2}\\
=&C_4(1+|s|)^{t_1}\|w^{(j)}\|_{1+|n|+t_1+t_2+2r_1+r_2}.
\end{aligned} \tag{3.11}$$

Here the inequality $l\leqslant\Delta_u+\Delta_{w^{(j)}}-1$ comes from the fact that every nonzero $Y_j(u,l)w^{(j)}$ must have non-negative energy. Similarly we have

$$\begin{aligned}
&\Big\|\sum_{l\geqslant 0}(-1)^{l}\binom{n}{l}Y_k(u,n-l)\mathcal{Y}_\alpha(w^{(i)},s+l)w^{(j)}\Big\|\\
\leqslant&\sum_{l\geqslant 0}C_5l^{|n|}\big\|Y_k(u,n-l)\mathcal{Y}_\alpha(w^{(i)},s+l)w^{(j)}\big\|\\
\leqslant&\sum_{l\geqslant 0}C_6l^{|n|}(1+|n-l|)^{t_3}\big\|\mathcal{Y}_\alpha(w^{(i)},s+l)w^{(j)}\big\|_{r_3}\\
\leqslant&\sum_{0\leqslant l\leqslant\Delta_{w^{(i)}}+\Delta_{w^{(j)}}-s-1}C_7l^{|n|}(1+l)^{t_3}(1+|s+l|)^{r_3+t_2}\|w^{(j)}\|_{r_3+r_2}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{0\leqslant l\leqslant \Delta_{w^{(i)}}+\Delta_{w^{(j)}}-s-1} C_7(1+|s|)^{r_3+t_2}(1+l)^{|n|+t_3+r_3+t_2}\|w^{(j)}\|_{r_3+r_2}\\
&\leqslant C_8(1+|s|)^{r_3+t_2}(1+\Delta_{w^{(j)}}+|s|)^{1+|n|+t_3+r_3+t_2}\|w^{(j)}\|_{r_3+r_2}\\
&\leqslant C_8(1+|s|)^{2r_3+2t_2+1+|n|+t_3}(1+\Delta_{w^{(j)}})^{1+|n|+t_3+r_3+t_2}\|w^{(j)}\|_{r_3+r_2}\\
&= C_8(1+|s|)^{2r_3+2t_2+1+|n|+t_3}\|w^{(j)}\|_{2r_3+r_2+t_2+1+|n|+t_3}.
\end{aligned}
\tag{3.12}
$$

The energy-boundedness of $\mathcal{Y}_\alpha(Y_i(u,n)w^{(i)},x)$ follows from these two inequalities. □

The following proposition is also very useful. One can prove it using the argument in [BS90] section 2.

Proposition 3.6. *If $v=\nu$ or $v\in V(1)$, then for any unitary V-module W_i, $Y_i(v,x)$ satisfies linear energy bounds.*

We summarize the results in this section as follows:

Corollary 3.7. *Let W_i, W_j, W_k be unitary V-modules, and $\mathcal{Y}_\alpha\in\mathcal{V}\binom{k}{i\ j}$.*
(a) Suppose that V is generated by a set E of homogeneous vectors. If for each $v\in E$, $Y_i(v,x)$ is energy-bounded, then Y_i is energy-bounded.
(b) If W_i is irreducible, Y_j, Y_k are energy-bounded, and there exists a nonzero homogeneous vector $w^{(i)}\in W_i$ such that $\mathcal{Y}_\alpha(w^{(i)},x)$ is energy-bounded, then $\mathcal{Y}_\alpha$ is energy-bounded.
(c) If $w^{(i)}\in W_i$ is homogeneous, and $\mathcal{Y}_\alpha(w^{(i)},x)$ is energy-bounded, then $\mathcal{Y}_{C^{\pm1}\alpha}(w^{(i)},x)$, $\mathcal{Y}_{\overline{\alpha}}(\overline{w^{(i)}},x)$, and $\mathcal{Y}_{\alpha*}(\overline{w^{(i)}},x)$ are energy-bounded.
(d) If $w^{(i)}\in W_i$ is quasi-primary, and $\mathcal{Y}_\alpha(w^{(i)},x)$ satisfies r-th order energy bounds. Then $\mathcal{Y}_{C^{\pm1}\alpha}(w^{(i)},x)$, $\mathcal{Y}_{\overline{\alpha}}(\overline{w^{(i)}},x)$, and $\mathcal{Y}_{\alpha*}(\overline{w^{(i)}},x)$ satisfy r-th order energy bounds.

Proof. (a) and (b) follow from proposition 3.5. (c) follows from propositions 3.3, 3.4, 3.5, and 3.6. (d) follows from propositions 3.3 and 3.4. □

3.2 Smeared intertwining operators

In this section, we construct smeared intertwining operators for energy-bounded intertwining operators, and prove the adjoint relation, the braid relations, the rotation covariance, and the intertwining property for these operators. The proof of the last property requires some knowledge of the strong commutativity of unbounded closed operators on a Hilbert space. We give a brief exposition of this theory in chapter B.

The unbounded operator $\mathcal{Y}_\alpha(w^{(i)},f)$

For any open subset I of S^1, we denote by $C_c^\infty(I)$ the set of all complex smooth functions on S^1 whose supports lie in I. If $I=\{e^{it}: a<t<b\}$ ($a,b\in\mathbb{R}, a<b<a+2\pi$), we say that I is an **open interval** of S^1. We let $\mathcal{J}$ be the set of all open intervals of S^1. In general, if U is an open subset of S^1, we let $\mathcal{J}(U)$ be the set of open intervals of S^1 contained in U.

If $I \in \mathcal{J}$, then its **complement** I^c is defined to be $S^1 \backslash \overline{I}$. If $I_1, I_2 \in \mathcal{J}$, we write $I_1 \subset\subset I_2$ if $\overline{I_1} \subset I_2$.

Let $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\,j}$ be unitary. (Recall that this means that W_i, W_j, W_k are unitary V-modules.) For any $w^{(i)} \in W_i, z \in \mathbb{C}^\times$, $\mathcal{Y}_\alpha(w^{(i)}, z)$ is a linear map $W_j \to \widehat{W}_k$. Therefore we can regard $\mathcal{Y}_\alpha(w^{(i)}, z)$ as a sesquilinear form $W_j \times W_k \to \mathbb{C}, (w^{(j)}, w^{(k)}) \mapsto \langle \mathcal{Y}_\alpha(w^{(i)}, z) w^{(j)} | w^{(k)} \rangle$.

We now define the smeared intertwining operators. Let $\text{đ}\theta = \frac{e^{i\theta}}{2\pi} d\theta$. For any $f \in C_c^\infty(S^1 \backslash \{-1\})$, we define a sesquilinear form

$$\mathcal{Y}_\alpha(w^{(i)}, f) : W_j \times W_k \to \mathbb{C}, \quad (w^{(j)}, w^{(k)}) \mapsto \langle \mathcal{Y}_\alpha(w^{(i)}, f) w^{(j)} | w^{(k)} \rangle$$

satisfying

$$\langle \mathcal{Y}_\alpha(w^{(i)}, f) w^{(j)} | w^{(k)} \rangle = \int_{-\pi}^{\pi} \langle \mathcal{Y}_\alpha(w^{(i)}, e^{i\theta}) w^{(j)} | w^{(k)} \rangle f(e^{i\theta}) \text{đ}\theta. \tag{3.13}$$

$\mathcal{Y}_\alpha(w^{(i)}, f)$ can be regarded as a linear map $W_j \to \widehat{W}_k$. In the following, we show that when $\mathcal{Y}_\alpha(w^{(i)}, x)$ is energy-bounded, $\mathcal{Y}_\alpha(w^{(i)}, f)$ is a preclosed unbounded operator.

To begin with, we note that for any $f \in C_c^\infty(S^1 \backslash \{-1\})$ and any $s \in \mathbb{R}$, the s-th mode of f is

$$\widehat{f}(s) = \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-is\theta} \cdot \frac{d\theta}{2\pi}. \tag{3.14}$$

Then we have

$$\mathcal{Y}_\alpha(w^{(i)}, f) = \sum_{s \in \mathbb{R}} \mathcal{Y}_\alpha(w^{(i)}, s) \widehat{f}(s). \tag{3.15}$$

Define

$$\begin{gathered}\mathcal{D}_V = \{\Delta_i + \Delta_j - \Delta_k : \ W_i, W_j, W_k \text{ are irreducible } V\text{-modules}\}, \\ \mathbb{Z}_V = \mathbb{Z} + \mathcal{D}_V.\end{gathered}$$

Then $\mathcal{Y}_\alpha(w^{(i)}, s) = 0$ except possibly when $s \in \mathbb{Z}_V$. Since V has finitely many equivalence classes of irreducible representations, the set $\mathcal{D}_V$ is finite. Now for any $t \in \mathbb{R}$ we define a norm $|\cdot|_{V,t}$ on $C_c^\infty(S^1 \backslash \{-1\})$ to be

$$|f|_{V,t} = \sum_{s \in \mathbb{Z}_V} (1 + |s|)^t |\widehat{f}(s)|, \tag{3.16}$$

which is easily seen to be finite. For each $r \in \mathbb{R}$, we define $e_r : S^1 \backslash \{-1\} \to \mathbb{C}$ to be

$$e_r(e^{i\theta}) = e^{ir\theta}, \quad (-\pi < \theta < \pi). \tag{3.17}$$

When $r \in \mathbb{Z}$, we regard e_r as a continuous function on S^1.

Lemma 3.8. *Suppose that $w^{(i)} \in W_i$ is homogeneous, and $\mathcal{Y}_\alpha(w^{(i)}, x)$ is energy-bounded and satisfies condition* (3.1).
(a) Let $p \in \mathbb{R}$. Then there exists $M_p \geqslant 0$, such that for any $f \in C_c^\infty(S^1 \backslash \{-1\})$, $w^{(j)} \in W_j$, we have $\mathcal{Y}_\alpha(w^{(i)}, f)w^{(j)} \in \mathcal{H}_k^\infty$, and

$$\big\|\mathcal{Y}_\alpha(w^{(i)}, f)w^{(j)}\big\|_p \leqslant M_p |f|_{V,|p|+t} \big\|w^{(j)}\big\|_{p+r}. \tag{3.18}$$

(b) For any $w^{(j)} \in W_j, w^{(k)} \in W_k$ we have

$$\langle w^{(k)}|\mathcal{Y}_\alpha(w^{(i)}, f)w^{(j)}\rangle = \sum_{m \geqslant 0} \frac{e^{-i\pi\Delta_{w^{(i)}}}}{m!} \langle \mathcal{Y}_{\alpha*}(\overline{L_1^m w^{(i)}}, \overline{e_{(m+2-2\Delta_{w^{(i)}})} f})w^{(k)}|w^{(j)}\rangle. \tag{3.19}$$

Proof. (a) We have

$$\mathcal{Y}_\alpha(w^{(i)}, f)w^{(j)} = \sum_{s \in \mathbb{Z}_V} \widehat{f}(s)\mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}. \tag{3.20}$$

Choose $M_p \geqslant 0$ such that (3.2) always holds. Then

$$\begin{aligned}
&\sum_{s \in \mathbb{Z}_V} \big\|\widehat{f}(s)\mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}\big\|_p \\
\leqslant &\sum_{s \in \mathbb{Z}_V} M_p \big|\widehat{f}(s)\big|(1+|s|)^{|p|+t}\|w^{(j)}\|_{p+r} \\
= &M_p |f|_{V,|p|+t} \big\|w^{(j)}\big\|_{p+r}.
\end{aligned} \tag{3.21}$$

In particular, $\mathcal{Y}_\alpha(w^{(i)}, f)w^{(j)} \in \mathcal{H}_k^\infty$.
(b) For any $w^{(j)} \in W_j, w^{(k)} \in W_k$, and $z \in \mathbb{C}^\times$ with argument $\arg z$, we have

$$\begin{aligned}
&\langle w^{(k)}|\mathcal{Y}_\alpha(w^{(i)}, z)w^{(j)}\rangle \\
= &\langle \mathcal{Y}_{\alpha*}(e^{\overline{z}L_1}(e^{-i\pi}\overline{z}^{-2})^{L_0}\overline{w^{(i)}}, \overline{z}^{-1})w^{(k)}|w^{(j)}\rangle \\
= &e^{-i\pi\Delta_{w^{(i)}}} \sum_{m \geqslant 0} \frac{\overline{z}^{m-2\Delta_{w^{(i)}}}}{m!} \langle \mathcal{Y}_{\alpha*}(\overline{L_1^m w^{(i)}}, \overline{z}^{-1})w^{(k)}|w^{(j)}\rangle.
\end{aligned} \tag{3.22}$$

Note also that $\overline{d\!\!\!/\theta} = e^{-2i\theta} d\!\!\!/\theta$. Therefore we have

$$\begin{aligned}
&\langle w^{(k)}|\mathcal{Y}_\alpha(w^{(i)}, f)w^{(j)}\rangle \\
= &\int_{-\pi}^{\pi} \langle w^{(k)}|\mathcal{Y}_\alpha(w^{(i)}, e^{i\theta})w^{(j)}\rangle \overline{f(e^{i\theta})d\!\!\!/\theta} \\
= &\sum_{m \geqslant 0} \int_{-\pi}^{\pi} \frac{e^{-i\pi\Delta_{w^{(i)}}}}{m!} \langle \mathcal{Y}_{\alpha*}(\overline{L_1^m w^{(i)}}, e^{i\theta})w^{(k)}|w^{(j)}\rangle e^{-i(m+2-2\Delta_{w^{(i)}})\theta} \overline{f(e^{i\theta})} d\!\!\!/\theta \\
= &\sum_{m \geqslant 0} \frac{e^{-i\pi\Delta_{w^{(i)}}}}{m!} \langle \mathcal{Y}_{\alpha*}(\overline{L_1^m w^{(i)}}, \overline{e_{(m+2-2\Delta_{w^{(i)}})} f})w^{(k)}|w^{(j)}\rangle.
\end{aligned} \tag{3.23}$$

□

By lemma 3.8, if $w^{(i)}$ is homogeneous and $\mathcal{Y}_\alpha(w^{(i)},x)$ is energy-bounded, then $\mathcal{Y}_\alpha(w^{(i)},f)$ can be viewed as an unbounded operator from $\mathcal{H}_j$ to $\mathcal{H}_k$ with domain W_j. Moreover, the domain of $\mathcal{Y}_\alpha(w^{(i)},f)^*$ contains a dense subspace of $\mathcal{H}_k$ (which is W_k). So $\mathcal{Y}_\alpha(w^{(i)},f)$ *is preclosed*. We let $\overline{\mathcal{Y}_\alpha(w^{(i)},f)}$ be its closure . By inequality (3.18), $\mathcal{H}^\infty_j$ is inside $\mathscr{D}(\overline{\mathcal{Y}_\alpha(w^{(i)},f)})$, the domain of $\overline{\mathcal{Y}_\alpha(w^{(i)},f)}$, and $\overline{\mathcal{Y}_\alpha(w^{(i)},f)}\mathcal{H}^\infty_j\subset\mathcal{H}^\infty_k$. In the following, we will always view $\mathcal{Y}_\alpha(w^{(i)},f):\mathcal{H}^\infty_j\to\mathcal{H}^\infty_k$ as the restriction of $\overline{\mathcal{Y}_\alpha(w^{(i)},f)}$ to $\mathcal{H}^\infty_j$. $\mathcal{Y}_\alpha(w^{(i)},f)$ is called a **smeared intertwining operator**. The closed operator $\mathcal{Y}_\alpha(w^{(i)},f)^*=\overline{\mathcal{Y}_\alpha(w^{(i)},f)}^*$ is the adjoint of $\mathcal{Y}_\alpha(w^{(i)},f)$. The **formal adjoint** of $\mathcal{Y}_\alpha(w^{(i)},f)$, which is denoted by $\mathcal{Y}_\alpha(w^{(i)},f)^\dagger$, is the restriction of $\mathcal{Y}_\alpha(w^{(i)},f)^*$ to $\mathcal{H}^\infty_k\to\mathcal{H}^\infty_j$.

The following proposition follows directly from lemma 3.8.

Proposition 3.9. *Suppose that $w^{(i)}\in W_i$ is homogeneous, $\mathcal{Y}_\alpha(w^{(i)},x)$ is energy-bounded and satisfies condition (3.1). Then for any $f\in C_c(S^1\backslash\{-1\})$, the following statements are true:*
(a) $\mathcal{Y}_\alpha(w^{(i)},f)\mathcal{H}^\infty_j\subset\mathcal{H}^\infty_k$. Moreover, for any $p\in\mathbb{R}$, there exists $M_p\geqslant 0$ independent of f, such that for any $\xi^{(j)}\in\mathcal{H}^\infty_j$, we have

$$\big\|\mathcal{Y}_\alpha(w^{(i)},f)\xi^{(j)}\big\|_p\leqslant M_p|f|_{V,|p|+t}\big\|\xi^{(j)}\big\|_{p+r}.\tag{3.24}$$

(b) $\mathcal{Y}_\alpha(w^{(i)},f):\mathcal{H}^\infty_j\to\mathcal{H}^\infty_k$ has the formal adjoint $\mathcal{Y}_\alpha(w^{(i)},f)^\dagger:\mathcal{H}^\infty_k\to\mathcal{H}^\infty_j$, which satisfies

$$\mathcal{Y}_\alpha(w^{(i)},f)^\dagger=\sum_{m\geqslant 0}\frac{e^{-i\pi\Delta_{w^{(i)}}}}{m!}\mathcal{Y}_{\alpha^*}(\overline{L_1^m w^{(i)}},\overline{e_{(m+2-2\Delta_{w^{(i)}})}f}).\tag{3.25}$$

In particular, if $w^{(i)}$ is quasi-primary, then we have the adjoint relation

$$\mathcal{Y}_\alpha(w^{(i)},f)^\dagger=e^{-i\pi\Delta_{w^{(i)}}}\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}},\overline{e_{(2-2\Delta_{w^{(i)}})}f}).\tag{3.26}$$

Hence the adjoint relation (3.26) for smeared intertwining operators is established.

Remark 3.10. If $\mathcal{Y}_\alpha\in\mathcal{V}\binom{k}{i\ j}$ is a unitary energy-bounded intertwining operator of V, $w^{(i)}\in W_i$ is not necessarily homogeneous, and $f\in C_c^\infty(S^1\backslash\{-1\})$, then by linearity, we can define a preclosed operator $\mathcal{Y}_\alpha(w^{(i)},f):\mathcal{H}^\infty_j\to\mathcal{H}^\infty_k$ to be $\mathcal{Y}_\alpha(w^{(i)},f)=\sum_{s\in\mathbb{R}}\mathcal{Y}_\alpha(P_s w^{(i)},f)$. Proposition 3.9-(a) still holds in this case.

Remark 3.11. If W_i is a unitary V-module, then $Y_i\in\mathcal{V}\binom{i}{0\ i}$. Choose any vector $v\in V$. Since the powers of x in $Y(v,x)$ are integers, for each $z\in\mathbb{C}^\times$, $Y_i(v,z)$ does not depend on $\arg z$. Therefore, for any $f\in C_c^\infty(S^1)$, we can defined a smeared vertex operator $Y_i(v,f)$: $\mathcal{H}^\infty_i\to\mathcal{H}^\infty_i$ using (3.13).

Braiding of smeared intertwining operators

The relation between products of smeared intertwining operators and correlation functions is indicated as follows.

Proposition 3.12. *Let $\mathcal{Y}_{\alpha_1}, \mathcal{Y}_{\alpha_2}, \dots, \mathcal{Y}_{\alpha_n}$ be a chain of unitary energy-bounded intertwining operators of V with charge spaces $W_{i_1}, W_{i_2}, \dots, W_{i_n}$ respectively. Let W_j be the source space of $\mathcal{Y}_{\alpha_1}$, and let W_k be the target space of $\mathcal{Y}_{\alpha_n}$. Choose mutually disjoint $I_1, I_2, \dots, I_n \in \mathcal{J}(S^1 \backslash \{-1\})$. For each $m = 1, 2, \dots, n$ we choose $w^{(i_m)} \in W_{i_m}$ and $f_m \in C_c^\infty(I_m)$. Then for any $w^{(j)} \in W_j$ and $w^{(k)} \in W_k$,*

$$\begin{aligned}&\langle \mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1) w^{(j)} | w^{(k)} \rangle \\ =& \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \langle \mathcal{Y}_{\alpha_n}(w^{(i_n)}, e^{i\theta_n}) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, e^{i\theta_1}) w^{(j)} | w^{(k)} \rangle f_1(e^{i\theta_1}) \cdots f_n(e^{i\theta_n}) \cdot d\!\!\!/\theta_1 \cdots d\!\!\!/\theta_n.\end{aligned} \tag{3.27}$$

Proof.

$$\begin{aligned}&\sum_{s_1,\dots,s_n \in \mathbb{R}} \left\| P_{s_n} \mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n) P_{s_{n-1}} \mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, f_{n-1}) P_{s_{n-2}} \cdots P_{s_1} \mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1) w^{(j)} \right\| \\ =& \sum_{t_1,\dots,t_n \in \mathbb{Z}_V} \left\| \mathcal{Y}_{\alpha_n}(w^{(i_n)}, t_n) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, t_1) w^{(j)} \right\| \cdot \left| \widehat{f}_1(t_1) \cdots \widehat{f}_n(t_n) \right|,\end{aligned} \tag{3.28}$$

which, by proposition 3.2, is finite. Hence, for all $r_1, \dots, r_n, r_1/r_2, \dots, r_{n-1}/r_n \in [1/2, 1]$, the following functions of $s_1, \dots, s_n$:

$$\begin{aligned}\Big| &\langle P_{s_n} \mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n) P_{s_{n-1}} \cdots P_{s_1} \mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1) w^{(j)} | w^{(k)} \rangle \\ &\cdot r_1^{-\Delta_{w^{(i_1)}}} \cdots r_n^{-\Delta_{w^{(i_n)}}} \left(\frac{r_1}{r_2}\right)^{s_1} \cdots \left(\frac{r_{n-1}}{r_n}\right)^{s_{n-1}} r_n^{\Delta_{w^{(k)}}} \Big|\end{aligned} \tag{3.29}$$

are bounded by a constant multiplied by

$$\left| \langle P_{s_n} \mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n) P_{s_{n-1}} \cdots P_{s_1} \mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1) w^{(j)} | w^{(k)} \rangle \right|, \tag{3.30}$$

the sum of which over $s_1, \dots, s_n$ is finite. Therefore, if we always assume that $r_1, \dots, r_n > 0$ and $0 < r_1/r_2 < \cdots < r_{n-1}/r_n \leqslant 1$, then by dominated convergence theorem and relation (1.26),

$$\begin{aligned}&\langle \mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1) w^{(j)} | w^{(k)} \rangle \\ =& \sum_{s_1,\dots,s_n \in \mathbb{R}} \langle P_{s_n} \mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n) P_{s_{n-1}} \cdots P_{s_1} \mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1) w^{(j)} | w^{(k)} \rangle \\ =& \sum_{s_1,\dots,s_n \in \mathbb{R}} \lim_{r_1,\dots,r_n \to 1} \Big(\langle P_{s_n} \mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n) P_{s_{n-1}} \cdots P_{s_1} \mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1) w^{(j)} | w^{(k)} \rangle \\ &\cdot r_1^{-\Delta_{w^{(i_1)}}} \cdots r_n^{-\Delta_{w^{(i_n)}}} \left(\frac{r_1}{r_2}\right)^{s_1} \cdots \left(\frac{r_{n-1}}{r_n}\right)^{s_{n-1}} r_n^{\Delta_{w^{(k)}}} \Big) \\ =& \lim_{r_1,\dots,r_n \to 1} \sum_{s_1,\dots,s_n \in \mathbb{R}} \Big(\langle P_{s_n} \mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n) P_{s_{n-1}} \cdots P_{s_1} \mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1) w^{(j)} | w^{(k)} \rangle\end{aligned}$$

$$
\begin{aligned}
&\cdot r_1^{-\Delta_{w^{(i_1)}}} \cdots r_n^{-\Delta_{w^{(i_n)}}} \left(\frac{r_1}{r_2}\right)^{s_1} \cdots \left(\frac{r_{n-1}}{r_n}\right)^{s_{n-1}} r_n^{\Delta_{w^{(k)}}}\Bigg) \\
=& \lim_{r_1,\dots,r_n\to 1} \sum_{s_1,\dots,s_n\in\mathbb{R}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \langle P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, e^{i\theta_n}) P_{s_{n-1}} \\
&\cdots P_{s_1}\mathcal{Y}_{\alpha_1}(w^{(i_1)}, e^{i\theta_1}) w^{(j)} | w^{(k)}\rangle r_1^{-\Delta_{w^{(i_1)}}} \cdots r_n^{-\Delta_{w^{(i_n)}}} \\
&\cdot \left(\frac{r_1}{r_2}\right)^{s_1} \cdots \left(\frac{r_{n-1}}{r_n}\right)^{s_{n-1}} r_n^{\Delta_{w^{(k)}}} f_1(e^{i\theta_1}) \cdots f_n(e^{i\theta_n}) d\!\!\!/\theta_1 \cdots d\!\!\!/\theta_n \\
=& \lim_{r_1,\dots,r_n\to 1} \sum_{s_1,\dots,s_n\in\mathbb{R}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \langle P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, r_n e^{i\theta_n}) P_{s_{n-1}} \\
&\cdots P_{s_1}\mathcal{Y}_{\alpha_1}(w^{(i_1)}, r_1 e^{i\theta_1}) w^{(j)} | w^{(k)}\rangle f_1(e^{i\theta_1}) \cdots f_n(e^{i\theta_n}) d\!\!\!/\theta_1 \cdots d\!\!\!/\theta_n.
\end{aligned} \tag{3.31}
$$

By theorem 2.2 and the discussion below, the sum and the integrals in (3.31) commute. Therefore (3.31) equals

$$
\begin{aligned}
& \lim_{r_1,\dots,r_n\to 1} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \sum_{s_1,\dots,s_n\in\mathbb{R}} \langle P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, r_n e^{i\theta_n}) P_{s_{n-1}} \\
&\cdots P_{s_1}\mathcal{Y}_{\alpha_1}(w^{(i_1)}, r_1 e^{i\theta_1}) w^{(j)} | w^{(k)}\rangle f_1(e^{i\theta_1}) \cdots f_n(e^{i\theta_n}) d\!\!\!/\theta_1 \cdots d\!\!\!/\theta_n \\
=& \lim_{r_1,\dots,r_n\to 1} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \langle \mathcal{Y}_{\alpha_n}(w^{(i_n)}, r_n e^{i\theta_n}) \\
&\cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, r_1 e^{i\theta_1}) w^{(j)} | w^{(k)}\rangle f_1(e^{i\theta_1}) \cdots f_n(e^{i\theta_n}) d\!\!\!/\theta_1 \cdots d\!\!\!/\theta_n.
\end{aligned} \tag{3.32}
$$

By continuity of correlation functions, the limit and the integrals in (3.32) commute. So (3.32) equals the right hand side of equation (3.27). Thus the proof is completed. □

Corollary 3.13. *Let $\mathcal{Y}_\alpha, \mathcal{Y}_{\alpha'}$ be unitary energy-bounded intertwining operators of V with common charge space W_i, and $\mathcal{Y}_\beta, \mathcal{Y}_{\beta'}$ be unitary energy-bounded intertwining operators of V with common charge space W_j. Choose $z_i, z_j \in S^1$ and assume that $\arg z_j < \arg z_i < \arg z_j + 2\pi$. Choose disjoint open intervals $I, J \in \mathcal{J}(S^1 \backslash \{-1\})$ such that I is anticlockwise to J. Suppose that for any $w^{(i)} \in W_i, w^{(j)} \in W_j$, the following braid relation holds:*

$$
\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}_\beta(w^{(j)}, z_j) = \mathcal{Y}_{\beta'}(w^{(j)}, z_j)\mathcal{Y}_{\alpha'}(w^{(i)}, z_i). \tag{3.33}
$$

Then for any $f \in C_c^\infty(I), g \in C_c^\infty(J)$, we have the braid relation for intertwining operators:

$$
\mathcal{Y}_\alpha(w^{(i)}, f)\mathcal{Y}_\beta(w^{(j)}, g) = \mathcal{Y}_{\beta'}(w^{(j)}, g)\mathcal{Y}_{\alpha'}(w^{(i)}, f). \tag{3.34}
$$

Note that if W_k is the source space of $\mathcal{Y}_\beta$, then both sides of equation (3.34) are understood to be acting on $\mathcal{H}_k^\infty$.

Remark 3.14. If $\mathcal{Y}_\alpha$ and $\mathcal{Y}_{\alpha'}$ (resp. $\mathcal{Y}_\beta$ and $\mathcal{Y}_{\beta'}$) are the vertex operator Y_k, then the above corollary still holds if we assume that $I \in \mathcal{J}$ (resp. $J \in \mathcal{J}$).

Rotation covariance of smeared intertwining operators

For each $t \in \mathbb{R}$, we define an action

$$\mathfrak{r}(t) : S^1 \to S^1, \quad \mathfrak{r}(t)(e^{i\theta}) = e^{i(\theta+t)}. \tag{3.35}$$

For any $g \in C_c^\infty(S^1)$, we let

$$\mathfrak{r}(t)g = g \circ \mathfrak{r}(-t). \tag{3.36}$$

Therefore, if $J \in \mathcal{J}$, then $\mathfrak{r}(t)C_c^\infty(J) = C_c^\infty(\mathfrak{r}(t)J)$. We also define $g' \in C_c^\infty(S^1)$ to be

$$g'(e^{i\theta}) = \frac{d}{d\theta}g(e^{i\theta}) \tag{3.37}$$

Rotation covariance is stated as follows.

Proposition 3.15. *Suppose that $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$ is unitary, $w^{(i)} \in W_j$ is homogeneous, $\mathcal{Y}(w^{(i)}, x)$ is energy bounded, and $J \in \mathcal{J}(S^1\backslash\{-1\})$. Choose $\varepsilon > 0$ such that $\mathfrak{r}(t)J \subset S^1\backslash\{-1\}$ for any $t \in (-\varepsilon, \varepsilon)$. Then for any $g \in C_c^\infty(J)$ and $t \in (-\varepsilon, \varepsilon)$, the following equations hold when both sides act on $\mathcal{H}_j^\infty$:*

$$[\overline{L_0}, \mathcal{Y}_\alpha(w^{(i)}, g)] = \mathcal{Y}_\alpha\big(w^{(i)}, (\Delta_{w^{(i)}} - 1)g + ig'\big), \tag{3.38}$$

$$e^{it\overline{L_0}}\mathcal{Y}_\alpha(w^{(i)}, g)e^{-it\overline{L_0}} = \mathcal{Y}_\alpha\big(w^{(i)}, e^{i(\Delta_{w^{(i)}}-1)t}\mathfrak{r}(t)g\big). \tag{3.39}$$

Proof. By equation (1.24), for any $z = e^{i\theta} \in J$ we have

$$\begin{aligned}
&[\overline{L_0}, \mathcal{Y}_\alpha(w^{(i)}, z)]\\
=&\Delta_{w^{(i)}}\mathcal{Y}_\alpha(w^{(i)}, z) + z\partial_z\mathcal{Y}_\alpha(w^{(i)}, z)\\
=&\Delta_{w^{(i)}}\mathcal{Y}_\alpha(w^{(i)}, e^{i\theta}) - i\partial_\theta\mathcal{Y}_\alpha(w^{(i)}, e^{i\theta})
\end{aligned}$$

when evaluated between vectors inside W_j and W_k. Thus we have

$$\begin{aligned}
[\overline{L_0}, \mathcal{Y}_\alpha(w^{(i)}, g)] &= \int_{-\pi}^{\pi} [\overline{L_0}, \mathcal{Y}_\alpha(w^{(i)}, e^{i\theta})]g(e^{i\theta})d\!\!\!{}^-\theta\\
=&\int_{-\pi}^{\pi} \big(\Delta_{w^{(i)}}\mathcal{Y}_\alpha(w^{(i)}, e^{i\theta}) - i\partial_\theta\mathcal{Y}_\alpha(w^{(i)}, e^{i\theta})\big)g(e^{i\theta})d\!\!\!{}^-\theta\\
=&\Delta_{w^{(i)}}\mathcal{Y}_\alpha(w^{(i)}, g) - i\int_{-\pi}^{\pi} \partial_\theta\mathcal{Y}_\alpha(w^{(i)}, e^{i\theta})g(e^{i\theta})\frac{e^{i\theta}}{2\pi}d\theta\\
=&\Delta_{w^{(i)}}\mathcal{Y}_\alpha(w^{(i)}, g) + i\int_{-\pi}^{\pi} \mathcal{Y}_\alpha(w^{(i)}, e^{i\theta})\frac{d}{d\theta}\Big(g(e^{i\theta})\frac{e^{i\theta}}{2\pi}\Big)d\theta\\
=&\Delta_{w^{(i)}}\mathcal{Y}_\alpha(w^{(i)}, g) + i\int_{-\pi}^{\pi} \mathcal{Y}_\alpha(w^{(i)}, e^{i\theta})\big(g'(e^{i\theta}) + ig(e^{i\theta})\big)\frac{e^{i\theta}}{2\pi}d\theta\\
=&(\Delta_{w^{(i)}} - 1)\mathcal{Y}_\alpha(w^{(i)}, g) + i\mathcal{Y}_\alpha(w^{(i)}, g').
\end{aligned}$$

This proves the first equation. To prove the second one, we first note that for any $\tau \geqslant 0$, when $h \in \mathbb{R}$ is small enough, the $|\cdot|_{V,\tau}$-norm of the function

$$\begin{gathered}e^{i(\Delta_{w^{(i)}}-1)(t+h)}r(t+h)g - e^{i(\Delta_{w^{(i)}}-1)t}\mathfrak{r}(t)g\\ -\big(i(\Delta_{w^{(i)}}-1)e^{i(\Delta_{w^{(i)}}-1)t}\mathfrak{r}(t)g - e^{i(\Delta_{w^{(i)}}-1)t}\mathfrak{r}(t)g'\big)h\end{gathered}$$

is $o(h)$. For any $\xi^{(j)} \in \mathcal{H}_j^\infty$, we define a function $\Xi(t)$ for $|t| < \varepsilon$ to be

$$\Xi(t) = e^{-it\overline{L_0}}\mathcal{Y}_\alpha(w^{(i)}, e^{i(\Delta_{w^{(i)}}-1)t}\mathfrak{r}(t)g)e^{it\overline{L_0}}\xi^{(j)}.$$

Now we can apply relation (3.38) and proposition 3.9 to see that the vector norm of $\Xi(t+h) - \Xi(t)$ is $o(h)$ for any $|t| < \varepsilon$. (In fact this is true for any Sobolev norm.) This shows that the derivative of $\Xi(t)$ exists and equals 0. So $\Xi(t)$ is a constant function. In particular, we have $\Xi(0) = \Xi(t)$, which implies (3.39). □

The strong intertwining property of smeared intertwining operators

Proposition 3.16. *Let $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$ be unitary, $w^{(i)} \in W_i$ be homogeneous, and $v \in V$ be quasi-primary. Suppose that $\theta v = v$, $\mathcal{Y}_\alpha(w^{(i)}, x)$ is energy bounded, and $Y_j(v,x), Y_k(v,x)$ satisfy linear energy bounds. Let $I \in \mathcal{J}$, $J \in \mathcal{J}(S^1\backslash\{-1\})$ be disjoint. Choose $f \in C_c^\infty(I), g \in C_c^\infty(J)$. Assume that f satisfies*

$$e^{i\pi\Delta_v/2}e_{1-\Delta_v}f = \overline{e^{i\pi\Delta_v/2}e_{1-\Delta_v}f}. \tag{3.40}$$

Then $Y_j(v,f)$ and $Y_k(v,f)$ are essentially self-adjoint, and for any $t \in \mathbb{R}$, we have

$$e^{it\overline{Y_j(v,f)}}\mathcal{H}_j^\infty \subset \mathcal{H}_j^\infty, \quad e^{it\overline{Y_k(v,f)}}\mathcal{H}_k^\infty \subset \mathcal{H}_k^\infty, \tag{3.41}$$

$$e^{it\overline{Y_k(v,f)}} \cdot \overline{\mathcal{Y}_\alpha(w^{(i)},g)} = \overline{\mathcal{Y}_\alpha(w^{(i)},g)} \cdot e^{it\overline{Y_j(v,f)}}. \tag{3.42}$$

Proof. Define the direct sum V-module $W_l = W_j \oplus^\perp W_k$ of W_j and W_k. Then $\mathcal{H}_l$ is the norm completion of W_l, $\mathcal{H}_l^\infty$ is the dense subspace of smooth vectors, and $Y_l(v,f) = \mathrm{diag}\big(Y_j(v,f), Y_k(v,f)\big)$. By equations (3.40) and (3.26), $Y_l(v,f)$ is symmetric (i.e., $Y_l(v,f)^\dagger = Y_l(v,f)$). Since $Y_l(v,x)$ satisfies linear energy bounds, by proposition 3.9-(a), relation (3.38), and lemma B.8, $Y_l(v,f)$ is essentially self-adjoint, and $e^{it\overline{Y_l(v,f)}}\mathcal{H}_l^\infty \subset \mathcal{H}_l^\infty$. This is equivalent to saying that $Y_j(v,f)$ and $Y_k(v,f)$ are essentially self-adjoint, and relation (3.41) holds.

Let $A = Y_l(v,f)$. Regard $B = \mathcal{Y}_\alpha(w^{(i)},g)$ as an unbounded operator on $\mathcal{H}_l$, being the original one when acting on $\mathcal{H}_j$, and zero when acting on $\mathcal{H}_k$. (So the domain of B is $\mathcal{H}_j^\infty \oplus^\perp \mathcal{H}_k$.) By propositions 2.13, 3.13, and remark 3.14, $AB = BA$ when both sides of the equation act on $\mathcal{H}_l^\infty$. By theorem B.9, $\overline{A}$ commutes strongly with $\overline{B}$. Therefore $e^{it\overline{A}} \cdot \overline{B} = \overline{B} \cdot e^{it\overline{A}}$, which is equivalent to equation (3.42). □

A Appendix for chapter 2

A.1 Uniqueness of formal series expansions

Using Cauchy's integral formula, the coefficients of a Laurent series $\sum_{n\geqslant N} a_n z^n$ are determined by the values of this series when z is near 0. This uniqueness property can be generalized to formal series, as we now see.

Let $\mathscr{G}_0$ be a *finite* subset of $\mathbb{R}$, and let $\mathscr{G} = \mathscr{G}_0 + \mathbb{Z}_{\geqslant 0} = \{\mu + m : \mu \in \mathscr{G}_0, m \in \mathbb{Z}_{\geqslant 0}\}$. It is clear that the series

$$f(z_1, \dots, z_n) = \sum_{\mu_1,\dots,\mu_n \in \mathscr{G}} c_{\mu_1,\dots,\mu_n} z_1^{\mu_1} \cdots z_n^{\mu_n} \tag{A.1}$$

converges absolutely if and only if for any $\mu_1, \dots, \mu_n \in \mathscr{G}_0$, the power series

$$\sum_{m_1,\dots,m_n \in \mathbb{Z}_{\geqslant 0}} c_{\mu_1+m_1,\dots,\mu_n+m_n} z_1^{\mu_1+m_1} \cdots z_n^{\mu_n+m_n}$$

converges absolutely. Hence, by root test, if $f(z_1, \dots, z_n)$ converges absolutely for some $z_1, \dots, z_n \neq 0$, then $f(\zeta_1, \dots, \zeta_n)$ converges absolutely whenever $0 < |\zeta_1| < |z_1|, \dots, 0 < |\zeta_n| < |z_n|$.

The uniqueness property is stated as follows:

Proposition A.1. *Let $r_1, \dots, r_n > 0$. For any $1 \leqslant l \leqslant n$, we choose a sequence of complex numbers $\{z_l(m_l) : 0 < |z_l(m_l)| < r_l\}_{m_l \in \mathbb{Z}_{>0}}$ such that $\lim_{m_l \to \infty} z_l(m_l) = 0$. Suppose that (A.1) converges absolutely when $0 < |z_1| < r_1, \dots, 0 < |z_n| < r_n$, and that for any $m_1, \dots, m_n$, we have $f\big(z_1(m_1), \dots, z_n(m_n)\big) = 0$. Then for any $\mu_1, \dots, \mu_n \in \mathscr{G}$, the coefficient $c_{\mu_1,\dots,\mu_n} = 0$.*

Proof. (cf. [Hua95] section 15.4) By induction, it suffices to prove the case when $n = 1$. Then the series can be written as $f(z) = \sum_{k \in \mathbb{Z}_{\geqslant 1}} c_{\mu_k} z^{\mu_k}$, where $\mu_{k+1} > \mu_k$ for any k, and we have a sequence of complex values $\{z_m\}$ converging to zero, on which the values of f vanish. Define a series $g(z) = \sum_{k \in \mathbb{Z}_{\geqslant 2}} c_{\mu_k} z^{\mu_k - \mu_2}$. Then the series $g(z)$ converges absolutely when $0 < |z| < r$, and $\limsup_{z \to 0} |g(z)| < +\infty$. Since $f(z) z^{-\mu_1} = c_{\mu_1} + z^{\mu_2 - \mu_1} g(z)$, we have $c_{\mu_1} = \lim_{m \to \infty} f\big(z(m)\big) z(m)^{-\mu_1} = 0$. This proves that $c_{\mu_1} = 0$. Repeat the same argument, we see that $c_{\mu_k} = 0$ for any k. □

A.2 Linear independence of products of intertwining operators

This section is devoted to the proof of proposition 2.3. First, we need the following lemma, the proof of which is an easy exercise.

Lemma A.2. *Let W_i be an irreducible V-module. Let $n = 1, 2, \dots$. Consider the V-module $W_i^{\oplus n} = \underbrace{W_i \oplus W_i \oplus \cdots \oplus W_i}_{n}$. Then for any V-module homomorphism $R : W_i \to W_i^{\oplus n}$, there exist complex numbers $\lambda_1, \dots, \lambda_n$ such that*

$$R(w^{(i)}) = (\lambda_1 w^{(i)}, \lambda_2 w^{(i)}, \dots, \lambda_n w^{(i)}) \qquad (w^{(i)} \in V). \tag{A.2}$$

Proof. For any $1 \leqslant m \leqslant n$, let p_m be the projection of $W_i^{\oplus n}$ onto its m-th component. Then $p_m R \in \mathrm{End}_V(W_i)$. Since W_i is irreducible, there exists $\lambda_m \in \mathbb{C}$ such that $p_m R = \lambda_m \mathrm{id}_{W_i}$. (A.2) now follows immediately. $\square$

Let W_i, W_j be two V-modules. For any $k \in \mathcal{E}$ we choose a basis $\{\mathcal{Y}_\alpha : \alpha \in \Theta_{ij}^k\}$ of $\mathcal{V}\binom{k}{i\ j}$. Consider the V-module $W_l = \bigoplus_{k\in\mathcal{E}} \big(\bigoplus_{\alpha\in\Theta_{ij}^k} W_k^\alpha\big)$, where each W_k^α is a V-module equivalent to W_k. It's contragredient module is $W_{\bar{l}} = \bigoplus_{k\in\mathcal{E}} \big(\bigoplus_{\alpha\in\Theta_{ij}^k} W_{\bar{k}}^\alpha\big)$, where $W_{\bar{k}}^\alpha$ is the contragredient module of W_k^α. Consider a type $\binom{l}{i\ j}$ intertwining operator $\mathcal{Y}$ defined as follows: for any $w^{(i)} \in W_i, w^{(j)} \in W_j$, we let

$$\mathcal{Y}(w^{(i)}, x)w^{(j)} = \bigoplus_{k\in\mathcal{E}} \bigg(\bigoplus_{\alpha\in\Theta_{ij}^k} \mathcal{Y}_\alpha(w^{(i)}, x)w^{(j)} \bigg), \tag{A.3}$$

i.e., the projection of $\mathcal{Y}(w^{(i)}, x)w^{(j)}$ to W_k^α is $\mathcal{Y}_\alpha(w^{(i)}, x)w^{(j)}$.

The following property is due to Huang. See [Hua95] lemma 14.9. The notations and terminologies in that article are different from ours, so we include a proof here.

Proposition A.3. *Choose $z \in \mathbb{C}^\times$ with argument $\arg z$. Let $w^{(\bar{l})} \in W_{\bar{l}}$. If for any $w^{(i)} \in W_i, w^{(j)} \in W_j$, we have*

$$\langle w^{(\bar{l})}, \mathcal{Y}(w^{(i)}, z)w^{(j)} \rangle = 0, \tag{A.4}$$

then $w^{(\bar{l})} = 0$.

Proof. Let W_1 be the subspace of all $w^{(\bar{l})} \in W_{\bar{l}}$ satisfying (A.4). We show that $W_1 = 0$.

Note that by relation (1.21), for any $u \in V, m \in \mathbb{Z}$ we have

$$Y_l(u, m)\mathcal{Y}(w^{(i)}, z) - \mathcal{Y}(w^{(i)}, z)Y_k(u, m) = \sum_{h\in\mathbb{Z}_{\geqslant 0}} \binom{m}{h} \mathcal{Y}(Y_i(u, h)w^{(i)}, z)z^{m-h}. \tag{A.5}$$

From this we see that W_1 is a V-submodule of $W_{\bar{l}}$. If $W_1 \neq 0$, then W_1 contains an irreducible submodule equivalent to $W_{\bar{k}}$ for some $k \in \mathcal{E}$. This implies that we have a non-zero V-module homomorphism $R : W_{\bar{k}} \to \bigoplus_{\alpha\in\Theta_{ij}^k} W_{\bar{k}}^\alpha \subset W_{\bar{l}}$, and that the image of R is inside W_1.

By lemma A.2, we can choose complex numbers $\{\lambda_\alpha : \alpha \in \Theta_{ij}^k\}$, not all of which are zero, such that for any $w^{(\bar{k})}$, $Rw^{(\bar{k})} = \bigoplus_{\alpha\in\Theta_{ij}^k} \lambda_\alpha w^{(\bar{k})}$. Hence for any $w^{(i)} \in W_i, w^{(j)} \in W_j, w^{(\bar{k})} \in W_k$, we have

$$\sum_{\alpha\in\Theta_{ij}^k} \lambda_\alpha \langle w^{(\bar{k})}, \mathcal{Y}_\alpha(w^{(i)}, z)w^{(j)} \rangle = 0.$$

Since 3-point correlation functions are determined by their values at the point z, we have

$$\sum_{\alpha\in\Theta_{ij}^k} \lambda_\alpha \langle w^{(\bar{k})}, \mathcal{Y}_\alpha(w^{(i)}, x)w^{(j)} \rangle = 0,$$

where x is a formal variable. But we know that $\{\mathcal{Y}_\alpha : \alpha \in \Theta_{ij}^k\}$ are linearly independent, which forces all the coefficients λ_α to be zero. Hence we have a contradiction. □

Corollary A.4. *Vectors of the form $\mathcal{Y}(w^{(i)}, s)w^{(j)}$ ($w^{(i)} \in W_i, w^{(j)} \in W_j, s \in \mathbb{R}$) span the vector space W_l.*

Proof. Choose any $w^{(\bar{l})} \in W_{\bar{l}}$ satisfying that for any $w^{(i)} \in W_i, w^{(j)} \in W_j, s \in \mathbb{R}$,

$$\langle w^{(\bar{l})}, \mathcal{Y}(w^{(i)}, s)w^{(j)}\rangle = 0. \tag{A.6}$$

Then for any $z \in \mathbb{C}^\times$, equation (A.4) holds. So $w^{(\bar{l})}$ must be zero. □

Proof of proposition 2.3. It is clear that Φ is surjective. So we only need to prove that Φ is injective. By induction, it suffices to prove that the linear map Ψ:

$$\bigoplus_{j\in\mathcal{E}} \left(\mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_2\ j} \otimes \mathcal{V}\binom{j}{i_1\ i_0}\right) \to \mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_1\ i_0},$$
$$\mathcal{X} \otimes \mathcal{Y}_\alpha \mapsto \mathcal{X}\mathcal{Y}_\alpha$$

is injective. To prove this, we choose, for any $j \in \mathcal{E}$, a linear basis $\{\mathcal{Y}_\alpha : \alpha \in \Theta_{i_1 i_0}^j\}$ of $\mathcal{V}\binom{j}{i_1\ i_0}$. If we can prove, for any $j \in \mathcal{E}, \alpha \in \Theta_{i_1 i_0}^j, \mathcal{X}_\alpha \in \mathcal{V}\binom{k}{i_n\ i_{n-1}\ \cdots\ i_2\ j}$, that

$$\sum_{j\in\mathcal{E}} \sum_{\alpha\in\Theta_{i_1 i_0}^j} \mathcal{X}_\alpha \mathcal{Y}_\alpha = 0 \tag{A.7}$$

always implies that $\mathcal{X}_\alpha = 0$ for all α, then the injectivity of Ψ follows immediately.

Now suppose that (A.7) is true. Then for any $w^{(i_0)} \in W_{i_0}, w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_n)} \in W_{i_n}, s \in \mathbb{R}$, and $z_2, \ldots, z_n$ satisfying $0 < |z_2| < \cdots < |z_n|$, we have, by proposition A.1,

$$\sum_{j\in\mathcal{E}} \sum_{\alpha\in\Theta_{i_1 i_0}^j} \mathcal{X}_\alpha(w^{(i_n)}, \ldots, w^{(i_2)}; z_n, \ldots, z_2)\mathcal{Y}_\alpha(w^{(i_1)}, s)w^{(i_0)} = 0. \tag{A.8}$$

By corollary A.4, for any $j \in \mathcal{E}, w^{(j)} \in W_j$ and $\alpha \in \Theta_{i_1 i_0}^j$, there exist $w_1^{(i_0)}, \ldots, w_m^{(i_0)} \in W_{i_0}, w_1^{(i_1)}, \ldots, w_m^{(i_1)} \in W_{i_1}, s_1, \ldots, s_m \in \mathbb{R}$, such that

$$\mathcal{Y}_\alpha(w_1^{(i_1)}, s_1)w_1^{(i_0)} + \cdots + \mathcal{Y}_\alpha(w_m^{(i_1)}, s_m)w_m^{(i_0)} = w^{(j)},$$

and that for any $\beta \neq \alpha$,

$$\mathcal{Y}_\beta(w_1^{(i_1)}, s_1)w_1^{(i_0)} + \cdots + \mathcal{Y}_\beta(w_m^{(i_1)}, s_m)w_m^{(i_0)} = 0.$$

Hence $\mathcal{X}_\alpha(w^{(i_n)}, \ldots, w^{(i_2)}; z_n, \ldots, z_2)w^{(j)} = 0$. □

A.3 General braiding and fusion relations

In this section, we prove all the results claimed in section 2.2. In the following proofs of absolute convergence, the idea of analytic continuation (lemma A.5) and induction is due to [HLZ11] proposition 12.7. The trick in step 1 of the proof of theorem 2.5 using $B_\pm$ to transform certain types of absolute convergence to other types can be found in [Hua95] proposition 14.1. What's new in our proofs is the change-of-variable trick: we replace the original complex variables $z_1, z_2, \dots$ with the moduli parameters $\omega_1, \omega_2, \dots$ which control the shape of gluing together Riemann spheres with three holes (pants).

We first introduce some temporary notations. For any $r > 0$, let $D(r) = \{z \in \mathbb{C} : |z| < r\}$, $D^\times(r) = D(r)\backslash\{0\}$, and $E(r) = D(r) \cap (0, +\infty)$. Then we have the following:

Lemma A.5. *Given a power series*

$$\sum_{n_0, n_1, \dots, n_l \in \mathbb{Z}_{\geqslant 0}} c_{n_0 n_1 \dots n_l} z_0^{n_0} z_1^{n_1} \cdots z_l^{n_l} \tag{A.9}$$

of the complex variables $z_0, z_1, \dots, z_l$, where $l \in \mathbb{Z}_{>0}$ and each $c_{n_0 n_1 \dots n_l} \in \mathbb{C}$. Suppose that there exist $r_0, r_1, \dots, r_l > 0$, such that for any n_0, the power series

$$g_{n_0}(z_1, \dots, z_l) = \sum_{n_1, \dots, n_l \in \mathbb{Z}_{\geqslant 0}} c_{n_0 n_1 \dots n_l} z_1^{n_1} \cdots z_l^{n_l} \tag{A.10}$$

converges absolutely on $D(r_1) \times \cdots \times D(r_l)$; that for any $z_1 \in E(r_1), \dots, z_l \in E(r_l)$,

$$f(z_0, z_1, \dots, z_l) = \sum_{n_0 \in \mathbb{Z}_{\geqslant 0}} g_{n_0}(z_1, \dots, z_n) z_0^n, \tag{A.11}$$

converges absolutely as a power series of z_0 on $D(r_0)$; and that f can be analytically continued to a multivalued holomorphic function on $D^\times(r_0) \times D^\times(r_1) \times \cdots \times D^\times(r_l)$. Then the power series (A.9) *converges absolutely on $D(r_0) \times D(r_1) \times \cdots \times D(r_l)$.*

Proof. Consider the multivalued holomorphic function f. From (A.11), we know that for any $z_1 \in E(r_1), \dots, z_l \in E(r_l)$, f is single-valued for $z_0 \in D^\times(r_0)$. So f is single-valued on z_0 for any $z_1 \in D^\times(r_1), \dots, z_l \in D^\times(r_l)$.

Now, for any $n_0 \in \mathbb{Z}$,

$$\widetilde{g}_{n_0}(z_1, \dots, z_n) = \oint_0 f(z_0, z_1, \dots, z_l) z_0^{-n-1} \frac{dz_0}{2i\pi} \tag{A.12}$$

is a multivalued holomorphic function on $D^\times(r_1) \times \cdots \times D^\times(r_l)$. If $n_0 \geqslant 0$, then by (A.11), we must have $\widetilde{g}_{n_0} = g_{n_0}$ on $E(r_1) \times \cdots \times E(r_l)$. Since g_{n_0} is holomorphic, $\widetilde{g}_{n_0} = g_{n_0}$ on $D^\times(r_1) \times \cdots \times D^\times(r_l)$. Hence $\widetilde{g}_{n_0}$ is single-valued. Similarly, when $n_0 < 0$, we have $\widetilde{g}_{n_0}(z_1, \dots, z_n) = 0$ on $E(r_1) \times \cdots \times E(r_l)$, and hence on $D^\times(r_1) \times \cdots \times D^\times(r_l)$. Therefore, $f(z_0, z_1, \dots, z_n) = \sum_{n_0 \in \mathbb{Z}} \widetilde{g}_{n_0}(z_1, \dots, z_n) z_0^{n_0}$ is single-valued on $D^\times(r_0) \times D^\times(r_1) \times \cdots \times$

$D^\times(r_n)$, and the Laurant series expansion of f near the origin has no negative powers of $z_0, z_1, \dots, z_n$. So f is a single-valued holomorphic function on $D(r_0) \times D(r_1) \times \cdots \times D(r_l)$ with power series expansion (A.9). We can thus conclude that (A.9) converges absolutely on $D(r_0) \times D(r_1) \times \cdots \times D(r_l)$. □

Recall that a series $f(z_1, \dots, z_n) = \sum_{s_1,\dots,s_n \in \mathbb{R}} c_{s_1 \dots s_n} z_1^{s_1} \cdots z_n^{s_n}$ is called a **quasi power series** of $z_1, \dots, z_n$, if f equals a power series multiplied by a monomial of $z_1, \dots, z_n$, i.e., if there exist $t_1, \dots, t_n \in \mathbb{C}$ such that $f(z_1, \dots, z_n) z_1^{t_1} \cdots z_n^{t_n} \in \mathbb{C}[[z_1, \dots, z_n]]$.

Proof of theorem 2.5. Step 1. We first prove the convergence. Let W_i be the charge space of $\mathcal{Y}_\gamma$. Then for any $w^{(i_0)} \in W_{i_0}, w^{(i)} \in W_i$, we have

$$
\begin{aligned}
&\mathcal{Y}_\gamma(w^{(i)}, x) w^{(i_0)} \\
=&\mathcal{Y}_{B_+ B_- \gamma}(w^{(i)}, x) w^{(i_0)} \\
=&e^{xL_{-1}} \mathcal{Y}_{B_- \gamma}(w^{(i_0)}, e^{i\pi} x) w^{(i)},
\end{aligned}
$$

where x is a formal variable. Then for any $w^{(\bar{k})} \in W_{\bar{k}}$, we have

$$
\begin{aligned}
&\langle \mathcal{Y}_\gamma(w^{(i)}, z_1) w^{(i_0)}, w^{(\bar{k})} \rangle \\
=&\langle \mathcal{Y}_\gamma(w^{(i)}, x) w^{(i_0)}, w^{(\bar{k})} \rangle \big|_{x=z_1} \\
=&\langle e^{xL_{-1}} \mathcal{Y}_{B_- \gamma}(w^{(i_0)}, e^{i\pi} x) w^{(i)}, w^{(\bar{k})} \rangle \big|_{x=z_1} \\
=&\langle \mathcal{Y}_{B_- \gamma}(w^{(i_0)}, e^{i\pi} x) w^{(i)}, e^{xL_1} w^{(\bar{k})} \rangle \big|_{x=z_1} \\
=&\langle \mathcal{Y}_{B_- \gamma}(w^{(i_0)}, e^{i\pi} z_1) w^{(i)}, e^{z_1 L_1} w^{(\bar{k})} \rangle.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\big\langle \mathcal{Y}_\gamma \big(P_{s_n} \mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1) P_{s_{n-1}} \mathcal{Y}_{\sigma_{n-1}}(w^{(i_{n-1})}, z_{n-1} - z_1) \\
&\qquad \cdots P_{s_2} \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1) w^{(i_1)}, z_1 \big) w^{(i_0)}, w^{(\bar{k})} \big\rangle \\
=&\langle \mathcal{Y}_{B_- \gamma}(w^{(i_0)}, e^{i\pi} z_1) P_{s_n} \mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1) P_{s_{n-1}} \mathcal{Y}_{\sigma_{n-1}}(w^{(i_{n-1})}, z_{n-1} - z_1) \\
&\qquad \cdots P_{s_2} \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1) w^{(i_1)}, e^{z_1 L_1} w^{(\bar{k})} \rangle.
\end{aligned} \tag{A.13}
$$

Hence, by theorem 2.2 and the discussion below, the sum of (A.13) over $s_2, s_3, \dots, s_n \in \mathbb{R}$ converges absolutely and locally uniformly.

Step 2. Assume that

$$
\begin{aligned}
&0 < |z_1| < |z_2| < \cdots < |z_n|, \\
&0 < |z_2 - z_1| < |z_3 - z_1| \cdots < |z_n - z_1| < |z_1|,
\end{aligned} \tag{A.14}
$$

and choose arguments $\arg z_1, \arg z_2, \dots, \arg z_n, \arg(z_2 - z_1), \dots, \arg(z_n - z_1)$. We prove, by induction on n, that (2.8) defined near the point $(z_1, z_2, \dots, z_n)$ is a correlation function, i.e., it can be written as a product of a chain of intertwining operators. The case $n = 2$ was proved in [Hua95] and [Hua05a]. Suppose this theorem holds for $n-1$, we now prove it for n. By analytic continuation, it suffices to assume also that

$$|z_1| + |z_2 - z_1| < |z_3|. \tag{A.15}$$

Let W_{j_2} be the target space of $\mathcal{Y}_{\sigma_2}$. By induction, there exists a chain of intertwining operators $\mathcal{Y}_\delta, \mathcal{Y}_{\alpha_3}, \mathcal{Y}_{\alpha_4}, \dots, \mathcal{Y}_{\alpha_n}$ with charge spaces $W_{j_2}, W_{i_3}, W_{i_4}, \dots, W_{i_n}$ respectively, such that W_{i_0} is the source space of $\mathcal{Y}_\delta$, that W_k is the target space of $\mathcal{Y}_{\alpha_n}$, and that for any $w^{(i_0)} \in W_{i_0}, w^{(j_2)} \in W_{j_2}, w^{(i_3)} \in W_{i_3}, w^{(i_4)} \in W_{i_4}, \dots, w^{(i_n)} \in W_{i_n}$, we have the fusion relation

$$\begin{aligned}
&\mathcal{Y}_\gamma\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1)\mathcal{Y}_{\sigma_{n-1}}(w^{(i_{n-1})}, z_{n-1} - z_1)\cdots\mathcal{Y}_{\sigma_3}(w^{(i_3)}, z_3 - z_1)w^{(j_2)}, z_1\big)w^{(i_0)}\\
=&\mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1})\cdots\mathcal{Y}_{\alpha_3}(w^{(i_3)}, z_3)\mathcal{Y}_\delta(w^{(j_2)}, z_1)w^{(i_0)}
\end{aligned} \tag{A.16}$$

near the point $(z_1, z_3, z_4, \dots, z_n)$.

There also exists a chain of intertwining operator $\mathcal{Y}_{\alpha_1}, \mathcal{Y}_{\alpha_2}$ with charge spaces W_{i_1}, W_{i_2}, such that the source space of $\mathcal{Y}_{\alpha_1}$ is W_{i_0}, that the target space of $\mathcal{Y}_{\alpha_2}$ equals that of $\mathcal{Y}_\delta$, and that the fusion relation

$$\mathcal{Y}_\delta\big(\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big) = \mathcal{Y}_{\alpha_2}(w^{(i_2)}, z_2)\mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1) \tag{A.17}$$

holds near the point (z_1, z_2). Now we compute, omitting the evaluation under any $w^{(\overline{k})} \in W_{\overline{k}}$, that

$$\begin{aligned}
&\mathcal{Y}_\gamma\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1)\mathcal{Y}_{\sigma_{n-1}}(w^{(i_{n-1})}, z_{n-1} - z_1)\cdots\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)w^{(i_0)}\\
=&\sum_{s_1\in\mathbb{R}}\mathcal{Y}_\gamma\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1)\mathcal{Y}_{\sigma_{n-1}}(w^{(i_{n-1})}, z_{n-1} - z_1)\cdots P_{s_1}\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)w^{(i_0)}\\
=&\sum_{s_1\in\mathbb{R}}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1})\cdots\mathcal{Y}_{\alpha_3}(w^{(i_3)}, z_3)\\
&\qquad\cdot\mathcal{Y}_\delta\big(P_{s_1}\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)w^{(i_0)}\\
=&\sum_{s_1\in\mathbb{R}}\sum_{s_2,\dots,s_{n-1}\in\mathbb{R}}\mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)P_{s_{n-1}}\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1})P_{s_{n-2}}\\
&\qquad\cdots P_{s_3}\mathcal{Y}_{\alpha_3}(w^{(i_3)}, z_3)P_{s_2}\mathcal{Y}_\delta\big(P_{s_1}\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)w^{(i_0)}.
\end{aligned} \tag{A.18}$$

If we can prove, for any $w^{(\overline{k})} \in W_{\overline{k}}$, and any $z_1, z_2, \dots, z_n$ satisfying

$$\begin{aligned}
0 < |z_2 - z_1| < |z_1| < |z_3| < |z_4| < \cdots < |z_n|,\\
|z_1| + |z_2 - z_1| < |z_3|,
\end{aligned} \tag{A.19}$$

that the expression

$$\begin{aligned}
&\langle\mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})}, z_{n-1})\cdots\mathcal{Y}_{\alpha_3}(w^{(i_3)}, z_3)\\
&\cdot\mathcal{Y}_\delta\big(\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)w^{(i_0)}, w^{(\overline{k})}\rangle
\end{aligned} \tag{A.20}$$

converges absolutely, i.e., the sum of the absolute values of

$$\begin{aligned}&\big\langle P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)},z_n)P_{s_{n-1}}\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})},z_{n-1})P_{s_{n-2}}\\&\cdots P_{s_3}\mathcal{Y}_{\alpha_3}(w^{(i_3)},z_3)P_{s_2}\mathcal{Y}_{\delta}\big(P_{s_1}\mathcal{Y}_{\sigma_2}(w^{(i_2)},z_2-z_1)w^{(i_1)},z_1\big)w^{(i_0)},w^{(\overline{k})}\big\rangle\end{aligned}\tag{A.21}$$

over $s_1,s_2,\dots,s_n\in\mathbb{R}$ is a finite number, then the two sums on the right hand side of (A.18) commute. Hence (A.18) equals

$$\begin{aligned}&\sum_{s_2,\dots,s_n\in\mathbb{R}}\sum_{s_1\in\mathbb{R}}P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)},z_n)P_{s_{n-1}}\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})},z_{n-1})P_{s_{n-2}}\\&\qquad\cdots P_{s_3}\mathcal{Y}_{\alpha_3}(w^{(i_3)},z_3)P_{s_2}\mathcal{Y}_{\delta}\big(P_{s_1}\mathcal{Y}_{\sigma_2}(w^{(i_2)},z_2-z_1)w^{(i_1)},z_1\big)w^{(i_0)}\\=&\sum_{s_2,\dots,s_n\in\mathbb{R}}\sum_{s_1\in\mathbb{R}}P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)},z_n)P_{s_{n-1}}\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})},z_{n-1})P_{s_{n-2}}\\&\qquad\cdots P_{s_3}\mathcal{Y}_{\alpha_3}(w^{(i_3)},z_3)P_{s_2}\mathcal{Y}_{\alpha_2}(w^{(i_2)},z_2)P_{s_1}\mathcal{Y}_{\alpha_1}(w^{(i_1)},z_1)w^{(i_0)}\\=&\mathcal{Y}_{\alpha_n}(w^{(i_n)},z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})},z_{n-1})\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)},z_1)w^{(i_0)}.\end{aligned}\tag{A.22}$$

Therefore, if the series (A.20) converges absolutely, then (2.8) defines an $(n+2)$-point correlation function of V. The converse statement (every $(n+2)$-point function can be written in the form (A.20)) can be proved in a similar way.

Step 3. We show that when (A.19) holds, (A.20) converges absolutely. Assume, without loss of generality, that all the intertwining operators in (A.20) are irreducible, and that all the vectors in (A.20) are homogeneous. Define a new set of variables $\omega_1,\omega_2,\dots,\omega_n$ by setting

$$\begin{gathered}z_m=\omega_m\omega_{m+1}\cdots\omega_n\quad(3\leqslant m\leqslant n),\\z_1=\omega_2\omega_3\cdots\omega_n,\\z_2-z_1=\omega_1\omega_2\cdots\omega_n.\end{gathered}$$

Then condition (A.19) is equivalent to the condition

$$\begin{gathered}0<|\omega_m|<1\quad(1\leqslant m\leqslant n-1),\\0<|\omega_n|,\\|\omega_2|(1+|\omega_1|)<1.\end{gathered}\tag{A.23}$$

It is clear that if $\mathring{\omega}_1,\mathring{\omega}_2,\dots\mathring{\omega}_n$ are complex numbers satisfying condition (A.23), then there exist positive numbers $r_1>|\mathring{\omega}_1|,r_2>|\mathring{\omega}_2|,\dots,r_n>|\mathring{\omega}_n|$, such that whenever $0<|\omega_m|<r_m$ $(1\leqslant m\leqslant n)$, condition (A.23) is satisfied. We now prove that the sum of (A.21) over $s_1,\dots,s_n$ converges absolutely on $\{0<|\omega_1|<r_1,\dots,0<|\omega_n|<r_n\}$.

Let

$$c_{s_1s_2\dots s_n}$$

$$
\begin{aligned}
=&\langle P_{s_n}\mathcal{Y}_{\alpha_n}(w^{(i_n)},1)P_{s_{n-1}}\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})},1)P_{s_{n-2}}\cdots\\
&\cdot P_{s_3}\mathcal{Y}_{\alpha_3}(w^{(i_3)},1)P_{s_2}\mathcal{Y}_{\delta}\big(P_{s_1}\mathcal{Y}_{\sigma_2}(w^{(i_2)},1)w^{(i_1)},1\big)w^{(i_0)},w^{(\overline{k})}\rangle,
\end{aligned}
\tag{A.24}
$$

where each $\mathcal{Y}_\cdot(\cdot,1)=\mathcal{Y}_\cdot(\cdot,x)\big|_{x=1}$. By relation (1.26), it is easy to see that (A.21) equals

$$
\begin{aligned}
&\langle P_{s_n}\omega_n^{L_0}\mathcal{Y}_{\alpha_n}(w^{(i_n)},1)P_{s_{n-1}}\omega_{n-1}^{L_0}\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})},1)P_{s_{n-2}}\cdots\\
&\cdot P_{s_3}\omega_3^{L_0}\mathcal{Y}_{\alpha_3}(w^{(i_3)},1)P_{s_2}\omega_2^{L_0}\mathcal{Y}_{\delta}\big(P_{s_1}\omega_1^{L_0}\mathcal{Y}_{\sigma_2}(w^{(i_2)},1)w^{(i_1)},1\big)w^{(i_0)},w^{(\overline{k})}\rangle\\
=&c_{s_1s_2\dots s_n}\omega_1^{s_1}\omega_2^{s_2}\cdots\omega_n^{s_n}
\end{aligned}
\tag{A.25}
$$

multiplied by a monomial $\omega_1^{r_1}\omega_2^{r_2}\cdots\omega_n^{r_n}$, where the powers $r_1,r_2,\dots,r_n\in\mathbb{R}$ are independent of $s_1,s_2,\dots,s_n$. Therefore, the absolute convergence of (A.20) is equivalent to the absolute convergence of the series

$$
\sum_{s_1,s_2,\dots,s_n\in\mathbb{R}}c_{s_1s_2\dots s_n}\omega_1^{s_1}\omega_2^{s_2}\cdots\omega_n^{s_n}
\tag{A.26}
$$

on $\{0<|\omega_1|<r_1,0<|\omega_2|<r_2,\dots,0<|\omega_n|<r_n\}$. Note that by irreducibility of the intertwining operators, (A.26) is a quasi power series of $\omega_1,\omega_2,\dots,\omega_n$. So we are going to prove the absolute convergence of (A.26) by checking that (A.26) satisfies all the conditions in lemma A.5.

Since (A.21) equals (A.25) multiplied by $\omega_1^{r_1}\omega_2^{r_2}\cdots\omega_n^{r_n}$, for each $s_2\in\mathbb{R}$, step 1 and theorem 2.2 imply that the series

$$
\sum_{s_1,s_3,s_4,\dots,s_n\in\mathbb{R}}c_{s_1s_2s_3\dots s_n}\omega_1^{s_1}\omega_3^{s_3}\omega_4^{s_4}\cdots\omega_n^{s_n}
\tag{A.27}
$$

converges absolutely on $\{0<|\omega_1|<r_1,0<|\omega_3|<r_3,0<|\omega_4|<r_4,\dots,0<|\omega_n|<r_n\}$. If we assume moreover that $0<\omega_1<r_1$, then $0<|\omega_2|<r_2$ clearly implies $0<|z_1|<|z_2|<\cdots<|z_n|$ and $0<|z_2-z_1|<|z_1|$. Hence, the following quasi power series of ω_2

$$
\begin{aligned}
&\omega_1^{r_1}\omega_2^{r_2}\cdots\omega_n^{r_n}\cdot\Bigg(\sum_{s_2\in\mathbb{R}}\Bigg(\sum_{s_1,s_3,\dots,s_n\in\mathbb{R}}c_{s_1s_2s_3\dots s_n}\omega_1^{s_1}\omega_3^{s_3}\cdots\omega_n^{s_n}\Bigg)\omega_2^{s_2}\Bigg)\\
=&\sum_{s_2\in\mathbb{R}}\langle\mathcal{Y}_{\alpha_n}(w^{(i_n)},z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})},z_{n-1})\cdots\mathcal{Y}_{\alpha_3}(w^{(i_3)},z_3)\\
&\qquad\cdot P_{s_2}\mathcal{Y}_{\delta}\big(\mathcal{Y}_{\sigma_2}(w^{(i_2)},z_2-z_1)w^{(i_1)},z_1\big)w^{(i_0)},w^{(\overline{k})}\rangle\\
=&\sum_{s_2\in\mathbb{R}}\langle\mathcal{Y}_{\alpha_n}(w^{(i_n)},z_n)\mathcal{Y}_{\alpha_{n-1}}(w^{(i_{n-1})},z_{n-1})\cdots\mathcal{Y}_{\alpha_3}(w^{(i_3)},z_3)\\
&\qquad\cdot P_{s_2}\mathcal{Y}_{\alpha_2}(w^{(i_2)},z_2)\mathcal{Y}_{\alpha_1}(w^{(i_1)},z_1)w^{(i_0)},w^{(\overline{k})}\rangle
\end{aligned}
\tag{A.28}
$$

must converge absolutely on $\{0<|\omega_2|<r_2\}$. By theorem 2.4, the function (A.28) defined on $\{0<\omega_1<r_1,0<|\omega_2|<r_2,\dots,0<|\omega_n|<r_n\}$ can be analytically continued to a multivalued holomorphic function on $\{0<|\omega_1|<r_1,0<|\omega_2|<r_2,\dots,0<|\omega_n|<r_n\}$. Hence by lemma A.5, the quasi power series (A.26) converges absolutely on $\{0<|\omega_1|<r_1,\dots,0<|\omega_n|<r_n\}$. □

Proof of theorem 2.6. The argument here is similar to step 3 of the proof of theorem 2.5. Assume, without loss of generality, that all the intertwining operators in (2.11) are irreducible, and all the vectors in it are homogeneous. We prove this theorem by induction on m. The case that $m = 1$ is proved in theorem 2.5. Suppose that the theorem holds for $m - 1$, we prove this for m.

Define a new set of variables $\{\omega_b^a : 1 \leqslant a \leqslant m, 1 \leqslant b \leqslant n_a\}$ in the following way: For any $1 \leqslant a \leqslant m$, we set

$$z_1^a = \omega_1^a \omega_1^{a+1} \cdots \omega_1^m, \tag{A.29}$$

and if $2 \leqslant b \leqslant n_a$, we set

$$z_b^a - z_1^a = \omega_1^a \omega_1^{a+1} \cdots \omega_1^m \cdot \omega_b^a \omega_{b+1}^a \cdots \omega_{n_a}^a. \tag{A.30}$$

Then the condition (1) and (2) on $\{z_b^a : 1 \leqslant a \leqslant m, 1 \leqslant b \leqslant n_a\}$ is equivalent to the condition

$$\begin{gathered} 0 < |\omega_b^a| < 1 \quad (1 \leqslant a \leqslant m, 2 \leqslant b \leqslant n_a), \\ 0 < |\omega_1^m|, \\ 0 < |\omega_1^a|\big(1 + (1 - \delta_{n_a,1})|\omega_{n_a}^a|\big) < 1 - \big(1 - \delta_{n_{a+1},1}\big)|\omega_{n_{a+1}}^{a+1}| \quad (1 \leqslant a \leqslant m - 1). \end{gathered} \tag{A.31}$$

It is clear that if $\{\mathring{\omega}_b^a : 1 \leqslant a \leqslant m, 1 \leqslant b \leqslant n_a\}$ are complex numbers satisfying condition (A.31), then there exist positive numbers $\{r_b^a > |\mathring{\omega}_b^a|\}$, such that whenever $0 < |\omega_b^a| < r_b^a$ for all a and b, then (A.31) is true. If, moreover, any ω_b^a except ω_1^1 satisfies $0 < \omega_b^a < r_b^a$, then condition (3) also also holds for $\{z_b^a : 1 \leqslant a \leqslant m, 1 \leqslant b \leqslant n_a\}$.

Let $\vec{s}$ be the sequence $\{s_b^a\}$, $\vec{\omega}$ be $\{\omega_b^a\}$, $\vec{s}\backslash s_1^1$ be $\{$all s_b^a except $s_1^1\}$, and $\vec{\omega}\backslash\omega_1^1$ be $\{$all ω_b^a except $\omega_1^1\}$. We let $\vec{\omega}^{\vec{s}} = \prod_{1\leqslant a\leqslant m, 1\leqslant b\leqslant n_a} (\omega_b^a)^{s_b^a}$. For each $\vec{s}$, we define

$$c_{\vec{s}} = \Big\langle \Big[\prod_{m\geqslant a\geqslant 1} P_{s_1^a} \mathcal{Y}_{\alpha^a} \Big(\Big(\prod_{n_a\geqslant b\geqslant 2} P_{s_b^a} \mathcal{Y}_{\alpha_b^a}(w_b^a, 1) \Big) w_1^a, 1 \Big) \Big] w^i, w^{\overline{k}} \Big\rangle, \tag{A.32}$$

where each $\mathcal{Y}_\cdot(\cdot, 1)$ means $\mathcal{Y}_\cdot(\cdot, x)|_{x=1}$. Then by (1.26), the expression

$$\Big\langle \Big[\prod_{m\geqslant a\geqslant 1} P_{s_1^a} \mathcal{Y}_{\alpha^a} \Big(\Big(\prod_{n_a\geqslant b\geqslant 2} P_{s_b^a} \mathcal{Y}_{\alpha_b^a}(w_b^a, z_b^a - z_1^a) \Big) w_1^a, z_1^a \Big) \Big] w^i, w^{\overline{k}} \Big\rangle \tag{A.33}$$

equals $c_{\vec{s}} \cdot \vec{\omega}^{\vec{s}}$ multiplied by a monomial of $\vec{\omega}$ whose power is independent of $\vec{s}$. By induction, we can show that for each $s_1^1 \in \mathbb{R}$, the series $\sum_{\vec{s}\backslash s_1^1} c_{\vec{s}} \cdot \vec{\omega}^{\vec{s}} \cdot (\omega_1^1)^{-s_1^1}$ of $\vec{\omega}\backslash\omega_1^1$ converges absolutely on $\{\vec{\omega}\backslash\omega_1^1 : 0 < |\omega_b^a| < r_b^a\}$; that for all $\vec{\omega}\backslash\omega_1^1$ satisfying $0 < \omega_b^a < r_b^a$,

$$\sum_{s_1^1\in\mathbb{R}} \sum_{\vec{s}\backslash s_1^1} c_{\vec{s}} \cdot \vec{\omega}^{\vec{s}}, \tag{A.34}$$

as a series of ω_1^1, converges absolutely on $\{\omega_1^1 : 0 < |\omega_1^1| < r_1^1\}$; and that as a function of $\vec{\omega}$, (A.34) can be analytically continued to a multivalued holomorphic function on $\{\vec{\omega} : 0 <$

$|\omega_b^a| < r_b^a\}$. Hence, by lemma A.5, the quasi power series $\sum_{\vec{s}} c_{\vec{s}} \cdot \vec{\omega}^{\vec{s}}$ converges absolutely on $\{\vec{\omega} : 0 < |\omega_b^a| < r_b^a\}$. If, moreover, $\{z_b^a\}$ satisfy condition (3), then by induction and the argument in step 2 of the proof of theorem 2.5, (2.11) can be written as a product of a chain of intertwining operators. So it is a correlation function defined near $\{z_b^a\}$. □

Proof of corollary 2.7. One can prove this corollary, either by theorem 2.6 and the argument in step 1 of the proof of theorem 2.5, or by induction and the argument in step 3 of the proof of theorem 2.5. We leave the details to the reader. □

Proof of proposition 2.9. Fix $z_i \in \mathbb{C}^\times$. Let w_1 (resp. w_2) be a vector in the source space (resp. in the contragredient module of the target space) of $\mathcal{Y}_\delta$. Let $x_i, x_{ji}, \widetilde{x}_{ji}$ be commuting independent formal variables. It is easy to check that for any $w^{(k)} \in W_k$,

$$\begin{aligned}&\big\langle \mathcal{Y}_\delta\big(e^{\widetilde{x}_{ji}L_{-1}}w^{(k)}, x_i\big)w_1, w_2\big\rangle = \big\langle \mathcal{Y}_\delta\big(w^{(k)}, x_i + \widetilde{x}_{ji}\big)w_1, w_2\big\rangle \\ :=& \sum_{s\in\mathbb{R}, l\in\mathbb{Z}_{\geqslant 0}} \big\langle \mathcal{Y}_\delta(w^{(k)}, s)w_1, w_2\big\rangle \binom{-s-1}{l} x_i^{-s-1-l}\widetilde{x}_{ji}^{l}.\end{aligned} \tag{A.35}$$

Put $x_i = z_i$, we have

$$\begin{aligned}&\big\langle \mathcal{Y}_\delta\big(e^{\widetilde{x}_{ji}L_{-1}}w^{(k)}, z_i\big)w_1, w_2\big\rangle = \big\langle \mathcal{Y}_\delta\big(w^{(k)}, z_i + \widetilde{x}_{ji}\big)w_1, w_2\big\rangle \\ :=& \sum_{s\in\mathbb{R}, l\in\mathbb{Z}_{\geqslant 0}} \big\langle \mathcal{Y}_\delta(w^{(k)}, s)w_1, w_2\big\rangle \binom{-s-1}{l} z_i^{-s-1-l}\widetilde{x}_{ji}^{l}.\end{aligned} \tag{A.36}$$

Clearly

$$\big\langle \mathcal{Y}_\delta\big(\mathcal{Y}_\gamma(w^{(i)}, e^{\pm i\pi}z_{ji})w^{(j)}, z_i + \widetilde{z}_{ji}\big)w_1, w_2\big\rangle \tag{A.37}$$

is a multivalued holomorphic function of $z_{ji}, \widetilde{z}_{ji}$ when $0 < |z_{ji}|, |\widetilde{z}_{ji}| < \frac{1}{2}|z_i|$. Since the series

$$\sum_{s\in\mathbb{R}} \big\langle \mathcal{Y}_\delta\big(P_s\mathcal{Y}_\gamma(w^{(i)}, e^{\pm i\pi}z_{ji})w^{(j)}, z_i + \widetilde{z}_{ji}\big)w_1, w_2\big\rangle \tag{A.38}$$

converges absolutely and locally uniformly, the infinite sum commutes with Cauchy's integrals around the pole $\widetilde{z}_{ji} = 0$. From this we see that (A.37) has the series expansion

$$\big\langle \mathcal{Y}_\delta\big(\mathcal{Y}_\gamma(w^{(i)}, e^{\pm i\pi}x_{ji})w^{(j)}, z_i + \widetilde{x}_{ji}\big)w_1, w_2\big\rangle\Big|_{x_{ji}=z_{ji}, \widetilde{x}_{ji}=\widetilde{z}_{ji}}, \tag{A.39}$$

which must be absolute convergent, and also equals

$$\big\langle \mathcal{Y}_\delta\big(e^{\widetilde{x}_{ji}L_{-1}}\mathcal{Y}_\gamma(w^{(i)}, e^{\pm i\pi}x_{ji})w^{(j)}, z_i\big)w_1, w_2\big\rangle\Big|_{x_{ji}=z_{ji}, \widetilde{x}_{ji}=\widetilde{z}_{ji}}. \tag{A.40}$$

Therefore, when $0 < |z_j - z_i| < \frac{1}{2}|z_i|$, the series

$$\sum_{r,s\in\mathbb{R}} \big\langle \mathcal{Y}_\delta\big(P_r e^{(z_j-z_i)L_{-1}} P_s\mathcal{Y}_\gamma(w^{(i)}, e^{\pm i\pi}(z_j - z_i))w^{(j)}, z_i\big)w_1, w_2\big\rangle \tag{A.41}$$

converges absolutely and equals (A.37) with $z_{ji} = \widetilde{z}_{ji} = z_j - z_i$.

One the other hand,

$$\begin{aligned}
&\big\langle \mathcal{Y}_\delta\big(\mathcal{Y}_{B_\pm\gamma}(w^{(j)}, z_j - z_i)w^{(i)}, z_i\big)w_1, w_2\big\rangle\\
=&\sum_{r\in\mathbb{R}}\big\langle \mathcal{Y}_\delta\big(P_r\mathcal{Y}_{B_\pm\gamma}(w^{(j)}, z_j - z_i)w^{(i)}, z_i\big)w_1, w_2\big\rangle\\
=&\sum_{r\in\mathbb{R}}\big\langle \mathcal{Y}_\delta\big(P_r e^{(z_j-z_i)L_{-1}}\mathcal{Y}_\gamma(w^{(i)}, e^{\pm i\pi}(z_j - z_i))w^{(j)}, z_i\big)w_1, w_2\big\rangle,
\end{aligned}$$

which is just (A.41). So it also equals (A.37) with $z_{ji} = \widetilde{z}_{ji} = z_j - z_i$. This proves relation (2.18) when $0 < |z_j - z_i| < \frac{1}{2}|z_i|$. The general case follows from analytic continuation. $\square$

Proof of theorem 2.8. The case $n = 2$ follows immediately from proposition 2.9 and the fusion relations of two intertwining operators. We now prove the general case.

Since S_n is generated by adjacent transpositions, we can assume that ς exchanges $m, m+1$ and fixes the other elements in $\{1, 2, \dots, n\}$. Write

$$\begin{aligned}
\mathcal{X}_1 &= \mathcal{Y}_{\alpha_{m-1}}(w^{(i_{m-1})}, z_{m-1})\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1),\\
\mathcal{X}_2 &= \mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n)\cdots\mathcal{Y}_{\alpha_{m+2}}(w^{(i_{m+2})}, z_{m+2}).
\end{aligned}$$

To proof the braid relation in this case, it is equivalent to showing that if $0 < |z_1| < \cdots < |z_{m-1}| < |z_{m+1}| < |z_m| < |z_{m+2}| < \cdots < |z_n|$, and if we move z_m, z_{m+1} to satisfy $0 < |z_1| < \cdots < |z_{m-1}| < |z_m| < |z_{m+1}| < |z_{m+2}| < \cdots < |z_n|$ by scaling the norms of z_m, z_{m+1}, then we can find intertwining operators $\mathcal{Y}_{\beta_m}, \mathcal{Y}_{\beta_{m+1}}$ independent of the choice of vectors, such that

$$\langle \mathcal{X}_2\mathcal{Y}_{\alpha_m}(w^{(i_m)}, z_m)\mathcal{Y}_{\alpha_{m+1}}(w^{(i_{m+1})}, z_{m+1})\mathcal{X}_1 w^{(i_0)}, w^{(\overline{k})}\rangle \tag{A.42}$$

can be analytically continued to

$$\langle \mathcal{X}_2\mathcal{Y}_{\beta_{m+1}}(w^{(i_{m+1})}, z_{m+1})\mathcal{Y}_{\beta_m}(w^{(i_m)}, z_m)\mathcal{X}_1 w^{(i_0)}, w^{(\overline{k})}\rangle. \tag{A.43}$$

By analytic continuation, we can also assume that during the process of moving z_m, z_{m+1}, conditions $0 < |z_1| < \cdots < |z_{m-1}| < |z_m|, |z_{m+1}| < |z_{m+2}| < \cdots < |z_n|$ and $0 < |z_m - z_{m+1}| < |z_{m+1}|$ are always satisfied.

Let W_{j_1} be the source space of $\mathcal{Y}_{\alpha_{m+1}}$ and W_{j_2} be the target space of $\mathcal{Y}_{\alpha_m}$. By braiding of two intertwining operators, there exists a chain of intertwining operators $\mathcal{Y}_{\beta_m}, \mathcal{Y}_{\beta_{m+1}}$ with charge spaces $W_{i_m}, W_{i_{m+1}}$ respectively, such that the source space of $\mathcal{Y}_{\beta_m}$ is W_{j_1}, that the target space of $\mathcal{Y}_{\beta_{m+1}}$ is W_{j_2}, and that for any $w^{(j_1)} \in W_{j_1}, w^{(i_m)} \in W_{i_m}, w^{(i_{m+1})} \in W_{i_{m+1}}, w^{(\overline{j_2})} \in W_{\overline{j_2}}$, the expression

$$\langle \mathcal{Y}_{\alpha_m}(w^{(i_m)}, z_m)\mathcal{Y}_{\alpha_{m+1}}(w^{(i_{m+1})}, z_{m+1})w^{(j_1)}, w^{(\overline{j_2})}\rangle \tag{A.44}$$

defined on $0 < |z_{m+1}| < |z_m|$ can be analytically continued to

$$\langle \mathcal{Y}_{\beta_{m+1}}(w^{(i_{m+1})}, z_{m+1})\mathcal{Y}_{\beta_m}(w^{(i_m)}, z_m)w^{(j_1)}, w^{(\overline{j_2})}\rangle \tag{A.45}$$

defined on $0 < |z_m| < |z_{m+1}|$ by scaling the norms of z_m and z_{m+1}.

Now, by fusion of intertwining operators, there exist intertwining operators $\mathcal{Y}_\delta, \mathcal{Y}_\gamma$ with suitable charge spaces, source spaces, and target spaces, such that (A.44) equals

$$\langle \mathcal{Y}_\delta\big(\mathcal{Y}_\gamma(w^{(i_m)}, z_m - z_{m+1})w^{(i_{m+1})}, z_{m+1}\big)w^{(j_1)}, w^{(\overline{j_2})}\rangle \tag{A.46}$$

when $|z_{m+1}| < |z_m|$. Then (A.45) equals (A.46) when $|z_m| < |z_{m+1}|$. By theorem 2.6, the expression

$$\langle \mathcal{X}_2\mathcal{Y}_\delta\big(\mathcal{Y}_\gamma(w^{(i_m)}, z_m - z_{m+1})w^{(i_{m+1})}, z_{m+1}\big)\mathcal{X}_1 w^{(i_0)}, w^{(\overline{k})}\rangle \tag{A.47}$$

converges absolutely and locally uniformly. Hence it is a locally defined holomorphic function when $0 < |z_1| < \cdots < |z_{m-1}| < |z_m|, |z_{m+1}| < |z_{m+2}| < \cdots < |z_n|$. Therefore (A.42) can be analytically continued to (A.43) from $\{0 < |z_{m+1}| < |z_m|\}$ to $\{0 < |z_m| < |z_{m+1}|\}$. $\square$

B Appendix for chapter 3

B.1 von Neumann algebras generated by closed operators

Let A be a (densely defined) unbounded operator on $\mathcal{H}$ with domain $\mathscr{D}(A)$. Choose $x \in B(\mathcal{H})$, i.e., let x be a bounded operator on $\mathcal{H}$. Recall that the notation $xA \subset Ax$ means that $x\mathscr{D}(A) \subset \mathscr{D}(A)$, and $xA\xi = Ax\xi$ for any $\xi \in \mathscr{D}(A)$. The following proposition is easy to show.

Proposition B.1. *Let A be a preclosed operator on $\mathcal{H}$ with closure $\overline{A}$.*
(1) If $x \in B(\mathcal{H})$ and $xA \subset Ax$, then we have $x^*A^* \subset A^*x^*$ and $x\overline{A} \subset \overline{A}x$.
(2) If A is closed, then the set of all $x \in B(\mathcal{H})$ satisfying $xA \subset Ax$ form a strongly closed subalgebra of $B(\mathcal{H})$.

Proof. If $xA \subset Ax$ then $(Ax)^* \subset (xA)^*$. Recall that in general, if A, B are two densely defined unbounded operators on $\mathcal{H}$, and if AB has dense domain, then $B^*A^* \subset (AB)^*$. If A is bounded, then $B^*A^* = (AB)^*$. Thus we have $x^*A^* \subset (Ax)^* \subset (xA)^* = A^*x^*$. Apply this relation to x^*, A^*, and note that $A^{**} = \overline{A}$, then we have $x\overline{A} \subset \overline{A}x$. This proves part (1). Part (2) is a routine check. $\square$

Definition B.2. Let A be a closed operator on a Hilbert space $\mathcal{H}$ with domain $\mathscr{D}(A)$, and let $x \in B(\mathcal{H})$. We say that A and x **commute strongly**[17], if the following relations hold:

$$xA \subset Ax, \quad x^*A \subset Ax^*. \tag{B.1}$$

Corollary B.3. *Suppose that $\mathfrak{S}$ is a collection of closed operators on $\mathcal{H}$. We define its* ***commutant*** *$\mathfrak{S}'$ to be the set of all bounded operators on $\mathcal{H}$ which commute strongly with any element of $\mathfrak{S}$. Then $\mathfrak{S}'$ is a von Neumann algebra. It's double commutant $\mathfrak{S}''$, which is the commutant of $\mathfrak{S}'$, is called the* ***von Neumann algebra generated by*** *$\mathfrak{S}$.*

[17]Our definition follows [Neu16] chapter XIV, in which the strong commutativity of an unbounded operator with a bounded one is called adjoint commutativity.

Lemma B.4. *Suppose that A is a closed operator on $\mathcal{H}$, and $v \in B(\mathcal{H})$ is a unitary operator. Let $A = uH$ (resp. Hu) be the left (resp. right) polar decomposition of A, such that u the partial isometry and H the self adjoint opertor. Then the following conditions are equivalent:*

(a) v commutes strongly with A.

(b) $vA = Av$. (B.2)

(c) $[u, v] = 0$, and $[e^{itH}, v] = 0$ for any $t \in \mathbb{R}$. (B.3)

Proof. We prove this for the left polar decomposition. The other case can be proved in the same way.

(a)⇒(b): Since v commutes strongly with A, we have $vA \subset Av$ and $v^{-1}A \subset Av^{-1}$. Therefore, $v\mathscr{D}(A) \subset \mathscr{D}(A)$ and $v^{-1}\mathscr{D}(A) \subset \mathscr{D}(A)$. So we must have $v\mathscr{D}(A) = \mathscr{D}(A)$, and hence $vA = Av$.

(b)⇒(a): If $vA = Av$, then $vAv^{-1} = A$. So $Av^{-1} = v^{-1}A$, which proves (a).

(b)⇒(c): We have $vAv^{-1} = A$. Thus by uniqueness of left polar decompositions, we have $vuv^{-1} = u$ and $vHv^{-1} = H$. Hence for any $t \in \mathbb{R}$ we have

$$ve^{itH}v^{-1} = e^{iv(tH)v^{-1}} = e^{itH}.$$

This proves (c).

(c)⇒(b): Suppose that we have (B.3). Then $vuv^{-1} = u$ and $ve^{itH}v^{-1} = e^{itH}$. On the other hand, we always have $ve^{itH}v^{-1} = e^{itvHv^{-1}}$ in general. So vHv^{-1} and H are both generators of the one parameter unitary group $ve^{itH}v^{-1}$. Hence we must have $vHv^{-1} = H$. This implies that $vA = Av$. Therefore (b) is true. □

Proposition B.5. *Let $\mathfrak{S}$ be a set of closed operators on $\mathcal{H}$. For each $A \in \mathfrak{S}$, we either let $A = u_A H_A$ be the left polar decomposition of A, or let $A = H_A u_A$ be the right polar decomposition of A. Then $\mathfrak{S}''$ is the von Neumann algebra generated by the bounded operators $\{u_A, e^{itH_A} : t \in \mathbb{R}, A \in \mathfrak{S}\}$.*

Proof. Let $\mathcal{M}$ be the von Neumann algebras generated by those u_A and e^{itH_A}. We show that $\mathcal{M} = \mathfrak{S}''$.

Let $\mathcal{U}(\mathfrak{S}')$ be the set of unitary operators in $\mathfrak{S}'$. We know that $\mathcal{U}(\mathfrak{S}')$ generates $\mathfrak{S}'$. So $\mathfrak{S}'' = \mathcal{U}(\mathfrak{S}')'$. By lemma B.4 (a)⇒(c) we see that $\mathcal{M}$ commutes with $\mathcal{U}(\mathfrak{S}')$. Hence $\mathcal{M} \subset \mathcal{U}(\mathfrak{S}')' = \mathfrak{S}''$.

Let $\mathcal{U}(\mathcal{M}')$ be the set of unitary operators in $\mathcal{M}'$, the commutant of $\mathcal{M}$. Then by lemma B.4 (c)⇒(a) we also have $\mathcal{U}(\mathcal{M}') \subset \mathfrak{S}'$. Hence $\mathcal{M}' \subset \mathfrak{S}'$, which implies that $\mathcal{M} \supset \mathfrak{S}''$. Thus we've proved that $\mathcal{M} = \mathfrak{S}''$. □

Corollary B.6. *Assume that A is a closed operator on $\mathcal{H}$ and $x \in B(\mathcal{H})$. Let $A = uH$ (resp. Hu) be the left (resp. right) polar decomposition of A with u the partial isometry and H the self adjoint opertor. Then x commutes strongly with A if and only if $[u, x] = 0$ and $[e^{itH}, x] = 0$ for any $t \in \mathbb{R}$.*

Proof. Let $\mathfrak{S} = \{A\}$. Then by proposition B.5, $\mathfrak{S}''$ is generated by u and all e^{itH}. Thus $x \in \mathfrak{S}'$ if and only if x commutes with u and all e^{itH}. □

Definition B.7. Let A and B be two closed operators on a Hilbert space $\mathcal{H}$. We say that A and B **commute strongly**, if the von Neumann algebra generated by A commutes with the one generated by B.

If $\mathcal{M}$ is a von Neumann algebra on $\mathcal{H}$ and A is a closed operator on $\mathcal{H}$. We say that A is **affiliated with** $\mathcal{M}$, if the von Neumann algebra generated by the single operator A is inside $\mathcal{M}$. Now suppose that $\mathcal{N}$ is another von Neumann algebra on a Hilbert space $\mathcal{K}$, and $\pi : \mathcal{M} \to \mathcal{N}$ is a normal (i.e. $\sigma-$weakly continuous) unital *-homomorphism. We define $\pi(A)$ to be a closed operator on $\mathcal{K}$ affiliated with $\mathcal{N}$ in the following way: Let $A = uH$ be its left polar decomposition. Define $\pi(H)$ to be the generator of the one parameter unitary group $\pi(e^{itH})$ acting on $\mathcal{H}$, i.e., the unique self-adjoint operator on $\mathcal{K}$ satisfying

$$e^{it\pi(H)} = \pi(e^{itH}) \quad (t \in \mathbb{R}). \tag{B.4}$$

We then define

$$\pi(A) = \pi(u)\pi(H). \tag{B.5}$$

We can also define $\pi(A)$ using the right polar decomposition of A. It is easy to show that these two definitions are the same.

B.2 A criterion for strong commutativity

A famous example of Nelson (cf. [Nel59]) shows that two unbounded self-adjoint operators commuting on a common invariant core might not commute strongly. In this section, we give a criterion on the strong commutativity of unbounded closed operators. Our approach follows [TL99] and [TL04]. See also [GJ12] section 19.4 for related materials.

Suppose that D is a self-adjoint positive operator on a Hilbert space $\mathcal{H}$. For any $r \in \mathbb{R}$, we let $\mathcal{H}^r$ be the domain of $(1+D)^r$. It is clear that $\mathcal{H}^{r_1} \supset \mathcal{H}^{r_2}$ if $r_1 < r_2$. We let $\mathcal{H}^\infty = \bigcap_{r \geqslant 0} \mathcal{H}^r$. Define a norm $\|\cdot\|_r$ on $\mathcal{H}_r$ to be $\|\xi\|_r = \|(1+D)^r\xi\|$. Suppose that K is an unbounded operator on $\mathcal{H}$ with invariant domain $\mathcal{H}^\infty$ ("invariant" means that $K\mathcal{H}^\infty \subset \mathcal{H}^\infty$), that K is symmetric, i.e., for any $\xi, \eta \in \mathcal{H}^\infty$ we have

$$\langle K\xi|\eta\rangle = \langle \xi|K\eta\rangle, \tag{B.6}$$

and that for any $n \in \mathbb{Z}_{\geqslant 0}$ there exist positive numbers $|K|_{n+1}$ and $|K|_{D,n+1}$, such that for any $\xi \in \mathcal{H}^\infty$ we have

$$\|K\xi\|_n \leqslant |K|_{n+1}\|\xi\|_{n+1}, \tag{B.7}$$

$$\|[D,K]\xi\|_n \leqslant |K|_{D,n+1}\|\xi\|_{n+1}. \tag{B.8}$$

Since K is symmetric, it is obviously preclosed. We let $\overline{K}$ denote the closure K. The following lemma is due to Toledano-Laredo (cf. [TL99] proposition 2.1[18] and corollary 2.2).

[18]Toledano-Laredo's proof of this proposition was based on a trick in [FL74] theorem 2.

Lemma B.8. *$\overline{K}$ is self-adjoint. Moreover, the following statements are true:*
(1) For any $n \in \mathbb{Z}_{\geqslant 0}$ and $t \in \mathbb{R}$, the unitary operator $e^{it\overline{K}}$ restricts to a bounded linear map $\mathcal{H}^n \to \mathcal{H}^n$ with

$$\|e^{it\overline{K}}\xi\|_n \leqslant e^{2nt|K|_{D,n}}\|\xi\|_n, \quad \xi \in \mathcal{H}^n. \tag{B.9}$$

(2) For any $\xi \in \mathcal{H}^\infty$, $h \in \mathbb{R}$ and $k = 1, 2, \ldots$, we have

$$e^{i(t+h)\overline{K}}\xi = e^{it\overline{K}}\xi + \cdots + \frac{h^k}{k!}K^k e^{it\overline{K}}\xi + R(h), \tag{B.10}$$

where all terms are in $\mathcal{H}^\infty$ and $R(h) = o(h^k)$ in each $\|\cdot\|_n$ norm, i.e., $\|R(h)\|_n h^{-k} \to 0$ as $h \to 0$.

This lemma may help us prove the following important criterion for strong commutativity of unbounded closed operators.

Theorem B.9. *Let T be another unbounded operator on $\mathcal{H}$ with invariant domain $\mathcal{H}^\infty$. Suppose that T satisfies the following conditions:*
(1) There exists $m \in \mathbb{Z}_{\geqslant 0}$, such that for any $n \in \mathbb{Z}_{\geqslant 0}$, we can find a positive number $|T|_{n+m}$, such that

$$\|T\xi\|_n \leqslant |T|_{n+m}\|\xi\|_{n+m} \quad (\xi \in \mathcal{H}^\infty). \tag{B.11}$$

(2) T is a preclosed operator on $\mathcal{H}$.
(3) $KT\xi = TK\xi$ for any $\xi \in \mathcal{H}^\infty$.
Then the self-adjoint operator $\overline{K}$ commutes strongly with $\overline{T}$, the closure of T.

Proof. By lemma B.8, for each $t \in \mathbb{R}$, $e^{it\overline{K}}$ leaves $\mathcal{H}^\infty$ invariant. We want to show that

$$e^{it\overline{K}}Te^{-it\overline{K}} = T \quad \text{on } \mathcal{H}^\infty. \tag{B.12}$$

For any $\xi \in \mathcal{H}^\infty$ we define a $\mathcal{H}^\infty$-valued function Ξ on $\mathbb{R}$ by

$$\Xi(t) = e^{it\overline{K}}Te^{-it\overline{K}}\xi. \tag{B.13}$$

If we can show that this function is constant, then we have $\Xi(t) = \Xi(0)$, which proves (B.12). To prove this, it suffices to show that the derivative of this function is always 0.

For any $t \in \mathbb{R}$, if $0 \neq h \in \mathbb{R}$, then

$$\begin{aligned}
\Xi(t+h) =& e^{i(t+h)\overline{K}}Te^{-i(t+h)\overline{K}}\xi && \text{(B.14)}\\
=& e^{i(t+h)\overline{K}}T\big((1-ihK)e^{-it\overline{K}}\xi + o(h)\big) && \text{(B.15)}\\
=& e^{i(t+h)\overline{K}}T(1-ihK)e^{-it\overline{K}}\xi + o(h) && \text{(B.16)}\\
=& e^{i(t+h)\overline{K}}Te^{-it\overline{K}}\xi - ihe^{i(t+h)\overline{K}}KTe^{-it\overline{K}}\xi + o(h) && \text{(B.17)}\\
=& [e^{it\overline{K}}(1+ihK)Te^{-it\overline{K}}\xi + o(h)]\\
& - ih[e^{it\overline{K}}(1+ihK)KTe^{-it\overline{K}}\xi + o(h)] + o(h) && \text{(B.18)}\\
=& e^{it\overline{K}}Te^{-it\overline{K}}\xi + o(h) = \Xi(t) + o(h), && \text{(B.19)}
\end{aligned}$$

where (B.15) and (B.18) follow from (B.10), and (B.17) follows from the relation $KT = TK$ on $\mathcal{H}^\infty$. We also used the fact that $To(h) = o(h)$ (which follows from (B.11)) in (B.16). Here the meaning of $o(h)$ is same as that in lemma B.8.

Hence we have shown that $\Xi'(t) = 0$ for any $t \in \mathbb{R}$, which proves (B.12). Now we regard T as an unbounded operator on $\mathcal{H}$. By passing to the closure, we have $e^{it\overline{K}}\overline{T}e^{-it\overline{K}} = \overline{T}$. This shows that $\overline{T}$ commutes strongly with $\overline{K}$. □

References

[BK01] Bakalov, B. and Kirillov, A.A., 2001. Lectures on tensor categories and modular functors (Vol. 21). American Mathematical Soc..

[BS90] Buchholz, D. and Schulz-Mirbach, H., 1990. Haag duality in conformal quantum field theory. Reviews in Mathematical Physics, 2(01), pp.105-125.

[CKLW15] Carpi S, Kawahigashi Y, Longo R, Weiner M. From vertex operator algebras to conformal nets and back. arXiv preprint arXiv:1503.01260. 2015 Mar 4.

[CKM17] Creutzig, T., Kanade, S. and McRae, R., 2017. Tensor categories for vertex operator superalgebra extensions. arXiv preprint arXiv:1705.05017.

[Con80] Connes, A., 1980. On the spatial theory of von Neumann algebras. Journal of Functional Analysis, 35(2), pp.153-164.

[DHR71] Doplicher, S., Haag, R. and Roberts, J.E., 1971. Local observables and particle statistics I. Communications in Mathematical Physics, 23(3), pp.199-230.

[DL14] Dong, C. and Lin, X., 2014. Unitary vertex operator algebras. Journal of algebra, 397, pp.252-277.

[EGNO04] Etingof, P.I., Gelaki, S., Nikshych, D. and Ostrik, V., 2015. Tensor categories (Vol. 205). Providence, RI: American Mathematical Society.

[FB04] Frenkel, E. and Ben-Zvi, D., 2004. Vertex algebras and algebraic curves (No. 88), 2nd edition. American Mathematical Soc..

[FHL93] Frenkel, I., Huang, Y.Z. and Lepowsky, J., 1993. On axiomatic approaches to vertex operator algebras and modules (Vol. 494). American Mathematical Soc..

[FL74] Faris, W.G. and Lavine, R.B., 1974. Commutators and self-adjointness of Hamiltonian operators. Communications in Mathematical Physics, 35(1), pp.39-48.

[FLM89] Frenkel, I., Lepowsky, J. and Meurman, A., 1989. Vertex operator algebras and the Monster (Vol. 134). Academic press.

[FRS89] Fredenhagen, K., Rehren, K.H. and Schroer, B., 1989. Superselection sectors with braid group statistics and exchange algebras. Communications in Mathematical Physics, 125(2), pp.201-226.

[Gal12] Galindo, C., 2012. On braided and ribbon unitary fusion categories. arXiv preprint arXiv:1209.2022.

[GJ12] Glimm, J. and Jaffe, A., 2012. Quantum physics: a functional integral point of view. Springer Science & Business Media.

[GW84] Goodman, R. and Wallach, N.R., 1984. Structure and unitary cocycle representations of loop groups and the group of diffeomorphisms of the circle. energy, 3, p.3.

[HK07] Huang, Y.Z. and Kong, L., 2007. Full field algebras. Communications in mathematical physics, 272(2), pp.345-396.

[HK10] Huang, Y.Z. and Kong, L., 2010. Modular invariance for conformal full field algebras. Transactions of the American Mathematical Society, 362(6), pp.3027-3067.

[HKL15] Huang, Y.Z., Kirillov, A. and Lepowsky, J., 2015. Braided tensor categories and extensions of vertex operator algebras. Communications in Mathematical Physics, 337(3), pp.1143-1159.

[HL94] Huang, Y.Z. and Lepowsky, J., 1994. Tensor products of modules for a vertex operator algebra and vertex tensor categories. Lie Theory and Geometry, in honor of Bertram Kostant, pp.349-383.

[HL95a] Huang, Y.Z. and Lepowsky, J., 1995. A theory of tensor products for module categories for a vertex operator algebra, I. Selecta Mathematica, 1(4), p.699.

[HL95b] Huang, Y.Z. and Lepowsky, J., 1995. A theory of tensor products for module categories for a vertex operator algebra, II. Selecta Mathematica, 1(4), p.757.

[HL95c] Huang, Y.Z. and Lepowsky, J., 1995. A theory of tensor products for module categories for a vertex operator algebra, III. Journal of Pure and Applied Algebra, 100(1-3), pp.141-171.

[HL13] Huang, Y.Z. and Lepowsky, J., 2013. Tensor categories and the mathematics of rational and logarithmic conformal field theory. Journal of Physics A: Mathematical and Theoretical, 46(49), p.494009.

[HLZ11] Huang, Y.Z., Lepowsky, J. and Zhang, L., 2011. Logarithmic tensor category theory, VIII: Braided tensor category structure on categories of generalized modules for a conformal vertex algebra. arXiv preprint arXiv:1110.1931.

[Hua95] Huang, Y.Z., 1995. A theory of tensor products for module categories for a vertex operator algebra, IV. Journal of Pure and Applied Algebra, 100(1-3), pp.173-216.

[Hua05a] Huang, Y.Z., 2005. Differential equations and intertwining operators. Communications in Contemporary Mathematics, 7(03), pp.375-400.

[Hua05b] Huang, Y.Z., 2005. Differential equations, duality and modular invariance. Communications in Contemporary Mathematics, 7(05), pp.649-706.

[Hua08a] Huang, Y.Z., 2008. Vertex operator algebras and the Verlinde conjecture. Communications in Contemporary Mathematics, 10(01), pp.103-154.

[Hua08b] Huang, Y.Z., 2008. Rigidity and modularity of vertex tensor categories. Communications in contemporary mathematics, 10(supp01), pp.871-911.

[KLM01] Kawahigashi, Y., Longo, R. and Müger, M., 2001. Multi-Interval Subfactors and Modularity of Representations in Conformal Field Theory. Communications in Mathematical Physics, 219(3), pp.631-669.

[Kac98] Kac, V.G., 1998. Vertex algebras for beginners (No. 10). American Mathematical Soc..

[Kaw15] Kawahigashi, Y., 2015. Conformal field theory, tensor categories and operator algebras. Journal of Physics A: Mathematical and Theoretical, 48(30), p.303001.

[Kong06] Kong, L., 2006. Full field algebras, operads and tensor categories. arXiv preprint math/0603065.

[Kong08] Kong, L., 2008. Cardy condition for open-closed field algebras. Communications in Mathematical Physics, 283(1), pp.25-92.

[LL12] Lepowsky, J. and Li, H., 2012. Introduction to vertex operator algebras and their representations (Vol. 227). Springer Science & Business Media.

[MS88] Moore, G. and Seiberg, N., 1988. Polynomial equations for rational conformal field theories. Physics Letters B, 212(4), pp.451-460.

[MS89] Moore, G. and Seiberg, N., 1989. Classical and quantum conformal field theory. Communications in Mathematical Physics, 123(2), pp.177-254.

[MS90] Moore, G. and Seiberg, N., 1990. Lectures on RCFT. In Physics, geometry and topology (pp. 263-361). Springer, Boston, MA.

[Muk10] Mukhopadhyay, S., 2010. Decomposition of conformal blocks (Doctoral dissertation, Master's thesis, University of North Carolina at Chapel Hill).

[Nel59] Nelson, E., 1959. Analytic vectors. Annals of Mathematics, pp.572-615.

[Neu16] Von Neumann, J., 2016. Functional Operators (AM-22), Volume 2: The Geometry of Orthogonal Spaces.(AM-22) (Vol. 2). Princeton University Press.

[OS73] Osterwalder, K. and Schrader, R., 1973. Axioms for Euclidean Green's functions. Communications in mathematical physics, 31(2), pp.83-112.

[Seg88] Segal, G.B., 1988. The definition of conformal field theory. In Differential geometrical methods in theoretical physics (pp. 165-171). Springer, Dordrecht.

[TL99] Toledano-Laredo, V., 1999. Integrating unitary representations of infinite-dimensional Lie groups. Journal of functional analysis, 161(2), pp.478-508.

[TL04] Toledano-Laredo, V., 2004. Fusion of positive energy representations of Lspin (2n). arXiv preprint math/0409044.

[Tur16] Turaev, V.G., 2016. Quantum invariants of knots and 3-manifolds (Vol. 18). Walter de Gruyter GmbH & Co KG.

[Ueno08] Ueno, K., 2008. Conformal field theory with gauge symmetry (Vol. 24). American Mathematical Soc..

[Was98] Wassermann, A., 1998. Operator algebras and conformal field theory III. Fusion of positive energy representations of LSU (N) using bounded operators. Inventiones Mathematicae, 133(3), pp.467-538.