Unitarity of the modular tensor categories associated to unitary vertex operator algebras, II

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Abstract

This is the second part in a two-part series of papers constructing a unitary structure for the modular tensor category (MTC) associated to a unitary rational vertex operator algebra (VOA). We define, for a unitary rational vertex operator algebra V, a non-degenerate sesquilinear form Λ on each vector space of intertwining operators. We give two sets of criteria for the positivity of Λ, both concerning the energy bounds condition of vertex operators and intertwining operators. These criteria can be applied to many familiar examples, including unitary Virasoro VOAs, unitary affine VOAs of type A, D, and more. Having shown that Λ is an inner product, we prove that Λ induces a unitary structure on the modular tensor category of V.

Contents

Introduction

In this article, we continue the study initiated in [Gui19a] of the unitarity structure on the modular tensor category (MTC) associated to a unitary "rational" vertex operator algebra (VOA) V. The precise meaning of the word "rational" will be given in the notation list. We will follow the notations and conventions in [Gui19a]. For simplicity, we assume that all V-modules are unitarizable. To prove the unitarity of the modular tensor category of V, we need to define, for any unitary V-modules W_i, W_j, a canonical unitary structure on the unitarizable V-module $W_i \boxtimes W_j$.

We first recall the definition of $W_i \boxtimes W_j$. For each equivalence class $[W_k]$ of irreducible unitary V-module, we choose a representing element W_k, and let $\mathcal{E}$ be the set of these W_k's. Then the cardinality of $\mathcal{E}$ is finite by the rationality of V. The tensor product of W_i, W_j is defined to be

$$W_{ij} \equiv W_i \boxtimes W_j = \bigoplus_{k\in\mathcal{E}} \mathcal{V}\binom{k}{i\ j}^* \otimes W_k.$$

where $\mathcal{V}\binom{k}{i\ j}^*$ is the dual vector space of $\mathcal{V}\binom{k}{i\ j}$, the finiteness of the dimension of which follows also from the rationality of V. The action of V on $W_i \boxtimes W_j$ is $\bigoplus_k \mathrm{id} \otimes Y_k$, where Y_k is the vertex operator describing the action of V on W_k. Since the unitary structure on each W_k is already chosen, to determine a suitable unitary on $W_i \boxtimes W_j$, it suffices to find such an inner product Λ on $\mathcal{V}\binom{k}{i\ j}^*$ for any $k \in \mathcal{E}$.

Definition of Λ

We first define $\Lambda = \Lambda(\cdot|\cdot)$ as a sesquilinear form on $\mathcal{V}\binom{k}{i\ j}^*$, antilinear on the second variable. Recall from part I that for any $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$, its adjoint intertwining operator $\mathcal{Y}_{\alpha^*} \in \mathcal{V}\binom{j}{\bar{i}\ k}$ is defined to satisfy that for any $w^{(i)} \in W_i$,

$$\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, x) = \mathcal{Y}_\alpha(e^{xL_1}(e^{-i\pi}x^{-2})^{L_0} w^{(i)}, x^{-1})^\dagger.$$

Here † is the formal adjoint operation, which means that for any $w^{(j)} \in W_j, w^{(k)} \in W_k$,

$$\langle \mathcal{Y}_{\alpha*}(\overline{w^{(i)}}, x)w^{(k)}|w^{(j)}\rangle = \langle w^{(k)}|\mathcal{Y}_\alpha(e^{xL_1}(e^{-i\pi}x^{-2})^{L_0}w^{(i)}, x^{-1})w^{(j)}\rangle.$$

The creation operator $\mathcal{Y}^i_{i0} \in \mathcal{V}\binom{i}{i\ 0}$ of W_i is defined so that for any $v \in V$ and $w^{(i)} \in W_i$,

$$\mathcal{Y}^i_{i0}(w^{(i)}, x)v = e^{xL_{-1}}Y_i(v, -x)w^{(i)}.$$

The annihilation operator $\mathcal{Y}^0_{\bar{i}i}$ is defined to be the adjoint intertwining operator of $\mathcal{Y}^i_{i0}$, which is of type $\binom{0}{\bar{i}\ i}$.

We now choose a basis $\{\mathcal{Y}_\alpha : \alpha \in \Theta^k_{ij}\}$ of the vector space $\mathcal{V}\binom{k}{i\ j}$, and let $\{\check{\mathcal{Y}}^\alpha : \alpha \in \Theta^k_{ij}\}$ be its dual basis in $\mathcal{V}\binom{k}{i\ j}^*$. By fusion relations, there exits, for each $k \in \mathcal{E}$, a complex matrix $\Lambda = \{\Lambda^{\alpha\beta}\}_{\alpha,\beta\in\Theta^k_{ij}}$, such that for any $z_1, z_2 \in \mathbb{C}^\times$ satisfying $0 < |z_2 - z_1| < |z_1| < |z_2|$ and $\arg z_1 = \arg z_2 = \arg(z_2 - z_1)$, we have

$$Y_j\big(\mathcal{Y}^0_{\bar{i}i}(\overline{w^{(i)}_2}, z_2 - z_1)w^{(i)}_1, z_1\big) = \sum_{k\in\mathcal{E}}\sum_{\alpha,\beta\in\Theta^k_{ij}} \Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w^{(i)}_2}, z_2)\mathcal{Y}_\alpha(w^{(i)}_1, z_1). \tag{0.7}$$

The fusion relation (0.7) is called *transport formula*, and the matrix Λ is called *transport matrix*.[1] We then define a sesquilinear form $\Lambda(\cdot|\cdot)$ on $\mathcal{V}\binom{k}{i\ j}^*$ to satisfy that

$$\Lambda(\check{\mathcal{Y}}^\alpha|\check{\mathcal{Y}}^\beta) = \Lambda^{\alpha\beta} \tag{0.8}$$

for any $\alpha, \beta \in \Theta^k_{ij}$. It is easy to see that this definition does not depend on the basis Θ^k_{ij} chosen.

The sesquilinear forms Λ on the dual vector spaces of intertwining operators induce a sesquilinear form on $W_i \boxtimes W_j$, also denoted by Λ. If one can prove that Λ is positive, then by the rigidity of the MTC of V, one can easily show that Λ is also non-degenerate. Therefore Λ becomes an inner product. It will then be not hard to show, as we shall see in chapter 7, that the MTC of V is unitary. However, the difficulty lies exactly in the proof of the positivity of Λ.

Positivity of Λ

To prove the positivity of Λ, we will use several analytic conditions on VOAs and their intertwining operators, so that certain results and techniques in conformal nets can be applied here. In this paper, we assume that the unitary rational VOA V satisfies condition A or B. The precise statements of these two conditions are in section 5.3. Here we make a brief and somewhat simplified description. Both conditions require that V is strongly local [CKLW18], so that we can construct a conformal net $\mathcal{M}_V$ using smeared vertex operators from V. Both require that there exists a “generating set” of intertwining operators,

[1] The primitive form of Λ appeared in [Was98]. See the discussion in remark 6.4.

which means, more precisely, that there exists a set $\mathcal{F} = \{W_i : i \in \mathcal{F}\}$ of irreducible V-modules such that any irreducible V-module is a submodule of a tensor product of these modules (i.e., the set $\mathcal{F}$ generate the monoidal category of V), and that for any $i \in \mathcal{F}$ and any irreducible V-module W_j, W_k, the type $\binom{k}{i\ j}$ intertwining operators are energy-bounded.[2] The difference between these two conditions is that condition A requires that there exists a generating set of quasi-primary vectors in V whose vertex operators satisfy 1-st order energy bounds, whereas in condition B, the 1-st order energy bounds condition is required for intertwining operators. The reason we need 1-st order energy bounds condition on either vertex operators or intertwining operators is to guarantee the *strong intertwining property*: causally disjoint smeared vertex operators and smeared intertwining operators commute, not only when acting on a common invariant core of them, but also strongly in the sense that the von Neumann algebras they generate commute. (See proposition 5.10.)

We now sketch the proof of the positivity of Λ on $W_i \boxtimes W_j$. We first assume that $i, j \in \mathcal{F}$. So any intertwining operator whose charge space is W_i or W_j is energy-bounded. In this special case, our proof is modeled on Wassermann's argument in [Was98]. We first recall some algebraic and analytic properties of smeared intertwining operators proved in part I. Choose disjoint open intervals $I, J \in S^1 \backslash \{-1\}$.

(a) *Braiding*: If $\mathcal{Y}_\alpha$ and $\mathcal{Y}_{\alpha'}$ are (energy bounded) intertwining operators with charge space W_i, $\mathcal{Y}_\beta$ and $\mathcal{Y}_{\beta'}$ are (energy bounded) intertwining operators with charge space W_j, and for any $w^{(i)} \in W_i, w^{(j)} \in W_j, z_1 \in I, z_2 \in J$ we have the braid relation

$$\mathcal{Y}_\alpha(w^{(i)}, z_1)\mathcal{Y}_\beta(w^{(j)}, z_2) = \mathcal{Y}_{\beta'}(w^{(j)}, z_2)\mathcal{Y}_{\alpha'}(w^{(i)}, z_1), \tag{0.9}$$

Then the smeared intertwining operators also have braiding

$$\mathcal{Y}_\alpha(w^{(i)}, f)\mathcal{Y}_\beta(w^{(j)}, g) = \mathcal{Y}_{\beta'}(w^{(j)}, g)\mathcal{Y}_{\alpha'}(w^{(i)}, f) \tag{0.10}$$

for any $f \in C_c^\infty(I), g \in C_c^\infty(J)$.

(b) *Adjoint relation*: If $w^{(i)}$ is quasi-primary with conformal dimension $\Delta_{w^{(i)}}$, then

$$\mathcal{Y}_\alpha(w^{(i)}, f)^\dagger = e^{-i\pi\Delta_{w^{(i)}}}\mathcal{Y}_{\alpha*}(\overline{w^{(i)}}, \overline{e_{(2-2\Delta_{w^{(i)}})}f}), \tag{0.11}$$

where for each $r \in \mathbb{R}$, e_r is a function on $S^1 \backslash \{-1\}$ satisfying $e_r(e^{i\theta}) = e^{ir\theta}$ $(-\pi < \theta < \pi)$.

(c) *Strong intertwining property*: If $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ l}$, then for any $f \in C_c^\infty(I)$ and $y \in \mathcal{M}_V(J)$,

$$\pi_k(y)\mathcal{Y}_\alpha(w^{(i)}, f) \subset \mathcal{Y}_\alpha(w^{(i)}, f)\pi_l(y). \tag{0.12}$$

[2]By proposition 3.4, if an intertwining operator is energy bounded, then so is its adjoint intertwining operator. Therefore, it suffices to require that $\mathcal{F} \cup \overline{\mathcal{F}}$, instead of $\mathcal{F}$, generates the monoidal category of V, where $\overline{\mathcal{F}} = \{W_{\bar{i}} : i \in \mathcal{F}\}$. Moreover, we are also interested in the general case where $\mathcal{F} \cup \overline{\mathcal{F}}$ generates, not the whole tensor category of V, but a smaller tensor subcategory. In this case, this tensor subcategory might not be modular, but only a ribbon fusion category. In section 5.3, the statement of conditions A and B will take care of this general case.

Here π_k and π_l are the representations of $\mathcal{M}_V$ on $\mathcal{H}_k, \mathcal{H}_l$ integrated from the V-modules W_k, W_l respectively, the existence of which will be discussed in section 4.2 (see also [CWX]).
(d) *Rotation covariance*: If $w^{(i)} \in W_i$ is a homogeneous vector with conformal weight $\Delta_{w^{(i)}}$, and for $t \in \mathbb{R}$, we define $\mathfrak{r}(t)f \in C^\infty(S^1)$ to satisfy $(\mathfrak{r}(t)f)(e^{i\theta}) = f(e^{i(\theta-t)})$, then

$$e^{it\overline{L_0}}\mathcal{Y}_\alpha(w^{(i)}, f)e^{-it\overline{L_0}} = \mathcal{Y}_\alpha\big(w^{(i)}, e^{i(\Delta_{w^{(i)}}-1)t}\mathfrak{r}(t)f\big). \tag{0.13}$$

Now the positivity of Λ on $W_i \boxtimes W_j$ can be argued as follows. Write $W_{ij} = W_i \boxtimes W_j$, and define a type $\binom{ij}{i\ j} = \binom{W_i \boxtimes W_j}{W_i\ W_j}$ intertwining operator $\mathcal{Y}_{i\boxtimes j}$, such that for any $w^{(i)} \in W_i, w^{(j)} \in W_j$,

$$\mathcal{Y}_{i\boxtimes j}(w^{(i)}, x)w^{(j)} = \sum_{k\in\mathcal{E}} \sum_{\alpha\in\Theta_{ij}^k} \check{\mathcal{Y}}^\alpha \otimes \mathcal{Y}_\alpha(w^{(i)}, x)w^{(j)}. \tag{0.14}$$

This definition is independent of the basis Θ_{ij}^k chosen. Using rotation covariance and lemma 6.3, we can show that the vectors of the form

$$\xi = \sum_{s=1}^N \mathcal{Y}_{i\boxtimes j}(w_s^{(i)}, f_s) \cdot \mathcal{Y}_{j0}^j(w_s^{(j)}, g_s)\Omega \tag{0.15}$$

form a dense subspace of the norm closure $\mathcal{H}_{ij}$ of $W_{ij} = W_i \boxtimes W_j$, where $N = 1, 2, \ldots, f_1, \ldots, f_N \in C_c^\infty(I), g_1, \ldots, g_N \in C_c^\infty(J), w_1^{(i)}, \ldots, w_N^{(i)} \in W_i, w_1^{(j)}, \ldots, w_N^{(j)} \in W_j$ are quasi-primary, and Ω is the vacuum vector in the vacuum module $W_0 = V$. Therefore, to prove the positivity of the sesquilinear form Λ on $W_i \boxtimes W_j$, it suffices to prove that $\Lambda(\xi|\xi) \geqslant 0$ for all such ξ. Note that the transport formula (0.7) can be written in the form of braiding:

$$\begin{aligned}
&\mathcal{Y}_{j0}^j(w^{(j)}, z_0)\mathcal{Y}_{\bar{i}i}^0(\overline{w_2^{(i)}}, z_2)\mathcal{Y}_{i0}^i(w_1^{(i)}, z_1)\\
=&\Big(\sum_{k\in\mathcal{E}} \sum_{\alpha,\beta\in\Theta_{ij}^*} \Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w_2^{(i)}}, z_2)\mathcal{Y}_\alpha(w_1^{(i)}, z_1)\Big)\mathcal{Y}_{j0}^j(w^{(j)}, z_0).
\end{aligned} \tag{0.16}$$

for any $z_1, z_2 \in I, z_0 \in J, w_1^{(i)}, w_2^{(i)} \in W_i, w^{(j)} \in W_j$. Then, using the smeared version of this braid relation, together with the adjoint relation, one is able to compute that

$$\Lambda(\xi|\xi) = \sum_{1\leqslant s,t\leqslant N} \langle \mathcal{Y}_{j0}^j(w_t^{(i)}, f_t)^\dagger \mathcal{Y}_{j0}^j(w_s^{(i)}, f_s)\mathcal{Y}_{i0}^i(w_t^{(j)}, g_t)^\dagger \mathcal{Y}_{i0}^i(w_s^{(j)}, g_s)\Omega|\Omega\rangle. \tag{0.17}$$

By the strong intertwining property, the right hand side of equation (0.17) can be approximated by

$$\sum_{1\leqslant s,t\leqslant N} \langle \mathfrak{A}_t^*\mathfrak{A}_s\mathfrak{B}_t^*\mathfrak{B}_s\Omega|\Omega\rangle, \tag{0.18}$$

where for each s,t, $\mathfrak{A}_s\in\mathrm{Hom}_{\mathcal{M}_V(I^c)}(\mathcal{H}_0,\mathcal{H}_i)$ and $\mathfrak{B}_t\in\mathrm{Hom}_{\mathcal{M}_V(J^c)}(\mathcal{H}_0,\mathcal{H}_j)$ are bounded operators, where I^c and J^c are the interiors of the complements of I and J in S^1 respectively, and $\mathcal{H}_0,\mathcal{H}_i,\mathcal{H}_j$ are the $\mathcal{M}_V$-modules integrated from W_0,W_i,W_j respectively. But (0.18) equals

$$\Big\|\sum_{1\leqslant s\leqslant N}\mathfrak{A}_s\Omega\boxtimes\mathfrak{B}_s\Omega\Big\|^2,\tag{0.19}$$

where $\boxtimes$ is the *Connes fusion product* [Con80]. So $\Lambda(\xi|\xi)$ must be non-negative, and hence Λ is positive on $W_i\boxtimes W_j$.

Generalized (smeared) intertwining operators

Now consider the general case when W_i and W_j are not necessarily in $\mathcal{F}$. Then the energy bounds condition of the intertwining operators with charge spaces W_i or W_j is unknown. But since $\mathcal{F}$ generates the monoidal category of V, there exist $W_{i_1},\dots,W_{i_m}\in\mathcal{F}$ such that W_i is equivalent to a submodule of $W_{i_m}\boxtimes\cdots\boxtimes W_{i_1}$. Equivalently, there exist irreducible V-modules $W_{r_2},\dots,W_{r_{m-1}}$ and non-zero (energy-bounded) intertwining operators $\mathcal{Y}_{\sigma_2}\in\mathcal{V}{r_2\choose i_2\ i_1},\mathcal{Y}_{\sigma_3}\in\mathcal{V}{r_3\choose i_3\ r_2},\dots\mathcal{Y}_{\sigma_m}\in\mathcal{V}{i\choose i_m\ r_{m-1}}$. Choose mutually disjoint open intervals $I_1,I_2,\dots,I_m\subset I$. Then we define a *generalized intertwining operator* $\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}$ (acting on the source space W_j of $\mathcal{Y}_\alpha$), such that for any $z_1\in I_1,z_2\in I_2,\dots,z_m\in I_m,w^{(i_1)}\in W_{i_1},w^{(i_2)}\in W_{i_2},\dots,w^{(i_m)}\in W_{i_m}$,

$$\begin{aligned}&\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)},z_n;\dots;w^{(i_2)},z_2;w^{(i_1)},z_1)\\=&\mathcal{Y}_\alpha\big(\mathcal{Y}_{\sigma_m}(w^{(i_n)},z_n-z_1)\cdots\mathcal{Y}_{\sigma_2}(w^{(i_2)},z_2-z_1)w^{(i_1)},z_1\big).\end{aligned}\tag{0.20}$$

Now, for each $f_1\in C_c^\infty(I_1),\dots,f_m\in C_c^\infty(I_m)$, we define a *generalized smeared interwining operator*

$$\begin{aligned}&\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha}(w^{(i_m)},f_m;\dots;w^{(i_1)},f_1)\\=&\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha}(w^{(i_m)},e^{i\theta_m};\dots;w^{(i_1)},e^{i\theta_1})\cdot f_1(e^{i\theta_1})\cdots f_m(e^{i\theta_m})d\!\!\!{}^-\theta_1\cdots d\!\!\!{}^-\theta_m,\end{aligned}$$

where $d\!\!\!{}^-\theta=e^{i\theta}d\theta/2\pi$. We can also define, for any $\mathcal{Y}_\beta$ with charge space W_j, mutually disjoint $J_1,\dots,J_n\subset J$, and $g_1\in C_c^\infty(J_1),\dots,g_n\in C_c^\infty(J_n)$, a generalized smeared intertwining operator $\mathcal{Y}_{\rho_n\cdots\rho_2,\beta}(w^{(j_n)},g_n;\dots;w^{(j_1)},g_1)$ in a similar way. Then we can prove the positivity of Λ on $W_i\boxtimes W_j$ in a similar way as above, once we've established the braid relations, the adjoint relation, the strong intertwining property, and the rotation covariance of generalized smeared intertwining operators. The last two properties follow easily from those of the smeared intertwining operators. The first two, especially the adjoint relation, is much harder to prove. All these properties will be treated in this paper.

Outline of this paper

In chapter 4 we discuss some of the relations between a strongly-local unitary rational VOA V and its conformal net $\mathcal{M}_V$. A unitary V-module W_i is called strongly integrable, if it can be integrated to an $\mathcal{M}_V$-module $\mathcal{H}_i$. In section 4.1, we study the relation between the abelian category of strongly integrable unitary V-modules and the one of $\mathcal{M}_V$-modules. In section 4.2 we give a useful criterion for the strong integrability of unitary V-modules based on the energy bounds condition of intertwining operators.

Chapter 5 is devoted to the study of generalized intertwining operators and their smeared versions. In particular, we prove (in section 5.3) the braid relations, the adjoint relation, the strong intertwining property, and the rotation covariance of generalized smeared intertwining operators. Since the proof of the first two properties are harder, and since the difficulty is mostly on the unsmeared side, in sections 5.1 and 5.2 we prove the braid relations and the adjoint relation of generalized (unsmeared) intertwining operators.

Chapter 6 is the climax of our series of papers. Recall that for any $w^{(i)} \in W_i$ and $z \in \mathbb{C}\backslash\{0\}$, $\mathcal{Y}_{i\boxtimes j}(w^{(i)}, z)W_j$ is dense in the algebraic completion $\widehat{W}_{ij}$ of $W_{ij} = W_i \boxtimes W_j$ by proposition A.3 in part I, where $\mathcal{Y}_{i\boxtimes j}$ is defined by equation (0.14). In section 6.1 we first prove a similar density result for generalized intertwining operators. Its smeared version is also proved with the help of rotation covariance. In section 6.2 we define the sesquilinear form Λ. The positive-definiteness of Λ is proved in section 6.3 using all the results we have achieved so far.

In chapter 7 we show that Λ defines a unitary structure on the MTC of V. Our result is applied to unitary Virasoro VOAs and many unitary affine VOAs in sections 8.1 and 8.2. The sesquilinear form Λ is closely related to the non-degenerate bilinear form considered in [HK07]. We will explain this relation in section 8.3.

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Conventions and Notations

We follow the notations and the conventions in part I [Gui19a] (see conventions 1.12, 2.1, 2.19, and definition 1.13). In particular, if $s \in \mathbb{R}$, we always assume $\arg e^{is} = s$. If $z \in \mathbb{C}^\times$ and $\arg z$ is chosen, then we let $\arg \overline{z} = -\arg z$ and $\arg z^s = s \arg z$. If $z_1, z_2 \in \mathbb{C}^\times$ and $\arg z_1, \arg z_2$ are chosen, then we set $\arg(z_1 z_2) = \arg z_1 + \arg z_2$. We also understand z_1/z_2 as $z_1 z_2^{-1}$. Therefore $\arg(z_1/z_2) = \arg(z_1 z_2^{-1}) = \arg z_1 + \arg z_2^{-1} = \arg z_1 - \arg z_2$.

Arguments of explicit positive *real* numbers (e.g. $1, \sqrt{2}, \pi$) are assumed to be 0 unless otherwise stated. (This is also assumed but not explicitly mentioned in [Gui19a].) In this article as well as [Gui19a], symbols like r, s, t (sometimes with subscripts) are used as **real variables**. When they take positive values we also assume their arguments to be

0. $e^{2i\pi}$ is regarded not as a positive real number but as a positive *complex* number. As a consequence, its argument is not 0 but 2π. Symbols like z, ζ are regarded as **complex variables**. Therefore, even if they can take positive values, their arguments are still not necessarily 0.

Now assume that U is an open subset of $\mathbb{C}$, $f : U \to \mathbb{C}^\times$ is continuous, $z_1, z_2 \in U$, and the interval E connecting z_1, z_2 is inside U. Choose arguments $\arg f(z_1), \arg f(z_2)$ of $f(z_1), f(z_2)$. Following definition 1.13, we say that $\arg f(z_2)$ is **close to** $\arg f(z_1)$ **as** $z_2 \to z_1$, if there exists a (unique) continuous function $A : [0,1] \to \mathbb{R}$ satisfying that: (a) $A(0) = \arg z_1, A(1) = \arg z_2$. (b) $A(t)$ is an argument of $f(tz_1 + (1-t)z_2)$ for any $t \in [0,1]$.

We always assume V to be a VOA of CFT type satisfying the following condition:

(α) V is isomorphic to V'.

(β) Every $\mathbb{N}$-gradable weak V-module is completely reducible.

(γ) V is C_2-cofinite.

The precise meanings of these conditions, which are not quite important to our theory, can be found in [Hua05b]. The importance of these conditions is to guarantee the existence of a MTC of V due to [Hua08b].

The following notation list, which is an expansion of the one in part I, is used throughout this paper.

A^{t}: the transpose of the linear operator A.
$A^\dagger$: the formal adjoint of the linear operator A.
A^*: the ajoint of the possibly unbounded linear operator A.
$\overline{A}$: the closure of the pre-closed linear operator A.
C_i: the antiunitary map $W_i \to W_{\bar{i}}$.
$\mathbb{C}^\times = \{z \in \mathbb{C} : z \neq 0\}$.
$\mathrm{Conf}_n(\mathbb{C}^\times)$: the n-th configuration space of $\mathbb{C}^\times$.
$\mathscr{D}(A)$: the domain of the possibly unbounded operator A.
$\text{đ}\theta = \frac{e^{i\theta}}{2\pi} d\theta$.
$e_r(e^{i\theta}) = e^{ir\theta} \quad (-\pi < \theta < \pi)$.
$E^1(W_i), E^1(V)$: see section 5.3.
$E^1(V)_{\mathbb{R}}$: the real subspace of the vectors in $E^1(V)$ that are fixed by the CPT operator θ.
$\mathcal{E}$: a complete list of mutually inequivalent irreducible V-modules.
$\mathcal{E}^{\mathrm{u}}$: the set of unitary V-modules in $\mathcal{E}$.
$\mathcal{F}$: a non-empty set of non-zero irreducible unitary V-modules.
$\mathcal{F}^{\boxtimes}$: see section 5.3.
$\mathrm{Hom}_V(W_i, W_j)$: the vector space of V-module homomorphisms from W_i to W_j.
$\mathrm{Hom}_{\mathcal{M}_V}(\mathcal{H}_i, \mathcal{H}_j)$: the vector space of bounded $\mathcal{M}_V$-module homomorphism from $\mathcal{H}_i$ to $\mathcal{H}_j$.
$\mathrm{Hom}_{\mathcal{M}_V(I)}(\mathcal{H}_i, \mathcal{H}_j)$: the vector space of bounded operators $\mathcal{H}_i \to H_j$ intertwining the action of elements in $\mathcal{M}_V(I)$.
$\mathcal{H}_i$: the norm completion of the vector space W_i. If the V-module W_i is a unitary, energy-

bounded, and strongly integrable, then $\mathcal{H}_i$ is the $\mathcal{M}_V$-module associated with W_i.
$\mathcal{H}_i^r$: the vectors of $\mathcal{H}_i$ that are inside $\mathscr{D}((1+\overline{L_0})^r)$.
$\mathcal{H}_i^\infty = \bigcap_{r\geqslant 0}\mathcal{H}_i^r$.
i: either the index of a V-module, or $\sqrt{-1}$.
$I^c = S^1\backslash\overline{I}$.
$I_1 \subset\subset I_2$: $I_1, I_2 \in \mathcal{J}$ and $\overline{I_1}\subset I_2$.
$\mathrm{id}_i = \mathrm{id}_{W_i}$: the identity operator of W_i.
$\mathcal{J}$: the set of (non-empty, non-dense) open intervals of S^1.
$\mathcal{J}(U)$: the set of open intervals of S^1 contained in the open set U.
$\mathcal{M}_V$: the conformal net constructed from V.
$\mathcal{M}_V(I)_\infty$: the set of smooth operators in $\mathcal{M}_V(I)$.
$\mathscr{O}_n(I)$: see the beginning of chapter 5.
$\mathfrak{O}_n(I)$: see section 5.3.
P_s: the projection operator of W_i onto $W_i(s)$.
$\mathfrak{r}(t): S^1\to S^1$: $\mathfrak{r}(t)(e^{i\theta}) = e^{i(\theta+t)}$.
$\mathfrak{r}(t): C^\infty(S^1)\to C^\infty(S^1)$: $\mathfrak{r}(t)h = h\circ\mathfrak{r}(-t)$.
$\mathrm{Rep}(V)$: the modular tensor category of the representations of V.
$\mathrm{Rep}^{\mathrm{u}}(V)$: the category of the unitary representations of V.
$\mathrm{Rep}^{\mathrm{u}}_{\mathcal{G}}(V)$: When $\mathcal{G}$ is additively closed, it is the subcategory of $\mathrm{Rep}^{\mathrm{u}}(V)$ whose objects are unitary V-modules in $\mathcal{G}$. When $\mathcal{G}$ is multiplicatively closed, then it is furthermore equipped with the structure of a ribbon tensor category.
$S^1 = \{z\in\mathbb{C} : |z| = 1\}$.
$\mathcal{S}$: the collection of strongly integrable energy-bounded unitary V-modules.
$\mathcal{V}\binom{k}{i\,j}$: the vector space of type $\binom{k}{i\,j}$ intertwining operators.
$W_0 = V$, the vacuum module of V.
W_i: a V-module.
$\widehat{W}_i$: the algebraic completion of W_i.
$W_{\bar{i}} \equiv W_i'$: the contragredient module of W_i.
$W_{ij}\equiv W_i\boxtimes W_j$: the tensor product of W_i, W_j.
$w^{(i)}$: a vector in W_i.
$\overline{w^{(i)}} = C_i w^{(i)}$.
x: a formal variable, or an element inside $\mathcal{M}_V(I)$.
Y_i: the vertex operator of W_i.
$\mathcal{Y}_\alpha$: an intertwining operator of V.
$\mathcal{Y}_{\overline{\alpha}}\equiv\overline{\mathcal{Y}_\alpha}$: the conjugate intertwining operator of $\mathcal{Y}_\alpha$.
$\mathcal{Y}_{\alpha^*}\equiv\mathcal{Y}_\alpha^\dagger$: the adjoint intertwining operator of $\mathcal{Y}_\alpha$.
$\mathcal{Y}_{B_\pm\alpha}\equiv B_\pm\mathcal{Y}_\alpha$: the braided intertwining operators of $\mathcal{Y}_\alpha$.
$\mathcal{Y}_{C\alpha}\equiv C\mathcal{Y}_\alpha$: the contragredient intertwining operator of $\mathcal{Y}_\alpha$.
$\mathcal{Y}_{i0}^i = B_\pm Y_i$, the creation operator of W_i.
$\mathcal{Y}_{\bar{i}i}^0 = C^{-1}\mathcal{Y}_{\bar{i}0}^{\bar{i}} = (\mathcal{Y}_{i0}^i)^\dagger$, the annihilation operator of W_i.
$\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}$: a generalized intertwining operator of V.
Δ_i: the conformal weight of W_i.

Δ_w: the conformal weight (the energy) of the homogeneous vector w.
Θ^k_{ij}: a set of linear basis of $\mathcal{V}\binom{k}{i\ j}$.
$\Theta^k_{i*} = \coprod_{j\in\mathcal{E}} \Theta^k_{ij}, \Theta^k_{*j} = \coprod_{i\in\mathcal{E}} \Theta^k_{ij}, \Theta^*_{ij} = \coprod_{k\in\mathcal{E}} \Theta^k_{ij}$.
$\Theta^*_{i*} = \coprod_{s,t\in\mathcal{E}\cap\mathcal{F}^\boxtimes} \Theta^t_{is}, \Theta^*_{*j} = \coprod_{s,t\in\mathcal{E}\cap\mathcal{F}^\boxtimes} \Theta^t_{sj}$.
θ: the PCT operator of V, or a real variable.
ϑ_i: the twist of W_i.
Λ: the sesquilinear form defined by transport matrices on $\mathcal{V}\binom{k}{i\ j}^*$ or on $W_i \boxtimes W_j$.
ν: the conformal vector of V.
$\sigma_{i,j}$: the braid operator $\sigma_{i,j} : W_i \boxtimes W_j \to W_j \boxtimes W_i$.
$\Upsilon^0_{i\bar{i}} = C\mathcal{Y}^i_{i0}$.
Ω: the vacuum vector of V.

4 From unitary VOAs to conformal nets

In this chapter, we assume that V is unitary and energy-bounded. A net $\mathcal{M}_V$ of von Neumann algebras on the circle can be defined using smeared vertex operators of V. If $\mathcal{M}_V$ is a conformal net, then V is called strongly local. A theorem in [CKLW18] shows that when V is generated by a set of quasi-primary vectors whose field operators satisfy linear energy bounds, then V is strongly local. This is discussed in section 4.1.

Let W_i be an energy-bounded unitary V-module. If this representation of V can be integrated to a representation of the conformal net $\mathcal{M}_V$, we say that W_i is strongly integrable. In section 4.1, we show that the abelian category of energy-bounded strongly-integrable unitary V-modules is equivalent to the category of the corresponding integrated $\mathcal{M}_V$-modules. A similar topic is treated in [CWX].

There are two major ways to prove the strong integrability of a unitary V-modules W_i. First, if the action of V on W_i is restricted from the inclusion of V in a larger energy-bounded strongly-local unitary VOA, then W_i is strongly local. This result is proved in [CWX], and will not be used in our paper. In section 4.2, we give a different criterion using the energy bounds condition of intertwining operators.

4.1 Unitary VOAs, conformal nets, and their representations

We first review the definition of conformal nets. Standard references are [CKLW18, Car04, GF93, GL96, KL04]. Conformal nets are based on the theory of von Neumann algebras. For an outline of this theory, we recommend [Con80] chapter 5. More details can be found in [Jon03, Tak02, Tak13, KR83, KR15].

Let $\mathrm{Diff}(S^1)$ be the group of orientation-preserving diffeomorphisms of S^1. Convergence in $\mathrm{Diff}(S^1)$ means uniform convergence of all derivatives. Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{U}(\mathcal{H})$ let the group of unitary operators on $\mathcal{H}$, equipped with the strong (operator) topology. $P\mathcal{U}(\mathcal{H})$ is the quotient topology group of $\mathcal{U}(\mathcal{H})$, defined by identifying x with λx when $x \in \mathcal{U}(\mathcal{H}), \lambda \in S^1$. A **strongly continuous projective representation** of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ is, by definition, a continuous homomorphism from $\mathrm{Diff}(S^1)$ into $P\mathcal{U}(\mathcal{H})$.

$\mathrm{Diff}(S^1)$ contains the subgroup $\mathrm{PSU}(1,1)$ of Möbius transformations of S^1. Elements in $\mathrm{PSU}(1,1)$ are of the form

$$z \mapsto \frac{\lambda z + \mu}{\overline{\mu} z + \overline{\lambda}} \quad (z \in S^1), \tag{4.1}$$

where $\lambda, \mu \in \mathbb{C}, |\lambda|^2 - |\mu|^2 = 1$. $\mathrm{PSU}(1,1)$ contains the subgroup $S^1 = \{\mathfrak{r}(t) : t \in \mathbb{R}\}$ of rotations of S^1.

A **conformal net** $\mathcal{M}$ associates to each $I \in \mathcal{J}$ a von Neumann algebra $\mathcal{M}(I)$ acting on a fixed Hilbert space $\mathcal{H}_0$, such that the following conditions hold:
(a) (Isotony) If $I_1 \subset I_2 \in \mathcal{J}$, then $\mathcal{M}(I_1)$ is a von Neumann subalgebra of $\mathcal{M}(I_2)$.
(b) (Locality) If $I_1, I_2 \in \mathcal{J}$ are disjoint, then $\mathcal{M}(I_1)$ and $\mathcal{M}(I_2)$ commute.
(c) (Conformal covariance) We have a strongly continuous projective unitary representation U of $\mathrm{Diff}(S^1)$ on $\mathcal{H}_0$, such that for any $g \in \mathrm{Diff}(S^1), I \in \mathcal{J}$,

$$U(g)\mathcal{M}(I)U(g)^* = \mathcal{M}(gI).$$

Moreover, if g fixes the points in I, then for any $x \in \mathcal{M}(I)$,

$$U(g)xU(g)^* = x.$$

(d) (Positivity of energy) The generator of the restriction of U to S^1 is positive.
(e) There exists a unique (up to scalar) unit vector $\Omega \in \mathcal{H}_0$ (the vacuum vector), such that $U(g)\Omega \in \mathbb{C}\Omega$ for any $g \in \mathrm{PSU}(1,1)$. Moreover, Ω is cyclic under the action of $\bigvee_{I \in \mathcal{J}} \mathcal{M}(I)$ (the von Neumann algebra generated by all $\mathcal{M}(I)$).

The following properties are satisfied by a conformal net, and will be used in our theory:
(1) (Additivity) If $\{I_a : a \in \mathcal{A}\}$ is a collection of open intervals in $\mathcal{J}$, $I \in \mathcal{J}$, and $I = \bigcup_{a \in \mathcal{A}} I_a$, then $\mathcal{M}(I) = \bigvee_{a \in \mathcal{A}} \mathcal{M}(I_a)$.
(2) (Haag duality) $\mathcal{M}(I)' = \mathcal{M}(I^c)$, where $\mathcal{M}' = \mathrm{End}_{\mathcal{M}}(\mathcal{H}_0)$ is the commutant of $\mathcal{M}$.
(3) $\mathcal{M}(I)$ is a type III factor. (Indeed, it is of type III$_1$.)
Properties (2) and (3) are natural consequences of Bisognano-Wichmann theorem, cf. [BGL93, GF93].

Following [CKLW18], we now show how to construct a conformal net $\mathcal{M}_V$ from V. Let the Hilbert space $\mathcal{H}_0$ be the norm completion of V. For any $I \in \mathcal{J}$ we define $\mathcal{M}_V(I)$ to be the von Neumann algebra on $\mathcal{H}_0$ generated by closed operators of the form $\overline{Y(v,f)}$, where $v \in V$ and $f \in C_c^\infty(I)$. Thus we've obtained a net of von Neumann algebras $I \in \mathcal{J} \mapsto \mathcal{M}_V(I)$ and denote it by $\mathcal{M}_V$. The vacuum vector Ω in $\mathcal{H}_0$ is the same as that of V. The projective representation U of $\mathrm{Diff}(S^1)$ is obtained by integrating the action of the real part of the Virasoro algebra on V. The representation of $\mathrm{PSU}(1,1)$ is determined by the action of $L_{\pm 1}, L_0$ on V. *All the axioms of conformal nets, except possibly locality, are satisfied for* $\mathcal{M}_V$.

Locality of $\mathcal{M}_V$, however, is much harder to prove. To be sure, for any disjoint $I, J \in \mathcal{J}$, and any $u, v \in V$, we can use proposition 2.13, corollary 3.13, and proposition 3.9 to show that

$$Y(u,f)Y(v,g) = Y(v,g)Y(u,f), \tag{4.2}$$

$$Y(u,f)^\dagger Y(v,g) = Y(v,g)Y(u,f)^\dagger, \tag{4.3}$$

where both sides act on $\mathcal{H}_0^\infty$. The commutativity of closed operators on a common invariant core, however, does not imply the strong commutativity of these two operators, as indicated by the example of Nelson (cf. [Nel59]). So far, the best result we have for the locality of $\mathcal{M}_V$ is the following:

Theorem 4.1. *Suppose that V is generated by a set E of quasi-primary vectors, and that for any $v \in E$, $Y(v,x)$ satisfies linear energy bounds. Then the net $\mathcal{M}_V$ satisfies the locality condition, and is therefore a conformal net. Moreover, if we let $E_{\mathbb{R}} = \{v + \theta v, i(v - \theta v) : v \in E\}$, then for any $I \in \mathcal{J}$, $\mathcal{M}_V(I)$ is generated by the closed operators $\overline{Y(u,f)}$, where $u \in E_{\mathbb{R}}$, and $f \in C_c^\infty(I)$ satisfies $e^{i\pi\Delta_u/2}e_{1-\Delta_u}f = \overline{e^{i\pi\Delta_u/2}e_{1-\Delta_u}f}$.*

Proof. Clearly $E_{\mathbb{R}}$ generates V. From the proof of [CKLW18] theorem 8.1, it suffices to prove, for any disjoint $I, J \in \mathcal{J}$, $u, v \in E_{\mathbb{R}}$, and $f \in C_c^\infty(I), g \in C_c^\infty(J)$ satisfying $e^{i\pi\Delta_u/2}e_{1-\Delta_u}f = \overline{e^{i\pi\Delta_u/2}e_{1-\Delta_u}f}, e^{i\pi\Delta_v/2}e_{1-\Delta_v}g = \overline{e^{i\pi\Delta_v/2}e_{1-\Delta_v}g}$, that $\overline{Y(u,f)}$ and $\overline{Y(v,g)}$ commute strongly. By proposition 3.9-(b), $Y(u,f)$ and $Y(v,g)$ are symmetric operators. Hence by equation (3.38), proposition 3.9-(a), equation (4.2), Lemma B.8, and theorem B.9, $\overline{Y(u,f)}$ and $\overline{Y(v,g)}$ are self-adjoint operators, and they commute strongly with each other. □

We say that a unitary energy-bounded strongly local VOA V is **strongly local**, if $\mathcal{M}_V$ satisfies the locality condition.

Suppose that V is strongly local. We now discuss representations of the conformal net $\mathcal{M}_V$. Let $\mathcal{H}_i$ be a Hilbert space (currently not yet related to W_i). Suppose that for any $I \in \mathcal{J}$, we have a (normal unital *-) representation $\pi_{i,I} : \mathcal{M}_V(I) \to B(\mathcal{H}_i)$, such that for any $I_1, I_2 \in \mathcal{J}$ satisfying $I_1 \subset I_2$, and any $x \in \mathcal{M}_V(I_1)$, we have $\pi_{i,I_1}(x) = \pi_{i,I_2}(x)$. Then $(\mathcal{H}_i, \pi_i)$ (or simply $\mathcal{H}_i$) is called a (locally normal) **represention** of the $\mathcal{M}_V$ (or an $\mathcal{M}_V$**-module**). We shall write $\pi_{i,I}(x)$ as $\pi_i(x)$ when we have specified that $x \in \mathcal{M}_V(I)$. If moreover $\xi^{(i)} \in \mathcal{H}_i$, we also write $x\xi^{(i)}$ for $\pi_i(x)\xi^{(i)} = \pi_{i,I}(x)\xi^{(i)}$.

The $\mathcal{M}_V$-modules we are interested in are those arising from unitary V-modules. Let W_i be an energy-bounded unitary V-module, and let $\mathcal{H}_i$ be the norm completion of the inner product space W_i. Assume that we have a representation π_i of $\mathcal{M}_V$ on $\mathcal{H}_i$. Then we say that $(\mathcal{H}_i, \pi_i)$ is **associated with the V-module** (W_i, Y_i), if for any $I \in \mathcal{J}$, $v \in V$, and $f \in C_c^\infty(I)$, we have

$$\pi_{i,I}\big(\overline{Y(v,f)}\big) = \overline{Y_i(v,f)}. \tag{4.4}$$

(See section B.1 for the definition of $\pi_{i,I}$ acting on unbounded closed operators affiliated with $\mathcal{M}_V(I)$.) A $\mathcal{M}_V$-module associated with W_i, if exists, must be unique. We say that an energy-bounded unitary V-module W_i is **strongly integrable** if there exists a $\mathcal{M}_V$-module $(\mathcal{H}_i, \pi_i)$ associated with W_i. Let $\mathcal{S}$ be the collection of strongly integrable energy-bounded unitary V-modules. Obviously $V \in \mathcal{S}$. It is easy to show that S is additively complete.

We now introduce a very useful density property. For any $I \in \mathcal{J}$, we define $\mathcal{M}_V(I)_\infty$ to be the set of **smooth operators** in $\mathcal{M}_V(I)$, i.e., the set of all $x \in \mathcal{M}_V(I)$ satisfying that for any unitary V-module W_i inside $\mathcal{S}$,

$$x\mathcal{H}_i^\infty \subset \mathcal{H}_i^\infty, \quad x^*\mathcal{H}_i^\infty \subset \mathcal{H}_i^\infty. \tag{4.5}$$

Proposition 4.2. *If V is unitary, energy-bounded, and strongly local, then $\mathcal{M}_V(I)_\infty$ is a strongly dense self-adjoint subalgebra of $\mathcal{M}_V(I)$.*

Proof. By additivity or by the construction of $\mathcal{M}_V$, we have $\mathcal{M}_V(I) = \bigvee_{J \subset\subset I} \mathcal{M}_V(J)$. ($J \subset\subset I$ means that $J \in \mathcal{J}$ and $\overline{J} \subset I$.) For each $J \subset\subset I$ and $x \in \mathcal{M}_V(J)$, we choose $\epsilon > 0$ such that $\mathfrak{r}(t)J \subset I$ whenever $t \in (-\epsilon, \epsilon)$. For each $h \in C_c^\infty(-\epsilon, \epsilon)$ such that $\int_{-\epsilon}^{\epsilon} h(t)dt = 1$, define

$$x_h = \int_{-\epsilon}^{\epsilon} e^{it\overline{L_0}} x e^{-it\overline{L_0}} h(t)dt.$$

Then by (3.39), $x_h \in \mathcal{M}_V(I)$. For each W_i inside $\mathcal{S}$, equations (3.39) and (4.4) imply that

$$\pi_i(e^{it\overline{L_0}} x e^{-it\overline{L_0}}) = e^{it\overline{L_0}} \pi_i(x) e^{-it\overline{L_0}}. \tag{4.6}$$

So we have

$$\pi_i(x_h) = \int_{-\epsilon}^{\epsilon} e^{it\overline{L_0}} \pi_i(x) e^{-it\overline{L_0}} h(t)dt,$$

which implies that

$$e^{it\overline{L_0}} \pi_i(x_h)\xi^{(i)} = \pi_i(x_{h_t}) e^{it\overline{L_0}} \xi^{(i)}, \tag{4.7}$$

where $h_t(s) = h(s-t)$. From this equation, we see that the derivative of $e^{it\overline{L_0}}\xi^{(i)} \in \mathcal{H}_i^\infty$ at $t = 0$ exists and equals

$$-\pi_i(x_{h'})\xi^{(i)} + i\pi_i(x_h)\overline{L_0}\xi^{(i)}. \tag{4.8}$$

This implies that $\pi_i(x_h)\xi^{(i)} \in \mathcal{H}_i^1$ and $i\overline{L_0}\pi_i(x_h)\xi^{(i)}$ equals (4.8). Using the same argument, we see that for each $n \in \mathbb{Z}_{\geqslant 0}$, the following Leibniz rule holds:

$$\pi_i(x_h)\xi^{(i)} \in \mathscr{D}(\overline{L_0}^n) = \mathcal{H}_i^n,$$
$$\overline{L_0}^n \pi_i(x_h)\xi^{(i)} = \sum_{m=0}^{n} \binom{n}{m} i^m \pi_i(x_{h^{(m)}}) \cdot \overline{L_0}^{n-m}\xi^{(i)},$$

where $h^{(m)}$ is the m-th derivative of h. This proves that $\pi_i(x_h)\mathcal{H}_i^\infty \subset \mathcal{H}_i^\infty$.

Since $(x_h)^* = (x^*)_{\overline{h}}$, we also have $x_h^*\mathcal{H}_i^\infty \subset \mathcal{H}_i^\infty$. So $x_h \in \mathcal{M}_V(I)_\infty$. Clearly $x_h \to x$ strongly as h converges to the δ-function at 0. We thus conclude that any $x \in \mathcal{M}_V(J)$ can be strongly approximated by elements in $\mathcal{M}_V(I)_\infty$. Hence the proof is finished. $\square$

We study the relation between the representation categories of $\mathcal{M}_V$ and V. Assume, as before, that V is unitary, energy-bounded, and strongly local. We define an additive category $\mathrm{Rep}_{\mathcal{S}}(\mathcal{M}_V)$ as follows: The objects are $\mathcal{M}_V$-modules of the form $\mathcal{H}_i$, where W_i is an element inside $\mathcal{S}$. If W_i, W_j are inside $\mathcal{S}$, then the vector space of morphisms $\mathrm{Hom}_{\mathcal{M}_V}(\mathcal{H}_i, \mathcal{H}_j)$ consists of bounded linear operators $R : \mathcal{H}_i \to \mathcal{H}_j$, such that for any $I \in \mathcal{J}, x \in \mathcal{M}_V(I)$, the relation $R\pi_i(x) = \pi_j(x)R$ holds.

Define a functor $\mathfrak{F} : \mathrm{Rep}^{\mathrm{u}}_{\mathcal{S}}(V) \to \mathrm{Rep}_{\mathcal{S}}(\mathcal{M}_V)$ in the following way: If W_i is a unitary V-module in $\mathcal{S}$, then we let $\mathfrak{F}(W_i)$ be the $\mathcal{M}_V$-module $\mathcal{H}_i$. If W_i, W_j are in $\mathcal{S}$ and $R \in \mathrm{Hom}_V(W_i, W_j)$, then by lemma 2.20, R is bounded, and hence can be extended to a bounded linear map $R : \mathcal{H}_i \to \mathcal{H}_j$. It is clear that R is an element in $\mathrm{Hom}_{\mathcal{M}_V}(\mathcal{H}_i, \mathcal{H}_j)$. We let $\mathfrak{F}(R)$ be this $\mathcal{M}_V$-module homomorphism. Clearly $\mathfrak{F} : \mathrm{Hom}_V(W_i, W_j) \to \mathrm{Hom}_{\mathcal{M}_V}(\mathcal{H}_i, \mathcal{H}_j)$ is linear. We show that $\mathfrak{F}$ is an isomorphism.

Theorem 4.3. [3] *Let V be unitary, energy-bounded, and strongly local. For any W_i, W_j in $\mathcal{S}$, the linear map $\mathfrak{F} : \mathrm{Hom}_V(W_i, W_j) \to \mathrm{Hom}_{\mathcal{M}_V}(\mathcal{H}_i, \mathcal{H}_j)$ is an isomorphism. Therefore, $\mathfrak{F} : \mathrm{Rep}^{\mathrm{u}}_{\mathcal{S}}(V) \to \mathrm{Rep}_{\mathcal{S}}(\mathcal{M}_V)$ is an equivalence of additive categories.*

Proof. The linear map $\mathfrak{F} : \mathrm{Hom}_V(W_i, W_j) \to \mathrm{Hom}_{\mathcal{M}_V}(\mathcal{H}_i, \mathcal{H}_j)$ is clearly injective. We only need to prove that $\mathfrak{F}$ is surjective. Choose $R \in \mathrm{Hom}_{\mathcal{M}_V}(\mathcal{H}_i, \mathcal{H}_j)$. Define an orthogonal direct sum module $W_k = W_i \oplus^{\perp} W_j$. Then $\mathcal{H}_k$ is the orthogonal direct sum $\mathcal{M}_V$-module of $\mathcal{H}_i, \mathcal{H}_j$. Regard R as an element in $\mathrm{End}_{\mathcal{M}_V}(\mathcal{H}_k)$, which is the original operator when acting on $\mathcal{H}_i$, and is 0 when acting on $\mathcal{H}_j$. Then for any $I \in \mathcal{J}, x \in \mathcal{M}_V(I)$, R commutes with $\pi_k(x), \pi_k(x^*)$. Therefore, for any homogeneous $v \in V$ and $f \in C^\infty_c(I)$, R commutes strongly with $\pi_k\big(\overline{Y(v,f)}\big) = \overline{Y_k(v,f)}$.

We first show that $RW_i \subset W_j$. Choose $I_1, I_2 \in \mathcal{J}$ and $f_1 \in C^\infty_c(I_1, \mathbb{R}), f_2 \in C^\infty_c(I_2, \mathbb{R})$ such that $f_1 + f_2 = 1$. Regard L_0 as an unbounded operator on $\mathcal{H}_k$ with domain W_k. Then L_0 is the restriction of the smeared vertex operator $Y_k(\nu, e_1)$ to W_k. (Recall that by our notation of e_r, $e_1(e^{i\theta}) = e^{i\theta}$.) Therefore,

$$L_0 \subset Y_k(\nu, e_1 f_1) + Y_k(\nu, e_1 f_2),$$

and hence

$$\overline{L_0} \subset \overline{Y_k(\nu, e_1 f_1) + Y_k(\nu, e_1 f_2)} \subset \overline{\overline{Y_k(\nu, e_1 f_1)} + \overline{Y_k(\nu, e_1 f_2)}}.$$

Recall that ν is quasi-primary and $\Delta_\nu = 2$. Therefore, by equation (3.25), $\overline{Y_k(\nu, e_1 f_1)}$ and $\overline{Y_k(\nu, e_1 f_2)}$ are symmetric operators. It follows that $A = \overline{\overline{Y_k(\nu, e_1 f_1)} + \overline{Y_k(\nu, e_1 f_2)}}$ is symmetric. Note that $\overline{L_0}$ is self adjoint. Thus we have

$$\overline{L_0} \subset A \subset A^* \subset \overline{L_0}^* = \overline{L_0},$$

which implies that

$$\overline{L_0} = \overline{\overline{Y_k(\nu, e_1 f_1)} + \overline{Y_k(\nu, e_1 f_2)}}.$$

[3] This theorem is also proved in [CWX]. We would like to thank Sebastiano Carpi for letting us know this fact.

Therefore, since R commutes strongly with $\overline{Y_k(\nu, e_1 f_1)}$ and $\overline{Y_k(\nu, e_1 f_2)}$, R also commutes strongly with $\overline{L_0}$. In particular, R preserves every eigenspace of $\overline{L_0}$ in $\mathcal{H}_k$. This implies that $RW_i(s) \subset W_j(s)$ for any $s \in \mathbb{R}$, and hence that $RW_i \subset W_j$.

Now, for any $n \in \mathbb{Z}$, $w^{(i)} \in W_i$, and $v \in V$, we have

$$Y_k(v, n)w^{(i)} = Y_k(v, e_n)w^{(i)} = \overline{Y_k(v, e_n f_1)}w^{(i)} + \overline{Y_k(v, e_n f_2)}w^{(i)}.$$

Since R commutes strongly with $\overline{Y_i(v, e_n f_1)}, \overline{Y_i(v, e_n f_2)}$, we have $RY_k(v, e_n)w^{(i)} = Y_k(v, e_n)Rw^{(i)}$, which implies that $RY_i(v, n)w^{(i)} = Y_j(v, n)Rw^{(i)}$. Therefore, $R \in \mathrm{Hom}_V(W_i, W_j)$. □

Corollary 4.4. *If W_i is a unitary V-module in $\mathcal{S}$, and $\mathcal{H}_1$ is a (norm-)closed $\mathcal{M}_V$-invariant subspace of $\mathcal{H}_i$, then there exists a V-invariant subspace W_1 of W_i, such that $\mathcal{H}_1$ is the norm closure of W_1.*

Proof. Let e_1 be the orthogonal projection of $\mathcal{H}_i$ onto $\mathcal{H}_1$. Then $e_1 \in \mathrm{End}_{\mathcal{M}_V}(\mathcal{H}_i)$. By theorem 4.3, e_1 restricts to an element in $\mathrm{End}_V(W_i)$. So $W_1 = e_1 W_i$ is a V-invariant subspace of W_i, and $e_1 L_0 = L_0 e_1$ when both sides act on W_i. Therefore e_1 commutes strongly with $\overline{L_0}$. Let P_s be the projection of $\mathcal{H}_i$ onto $W_i(s)$. Then P_s is a spectral projection of $\overline{L_0}$. Hence $eP_s = P_s e$ for any $s \geqslant 0$.

Choose any $\xi \in \mathcal{H}_1$. Then $\xi = \sum_{s \geqslant 0} P_s \xi$. Since for any $s \geqslant 0$ we have $P_s \xi = P_s e_1 \xi = e_1 P_s \xi \in e_1 W_i = W_1$, we see that ξ can be approximated by vectors in W_1. This proves that $\mathcal{H}_1$ is the norm closure of W_1. □

4.2 A criterion for strong integrability

Assume that V is unitary, energy bounded, and strongly local. In this section, we give a criterion for the strong integrability of energy-bounded unitary V-modules.

Proposition 4.5. *Let W_i be a non-trivial energy-bounded unitary V-module. Then W_i is strongly integrable if and only if for any $I \in \mathcal{J}$, there exists a unitary operator $U_I : \mathcal{H}_0 \to \mathcal{H}_i$, such that any $v \in V$ and $f \in C_c^\infty(I)$ satisfy*

$$\overline{Y_i(v, f)} = U_I \overline{Y(v, f)} U_I^*. \tag{4.9}$$

Proof. "If part": For any $I \in \mathcal{J}(I)$, we define a representation $\pi_{i,I}$ of $\mathcal{M}_V(I)$ on $\mathcal{H}_i$ to be

$$\pi_{i,I}(x) = U_I x U_I^* \qquad (x \in \mathcal{M}_V(I)). \tag{4.10}$$

If $J \in \mathcal{J}(I)$ and $I \subset J$, then by equation (4.9), $U_J^* U_I$ commutes strongly with every $\overline{Y(v, f)}$ where $v \in V$ and $f \in C_c^\infty(I)$. So $U_J^* U_I$ commutes with $\mathcal{M}_V(I)$, which implies that $\pi_{i,I}$ is the restriction of $\pi_{i,J}$ on $\mathcal{M}_V(I)$. So π_i is a representation of the conformal net $\mathcal{M}_V$. It is obvious that π_i is associated with W_i. So W_i is strongly integrable.

"Only if part": Suppose that W_i is strongly integrable. We let $(\mathcal{H}_i, \pi_i)$ be the $\mathcal{M}_V$-module associated with W_i. For each $I \in \mathcal{M}_I$, $\pi_{i,I}$ is a non-trivial representation of $\mathcal{M}_V(I)$

on $\mathcal{H}_i$. Since the Hilbert spaces $\mathcal{H}_0, \mathcal{H}_i$ are separable, and $\mathcal{M}_V(I)$ is a type III factor, $\pi_{i,I}$ is (unitary) equivalent to the representation $\pi_{0,I}$ of $\mathcal{M}_V(I)$ on $\mathcal{H}_0$. So there exits a unitary $U_I : \mathcal{H}_i \to \mathcal{H}_i$ such that equation (4.9) always holds. □

Remark 4.6. Equation (4.9) is equivalent to one of the following equivalent relations:

$$U_I\overline{Y(v,f)} \subset \overline{Y_i(v,f)}U_I, \tag{4.11}$$

$$U_I^*\overline{Y_i(v,f)} \subset \overline{Y(v,f)}U_I^*. \tag{4.12}$$

Proposition 4.7. *Let W_j, W_k be non-trivial energy-bounded unitary V-modules. Assume that W_j is strongly integrable. If for any $I \in \mathcal{J}$ there exits a collection $\{T_a : a \in \mathcal{A}\}$ of bounded linear operators from $\mathcal{H}_j$ to $\mathcal{H}_k$, such that $\bigvee_{a\in\mathcal{A}} T_a\mathcal{H}_j$ is dense in $\mathcal{H}_k$, and that for any $a \in \mathcal{A}, v \in V, f \in C_c^\infty(I)$, we have*

$$T_a\overline{Y_j(v,f)} \subset \overline{Y_k(v,f)}T_a, \tag{4.13}$$

$$T_a^*\overline{Y_k(v,f)} \subset \overline{Y_j(v,f)}T_a^*, \tag{4.14}$$

then W_k is strongly integrable.

We remark that when T_α is not unitary, equations (4.13) and (4.14) do not imply each other.

Proof. Let $W_l = W_j \oplus^\perp W_k$ be the direct sum module of W_j and W_k, and extend each T_a to a bounded linear operator on $\mathcal{H}_l$, such that T_a equals zero on the subspace $\mathcal{H}_k$. Choose any $I \in \mathcal{J}$. Since $\overline{Y_l(v,f)} = \mathrm{diag}(\overline{Y_j(v,f)}, \overline{Y_k(v,f)})$, equations (4.13) and (4.14) are equivalent to that T_a commutes strongly with $\overline{Y_l(v,f)}$ for any $v \in V, f \in C_c^\infty(I)$. We construct a unitary operator $U_I : \mathcal{H}_j \to \mathcal{H}_k$ such that

$$\overline{Y_k(v,f)} = U_I\overline{Y_j(v,f)}U_I^* \tag{4.15}$$

for any $v \in V, f \in C_c^\infty(I)$. Then the strong integrability of W_k will follow immediately from proposition 4.5 and the strong integrability of W_j.

Let $\{U_b : b \in \mathcal{B}\}$ be a maximal collection of non-zero partial isometries from $\mathcal{H}_j$ to $\mathcal{H}_k$ satisfying the following conditions:
(a) For any $b \in \mathcal{B}, v \in V, f \in C_c^\infty(I)$, U_b commutes strongly with $\overline{Y_l(v,f)}$.
(b) The projections $\{e_b = U_bU_b^* : b \in \mathcal{B}\}$ are pairwise orthogonal.
Note that similar to T_a, each U_b is extended to a partial isometry on $\mathcal{H}_l$, being zero when acting on $\mathcal{H}_k$.

Let $e = \sum_{b\in\mathcal{B}} e_b$. We prove that $e = \mathrm{id}_{\mathcal{H}_k}$. Let $e' = \mathrm{id}_{\mathcal{H}_k} - e$. If $e' \neq 0$, then by the density of $\bigvee_{a\in\mathcal{A}} T_a\mathcal{H}_j$ in $\mathcal{H}_k$, there exists $a \in \mathcal{A}$ such that $e'T_a \neq 0$. Take the left polar decomposition $e'T_a = U_aH_a$ of $e'T_a$, where U_a is the partial isometry part. Then $U_aU_a^*$ is the projection of $\mathcal{H}_l$ onto the range of $e'T_a$, which is nonzero and orthogonal to each e_b. For each $v \in V, f \in C_c^\infty(I)$, since e' and T_a commute strongly with $\overline{Y_l(v,f)}$, U_a also commutes strongly with $\overline{Y_l(v,f)}$. Therefore, $\{U_b : b \in \mathcal{B}\} \cup \{U_a\}$ is a collection of partial isometries from $\mathcal{H}_j$ to $\mathcal{H}_k$ satisfying conditions (a) and (b), and $\{U_b : b \in \mathcal{B}\}$ is its proper sub-collection. This contradicts the fact that $\{U_b : b \in \mathcal{B}\}$ is maximal. So $e' = 0$, and hence $e = \mathrm{id}_{\mathcal{H}_k}$.

For each $b \in \mathcal{B}$ we let $p_b = U_b^* U_b$, which is a non-zero projection on $\mathcal{H}_j$. We now restrict ourselves to operators on $\mathcal{H}_j$. Then p_b commutes strongly with each $\overline{Y_j(v,f)}$, which, by the strong integrability of W_j, is equivalent to that $p_b \in \pi_{j,I}(\mathcal{M}_V(I))'$. Note that $\mathcal{B}$ must be countable. We choose a countable collection $\{q_b : b \in \mathcal{B}\}$ of non-zero orthogonal projections on $\mathcal{H}_j$ satisfying that $\sum_{b\in\mathcal{B}} q_b = \mathrm{id}_{\mathcal{H}_j}$, and that each $q_b \in \pi_{j,I}(\mathcal{M}_V(I))'$. Since $\pi_{j,I}(\mathcal{M}_V(I))'$ is a type III factor, for each b there exists a partial isometry $\widetilde{U}_b \in \pi_{j,I}(\mathcal{M}_V(I))'$ satisfying $\widetilde{U}_b\widetilde{U}_b^* = p_b, \widetilde{U}_b^*\widetilde{U}_b = q_b$.

We turn our attention back to operators on $\mathcal{H}_l$. Since $\widetilde{U}_b \in \pi_{j,I}(\mathcal{M}_V(I))'$, $\widetilde{U}_b$ commutes strongly with each $\overline{Y_l(v,f)}$. Let $U_I = \sum_{b\in\mathcal{B}} U_b\widetilde{U}_b$. Then U_I is a unitary operator from $\mathcal{H}_j$ to $\mathcal{H}_k$ satisfying relation (4.15) for any $v \in V, f \in C_c^\infty(I)$. Thus our proof is finished. □

We now prove the strong integrability of an energy-bounded unitary V-module using the linear energy-boundedness of intertwining operators.

Theorem 4.8. *Let W_i, W_j, W_k be non-zero unitary irreducible V-modules. Assume that W_j and W_k are energy-bounded, that W_j is strongly integrable, and that there exist a non-zero quasi-primary vector $w_0^{(i)} \in W_i$ and a non-zero intertwining operator $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$, such that $\mathcal{Y}_\alpha(w_0^{(i)}, x)$ satisfies linear energy bounds. Then W_k is strongly integrable.*

Proof. Step 1. Fix any $J \in \mathcal{J}(S^1\backslash\{-1\})$, and let $\mathcal{W}_J$ be the subspace of $\mathcal{H}_k$ spanned by the vectors $\mathcal{Y}_\alpha(w_0^{(i)}, g)w^{(j)}$ where $g \in C_c^\infty(J)$ and $w^{(j)} \in W_j$. We show that $\mathcal{W}_J$ is a dense subspace of $\mathcal{H}_k$.

Our proof is similar to that of Reeh-Schlieder theorem (cf. [RS61]). Choose $\xi^{(k)} \in \mathcal{W}_J^\perp$. Note that for each $\eta^{(k)} \in \mathcal{H}^k$, the multivalued function

$$z \mapsto z^{\overline{L_0}}\eta^{(k)} = \sum_{s\geqslant 0} z^s P_s \eta^{(k)} \tag{4.16}$$

is continuous on $\overline{D}^\times(1) = \{\zeta \in \mathbb{C} : 0 < |\zeta| \leqslant 1\}$ and holomorphic on its interior $D^\times(1)$. So we have a multivalued holomorphic function of z:

$$\langle z^{\overline{L_0}}\mathcal{Y}_\alpha(w_0^{(i)}, g)w^{(j)}|\xi^{(k)}\rangle, \tag{4.17}$$

which is continuous on $\overline{D}^\times(1)$ and holomorphic on $D^\times(1)$. Choose $\varepsilon > 0$ such that the support of $g^t = \exp(it(\Delta_{w_0^{(i)}} - 1))\mathfrak{r}(t)g$ is inside J for any $t \in (-\varepsilon, \varepsilon)$. Then, by proposition 3.15, we have

$$\langle e^{it\overline{L_0}}\mathcal{Y}_\alpha(w_0^{(i)}, g)w^{(j)}|\xi^{(k)}\rangle = \langle\mathcal{Y}_\alpha(w_0^{(i)}, g^t)e^{it\overline{L_0}}w^{(j)}|\xi^{(k)}\rangle, \tag{4.18}$$

which must be zero when $t \in (-\delta, \delta)$.

By Schwarz reflection principle, the value of function (4.17) is zero for any $z \in \overline{D}^{\times}(r)$. In particular, it is zero for any $z \in S^1$. This shows that (4.18) is zero for any $t \in \mathbb{R}$. Here, when we define the smeared intertwining operator, we allow the arguments to exceed the region $(-\pi, \pi)$ under the action of $\mathfrak{r}(t)$. So the right hand side of equation (4.18) becomes

$$\sum_{s\in\mathbb{R}} \int_{t-\pi}^{t+\pi} \langle \mathcal{Y}_\alpha(w_0^{(i)}, e^{i\theta}) e^{it\overline{L_0}} w^{(j)} | P_s \xi^{(k)} \rangle \cdot \exp(it(\Delta_{w_0^{(i)}} - 1)) g(e^{i(\theta - t)}) \,đ\theta, \tag{4.19}$$

which is 0 for any $t \in \mathbb{R}$. (Recall our notation that $đ\theta = e^{i\theta} d\theta / 2\pi$.) Since W_i, W_j, W_k are irreducible, we let $\Delta_i, \Delta_j, \Delta_k$ be their conformal dimensions, and set $\Delta_\alpha = \Delta_i + \Delta_j - \Delta_k$. Then by equation (1.25),

$$\mathcal{Y}_\alpha(w_0^{(i)}, z) z^{\Delta_\alpha} = \sum_{n\in\mathbb{Z}} \mathcal{Y}_\alpha(w_0^{(i)}, \Delta_\alpha - 1 - n) z^n \tag{4.20}$$

is a single valued holomorphic function for $z \in \mathbb{C}^\times$. So the fact that (4.19) always equals 0 implies that

$$\sum_{s\in\mathbb{R}} \int_{-\pi}^{\pi} \langle \mathcal{Y}_\alpha(w_0^{(i)}, e^{i\theta}) w^{(j)} | P_s \xi^{(k)} \rangle e^{i\Delta_\alpha \theta} \cdot h(e^{i\theta}) d\theta = 0 \tag{4.21}$$

for any $w^{(j)} \in W_j, I \in \mathcal{J}$ and $h \in C_c^\infty(I)$. By partition of unity on S^1, we see that equation (4.21) holds for any $h \in C^\infty(S^1)$.

For any $m \in \mathbb{Z}$, we choose $h(e^{i\theta}) = e^{-im\theta}$. Then the left hand side of equation (4.21) becomes

$$\begin{aligned}
&\sum_{s\in\mathbb{R}} \int_{-\pi}^{\pi} \langle \mathcal{Y}_\alpha(w_0^{(i)}, e^{i\theta}) w^{(j)} | P_s \xi^{(k)} \rangle e^{i\Delta_\alpha \theta} \cdot e^{-im\theta} d\theta \\
=&\sum_{s\in\mathbb{R}} \int_{-\pi}^{\pi} \sum_{n\in\mathbb{Z}} \langle \mathcal{Y}_\alpha(w_0^{(i)}, \Delta_\alpha - 1 - n) w^{(j)} | P_s \xi^{(k)} \rangle \cdot e^{i(n-m)\theta} d\theta \\
=&\sum_{s\in\mathbb{R}} \sum_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} \langle \mathcal{Y}_\alpha(w_0^{(i)}, \Delta_\alpha - 1 - n) w^{(j)} | P_s \xi^{(k)} \rangle \cdot e^{i(n-m)\theta} d\theta \\
=&2\pi \sum_{s\in\mathbb{R}} \langle \mathcal{Y}_\alpha(w_0^{(i)}, \Delta_\alpha - 1 - m) w^{(j)} | P_s \xi^{(k)} \rangle \\
=&2\pi \langle \mathcal{Y}_\alpha(w_0^{(i)}, \Delta_\alpha - 1 - m) w^{(j)} | \xi^{(k)} \rangle,
\end{aligned} \tag{4.22}$$

which by equation (4.21) must be zero. By corollary 2.15 and the proof of corollary A.4, vectors of the form $\mathcal{Y}_\alpha(w_0^{(i)}, s) w^{(j)}$ (where $s \in \mathbb{R}, w^{(j)} \in W_j$) span W_k, which is a dense subspace of $\mathcal{H}_j$. So $\xi^{(k)} = 0$.

Step 2. Choose any $I \in \mathcal{J}$, and let $J \in \mathcal{J}(I^c \backslash \{-1\})$. Take $W_l = W_j \oplus^\perp W_k$. Then for each $v \in V, f \in C_c^\infty(I)$ we have $\overline{Y_l(v, f)} = \mathrm{diag}(\overline{Y_j(v, f)}, \overline{Y_k(v, f)})$. For each $g \in C_c^\infty(J)$, we extend $\mathcal{Y}_\alpha(w_0^{(i)}, g)$ to an operator on $\mathcal{H}_l^\infty$ whose restriction to $\mathcal{H}_k^\infty$ is zero. We also regard $A = \mathcal{Y}_\alpha(w_0^{(i)}, g)$ as an unbounded operator on $\mathcal{H}_l$ with domain $\mathcal{H}_l^\infty$. Let $\mathcal{N}(I)$ be the von Neumann algebra on $\mathcal{H}_j$ generated by the operators $\overline{Y_l(v,f)}$ where $v \in V, f \in C_c^\infty(I)$, and let $\mathcal{N}(I)_\infty$ be the set of all $x \in \mathcal{N}(I)$ satisfying $x\mathcal{H}_l^\infty \subset \mathcal{H}_l^\infty, x^*\mathcal{H}_l^\infty \subset \mathcal{H}_l^\infty$. Then as in the proof of proposition 4.2, $\mathcal{N}(I)_\infty$ is a strongly dense self-adjoint subalgebra of $\mathcal{N}(I)$. Let $H = (A + A^\dagger)/2$ and $K = (A - A^\dagger)/(2i)$ be symmetric unbounded operators on $\mathcal{H}_l$ with domain $\mathcal{H}_l^\infty$. Then by proposition 2.13, corollary 3.13, remark 3.14, and equation (3.26), for any $v \in V$ and $f \in C_c^\infty(I)$, $Y_l(v,f)$ commutes with H and K when acting on $\mathcal{H}_l^\infty$. By lemma B.8 and relations (3.38), (3.26), $\overline{H}$ and $\overline{K}$ are self adjoint, and by theorem B.9, $\overline{Y_l(v,f)}$ commutes strongly with $\overline{H}$ and $\overline{K}$. Hence any $x \in \mathcal{N}(I)$ commutes strongly with $\overline{H}$ and $\overline{K}$. In particular, if $x \in \mathcal{N}(I)_\infty$, we have $xH = Hx, xK = Kx$ when both sides of the equations act on $\mathcal{H}_l^\infty$. So $x(H + iK) = (H + iK)x$ when acting on $\mathcal{H}_l^\infty$. Therefore, $x\overline{\mathcal{Y}_\alpha(w_0^{(i)}, g)} \subset \overline{\mathcal{Y}_\alpha(w_0^{(i)}, g)}x$ for any $x \in \mathcal{N}(I)_\infty$, which implies that $\mathcal{N}(I)$ commutes strongly with $\overline{\mathcal{Y}_\alpha(w_0^{(i)}, g)}$. Thus $\overline{Y_l(v,f)}$ commutes strongly with $\overline{\mathcal{Y}_\alpha(w_0^{(i)}, g)}$.

Let $\overline{\mathcal{Y}_\alpha(w_0^{(i)}, g)} = T_g H_g$ be the left polar decomposition of $\overline{\mathcal{Y}_\alpha(w_0^{(i)}, g)}$, where T_g is the partial isometry. Then T_g commutes strongly with each $\overline{Y_l(v,f)}$. By step 1, $\{T_g : g \in C_c^\infty(J)\}$ form a collection of bounded operators from $\mathcal{H}_j$ to $\mathcal{H}_k$ satisfying the conditions in proposition 4.7. Therefore, by that proposition, W_k is strongly integrable. □

5 Generalized intertwining operators

Generalized intertwining operators are nothing but genus 0 correlation functions written in a particular way. Suppose that $\mathcal{Y}_{\sigma_2}, \dots, \mathcal{Y}_{\sigma_n}$ is a chain of intertwining operators with charge spaces $W_{i_2}, \dots, W_{i_n}$ respectively, such that the source space of $\mathcal{Y}_{\sigma_2}$ is W_{i_1}, and the target space of $\mathcal{Y}_{\sigma_n}$ is W_i. Choose $\mathcal{Y}_\alpha \in \mathcal{Y}\binom{k}{i\ j}$. Choose $(z_1, \dots, z_n) \in \mathrm{Conf}_n(\mathbb{C}^\times)$, and choose arguments $\arg z_1, \arg(z_2 - z_1), \dots, \arg(z_n - z_1)$. A **generalized intertwining operator** $\mathcal{Y}_{\sigma_n \cdots \sigma_n, \alpha}$ is defined near $(z_1, \dots, z_n)$ in the following two situations.

The first case is when $(z_1, \dots, z_n)$ satisfies $0 < |z_2 - z_1| < \cdots < |z_n - z_1| < |z_1|$. We define a $(W_j \otimes W_{i_1} \otimes \cdots \otimes W_{i_n} \otimes W_{\overline{k}})^*$-valued holomorphic function $\mathcal{Y}_{\sigma_n \cdots \sigma_n, \alpha}$ near $(z_1, \dots, z_n)$ to satisfy that for any $w^{(j)} \in W_j, w^{(i_1)} \in W_{i_1}, \dots, w^{(i_n)} \in W_{i_n}, w^{(\overline{k})} \in W_{\overline{k}}$,

$$\begin{aligned}&\langle \mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, z_n; \dots; w^{(i_2)}, z_2; w^{(i_1)}, z_1) w^{(j)}, w^{(\overline{k})} \rangle \\ =&\big\langle \mathcal{Y}_\alpha\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1) \cdots \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1) w^{(i_1)}, z_1\big) w^{(j)}, w^{(\overline{k})} \big\rangle. \qquad (5.1)\end{aligned}$$

The V-modules $W_{i_1}, \dots, W_{i_n}$ are called the **charge spaces** of $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}$. W_j is called the **source space** of $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}$, and W_k is called the **target space** of $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}$. The vector space of generalized intertwining operators with charge spaces $W_{i_1}, \dots, W_{i_n}$, source space W_j, and target space W_k is also denoted by $\mathcal{V}\binom{k}{i_n\ \dots\ i_1\ j}$.

In the second case, we choose $I \in \mathcal{J}$, and choose an arbitrary continuous argument function $\arg_I$ on I. We define $\mathcal{O}_n(I)$ to be the set of all $(z_1, \dots, z_n) \in \mathrm{Conf}_n(\mathbb{C}^\times) \cap I^n$

satisfying that for any $2 \leqslant l < m \leqslant n$, either $\arg_I(z_l z_1^{-1}) \arg_I(z_m z_1^{-1}) < 0$, or $|\arg_I(z_l z_1^{-1})| < |\arg_I(z_m z_1^{-1})|$. Our definition is clearly independent of the choice of $\arg_I$, and $\mathscr{O}_n(I)$ is a finite disconnected union of simply-connected sets.

We want to define our generalized intertwining operators near any $(z_1, \dots, z_n) \in \mathscr{O}_n(I)$. To do this, we rotate $z_1, \dots, z_n$ along I without meeting each other, until these points satisfy $0 < |z_2 - z_1| < \cdots < |z_n - z_1| < |z_1| = 1$. The arguments of $z_1, z_2 - z_1, \dots, z_n - z_1$ are changed continuously. We first define $\mathcal{Y}_{\sigma_n \cdots \sigma_n, \alpha}$ near the new point $(z_1, \dots, z_n)$ using equation (5.1). Then we reverse this process of rotating $z_1, \dots, z_n$, and change $\mathcal{Y}_{\sigma_n \cdots \sigma_n, \alpha}$ continuously so as to be defined near the original point.

We now define the product of two generalized intertwining operators defined near S^1. Products of more than two generalized intertwining operators are defined in a similar way. Choose disjoint $I, J \in \mathcal{J}$, choose $(z_1, \dots, z_m) \in \mathscr{O}_m(I), (\zeta_1, \dots, \zeta_n) \in \mathscr{O}_n(J)$, and choose arguments $\arg z_1, \arg(z_2 - z_1), \dots, \arg(z_m - z_1), \arg \zeta_1, \arg(\zeta_2 - \zeta_1), \dots, \arg(\zeta_n - \zeta_1)$. Choose generalized intertwining operators $\mathcal{Y}_{\sigma_m \cdots \sigma_1, \alpha} \in \mathcal{V}\binom{k}{i_m \cdots i_1 i_0}, \mathcal{Y}_{\rho_n \cdots \rho_1, \beta} \in \mathcal{V}\binom{i_0}{j_n \cdots j_1 j_0}$. If we choose $\arg z_2, \dots, \arg z_m, \arg \zeta_2, \dots, \arg \zeta_n$, then we can find uniquely chains of intertwining operators $\mathcal{Y}_{\alpha_1}, \dots, \mathcal{Y}_{\alpha_m}$ with charge spaces $W_{i_1}, \dots, W_{i_m}$ respectively, and $\mathcal{Y}_{\beta_1}, \dots, \mathcal{Y}_{\beta_n}$ with charge spaces $W_{j_1}, \dots, W_{j_n}$ respectively, such that the source space of $\mathcal{Y}_{\beta_1}$ is W_{j_0}, that the source space of $\mathcal{Y}_{\alpha_1}$ and the target space of $\mathcal{Y}_{\beta_n}$ are W_{i_0}, that the target space of $\mathcal{Y}_{\alpha_m}$ is W_k, and that for any $w^{(j_1)} \in W_{j_1}, \dots, w^{(j_n)} \in W_{j_n}, w^{(i_1)} \in W_{i_1}, \dots, w^{(i_m)} \in W_{i_m}$, we have the fusion relations

$$\mathcal{Y}_{\sigma_m \cdots \sigma_2, \alpha}(w^{(i_m)}, z_m; \dots; w^{(i_1)}, z_1) = \mathcal{Y}_{\alpha_m}(w^{(i_m)}, z_m) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1), \tag{5.2}$$

$$\mathcal{Y}_{\rho_n \cdots \rho_2, \beta}(w^{(j_n)}, \zeta_n; \dots; w^{(j_1)}, \zeta_1) = \mathcal{Y}_{\beta_n}(w^{(j_n)}, \zeta_n) \cdots \mathcal{Y}_{\beta_1}(w^{(j_1)}, \zeta_1). \tag{5.3}$$

We then define a $(W_{j_0} \otimes W_{j_1} \otimes \cdots \otimes W_{j_n} \otimes W_{i_1} \otimes \cdots \otimes W_{i_m} \otimes W_{\overline{k}})^*$-valued holomorphic function $\mathcal{Y}_{\sigma_m \cdots \sigma_1, \alpha} \mathcal{Y}_{\rho_n \cdots \rho_1, \beta}$ near $(\zeta_1, \dots, \zeta_n, z_1, \dots, z_m)$ to satisfy that for any $w^{(j_0)} \in W_{j_0}, w^{(j_1)} \in W_{j_1}, \dots, w^{(j_n)} \in W_{j_n}, w^{(i_1)} \in W_{i_1}, \dots, w^{(i_m)} \in W_{i_m}, w^{(\overline{k})} \in W_k$,

$$\begin{aligned}
&\langle \mathcal{Y}_{\sigma_m \cdots \sigma_2, \alpha}(w^{(i_m)}, z_m; \dots; w^{(i_1)}, z_1) \mathcal{Y}_{\rho_n \cdots \rho_2, \beta}(w^{(j_n)}, \zeta_n; \dots; w^{(j_1)}, \zeta_1) w^{(j_0)}, w^{(\overline{k})} \rangle \\
=&\langle \mathcal{Y}_{\alpha_m}(w^{(i_m)}, z_m) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1) \mathcal{Y}_{\beta_n}(w^{(j_n)}, \zeta_n) \cdots \mathcal{Y}_{\beta_1}(w^{(j_1)}, \zeta_1) w^{(j_0)}, w^{(\overline{k})} \rangle.
\end{aligned} \tag{5.4}$$

Remark 5.1. It is clear that our definition does not depend on the choice of $\arg z_2, \dots, \arg z_m, \arg \zeta_2, \dots, \arg \zeta_n$. Moreover, if we choose $\varsigma \in S_m, \varpi \in S_n$, and *real variables* $\lambda_1, \dots, \lambda_n, r_1, \dots, r_m$ defined near 1 and satisfying $0 < \lambda_{\varpi(1)} < \cdots, \lambda_{\varpi(n)} < r_{\varsigma(1)} < \cdots < r_{\varsigma(m)}$, then the following series

$$\sum_{s \in \mathbb{R}} \langle \mathcal{Y}_{\sigma_m \cdots \sigma_2, \alpha}(w^{(i_m)}, r_m z_m; \dots; w^{(i_1)}, r_1 z_1) P_s \mathcal{Y}_{\rho_n \cdots \rho_2, \beta}(w^{(j_n)}, \lambda_n \zeta_n; \dots; w^{(j_1)}, \lambda_1 \zeta_1) w^{(j_0)}, w^{(\overline{k})} \rangle \tag{5.5}$$

of s converges absolutely, and by proposition 2.11, as $r_1, \dots, r_m, \lambda_1, \dots, \lambda_n \to 1$, the limit of (5.5) exists and equals the left hand side of equation (5.4).

5.1 Braiding of generalized intertwining operators

Theorem 5.2. *Choose disjoint* $I, J \in \mathcal{J}$. *Choose* $(z_1, \ldots, z_m) \in \mathscr{O}_m(I), (\zeta_1, \ldots, \zeta_n) \in \mathscr{O}_n(J)$. *Choose arguments* $\arg z_1, \arg \zeta_1, \arg(z_2 - z_1), \ldots, \arg(z_m - z_1), \arg(\zeta_n - \zeta_1), \ldots, \arg(\zeta_n - \zeta_1)$. *Let* $W_i, W_j, W_{i_1}, W_{i_2}, \ldots, W_{i_m}, W_{j_1}, W_{j_2}, \ldots, W_{j_n}$ *be* V*-modules. Assume that for any* $w^{(i)} \in W_i, w^{(j)} \in W_j$*, the braid relation*

$$\mathcal{Y}_\alpha(w^{(i)}, z_1)\mathcal{Y}_\beta(w^{(j)}, \zeta_1) = \mathcal{Y}_{\beta'}(w^{(j)}, \zeta_1)\mathcal{Y}_{\alpha'}(w^{(i)}, z_1) \tag{5.6}$$

holds. Then for any intertwining operators $\mathcal{Y}_{\sigma_2}, \ldots, \mathcal{Y}_{\sigma_m}, \mathcal{Y}_{\rho_2}, \ldots, \mathcal{Y}_{\rho_n}$*, any* $w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_m)} \in W_{i_m}, w^{(j_1)} \in W_{j_1}, \ldots, w^{(j_n)} \in W_{j_n}$*, we have the generalized braid relation*

$$\begin{aligned}&\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha}(w^{(i_m)}, z_m; \ldots; w^{(i_1)}, z_1)\mathcal{Y}_{\rho_n\cdots\rho_2,\beta}(w^{(j_n)}, \zeta_n; \ldots; w^{(j_1)}, \zeta_1)\\ =&\mathcal{Y}_{\rho_n\cdots\rho_2,\beta'}(w^{(j_n)}, \zeta_n; \ldots; w^{(j_1)}, \zeta_1)\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha'}(w^{(i_m)}, z_n; \ldots; w^{(i_1)}, z_1).\end{aligned} \tag{5.7}$$

(Note that here, as before, we follow convention 2.19 to simplify our statement.)

Proof. By analytic continuation, it suffices to assume that $|z_1 - \zeta_1|$ is small enough with respect to 1, and $|z_2 - z_1|, \ldots, |z_m - z_1|, |\zeta_2 - \zeta_1|, \ldots, |\zeta_n - \zeta_1|$ are small enough with respect to $|z_1 - \zeta_1|$, such that for any *real variables* $r, \lambda > 0$ satisfying $\frac{2}{3} < \frac{r}{\lambda} < \frac{3}{2}$, the following inequalities are satisfied:

$$|\zeta_n - \zeta_1| + |z_m - z_1| < 1/4, \tag{5.8}$$

$$0 < |\lambda\zeta_2 - \lambda\zeta_1| < |\lambda\zeta_3 - \lambda\zeta_1| < \cdots < |\lambda\zeta_n - \lambda\zeta_1| < |rz_1 - \lambda\zeta_1| - |rz_m - rz_1|, \tag{5.9}$$

$$0 < |rz_2 - rz_1| < |rz_3 - rz_1| < \cdots < |rz_m - rz_1| < |rz_1 - \lambda\zeta_1| < \lambda - |rz_m - rz_1|. \tag{5.10}$$

Choose $\arg(z_1 - \zeta_1)$. Since $|z_1 - \zeta_1| < 1$, there exist intertwining operators $\mathcal{Y}_\gamma$ and $\mathcal{Y}_\delta$ such that for any $w^{(i)} \in W_i, w^{(j)} \in W_j$, we have

$$\mathcal{Y}_\alpha(w^{(i)}, z_1)\mathcal{Y}_\beta(w^{(j)}, \zeta_1) = \mathcal{Y}_\delta(\mathcal{Y}_\gamma(w^{(i)}, z_1 - \zeta_1)w^{(j)}, \zeta_1) = \mathcal{Y}_{\beta'}(w^{(j)}, \zeta_1)\mathcal{Y}_{\alpha'}(w^{(i)}, z_1). \tag{5.11}$$

Choose *real variables* $r_m > \cdots > r_1 > \lambda_n > \cdots > \lambda_1 > 0$ satisfying $2/3 < r_1/\lambda_1 < 3/2$. (Recall that the arguments of real variables are chosen to be 0.) When $r_2/r_1, \ldots, r_m/r_1, \lambda_2/\lambda_1, \ldots, \lambda_n/\lambda_1$ are close to 1, by corollary 2.7, the right hand side of the equation

$$\begin{aligned}&\mathcal{Y}_\alpha\big(\mathcal{Y}_{\sigma_m}(w^{(i_m)}, r_mz_m - r_1z_1)\cdots\mathcal{Y}_{\sigma_2}(w^{(i_2)}, r_2z_2 - r_1z_1)w^{(i_1)}, r_1z_1\big)\\ &\cdot\mathcal{Y}_\beta\big(\mathcal{Y}_{\rho_n}(w^{(j_n)}, \lambda_n\zeta_n - \lambda_1\zeta_1)\cdots\mathcal{Y}_{\rho_2}(w^{(j_2)}, \lambda_2\zeta_2 - \lambda_1\zeta_1)w^{(j_1)}, \lambda_1\zeta_1\big)\\ =&\mathcal{Y}_\delta\bigg(\mathcal{Y}_\gamma\Big(\mathcal{Y}_{\sigma_m}(w^{(i_m)}, r_mz_m - r_1z_1)\cdots\mathcal{Y}_{\sigma_2}(w^{(i_2)}, r_2z_2 - r_1z_1)w^{(i_1)}, r_1z_1 - \lambda_1\zeta_1\Big)\\ &\quad\cdot\mathcal{Y}_{\rho_n}(w^{(j_n)}, \lambda_n\zeta_n - \lambda_1\zeta_1)\cdots\mathcal{Y}_{\rho_2}(w^{(j_2)}, \lambda_2\zeta_2 - \lambda_1\zeta_1)w^{(j_1)}, \lambda_1\zeta_1\bigg)\end{aligned} \tag{5.12}$$

converges absolutely and locally uniformly. If moreover $r_1/\lambda_1 = 4/3$, then by theorem 2.6, the left hand side of equation (5.12) also converges absolutely and locally uniformly, and hence equation (5.12) holds.

Now we let $r_1, \ldots, r_m, \lambda_1, \ldots, \lambda_n \to 1$, then the left hand side of equation (5.12) converges to the left hand side of equation (5.7), and the right hand side of (5.12) converges to

$$\begin{aligned}\mathcal{Y}_\delta\bigg(&\mathcal{Y}_\gamma\Big(\mathcal{Y}_{\sigma_m}(w^{(i_m)}, z_m - z_1)\cdots\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1 - \zeta_1\Big)\\ &\cdot\mathcal{Y}_{\rho_n}(w^{(j_n)}, \zeta_n - \zeta_1)\cdots\mathcal{Y}_{\rho_2}(w^{(j_2)}, \zeta_2 - \zeta_1)w^{(j_1)}, \zeta_1\bigg).\end{aligned} \tag{5.13}$$

Therefore, the left hand side of equation (5.7) equals (5.13). The same argument shows that the right hand side of (5.7) also equals (5.13). This finishes our proof. □

Note that it is easy to generalize proposition 2.11 to generalized intertwining operators.

5.2 The adjoint relation for generalized intertwining operators

This section is devoted to the proof of the adjoint relation for generalized intertwining operators (5.34). We first recall that if $\mathcal{Y}_\alpha$ is a unitary intertwining operator of a unitary V, $z \in S^1$ with chosen argument, and $w^{(i)} \in W_i$ is quasi-primary, then by relation (1.34),

$$\mathcal{Y}_\alpha(w^{(i)}, z)^\dagger = e^{-i\pi\Delta_{w^{(i)}}} z^{2\Delta_{w^{(i)}}} \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, z). \tag{5.14}$$

We want to obtain a similar relation for generalized intertwining operators. To achieve this goal, we first need an auxiliary fusion relation. Recall that for any V-module W_i, we have the creation operator $\mathcal{Y}^i_{i0} = B_\pm Y_i$ of W_i, and the annihilation operator $\mathcal{Y}^0_{i\bar{i}} = C^{-1}\mathcal{Y}^i_{i0}$ of $W_{\bar{i}}$. We set $\Upsilon^0_{i\bar{i}} = C\mathcal{Y}^i_{i0}$. Then similar to equation (1.40), for any $w^{(i)}_1 \in W_i, w^{(\bar{i})}_2 \in W_{\bar{i}}$ we have

$$\langle \Upsilon^0_{i\bar{i}}(w^{(i)}_1, x)w^{(\bar{i})}_2, \Omega\rangle = \langle e^{x^{-1}L_1}w^{(\bar{i})}_2, (e^{-i\pi}x^{-2})^{L_0}e^{-x^{-1}L_1}w^{(i)}_1\rangle. \tag{5.15}$$

Proposition 5.3 (Fusion with annihilation operators). *Let $z_1, z_2 \in \mathbb{C}^\times$ satisfy $0 < |z_1|, |z_1 - z_2| < |z_2|$. Choose $\arg z_2$, let $\arg z_1$ be close to $\arg z_2$ as $z_1 \to z_2$, and let $\arg(z_2 - z_1)$ be close to $\arg z_2$ as $z_1 \to 0$. Then for any $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}, w^{(i)} \in W_i$ and $w^{(j)} \in W_j$, we have the fusion relation*

$$\Upsilon^0_{k\bar{k}}\big(\mathcal{Y}_\alpha(w^{(i)}, e^{i\pi}(z_2 - z_1))w^{(j)}, z_2\big) = \Upsilon^0_{j\bar{j}}(w^{(j)}, z_2)\mathcal{Y}_{C\alpha}(w^{(i)}, z_1). \tag{5.16}$$

Proof. Let us assume that $z_1, z_2 \in \mathbb{R}_{>0}$ and $0 < z_2 - z_1 < z_1 < z_2$. If the proposition is proved for this special case, then by analytic continuation, it also holds in general.

Therefore, we assume that $\arg z_1 = \arg z_2 = \arg(z_2 - z_1) = 0$. Let $\arg(z_1^{-1} - z_2^{-1})$ be close to $\arg(z_1^{-1}) = -\arg z_1$ as $z_2^{-1} \to 0$. (The reason for choosing this argument is to use lemma 2.16-(1). Recall also convention 1.12 in [Gui19a].) Then it is obvious that $\arg(z_1^{-1} - z_2^{-1}) = 0 = \arg(\frac{z_2 - z_1}{z_1 z_2})$.

We now use equation (5.15) and the definition of $C\alpha$ to compute that

$$\begin{aligned}
&\langle \Upsilon^0_{j\bar j}(w^{(j)},z_2)\mathcal{Y}_{C\alpha}(w^{(i)},z_1)w^{(\bar k)},\Omega\rangle\\
=&\sum_{s\in\mathbb{R}}\langle \Upsilon^0_{j\bar j}(w^{(j)},z_2)P_s\mathcal{Y}_{C\alpha}(w^{(i)},z_1)w^{(\bar k)},\Omega\rangle\\
=&\sum_{s\in\mathbb{R}}\langle e^{z_2^{-1}L_1}P_s\mathcal{Y}_{C\alpha}(w^{(i)},z_1)w^{(\bar k)},(e^{-i\pi}z_2^{-2})^{L_0}e^{-z_2^{-1}L_1}w^{(j)}\rangle\\
=&\sum_{s\in\mathbb{R}}\big\langle w^{(\bar k)},\mathcal{Y}_\alpha\big(e^{z_1L_1}(e^{-i\pi}z_1^{-2})^{L_0}w^{(i)},z_1^{-1}\big)P_se^{z_2^{-1}L_{-1}}(e^{-i\pi}z_2^{-2})^{L_0}e^{-z_2^{-1}L_1}w^{(j)}\big\rangle,
\end{aligned}\tag{5.17}$$

which, according to lemma 2.16-(1), converges absolutely and equals

$$\Big\langle w^{(\bar k)},e^{z_2^{-1}L_{-1}}\mathcal{Y}_\alpha\Big(e^{z_1L_1}(e^{-i\pi}z_1^{-2})^{L_0}w^{(i)},\frac{z_2-z_1}{z_1z_2}\Big)(e^{-i\pi}z_2^{-2})^{L_0}e^{-z_2^{-1}L_1}w^{(j)}\Big\rangle.\tag{5.18}$$

By (1.26) and (1.30), the above formula equals

$$\begin{aligned}
&\Big\langle w^{(\bar k)},e^{z_2^{-1}L_{-1}}(e^{-i\pi}z_2^{-2})^{L_0}\\
&\qquad\cdot\mathcal{Y}_\alpha\Big((e^{i\pi}z_2^2)^{L_0}e^{z_1L_1}(e^{-i\pi}z_1^{-2})^{L_0}w^{(i)},e^{i\pi}(z_2-z_1)\frac{z_2}{z_1}\Big)e^{-z_2^{-1}L_1}w^{(j)}\Big\rangle\\
=&\Big\langle w^{(\bar k)},e^{z_2^{-1}L_{-1}}(e^{-i\pi}z_2^{-2})^{L_0}\\
&\qquad\cdot\mathcal{Y}_\alpha\Big(e^{-z_1z_2^{-2}L_1}\Big(\frac{z_2}{z_1}\Big)^{2L_0}w^{(i)},e^{i\pi}(z_2-z_1)\frac{z_2}{z_1}\Big)e^{-z_2^{-1}L_1}w^{(j)}\Big\rangle.
\end{aligned}\tag{5.19}$$

On the other hand, we have

$$\begin{aligned}
&\big\langle \Upsilon^0_{k\bar k}\big(\mathcal{Y}_\alpha(w^{(i)},e^{i\pi}(z_2-z_1))w^{(j)},z_2\big)w^{(\bar k)},\Omega\big\rangle\\
=&\sum_{s\in\mathbb{R}}\big\langle \Upsilon^0_{k\bar k}\big(P_s\mathcal{Y}_\alpha(w^{(i)},e^{i\pi}(z_2-z_1))w^{(j)},z_2\big)w^{(\bar k)},\Omega\big\rangle\\
=&\sum_{s\in\mathbb{R}}\langle w^{(\bar k)},e^{z_2^{-1}L_{-1}}(e^{-i\pi}z_2^{-2})^{L_0}e^{-z_2^{-1}L_1}P_s\mathcal{Y}_\alpha(w^{(i)},e^{i\pi}(z_2-z_1))w^{(j)}\rangle.
\end{aligned}\tag{5.20}$$

Note that $|-z_2^{-1}|<|e^{i\pi}(z_2-z_1)|^{-1}$. Let $\arg(1-e^{i\pi}(z_2-z_1)\cdot(-z_2^{-1}))$ be close to $\arg(1-e^{i\pi}(z_2-z_1)\cdot 0)=0$ as $-z_2^{-1}\to 0$. Then clearly $\arg(1-e^{i\pi}(z_2-z_1)\cdot(-z_2^{-1}))=0=\arg(\frac{z_1}{z_2})$. We can use lemma 2.16-(2) to compute that (5.20) equals (5.19). This proves equation (5.16) when both sides act on Ω. By the proof of corollary 2.15, equation (5.16) holds when acting on any vector inside V. □

Remark 5.4. By proposition 2.9 and the above property, we have the fusion relation

$$\Upsilon^0_{k\bar k}\big(\mathcal{Y}_{B_+\alpha}(w^{(j)},z_2-z_1)w^{(i)},z_1\big)=\Upsilon^0_{j\bar j}(w^{(j)},z_2)\mathcal{Y}_{C\alpha}(w^{(i)},z_1)\tag{5.21}$$

when $0<|z_2-z_1|<|z_1|<|z_2|$, $\arg z_1$ is close to $\arg z_2$ as $z_1\to z_2$, and $\arg(z_2-z_1)$ is close to $\arg z_2$ as $z_1\to 0$. Similarly, we can also show that

$$\mathcal{Y}^0_{k\bar k}\big(\mathcal{Y}_{B_-\alpha}(w^{(j)},z_2-z_1)w^{(i)},z_1\big)=\mathcal{Y}^0_{j\bar j}(w^{(j)},z_2)\mathcal{Y}_{C^{-1}\alpha}(w^{(i)},z_1).\tag{5.22}$$

Theorem 5.5 (Fusion of contragredient intertwining operators). *Let $z_1, \dots, z_n, z'_1, \dots, z'_n \in \mathbb{C}^\times$ satisfy the following conditions:*
(1) $0 < |z_1| < |z_2| < \cdots < |z_n|$ and $0 < |z_2 - z_1| < \cdots < |z_n - z_1| < |z_1|$;
(1′) $|z'_1| > |z'_2| > \cdots > |z'_n| > 0$ and $0 < |z'_2 - z'_1| < \cdots < |z'_n - z'_1| < |z'_1|$.
Choose arguments $\arg z_1, \arg z'_1$. For each $2 \leqslant m \leqslant n$, we choose arguments $\arg(z_m - z_1), \arg(z'_m - z'_1)$. Let $\arg z_m$ be close to $\arg z_1$ as $z_m \to z_1$, and let $\arg z'_m$ be close to $\arg z'_1$ as $z'_m \to z'_1$.

Let $W_{i_1}, \dots, W_{i_n}$, and W_i be V-modules, and let $\mathcal{Y}_{\sigma_2}, \dots, \mathcal{Y}_{\sigma_n}$ be a chain of intertwining operators of V satisfying the following conditions:
(a) for each $2 \leqslant m \leqslant n$, the charge space of $\mathcal{Y}_{\sigma_m}$ is W_{i_m};
(b) the source space of $\mathcal{Y}_{\sigma_2}$ is W_{i_1};
(c) the target space of $\mathcal{Y}_{\sigma_n}$ is W_i.
Then there exists a chain of intertwining operators $\mathcal{Y}_{\sigma'_2}, \dots, \mathcal{Y}_{\sigma'_n}$, whose types are the same as those of $\mathcal{Y}_{\sigma_2}, \dots, \mathcal{Y}_{\sigma_n}$ respectively, such that for any $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$, if $\mathcal{Y}_{\alpha_1}, \mathcal{Y}_{\alpha_2}, \dots, \mathcal{Y}_{\alpha_n}$ is a chain of intertwining operators of V satisfying the following conditions:
(i) for each $1 \leqslant m \leqslant n$, the charge space of $\mathcal{Y}_{\alpha_m}$ is W_{i_m};
(ii) the source space of $\mathcal{Y}_{\alpha_1}$ is W_j;
(iii) the target space of $\mathcal{Y}_{\alpha_n}$ is W_k;
(iv) for any $w^{(i_1)} \in W_{i_1}, \dots, w^{(i_n)} \in W_{i_n}$, we have the fusion relation

$$\begin{aligned}&\mathcal{Y}_\alpha\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1) \cdots \mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1) w^{(i_1)}, z_1\big)\\ =&\mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1),\end{aligned} \tag{5.23}$$

then the following fusion relation also holds:

$$\begin{aligned}&\mathcal{Y}_{C\alpha}\big(\mathcal{Y}_{\sigma'_n}(w^{(i_n)}, z'_n - z'_1) \cdots \mathcal{Y}_{\sigma'_2}(w^{(i_2)}, z'_2 - z'_1) w^{(i_1)}, z'_1\big)\\ =&\mathcal{Y}_{C\alpha_1}(w^{(i_1)}, z'_1) \cdots \mathcal{Y}_{C\alpha_n}(w^{(i_n)}, z'_n).\end{aligned} \tag{5.24}$$

Proof. Let $W_{j_1}, \dots, W_{j_{n-1}}$ be the target spaces of $\mathcal{Y}_{\alpha_1}, \dots, \mathcal{Y}_{\alpha_{n-1}}$ respectively. Choose $\zeta'_0, \zeta'_1, \cdots, \zeta'_n \in \mathbb{R}_{<0}$ satisfying $\zeta'_0 < \zeta'_1 < \cdots < \zeta'_n < 0$ and $|\zeta'_0 - \zeta'_1| > |\zeta'_1 - \zeta'_n|$. Let $\zeta_1 = \zeta'_1 - \zeta'_0, \dots, \zeta_n = \zeta'_n - \zeta'_0$. Let $\arg \zeta'_0 = \arg \zeta'_1 = \cdots = \arg \zeta'_n = -\pi, \arg \zeta_1 = \arg(\zeta'_1 - \zeta'_0) = 0, \dots, \arg \zeta_n = \arg(\zeta'_n - \zeta'_0) = 0$. Note that for any $2 \leqslant m \leqslant n$, $\zeta_m - \zeta_1 = \zeta'_m - \zeta'_1$. We let $\arg(\zeta_m - \zeta_1) = \arg(\zeta'_m - \zeta'_1) = 0$.

We now rotate and stretch these points, so that for each $1 \leqslant m \leqslant n$, ζ_m is moved to $\widetilde{z}_m = z_m$, ζ'_m is moved to $\widetilde{z}'_m = z'_m$, $\arg \zeta_m$ becomes $\arg \widetilde{z}_m = \arg z_m$, and $\arg \zeta'_m$ becomes $\arg \widetilde{z}'_m = \arg z'_m$. We assume that during this process, conditions (1) and (1′) are always satisfied. (Note that such process might not exist if the choice of $\arg z_2, \arg z_3, \dots$ and $\arg z'_2, \arg z'_3, \dots$ are arbitrary with respect to $\arg z_1$ and $\arg z'_1$.) Denote this process by $(\mathbf{P})$. Then under this process, for each $2 \leqslant m \leqslant n$, $\arg(\zeta_m - \zeta_1)$ is changed to an argument $\arg(\widetilde{z}_m - \widetilde{z}_1)$ of $\widetilde{z}_m - \widetilde{z}_1$, and $\arg(\zeta'_m - \zeta'_1)$ is changed to an argument $\arg(\widetilde{z}'_m - \widetilde{z}'_1)$ of $\widetilde{z}'_m - \widetilde{z}'_1$ accordingly. Since $\arg(\widetilde{z}_m - \widetilde{z}_1) \in \arg(z_m - z_1) + 2i\pi\mathbb{Z}$ and $\arg(\widetilde{z}'_m - \widetilde{z}'_1) \in \arg(z'_m - z'_1) + 2i\pi\mathbb{Z}$, there exist intertwining operators $\mathcal{Y}_{\widetilde{\sigma}_m}, \mathcal{Y}_{\sigma'_m}$ of the same type as that of $\mathcal{Y}_{\sigma_m}$, such that for any $w^{(i_m)} \in W_{i_m}$,

$$\mathcal{Y}_{\widetilde{\sigma}_m}(w^{(i_m)}, \widetilde{z}_m - \widetilde{z}_1) = \mathcal{Y}_{\sigma_m}(w^{(i_m)}, z_m - z_1),$$

$$\mathcal{Y}_{\sigma'_m}(w^{(i_m)}, z'_m - z'_1) = \mathcal{Y}_{\widetilde{\sigma}_m}(w^{(i_m)}, \widetilde{z}'_m - \widetilde{z}'_1).$$

Then equation (5.23) implies that

$$\begin{aligned}&\mathcal{Y}_\alpha\big(\mathcal{Y}_{\widetilde{\sigma}_n}(w^{(i_n)}, \widetilde{z}_n - \widetilde{z}_1)\cdots\mathcal{Y}_{\widetilde{\sigma}_2}(w^{(i_2)}, \widetilde{z}_2 - \widetilde{z}_1)w^{(i_1)}, \widetilde{z}_1\big)\\ =&\mathcal{Y}_{\alpha_n}(w^{(i_n)}, \widetilde{z}_n)\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, \widetilde{z}_1).\end{aligned} \tag{5.25}$$

By reversing process $(\mathbf{P})$, the above equation is analytically continued to the equation

$$\begin{aligned}&\mathcal{Y}_\alpha\big(\mathcal{Y}_{\widetilde{\sigma}_n}(w^{(i_n)}, \zeta'_n - \zeta'_1)\cdots\mathcal{Y}_{\widetilde{\sigma}_2}(w^{(i_2)}, \zeta'_2 - \zeta'_1)w^{(i_1)}, \zeta'_1 - \zeta'_0\big)\\ =&\mathcal{Y}_{\alpha_n}(w^{(i_n)}, \zeta'_n - \zeta'_0)\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, \zeta'_1 - \zeta'_0).\end{aligned} \tag{5.26}$$

For any $1 \leqslant m \leqslant n$, we let $\arg(\zeta'_0 - \zeta'_m)$ be close to $\arg\zeta'_0 = -\pi$ as $\zeta'_m \to 0$. Then $\arg(\zeta'_0 - \zeta'_m) = -\pi$, and hence $\zeta'_m - \zeta'_0 = e^{i\pi}(\zeta_0 - \zeta_m)$. Choose arbitrary $w^{(j)} \in W_j$. Then by lemma 5.3, we have

$$\begin{aligned}&\Upsilon^0_{j\overline{j}}(w^{(j)}, \zeta'_0)\mathcal{Y}_{C\alpha_1}(w^{(i_1)}, \zeta'_1)\cdots\mathcal{Y}_{C\alpha_n}(w^{(i_n)}, \zeta'_n)\\ =&\Upsilon^0_{j_1\overline{j_1}}\big(\mathcal{Y}_{\alpha_1}(w^{(i_1)}, \zeta'_1 - \zeta'_0)w^{(j)}, \zeta'_0\big)\mathcal{Y}_{C\alpha_2}(w^{(i_2)}, \zeta'_2)\\ &\cdot\mathcal{Y}_{C\alpha_3}(w^{(i_3)}, \zeta'_3)\cdots\mathcal{Y}_{C\alpha_n}(w^{(i_n)}, \zeta'_n)\\ =&\Upsilon^0_{j_2\overline{j_2}}\big(\mathcal{Y}_{\alpha_2}(w^{(i_2)}, \zeta'_2 - \zeta'_0)\mathcal{Y}_{\alpha_1}(w^{(i_1)}, \zeta'_1 - \zeta'_0)w^{(j)}, \zeta'_0\big)\\ &\cdot\mathcal{Y}_{C\alpha_3}(w^{(i_3)}, \zeta'_3)\cdots\mathcal{Y}_{C\alpha_n}(w^{(i_n)}, \zeta'_n)\\ &\qquad\vdots\\ =&\Upsilon^0_{k\overline{k}}\big(\mathcal{Y}_{\alpha_n}(w^{(i_n)}, \zeta'_n - \zeta'_0)\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, \zeta'_1 - \zeta'_0)w^{(j)}, \zeta'_0\big),\end{aligned} \tag{5.27}$$

where, by theorem 2.6, the expression in each step converges absolutely. By (5.26), expression (5.27) equals

$$\Upsilon^0_{k\overline{k}}\Big(\mathcal{Y}_\alpha\big(\mathcal{Y}_{\widetilde{\sigma}_n}(w^{(i_n)}, \zeta'_n - \zeta'_1)\cdots\mathcal{Y}_{\widetilde{\sigma}_2}(w^{(i_2)}, \zeta'_2 - \zeta'_1)w^{(i_1)}, \zeta'_1 - \zeta'_0\big)w^{(j)}, \zeta'_0\Big), \tag{5.28}$$

the absolute convergence of which is guaranteed by corollary 2.7. Again by proposition 5.3, equation (5.28) equals

$$\Upsilon^0_{j\overline{j}}(w^{(j)}, \zeta'_0)\mathcal{Y}_{C\alpha}\big(\mathcal{Y}_{\widetilde{\sigma}_n}(w^{(i_n)}, \zeta'_n - \zeta'_1)\cdots\mathcal{Y}_{\widetilde{\sigma}_2}(w^{(i_2)}, \zeta'_2 - \zeta'_1)w^{(i_1)}, \zeta'_1\big), \tag{5.29}$$

the absolute convergence of which follows from theorem 2.6. Therefore, the left hand side of equation (5.27) equals (5.29). By proposition 2.3, we obtain

$$\begin{aligned}&\mathcal{Y}_{C\alpha}\big(\mathcal{Y}_{\widetilde{\sigma}_n}(w^{(i_n)}, \zeta'_n - \zeta'_1)\cdots\mathcal{Y}_{\widetilde{\sigma}_2}(w^{(i_2)}, \zeta'_2 - \zeta'_1)w^{(i_1)}, \zeta'_1\big)\\ =&\mathcal{Y}_{C\alpha_1}(w^{(i_1)}, \zeta'_1)\cdots\mathcal{Y}_{C\alpha_n}(w^{(i_n)}, \zeta'_n).\end{aligned} \tag{5.30}$$

Now we do process $(\mathbf{P})$. Then (5.30) is analytically continued to the equation

$$\begin{aligned}&\mathcal{Y}_{C\alpha}\big(\mathcal{Y}_{\widetilde{\sigma}_n}(w^{(i_n)}, \widetilde{z}'_n - \widetilde{z}'_1)\cdots\mathcal{Y}_{\widetilde{\sigma}_2}(w^{(i_2)}, \widetilde{z}'_2 - \widetilde{z}'_1)w^{(i_1)}, \widetilde{z}'_1\big)\\ =&\mathcal{Y}_{C\alpha_1}(w^{(i_1)}, \widetilde{z}'_1)\cdots\mathcal{Y}_{C\alpha_n}(w^{(i_n)}, \widetilde{z}'_n),\end{aligned} \tag{5.31}$$

which implies (5.24). Thus the proof is completed. □

Remark 5.6. Choose (not necessarily disjoint) $I, J \in \mathcal{J}$, and choose $(z_1, \dots, z_n) \in \mathcal{O}_n(I), (z'_1, \dots, z'_n) \in \mathcal{O}_n(J)$. Choose continuous argument functions $\arg_I, \arg_J$ on I, J respectively, and let $\arg z_1 = \arg_I(z_1), \dots, \arg z_n = \arg_I(z_n), \arg z'_1 = \arg_J(z'_1), \dots, \arg z'_n = \arg_J(z'_n)$. For each $2 \leqslant m \leqslant n$ we choose arguments $\arg(z_m - z_1)$ and $\arg(z'_m - z'_1)$. Then by theorem 5.5, for any chain of intertwining operators $\mathcal{Y}_{\sigma_2}, \dots, \mathcal{Y}_{\sigma_n}$ satisfying conditions (a), (b), and (c) of theorem 5.5, there exists a chain of intertwining operators $\mathcal{Y}_{\sigma'_2}, \dots, \mathcal{Y}_{\sigma'_n}$ whose types are the same as those of $\mathcal{Y}_{\sigma_2}, \dots, \mathcal{Y}_{\sigma_n}$ respectively, such that

$$\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, z_n; \dots, w^{(i_1)}, z_1) = \mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1) \tag{5.32}$$

always implies

$$\mathcal{Y}_{\sigma'_n \cdots \sigma'_2, C\alpha}(w^{(i_n)}, z'_n; \dots, w^{(i_1)}, z'_1) = \mathcal{Y}_{C\alpha_1}(w^{(i_1)}, z'_1) \cdots \mathcal{Y}_{C\alpha_n}(w^{(i_n)}, z'_n). \tag{5.33}$$

Corollary 5.7 (Adjoint of generalized intertwining operators)**.** *Let V be unitary. Let $I \in \mathcal{J}$, choose $(z_1, \dots, z_n) \in \mathcal{O}_n(I)$, and choose arguments $\arg z_1, \arg(z_2 - z_1), \dots, \arg(z_n - z_1)$. Let $W_{i_1}, \dots, W_{i_n}$, and W_i be unitary V-modules, and let $\mathcal{Y}_{\sigma_2}, \dots, \mathcal{Y}_{\sigma_n}$ be a chain of unitary intertwining operators of V satisfying the following conditions:*
(a) for each $2 \leqslant m \leqslant n$, the charge space of $\mathcal{Y}_{\sigma_m}$ is W_{i_m};
(b) the source space of $\mathcal{Y}_{\sigma_2}$ is W_{i_1};
(c) the target space of $\mathcal{Y}_{\sigma_n}$ is W_i.
Then for each $2 \leqslant m \leqslant n$, there exists a unitary intertwining operator $\mathcal{Y}_{\widetilde{\sigma}_m}$ whose type is the same as that of $\mathcal{Y}_{\overline{\sigma_m}}$, such that for any unitary $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$, and any nonzero quasi-primary vectors $w^{(i_1)} \in W_{i_1}, \dots, w^{(i_n)} \in W_{i_n}$, we have

$$\begin{aligned}&\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}\big(w^{(i_n)}, z_n; \dots; w^{(i_1)}, z_1\big)^\dagger \\ =&e^{-i\pi\left(\Delta_{w^{(i_1)}} + \cdots + \Delta_{w^{(i_n)}}\right)} z_1^{2\Delta_{w^{(i_1)}}} \cdots z_n^{2\Delta_{w^{(i_n)}}} \cdot \mathcal{Y}_{\widetilde{\sigma}_n \cdots \widetilde{\sigma}_2, \alpha^*}\big(\overline{w^{(i_n)}}, z_n; \dots; \overline{w^{(i_1)}}, z_1\big),\end{aligned} \tag{5.34}$$

where the formal adjoint is defined for evaluations of the operators between the vectors inside W_j and W_k.

Proof. Let $\arg_I$ be the continuous argument function on I satisfying $\arg_I(z_1) = \arg z_1$. We let $\arg z_2 = \arg_I(z_2), \dots, \arg z_n = \arg_I(z_n)$. Recall that by convention 1.12, we have $\arg \overline{z_1} = -\arg z_1, \arg \overline{z_2} = -\arg z_2, \dots, \arg \overline{z_n} = -\arg z_n$. Let $\arg(\overline{z_2} - \overline{z_1}) = -\arg(z_2 - z_1), \dots, \arg(\overline{z_n} - \overline{z_1}) = -\arg(z_n - z_1)$. By remark 5.6, we can find a chain of unitary intertwining operators $\mathcal{Y}_{\sigma'_2}, \dots, \mathcal{Y}_{\sigma'_n}$ whose types are the same as those of $\mathcal{Y}_{\sigma_2}, \dots, \mathcal{Y}_{\sigma_n}$ respectively, such that for any chain of intertwining operators $\mathcal{Y}_{\alpha_1}, \dots, \mathcal{Y}_{\alpha_n}$ and any unitary $\mathcal{Y}_\alpha$, if equation (5.32) holds for any $w^{(i_1)} \in W_1, \dots, w^{(i_n)} \in W_{i_n}$, then

$$\mathcal{Y}_{\sigma'_n \cdots \sigma'_2, C\alpha}(w^{(i_n)}, \overline{z_n}; \dots, w^{(i_1)}, \overline{z_1}) = \mathcal{Y}_{C\alpha_1}(w^{(i_1)}, \overline{z_1}) \cdots \mathcal{Y}_{C\alpha_n}(w^{(i_n)}, \overline{z_n}). \tag{5.35}$$

Now assume that $w^{(i_1)}, \dots, w^{(i_n)}$ are quasi-primary. By equation (1.27), for any $1 \leqslant m \leqslant n$, we have

$$\mathcal{Y}_{C\alpha_m}(w^{(i_m)}, \overline{z_m}) = e^{-i\pi\Delta_{w^{(i_m)}}} z_m^{2\Delta_{w^{(i_m)}}} \mathcal{Y}_{\alpha_m}(w^{(i_m)}, z_m)^{\mathrm{t}}. \tag{5.36}$$

Therefore, by equation (5.32), we see that (5.35) equals

$$\begin{aligned}&e^{-i\pi\left(\Delta_{w^{(i_1)}}+\cdots+\Delta_{w^{(i_n)}}\right)}z_1^{2\Delta_{w^{(i_1)}}}\cdots z_n^{2\Delta_{w^{(i_n)}}}\left(\mathcal{Y}_{\alpha_n}(w^{(i_n)},z_n)\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)},z_1)\right)^{\mathrm{t}}\\=&e^{-i\pi\left(\Delta_{w^{(i_1)}}+\cdots+\Delta_{w^{(i_n)}}\right)}z_1^{2\Delta_{w^{(i_1)}}}\cdots z_n^{2\Delta_{w^{(i_n)}}}\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}\left(w^{(i_n)},z_n;\ldots;w^{(i_1)},z_1\right)^{\mathrm{t}}.\end{aligned}\tag{5.37}$$

Recall that $\alpha^*=\overline{C\alpha}$. It is obvious that equation

$$C_j^{-1}\mathcal{Y}_{\sigma_n'\cdots\sigma_2',C\alpha}(w^{(i_n)},\overline{z_n};\ldots,w^{(i_1)},\overline{z_1})C_k=\mathcal{Y}_{\overline{\sigma_n'}\cdots\overline{\sigma_2'},\alpha^*}(\overline{w^{(i_n)}},z_n;\ldots;\overline{w^{(i_1)}},z_1)\tag{5.38}$$

holds when $z_1,\ldots,z_n$ also satisfy $0<|z_2-z_1|<\cdots<|z_n-z_1|<|z_1|$. By analytic continuation, it holds for general $(z_1,\ldots,z_n)\in\mathscr{O}_n(I)$. Therefore, if we apply $C_j^{-1}(\cdot)C_k$ to the left hand side of equation (5.35) and the right hand side of equation (5.37), we obtain

$$\begin{aligned}&\mathcal{Y}_{\overline{\sigma_n'}\cdots\overline{\sigma_n'},\alpha^*}(\overline{w^{(i_n)}},z_n;\ldots;\overline{w^{(i_1)}},z_1)\\=&e^{i\pi\left(\Delta_{w^{(i_1)}}+\cdots+\Delta_{w^{(i_n)}}\right)}z_1^{-2\Delta_{w^{(i_1)}}}\cdots z_n^{-2\Delta_{w^{(i_n)}}}\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}\left(w^{(i_n)},z_n;\ldots;w^{(i_1)},z_1\right)^{\dagger}.\end{aligned}\tag{5.39}$$

So if we let $\mathcal{Y}_{\tilde\sigma_2}=\mathcal{Y}_{\overline{\sigma_2'}},\ldots,\mathcal{Y}_{\tilde\sigma_2}=\mathcal{Y}_{\overline{\sigma_2'}}$, then equation (5.34) is proved. □

5.3 Generalized smeared intertwining operators

In this section, we assume that V is unitary, energy-bounded, and strongly local. Let $\mathcal{F}$ be a non-empty set of non-zero irreducible unitary V-modules, and let $\overline{\mathcal{F}}=\{W_{\bar i}:i\in\mathcal{F}\}$. Let $\mathcal{F}^{\boxtimes}$ be the collection of unitary V-modules W_i, where W_i is equivalent to a finite direct sum of submodules of tensor products of some V-modules in $\mathcal{F}\cup\overline{\mathcal{F}}$. So $\mathcal{F}^{\boxtimes}$ is additively closed, and any irreducible element in $\mathcal{F}^{\boxtimes}$ is equivalent to a submodule of $W_{i_1}\boxtimes\cdots\boxtimes W_{i_n}$, where $i_1,\ldots,i_n\in\mathcal{F}\cup\overline{\mathcal{F}}$. If $i\in\mathcal{F}$, we let $E^1(W_i)$ be the vector space of all quasi-primary vectors $w^{(i)}\in W_i$ satisfying the condition that for any $j,k\in\mathcal{F}^{\boxtimes}$ and any $\mathcal{Y}_\alpha\in\mathcal{V}{k\choose i\,j}$, $\mathcal{Y}_\alpha(w^{(i)},x)$ satisfies linear energy bounds. $E^1(V)$ is defined in a similar way to be the set of all quasi-primary vectors $v\in V$, such that for any $k\in\mathcal{F}^{\boxtimes}$, $Y_k(v,x)$ satisfies linear energy bounds.

In this section, we always assume, unless otherwise stated, that $\mathcal{F}$ satisfies one of the following two conditions.

Condition A.
(a) Every irreducible submodule of a tensor product of V-modules in $\mathcal{F}\cup\overline{\mathcal{F}}$ is unitarizable.
(b) V is generated by $E^1(V)$.
(c) If $i\in\mathcal{F}$, $j,k\in\mathcal{F}^{\boxtimes}$, and $\mathcal{Y}_\alpha\in\mathcal{V}{k\choose i\,j}$, then $\mathcal{Y}_\alpha$ is energy-bounded.

Condition B.
(a) Every irreducible submodule of a tensor product of V-modules in $\mathcal{F}\cup\overline{\mathcal{F}}$ is unitarizable and energy-bounded.
(b) For any $i\in\mathcal{F}$, $E^1(W_i)$ contains at least one non-zero vector.

Note that if V is unitary and $\mathcal{F}$ satisfies condition A-(b), then by corollary 3.7 and theorem 4.1, V is energy bounded and strongly local. By corollary 3.7, Conditions A-(a),(b) $\Rightarrow$ condition B-(a), and condition B-(b) $\Rightarrow$ A-(c).

Remark 5.8. If $\mathcal{F}$ satisfies condition B, then by theorem 4.8, any unitary V-module W_i in $\mathcal{F}^{\boxtimes}$ is strongly integrable. Now we suppose that $\mathcal{F}$ satisfies condition A. Then, using the same argument as in the proof of theorem 4.8, one can show that any W_i in $\mathcal{F}^{\boxtimes}$ is **almost strongly integrable**, which means the following: Define a real vector subspace $E^1(V)_{\mathbb{R}} = \{v + \theta v, i(v - \theta v) : v \in E^1(V)\}$ of $E^1(V)$. Then there exists a representation π_i of the conformal net $\mathcal{M}_V$ on the $\mathcal{H}_i$, such that for any $I \in \mathcal{J}, v \in E^1(V)_{\mathbb{R}}$, and $f \in C_c^\infty(I)$ satisfying that

$$e^{i\pi\Delta_v/2}e_{1-\Delta_v}f = \overline{e^{i\pi\Delta_v/2}e_{1-\Delta_v}f}, \tag{5.40}$$

we have

$$\pi_{i,I}\big(\overline{Y(v,f)}\big) = \overline{Y_i(v,f)}. \tag{5.41}$$

Note that by theorem 4.1, the von Neumann algebra $\mathcal{M}_V(I)$ is generated by these $\overline{Y(v,f)}$'s. Therefore, such representation π_i, if exists, must be unique. In this way, we have a functor $\mathfrak{F} : \mathrm{Rep}^{\mathrm{u}}_{\mathcal{F}^{\boxtimes}}(V) \to \mathrm{Rep}_{\mathcal{F}^{\boxtimes}}(\mathcal{M}_V)$ sending the object (W_i, Y_i) to $(\mathcal{H}_i, \pi)$. By proposition 3.6, the conformal vector ν is inside $E^1(V)_{\mathbb{R}}$. Therefore, from their proof we see that theorem 4.3 and corollary 4.4 still hold, with $\mathcal{S}$ replaced by $\mathcal{F}^{\boxtimes}$.

We define $\mathcal{M}_V(I)_\infty$ to be the set of all $x \in \mathcal{M}_V(I)$ satisfying relation (4.5) for any $i \in \mathcal{F}^{\boxtimes}$. We can conclude that $\mathcal{M}_V(I)_\infty$ is a strongly dense self-adjoint subalgebra of $\mathcal{M}_V(I)$, either by using the same argument as in the proof of proposition 4.2, or by observing that every $e^{it\overline{Y(v,f)}}$ is inside $\mathcal{M}_V(I)_\infty$ (by Lemma B.8-(1)), where $t \in \mathbb{R}, v \in E^1(V)_{\mathbb{R}}$, and $f \in C_c^\infty(I)$ satisfies equation (5.40).

We now define generalized smeared intertwining operators. First, for any $I \in \mathcal{J}, n = 1, 2, \ldots$, we choose an arbitrary continuous argument function $\arg_I$ on I, and define $\mathfrak{O}_n(I)$ to be the set of all $(I_1, \ldots, I_n)$, where $I_1, \ldots, I_n \in \mathcal{J}(I)$ are mutually disjoint, and for any $2 \leqslant l < m \leqslant n$, either $\arg_I(z_l z_1^{-1}) \arg_I(z_m z_1^{-1}) < 0$ for all $z_m \in I_m, z_l \in I_l$, or $|\arg_I(z_l z_1^{-1})| < |\arg_I(z_m z_1^{-1})|$ for all $z_m \in I_m, z_l \in I_l$.

Let $\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}$ be a generalized intertwining operator in $\mathcal{V}\binom{k}{i_n\ \cdots\ i_1\ j}$. We say that $\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}$ is **controlled by** $\mathcal{F}$ if $i_1, \ldots, i_n \in \mathcal{F} \cup \overline{\mathcal{F}}$, and $j, k \in \mathcal{F}^{\boxtimes}$. Choose $I \in \mathcal{J}(S^1 \backslash \{-1\})$, $(I_1, \ldots, I_n) \in \mathfrak{O}_n(I)$ and $f_1 \in C_c^\infty(I_1), \ldots, f_n \in C_c^\infty(I_n)$. For any $w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_n)} \in W_{i_n}$, we define a sesquilinear form

$$\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1) : W_j \times W_k \to \mathbb{C},$$
$$(w^{(j)}, w^{(k)}) \mapsto \langle \mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1) w^{(j)} | w^{(k)} \rangle$$

using the equation

$$\langle \mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1) w^{(j)} | w^{(k)} \rangle$$

$$= \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \langle \mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, e^{i\theta_n}; \ldots; w^{(i_1)}, e^{i\theta_1}) w^{(j)} | w^{(k)} \rangle \cdot f_1(e^{i\theta_1}) \cdots f_n(e^{i\theta_n}) d\!\!\!{}^-\theta_1 \cdots d\!\!\!{}^-\theta_n, \tag{5.42}$$

where, for each $l = 2, 3, \ldots, n$, $\arg(e^{i\theta_l} - e^{i\theta_1})$ is close to $\theta_l = \arg e^{i\theta_l}$ as $e^{i\theta_1} \to 0$.

Proposition 5.9. *Assume that $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}$ is controlled by $\mathcal{F}$. Then the linear operator $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1) : W_j \to \widehat{W}_k$ maps W_j into $\mathcal{H}_k^\infty$. If we regard it as an unbounded operator $\mathcal{H}_j \to \mathcal{H}_k$ with domain W_j, then it is preclosed. The closure $\overline{\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)}$ maps $\mathcal{H}_j^\infty$ into $\mathcal{H}_k^\infty$, and its adjoint maps $\mathcal{H}_k^\infty$ into $\mathcal{H}_j^\infty$. Moreover, there exists $p \in \mathbb{Z}_{\geqslant 0}$, such that for any $l \in \mathbb{Z}_{\geqslant 0}$, we can find $C_{l+p} > 0$, such that the inequality*

$$\left\| \overline{\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)} \xi^{(j)} \right\|_l \leqslant C_{l+p} \| \xi^{(j)} \|_{l+p} \tag{5.43}$$

holds for any $\xi^{(j)} \in \mathcal{H}_j^\infty$.

Proof. Choose any $z_1 \in I_1, \ldots, z_n \in I_n$. Choose arguments $\arg z_1, \ldots, \arg z_n \in (-\pi, \pi)$. For each $l = 2, 3, \ldots, n$, we let $\arg(z_l - z_1)$ be close to $\arg z_l$ as $z_1 \to 0$. Suppose that for any $w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_n)} \in W_{i_n}$ we have the fusion relation

$$\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, z_n; \ldots; w^{(i_1)}, z_1) = \mathcal{Y}_{\alpha_n}(w^{(i_n)}, z_n) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1) \tag{5.44}$$

for a chain of intertwining operators $\mathcal{Y}_{\alpha_1}, \ldots, \mathcal{Y}_{\alpha_n}$. Then the source spaces and the charge spaces of these intertwining operators are unitary V-modules in $\mathcal{F}^{\boxtimes}$. By condition A-(c) and proposition 3.3, these intertwining operators are energy-bounded. It follows from proposition 3.12 that

$$\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1) = \mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n) \cdots \mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1) \tag{5.45}$$

when both sides act on W_j. Therefore, by equation (3.25), the adjoint of $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)$ has a dense domain containing $\mathcal{H}_k^\infty$, which proves that $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)$ is preclosed. By proposition 3.9, there exists $p \in \mathbb{Z}_{\geqslant 0}$, such that for any $l \in \mathbb{Z}_{\geqslant 0}$, there exists $C_{l+p} > 0$, such that inequality (5.43) holds for any $\xi^{(j)} \in W_j$. From this we know that $\mathcal{H}_j^\infty$ is inside the domain of $\overline{\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)}$, that this closed operator maps $\mathcal{H}_j^\infty$ into $\mathcal{H}_k^\infty$, and that inequality (5.43) holds for any $\xi^{(j)} \in \mathcal{H}_j^\infty$. Clearly we have

$$\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)^* \supset \mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1)^\dagger \cdots \mathcal{Y}_{\alpha_n}(w^{(i_n)}, f_n)^\dagger.$$

So $\overline{\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)}^*$ maps $\mathcal{H}_k^\infty$ into $\mathcal{H}_j^\infty$. □

We regard the linear operator $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1) : \mathcal{H}_j^\infty \to \mathcal{H}_k^\infty$ as the restriction of $\overline{\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)}$ to $\mathcal{H}_j^\infty$, and call it a **generalized smeared intertwining operator**. Then, if the fusion relation (5.44) holds, relation (5.45) holds when both sides act on $\mathcal{H}_j^\infty$. The **formal adjoint** $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)^\dagger : \mathcal{H}_k^\infty \to \mathcal{H}_j^\infty$ of $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)$ is defined to be the restriction of the closed operator $\mathcal{Y}_{\sigma_n \cdots \sigma_2, \alpha}(w^{(i_n)}, f_n; \ldots; w^{(i_1)}, f_1)^*$ to $\mathcal{H}_k^\infty$.

Proposition 5.10 (Strong intertwining property)**.** *Let $\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha} \in \mathcal{V}\binom{k}{i_n \ \cdots \ i_1 \ j}$ be controlled by $\mathcal{F}$, $w^{(i_1)} \in W_{i_1}, \dots, w^{(i_n)} \in W_{i_n}$, $I \in \mathcal{J}, J \in \mathcal{J}(S^1\backslash\{-1\})$ be disjoint, and $(J_1, \dots, J_n) \in \mathfrak{O}_n(J)$. If $\mathcal{F}$ satisfies condition A, then for any $x \in \mathcal{M}_V(I), w^{(i_1)} \in W_{i_1}, \dots, w^{(i_n)} \in W_{i_n}, g_1 \in C_c^\infty(J_1), \dots, g_n \in C_c^\infty(J_n)$, we have*

$$\pi_k(x) \cdot \overline{\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)}, g_n; \dots; w^{(i_1)}, g_1)} \subset \overline{\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)}, g_n; \dots; w^{(i_1)}, g_1)} \cdot \pi_j(x). \tag{5.46}$$

Relation (5.46) still holds if we assume that $\mathcal{F}$ satisfies condition B, and that $x \in \mathcal{M}_V(I), w^{(i_1)} \in E^1(W_{i_1}), \dots, w^{(i_n)} \in E^1(W_{i_n}), g_1 \in C_c^\infty(J_1), \dots, g_n \in C_c^\infty(J_n)$.

Proof. We assume that the fusion relation (5.44) holds when $z_1 \in J_1, \dots, z_n \in J_n$ and the arguments are chosen as in the proof of proposition 5.9.

First, suppose that $\mathcal{F}$ satisfies condition A. By theorem 4.1, the von Neumann algebra $\mathcal{M}_V(I)$ is generated by the bounded operators $e^{it\overline{Y(v,f)}}$, where $t \in \mathbb{R}, v \in E^1(V)_{\mathbb{R}}, f \in C_c^\infty(I)$, and $e^{i\pi\Delta_v/2}e_{1-\Delta_v}f = \overline{e^{i\pi\Delta_v/2}e_{1-\Delta_v}f}$. Now for $m = 1, 2, \dots, n$ we let $W_{j_{m-1}}$ and W_{j_m} be the source space and the target space of $\mathcal{Y}_{\alpha_m}$ respectively. Then by proposition 3.16 (and proposition B.1), for any $x \in \mathcal{M}_V(I), w^{(i_m)} \in W_{i_m}, g_m \in C_c^\infty(J_m)$, we have

$$\pi_{j_m}(x)\overline{\mathcal{Y}_{\alpha_m}(w^{(i_m)}, g_m)} \subset \overline{\mathcal{Y}_{\alpha_m}(w^{(i_m)}, g_m)}\pi_{j_{m-1}}(x). \tag{5.47}$$

Therefore, if $x \in \mathcal{M}_V(I)_\infty$, then equation

$$\pi_{j_m}(x)\mathcal{Y}_{\alpha_m}(w^{(i_m)}, g_m) = \mathcal{Y}_{\alpha_m}(w^{(i_m)}, g_m)\pi_{j_{m-1}}(x). \tag{5.48}$$

holds when both sides act on $\mathcal{H}_{j_{m-1}}^\infty$. Thus, by (5.45), for any $x \in \mathcal{M}_V(I)_\infty$, equation

$$\pi_k(x) \cdot \mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)}, g_n; \dots; w^{(i_1)}, g_1) = \mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)}, g_n; \dots; w^{(i_1)}, g_1) \cdot \pi_j(x) \tag{5.49}$$

also holds when both sides act on $\mathcal{H}_j^\infty$. This proves relation (5.46) for any $x \in \mathcal{M}_V(I)_\infty$, and hence for any $x \in \mathcal{M}_V(I)$.

Now we assume that $\mathcal{F}$ satisfies condition B. Then from step 2 of the proof of theorem 4.8, relation (5.47) holds for any $x \in \mathcal{M}_V(I)$. This again implies relation (5.46). Thus we are done with the proofs for both cases. □

Proposition 5.11 (Rotation covariance)**.** *Let $\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha} \in \mathcal{V}\binom{k}{i_n \ \cdots \ i_1 \ j}$ be controlled by $\mathcal{F}$, $w^{(i_1)} \in W_{i_1}, \dots, w^{(i_n)} \in W_{i_n}$ be homogeneous, $J \in S^1\backslash\{-1\}$, and $(J_1, \dots, J_n) \in \mathfrak{O}_n(J)$. Choose $\varepsilon > 0$ such that $\mathfrak{r}(t)J \subset S^1\backslash\{-1\}$. Then for any $g_1 \in C_c^\infty(J_1), \dots, g_n \in C_c^\infty(J_n)$, and $t \in (-\varepsilon, \varepsilon)$, we have*

$$\begin{aligned}&e^{it\overline{L_0}} \cdot \overline{\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}(w^{(i_n)}, g_n; \dots; w^{(i_1)}, g_1)} \cdot e^{-it\overline{L_0}}\\ =&\overline{\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}\big(w^{(i_n)}, e^{i(\Delta_{w^{(i_n)}}-1)t}\mathfrak{r}(t)g_n; \dots; w^{(i_1)}, e^{i(\Delta_{w^{(i_1)}}-1)t}\mathfrak{r}(t)g_1\big)}.\end{aligned} \tag{5.50}$$

Proof. This follows from relations (5.45) and (3.39). □

Theorem 5.12 (Braiding). *Let $I, J \in \mathcal{J}(S^1\backslash\{-1\})$ be disjoint. Choose $(I_1, \ldots, I_m) \in \mathfrak{O}_m(I), (J_1, \ldots, J_n) \in \mathfrak{O}_n(J)$. Choose $z \in I, \zeta \in J$, and let $-\pi < \arg z, \arg\zeta < \pi$. Let $\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha} \in \mathcal{V}\binom{k'}{i_m\ \ldots\ i_1\ k_1}, \mathcal{Y}_{\rho_n\cdots\rho_2,\beta} \in \mathcal{V}\binom{k_1}{j_n\ \ldots\ j_1\ k}, \mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha'} \in \mathcal{V}\binom{k_2}{i_m\ \ldots\ i_1\ k}, \mathcal{Y}_{\rho_n\cdots\rho_2,\beta'} \in \mathcal{V}\binom{k'}{j_n\ \ldots\ j_1\ k_2}$ be generalized intertwining operators of V controlled by $\mathcal{F}$. Suppose that W_i is the charge spaces of $\mathcal{Y}_\alpha$ and $\mathcal{Y}_{\alpha'}$, W_j is the charge space of $\mathcal{Y}_\beta$ and $\mathcal{Y}_{\beta'}$, and for any $w^{(i)} \in W_i, w^{(j)} \in W_j$, we have the braid relation*

$$\mathcal{Y}_\alpha(w^{(i)}, z)\mathcal{Y}_\beta(w^{(j)}, \zeta) = \mathcal{Y}_{\beta'}(w^{(j)}, \zeta)\mathcal{Y}_{\alpha'}(w^{(i)}, z). \tag{5.51}$$

Then for any $w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_m)} \in W_{i_m}, w^{(j_1)} \in W_{j_1}, \ldots, w^{(j_n)} \in W_{j_n}, f_1 \in C_c^\infty(I_1), \ldots f_m \in C_c^\infty(I_m), g_1 \in C_c^\infty(J_1), \ldots, g_n \in C_c^\infty(J_n)$, we have the braid relation

$$\begin{aligned}&\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha}(w^{(i_m)}, f_m; \ldots; w^{(i_1)}, f_1)\mathcal{Y}_{\rho_n\cdots\rho_2,\beta}(w^{(j_n)}, g_n; \ldots; w^{(j_1)}, g_1)\\ =&\mathcal{Y}_{\rho_n\cdots\rho_2,\beta'}(w^{(j_n)}, g_n; \ldots; w^{(j_1)}, g_1)\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha'}(w^{(i_m)}, f_m; \ldots; w^{(i_1)}, f_1).\end{aligned} \tag{5.52}$$

Proof. Choose $z_1 \in I_1, \ldots, z_m \in I_m, \zeta_1 \in J_1, \ldots, \zeta_n \in J_n$. Let $-\pi < \arg z_1, \ldots, \arg z_m, \arg\zeta_1, \ldots, \arg\zeta_n < \pi$, and let $\arg(z_2 - z_1), \ldots, \arg(z_m - z_1), \arg(\zeta_2 - \zeta_1), \ldots, \arg(\zeta_n - \zeta_1)$ be close to $\arg z_2, \ldots, \arg z_m, \arg\zeta_2, \ldots, \arg\zeta_n$ as $z_1, \ldots, z_1, \zeta_1, \ldots, \zeta_1 \to 0$ respectively. Suppose that for any $w^{(i_1)} \in W_{i_1}, \ldots, w^{(i_m)} \in W_{i_m}, w^{(j_1)} \in W_{j_1}, \ldots, w^{(j_n)} \in W_{j_n}$, we have the fusion relations

$$\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha}(w^{(i_m)}, z_m; \ldots; w^{(i_1)}, z_1) = \mathcal{Y}_{\alpha_m}(w^{(i_m)}, z_m)\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, z_1), \tag{5.53}$$

$$\mathcal{Y}_{\rho_n\cdots\rho_2,\beta}(w^{(j_n)}, \zeta_n; \ldots; w^{(j_1)}, \zeta_1) = \mathcal{Y}_{\beta_n}(w^{(j_n)}, \zeta_n)\cdots\mathcal{Y}_{\beta_1}(w^{(j_1)}, \zeta_1). \tag{5.54}$$

Then the source spaces and the target spaces of $\mathcal{Y}_{\alpha_1}, \ldots \mathcal{Y}_{\alpha_m}, \mathcal{Y}_{\beta_1}, \ldots, \mathcal{Y}_{\beta_n}$ are unitary V-modules inside $\mathcal{F}^{\boxtimes}$. So these intertwining operators are energy-bounded. By relation (5.45), we have

$$\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha}(w^{(i_m)}, f_m; \ldots; w^{(i_1)}, f_1) = \mathcal{Y}_{\alpha_m}(w^{(i_m)}, f_m)\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1), \tag{5.55}$$

$$\mathcal{Y}_{\rho_n\cdots\rho_2,\beta}(w^{(j_n)}, g_n; \ldots; w^{(j_1)}, g_1) = \mathcal{Y}_{\beta_n}(w^{(j_n)}, g_n)\cdots\mathcal{Y}_{\beta_1}(w^{(j_1)}, g_1). \tag{5.56}$$

Therefore, by proposition 3.12,

$$\begin{aligned}&\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha}(w^{(i_m)}, f_m; \ldots; w^{(i_1)}, f_1)\mathcal{Y}_{\rho_n\cdots\rho_2,\beta}(w^{(j_n)}, g_n; \ldots; w^{(j_1)}, g_1)\\ =&\mathcal{Y}_{\alpha_m}(w^{(i_m)}, f_m)\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, f_1)\mathcal{Y}_{\beta_n}(w^{(j_n)}, g_n)\cdots\mathcal{Y}_{\beta_1}(w^{(j_1)}, g_1)\\ =&\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\cdot\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\mathcal{Y}_{\alpha_m}(w^{(i_m)}, e^{i\theta_m})\cdots\mathcal{Y}_{\alpha_1}(w^{(i_1)}, e^{i\theta_1})\\ &\cdot\mathcal{Y}_{\beta_n}(w^{(j_n)}, e^{i\vartheta_n})\cdots\mathcal{Y}_{\beta_1}(w^{(j_1)}, e^{i\vartheta_1})f_1(e^{i\theta_1})\cdots f_m(e^{i\theta_m})\\ &\cdot g_1(e^{i\vartheta_1})\cdots g_n(e^{i\vartheta_n})đ\theta_1\cdots đ\theta_m đ\vartheta_1\cdots đ\vartheta_n\\ =&\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\cdot\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha}(w^{(i_m)}, e^{i\theta_m}; \cdots; w^{(i_1)}, e^{i\theta_1})\\ &\cdot\mathcal{Y}_{\rho_n\cdots\rho_2,\beta}(w^{(j_n)}, e^{i\vartheta_n}; \cdots; w^{(j_1)}, e^{i\vartheta_1})f_1(e^{i\theta_1})\cdots f_m(e^{i\theta_m})\end{aligned}$$

$$\cdot g_1(e^{i\vartheta_1})\cdots g_n(e^{i\vartheta_n})d\!\!\!{}^-\theta_1\cdots d\!\!\!{}^-\theta_m d\!\!\!{}^-\vartheta_1\cdots d\!\!\!{}^-\vartheta_n. \tag{5.57}$$

The same argument shows that

$$\begin{aligned}
&\mathcal{Y}_{\rho_n\cdots\rho_2,\beta'}(w^{(j_n)},g_n;\dots;w^{(j_1)},g_1)\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha'}(w^{(i_m)},f_m;\dots;w^{(i_1)},f_1)\\
=&\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\cdot\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\mathcal{Y}_{\rho_n\cdots\rho_2,\beta'}(w^{(j_n)},e^{i\vartheta_n};\cdots;w^{(j_1)},e^{i\vartheta_1})\\
&\cdot\mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha'}(w^{(i_m)},e^{i\theta_m};\cdots;w^{(i_1)},e^{i\theta_1})f_1(e^{i\theta_1})\cdots f_m(e^{i\theta_m})\\
&\cdot g_1(e^{i\vartheta_1})\cdots g_n(e^{i\vartheta_n})d\!\!\!{}^-\theta_1\cdots d\!\!\!{}^-\theta_m d\!\!\!{}^-\vartheta_1\cdots d\!\!\!{}^-\vartheta_n.
\end{aligned}\tag{5.58}$$

By theorem 5.2, the right hand sides of equations (5.57) and (5.58) are equal, which proves equation (5.52). □

Theorem 5.13 (Adjoint relation). *Choose $I\in\mathcal{J}(S^1\backslash\{-1\})$ and $(I_1,\dots,I_n)\in\mathfrak{O}_n(I)$. Let $W_{i_1},W_{i_2},\dots,W_{i_n}$ be unitary V-modules in $\mathcal{F}\cup\overline{\mathcal{F}}$, and let $\mathcal{Y}_{\sigma_2},\dots,\mathcal{Y}_{\sigma_n}$ be a chain of unitary intertwining operators of V with charge spaces $W_{i_2},\dots,W_{i_n}$ respectively, such that the source space of $\mathcal{Y}_{\sigma_2}$ is W_{i_1}. We let $W_i\in\mathcal{F}^{\boxtimes}$ be the target space of $\mathcal{Y}_{\sigma_n}$. Then for each $2\leqslant m\leqslant n$, there exists a unitary intertwining operator $\mathcal{Y}_{\widetilde{\sigma}_m}$ whose type is the same as that of $\mathcal{Y}_{\overline{\sigma_m}}$, such that for any unitary V-modules W_j,W_k in $\mathcal{F}^{\boxtimes}$, $\mathcal{Y}_\alpha\in\mathcal{V}\binom{k}{i\ j}$, $w^{(i_1)}\in W_{i_1},\dots,w^{(i_n)}\in W_{i_n}$ being quasi-primary, and $f_1\in C_c^\infty(I_1),\dots,f_n\in C_c^\infty(I_n)$, we have*

$$\begin{aligned}
&\mathcal{Y}_{\sigma_n\cdots\sigma_2,\alpha}\big(w^{(i_n)},f_n;\dots;w^{(i_1)},f_1\big)^\dagger\\
=&e^{-i\pi\left(\Delta_{w^{(i_1)}}+\cdots+\Delta_{w^{(i_n)}}\right)}\cdot\mathcal{Y}_{\widetilde{\sigma}_n\cdots\widetilde{\sigma}_2,\alpha^*}\Big(\overline{w^{(i_n)}},\overline{e_{(2-2\Delta_{w^{(i_n)}})}f_n};\dots;\overline{w^{(i_1)}},\overline{e_{(2-2\Delta_{w^{(i_1)}})}f_1}\Big).
\end{aligned}\tag{5.59}$$

Proof. This is obtained by multiplying both sides of equation (5.34) by the expression

$$\overline{f_1(e^{i\theta_1})}\cdots\overline{f_n(e^{i\theta_n})}e^{-2i(\theta_1+\cdots+\theta_n)}d\!\!\!{}^-\theta_1\cdots d\!\!\!{}^-\theta_n,$$

and then taking the integral. We leave the details to the reader. □

6 Defining an inner product Λ on $W_i\boxtimes W_j$

In this chapter, we define (in section 2) a sesquilinear form Λ on $W_{ij}=W_i\boxtimes W_j$ using transport matrices, and prove (in section 3) that these forms are inner products. As discussed in the introduction of part I, our strategy for proving the positivity of Λ is to identify the form Λ on a dense subspace of $\mathcal{H}_{ij}$ with the inner product on a subspace of the Connes fusion product $\mathcal{H}_i\boxtimes\mathcal{H}_j$ of the conformal net modules $\mathcal{H}_i$ and $\mathcal{H}_j$. In section 1, we prove a density property for constructing such a dense subspace.

Note that the Connes fusion product (Connes relative tensor product) is a motivation rather than a logistic background of our theory. So we don't assume the reader has any previous knowledge on this topic, nor shall we give a formal definition on Connes fusion in this paper. Those who are interested in this topic can read [Was98] section 30 for a brief introduction, or read [Con80] or [Tak13] section IX.3 for more details.

6.1 Density of the range of fusion product

Recall from section 3.2 that $W_{ij} = W_i \boxtimes W_j = \bigoplus_{k\in\mathcal{E}} \mathcal{V}{k \choose i\ j}^* \otimes W_k$ is the tensor product module of W_i, W_j. We now define a type ${ij \choose i\ j}$ intertwining operator $\mathcal{Y}_{i\boxtimes j} : W_i \otimes W_j \to W_{ij}\{x\}$ in the following way: If $\mathcal{Y}_\alpha \in \mathcal{V}{k \choose i\ j}$, $w^{(i)} \in W_i$, $w^{(j)} \in W_j$ and $w^{(\overline{k})} \in W_{\overline{k}}$, then

$$\langle \mathcal{Y}_\alpha \otimes w^{(\overline{k})}, \mathcal{Y}_{i\boxtimes j}(w^{(i)}, x)w^{(j)}\rangle = \langle w^{(\overline{k})}, \mathcal{Y}_\alpha(w^{(i)}, x)w^{(j)}\rangle. \tag{6.1}$$

For any $k \in \mathcal{E}$, we choose a basis $\{\mathcal{Y}_\alpha : \alpha \in \Theta^k_{ij}\}$ of $\mathcal{V}{k \choose i\ j}$, and let $\{\check{\mathcal{Y}}^\alpha : \alpha \in \Theta^k_{ij}\} \subset \mathcal{V}{k \choose i\ j}^*$ be the dual basis of Θ^k_{ij}. (i.e., if $\alpha, \beta \in \Theta^k_{ij}$, then $\langle \mathcal{Y}_\alpha, \check{\mathcal{Y}}^\beta\rangle = \delta_{\alpha,\beta}$.) Then for any $w^{(i)} \in W_i$ and $w^{(j)} \in W_j$ we have

$$\mathcal{Y}_{i\boxtimes j}(w^{(i)}, x)w^{(j)} = \sum_{k\in\mathcal{E}} \sum_{\alpha\in\Theta^k_{ij}} \check{\mathcal{Y}}^\alpha \otimes \mathcal{Y}_\alpha(w^{(i)}, x)w^{(j)} = \sum_{\alpha\in\Theta^*_{ij}} \check{\mathcal{Y}}^\alpha \otimes \mathcal{Y}_\alpha(w^{(i)}, x)w^{(j)}. \tag{6.2}$$

(See the beginning of section 2 for notations.)

The following density property generalizes proposition A.3.

Proposition 6.1. *Let $\mathcal{Y}_{\sigma_2}, \dots, \mathcal{Y}_{\sigma_n}$ be a chain of non-zero irreducible intertwining operators of V with charge spaces $W_{i_2}, \dots, W_{i_n}$ respectively. Let W_{i_1} be the source space of $\mathcal{Y}_{\sigma_2}$, and let W_i be the target space of $\mathcal{Y}_{\sigma_n}$. Choose a V-module W_j, non-zero vectors $w^{(i_1)}_0 \in W_{i_1}, \dots, w^{(i_n)}_0 \in W_{i_n}$, $I \in \mathcal{J}, (z_1, \dots, z_n) \in \mathcal{O}_n(I)$, and choose arguments $\arg z_1, \arg(z_2 - z_1), \dots, \arg(z_n - z_1)$. Fix $w^{(\overline{ij})} \in W_{\overline{ij}}$. Suppose that for any $w^{(j)} \in W_j$,*

$$\langle w^{(\overline{ij})}, \mathcal{Y}_{\sigma_n\cdots\sigma_2, i\boxtimes j}(w^{(i_n)}_0, z_n; \dots; w^{(i_1)}_0, z_1)w^{(j)}\rangle = 0, \tag{6.3}$$

then $w^{(\overline{ij})} = 0$.

Proof. Suppose that equation (6.3) holds. From the proof of corollary 2.15, we see that

$$\langle w^{(\overline{ij})}, \mathcal{Y}_{\sigma_n\cdots\sigma_2, i\boxtimes j}(w^{(i_n)}, z_n; \dots; w^{(i_1)}, z_1)w^{(j)}\rangle = 0 \tag{6.4}$$

for all $w^{(i_1)} \in W_{i_1}, \dots, w^{(i_n)} \in W_{i_n}, w^{(i)} \in W_i$. By theorem 2.4 and the discussion below, equation (6.4) holds for all $(z_1, \dots, z_n) \in \mathcal{O}_n(I)$ (the arguments $\arg z_1, \arg(z_2 - z_1), \dots, \arg(z_n - z_1)$ are changed continuously). In particular, for any $(z_1, \dots, z_n) \in \mathcal{O}_n(I)$ satisfying $0 < |z_2 - z_1| < |z_3 - z_1| < \cdots < |z_n - z_1| < |z_1|$, equation (6.4) reads

$$\Big\langle w^{(\overline{ij})}, \mathcal{Y}_{i\boxtimes j}\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1)\cdots\mathcal{Y}_{\sigma_2}(w^{(i_2)}, z_2 - z_1)w^{(i_1)}, z_1\big)w^{(j)}\Big\rangle = 0. \tag{6.5}$$

If we let z_2 be close to z_1, then by proposition A.1, for any $s_2 \in \mathbb{R}$, we have

$$\Big\langle w^{(\overline{ij})}, \mathcal{Y}_{i\boxtimes j}\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1)\cdots\mathcal{Y}_{\sigma_3}(w^{(i_3)}, z_3 - z_1)\mathcal{Y}_{\sigma_2}(w^{(i_2)}, s_2)w^{(i_1)}, z_1\big)w^{(j)}\Big\rangle = 0, \tag{6.6}$$

where $\mathcal{Y}_{\sigma_2}(w^{(i_2)}, s_2)$ is a mode of the intertwining operator $\mathcal{Y}_{\sigma_2}(w^{(i_2)}, x)$. Let W_{j_2} be the target space of $\mathcal{Y}_{\sigma_2}$ (which is also the source space of $\mathcal{Y}_{\sigma_3}$). Then by corollary A.4, vectors of the form $\mathcal{Y}_{\sigma_2}(w^{(i_2)}, s_2)w^{(i_1)}$ span the vector space W_{j_2}. Therefore, for any $w^{(j_2)} \in W_{j_2}$, we have

$$\left\langle w^{(\overline{ij})}, \mathcal{Y}_{i\boxtimes j}\big(\mathcal{Y}_{\sigma_n}(w^{(i_n)}, z_n - z_1) \cdots \mathcal{Y}_{\sigma_3}(w^{(i_3)}, z_3 - z_1)w^{(j_2)}, z_1\big)w^{(j)}\right\rangle = 0. \tag{6.7}$$

If we apply the same argument several times, then for any $w^{(i)} \in W_i, w^{(j)} \in W_j$,

$$\langle w^{(\overline{ij})}, \mathcal{Y}_{i\boxtimes j}(w^{(i)}, z_1)w^{(j)}\rangle = 0. \tag{6.8}$$

So by proposition A.3, $w^{(\overline{ij})}$ must be zero. □

A smeared version of the above proposition can be stated as follows.

Proposition 6.2. *Let V be unitary, energy-bounded, and strongly local. Let $\mathcal{F}$ be a non-empty set of non-zero irreducible unitary V-modules satisfying condition A or B in section 5.3. Let W_i, W_j be unitary V-modules in $\mathcal{F}^{\boxtimes}$, and assume that W_i is irreducible. Fix an arbitrary unitary structure on W_{ij}.*

Let $W_{i_1}, \dots, W_{i_n}$ be irreducible unitary V-modules in $\mathcal{F} \cup \overline{\mathcal{F}}$. Let $\mathcal{Y}_{\sigma_2}, \dots, \mathcal{Y}_{\sigma_n}$ be a chain of non-zero irreducible unitary intertwining operators of V with charge spaces $W_{i_2}, \dots, W_{i_n}$ respectively, such that W_{i_1} is the source space of $\mathcal{Y}_{\sigma_2}$, and W_i is the target space of $\mathcal{Y}_{\sigma_n}$. Choose $I \in \mathcal{J}(S^1\backslash\{-1\}), (I_1, \dots, I_n) \in \mathfrak{O}_n(I)$. Fix non-zero homogeneous vectors $w_0^{(i_1)} \in W_{i_1}, \dots, w_0^{(i_n)} \in W_{i_n}$. Then for any $l \in \mathbb{Z}_{\geqslant 0}$, vectors of the form

$$\pi_{ij}(x)\mathcal{Y}_{\sigma_n\cdots\sigma_2,i\boxtimes j}(w_0^{(i_n)}, f_n; \dots; w_0^{(i_1)}, f_1)w^{(j)}, \tag{6.9}$$

span a core for $\overline{L_0}^l$, where $x \in \mathcal{M}_V(I)_\infty, f_1 \in C_c^\infty(I_1), \dots, f_n \in C_c^\infty(I_n), w^{(j)} \in W_j$.

Proof. Let $\mathcal{W}_1$ be the subspace of $\mathcal{H}_{ij}^\infty$ spanned by vectors of the form (6.9). We first show that $\mathcal{W}_1$ is a dense subspace of $\mathcal{H}_{ij}$.

The first step is to show that $\mathcal{W}_1^\perp$ is invariant under the action of the conformal net $\mathcal{M}_V$. Choose an open interval $J \subset\subset I$, and choose $\delta > 0$ such that $\mathfrak{r}(t)J \subset I$ for any $t \in (-\delta, \delta)$. Fix $\xi^{(ij)} \in \mathcal{W}_1^\perp$. Then for any $w^{(j)} \in W_j, m \in \mathbb{Z}_{>0}, x_1, \dots, x_m \in \mathcal{M}_V(J)_\infty, f_1 \in C_c^\infty(I_1), \dots, f_n \in C_c^\infty(I_n)$, we have

$$\langle x_m \cdots x_2x_1\mathcal{Y}_{\sigma_n\cdots\sigma_2,i\boxtimes j}(w_0^{(i_n)}, f_n; \dots; w_0^{(i_1)}, f_1)w^{(j)}|\xi^{(ij)}\rangle = 0. \tag{6.10}$$

Choose $\varepsilon > 0$ such that the support of

$$f_a^t = \exp(it(\Delta_{w_0^{(i_a)}} - 1))\mathfrak{r}(t)f_a$$

is inside I_a for any $t \in (-\varepsilon, \varepsilon)$ and any $a = 1, 2, \dots, n$. Then, by proposition 5.11, for any $t \in \mathbb{R}$ we have

$$\begin{aligned}&\langle x_m \cdots x_1 \cdot e^{it\overline{L_0}}\mathcal{Y}_{\sigma_n\cdots\sigma_2,i\boxtimes j}(w_0^{(i_n)}, f_n; \dots; w_0^{(i_1)}, f_1)w^{(j)}|\xi^{(ij)}\rangle\\ =&\langle x_m \cdots x_1\mathcal{Y}_{\sigma_n\cdots\sigma_2,i\boxtimes j}(w_0^{(i_n)}, f_n^t; \dots; w_0^{(i_1)}, f_1^t)e^{it\overline{L_0}}w^{(j)}|\xi^{(ij)}\rangle,\end{aligned} \tag{6.11}$$

which must be zero when $t \in (-\varepsilon, \varepsilon)$. Therefore, as in step 1 of the proof of theorem 4.8, the Schwarz reflection principle implies that (6.11) equals zero for any $t \in \mathbb{R}$. (Note that when we define generalized smeared intertwining operators, the arguments are restricted to $(-\pi, \pi)$. Here we allow the arguments to exceed $(-\pi, \pi)$ and change continuously according to the action of $\mathfrak{r}(t)$.) Hence we conclude that equation (6.10) holds for any $t \in \mathbb{R}, w^{(j)} \in W_j, x_1, \ldots, x_m \in \mathcal{M}_V(J)_\infty, f_1 \in C_c^\infty(\mathfrak{r}(t)I_1), \ldots, f_n \in C_c^\infty(\mathfrak{r}(t)I_n)$.

We use similar argument once more. Choose any $w^{(j)} \in W_j, t_0, t \in \mathbb{R}, x_1, \ldots, x_m \in \mathcal{M}_V(J)_\infty, f_1 \in C_c^\infty(\mathfrak{r}(t_0)I_1), \ldots, f_n \in C_c^\infty(\mathfrak{r}(t_0)I_n)$. Then by proposition 5.11 and equation (4.6), we have

$$\begin{aligned}
&\langle x_m \cdots x_2 \cdot e^{it\overline{L_0}} \pi_{ij}(x_1) \mathcal{Y}_{\sigma_n \cdots \sigma_2, i\boxtimes j}(w_0^{(i_n)}, f_n; \ldots; w_0^{(i_1)}, f_1) w^{(j)} | \xi^{(ij)} \rangle \\
=&\langle x_m \cdots x_2 \cdot e^{it\overline{L_0}} \pi_{ij}(x_1) e^{-it\overline{L_0}} \mathcal{Y}_{\sigma_n \cdots \sigma_2, i\boxtimes j}(w_0^{(i_n)}, f_n^t; \ldots; w_0^{(i_1)}, f_1^t) e^{it\overline{L_0}} w^{(j)} | \xi^{(ij)} \rangle \\
=&\langle x_m \cdots x_2 \cdot \pi_{ij}(e^{it\overline{L_0}} x_1 e^{-it\overline{L_0}}) \mathcal{Y}_{\sigma_n \cdots \sigma_2, i\boxtimes j}(w_0^{(i_n)}, f_n^t; \ldots; w_0^{(i_1)}, f_1^t) e^{it\overline{L_0}} w^{(j)} | \xi^{(ij)} \rangle. \qquad (6.12)
\end{aligned}$$

If $t \in (-\delta, \delta)$, then $e^{it\overline{L_0}} x_1 e^{-it\overline{L_0}} \in \mathcal{M}_V(\mathfrak{r}(t)J)_\infty \subset \mathcal{M}_V(I)_\infty$, and hence (6.12) must be zero. So the value of (6.12) equals zero when $t \in (-\delta, \delta)$. By Schwarz reflection principle, (6.12) equals zero for any $t \in \mathbb{R}$. Since the choice of t_0 is arbitrary, we conclude that equation (6.10) holds for any $t_0, t_1 \in \mathbb{R}, x_1 \in \mathcal{M}_V(\mathfrak{r}(t_1)J)_\infty, x_2 \in \mathcal{M}_V(J)_\infty, \ldots, x_m \in \mathcal{M}_V(J)_\infty, w^{(j)} \in W_j, f_1 \in C_c^\infty(\mathfrak{r}(t_0)I_1), \ldots, f_n \in C_c^\infty(\mathfrak{r}(t_0)I_n)$. The same argument shows that equation 6.10 holds for any $t_0, t_1, t_2, \ldots, t_m \in \mathbb{R}, w^{(j)} \in W_j, x_1 \in \mathcal{M}_V(\mathfrak{r}(t_1)J)_\infty, x_2 \in \mathcal{M}_V(\mathfrak{r}(t_2)J)_\infty, \ldots, x_m \in \mathcal{M}_V(\mathfrak{r}(t_m)J)_\infty, f_1 \in C_c^\infty(\mathfrak{r}(t_0)I_1), \ldots, f_n \in C_c^\infty(\mathfrak{r}(t_0)I_n)$. Hence, by proposition 4.2 and the additivity of $\mathcal{M}_V$, the equation

$$\langle x_m \cdots x_1 \mathcal{Y}_{\sigma_n \cdots \sigma_2, i\boxtimes j}(w_0^{(i_n)}, f_n; \ldots; w_0^{(i_1)}, f_1) w^{(j)} | \xi^{(ij)} \rangle = 0 \qquad (6.13)$$

holds for any $m \in \mathbb{Z}_{\geqslant 0}, J_1, \ldots, J_m \in \mathcal{J}, x_1 \in \mathcal{M}_V(J_1), \ldots, x_m \in \mathcal{M}_V(J_m), f_1 \in C_c^\infty(I_1), \ldots, f_n \in C_c^\infty(I_n), w^{(j)} \in W_j, \xi^{(ij)} \in \mathcal{W}_1^\perp$. This proves that $\mathcal{W}_1^\perp$ is $\mathcal{M}_V$-invariant.

Now suppose that $\mathcal{W}_1^\perp$ is non-trivial. By corollary 4.4 and remark 5.8, $\mathcal{W}_1^\perp$ is the closure of a non-trivial V-submodule of W_{ij}. Thus there exists a non-zero vector $w^{(ij)} \in W_{ij} \cap \mathcal{W}_1^\perp$. For any $f_1 \in C_c^\infty(I_1), \ldots, f_n \in C_c^\infty(I_n), w^{(j)} \in W_j$, we have

$$\langle \mathcal{Y}_{\sigma_n \cdots \sigma_2, i\boxtimes j}(w_0^{(i_n)}, f_n; \ldots; w_0^{(i_1)}, f_1) w^{(j)} | w^{(ij)} \rangle = 0. \qquad (6.14)$$

Fix $z_1 \in I_1, \ldots, z_n \in I_n$. For each $1 \leqslant m \leqslant n$ we let f_m converge to the δ-function at z_m. Then we have

$$\langle \mathcal{Y}_{\sigma_n \cdots \sigma_2, i\boxtimes j}(w_0^{(i_n)}, z_n; \ldots; w_0^{(i_1)}, z_1) w^{(j)} | w^{(ij)} \rangle = 0 \qquad (6.15)$$

for any $w^{(j)} \in W_j$. By proposition 6.1, $w^{(ij)}$ equals zero, which is impossible. So $\mathcal{W}_1$ must be dense.

Now we show that $\mathcal{W}_1$ is a core for $\overline{L_0}^l$. Choose an open interval $K \subset\subset I$, and $(K_1, \ldots, K_n) \in \mathfrak{O}_n(K)$, such that $K_1 \subset\subset I_1, \ldots, K_n \subset\subset I_n$. Let $\mathcal{W}_2$ be the subspace of $\mathcal{H}_{ij}^\infty$ spanned by vectors of the form

$$\pi_{ij}(x) \mathcal{Y}_{\sigma_n \cdots \sigma_2, i\boxtimes j}(w_0^{(i_n)}, f_n; \ldots; w_0^{(i_1)}, f_1) w^{(j)},$$

where $x \in \mathcal{M}_V(K)_\infty, f_1 \in C_c^\infty(K_1), \dots, f_n \in C_c^\infty(K_n), w^{(j)} \in W_j$. Then clearly $\mathcal{W}_2$ is also dense in $\mathcal{H}_{ij}$. Choose $\epsilon > 0$ such that for any $t \in (-\epsilon, \epsilon)$, $\mathfrak{r}(t)K \subset I, \mathfrak{r}(t)K_1 \subset I_1, \dots, \mathfrak{r}(t)K_n \subset I_n$. Then by proposition 5.11, $e^{it\overline{L_0}}\mathcal{W}_2 \subset \mathcal{W}_1$. Hence, by the next lemma, $\mathcal{W}_1$ is a core for $\overline{L_0}^l$. □

Lemma 6.3 (cf. [CKLW18] lemma 7.2.). *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and let $U(t) = e^{itA}$, $t \in \mathbb{R}$ be the corresponding strongly-continuous one-parameter group of unitary operators on $\mathcal{H}$. For any $k \in \mathbb{Z}_{\geqslant 0}$, let $\mathcal{H}^k$ denote the domain of A^k, and let $\mathcal{H}^\infty = \bigcap_{k \in \mathbb{Z}_{\geqslant 0}} \mathcal{H}^k$. Assume that there exists a real number $\epsilon > 0$ and two dense linear subspaces $\mathscr{D}_\epsilon$ and $\mathscr{D}$ of $\mathcal{H}^\infty$ such that $U(t)\mathscr{D}_\epsilon \subset \mathscr{D}$ for any $t \in (-\epsilon, \epsilon)$. Then, for every positive integer k, $\mathscr{D}$ is a core for A^k.*

6.2 The sesquilinear form Λ on $W_i \boxtimes W_j$

Beginning with this section, we assume that V is unitary, energy bounded, and strongly local, and that there exists a non-empty set $\mathcal{F}$ of non-zero irreducible unitary V-modules satisfying condition A or B in section 5.3.

Choose unitary V-modules W_i, W_j in $\mathcal{F}^\boxtimes$. We now define, for any $k \in \mathcal{E}$, a sesquilinear form $\Lambda = \Lambda(\cdot|\cdot)$ on $\mathcal{V}\binom{k}{i\ j}^*$ (antilinear on the second variable). Choose a basis $\{\mathcal{Y}_\alpha : \alpha \in \Theta_{ij}^k\}$ of $\mathcal{V}\binom{k}{i\ j}$. Choose $z_1, z_2 \in \mathbb{C}^\times$ satisfying $0 < |z_2 - z_1| < |z_1| < |z_2|$. Choose $\arg z_2$, let $\arg z_1$ be close to $\arg z_2$ as $z_2 - z_1 \to 0$, and let $\arg(z_2 - z_1)$ be close to $\arg z_2$ as $z_1 \to 0$. By fusion of intertwining operators, there exists a complex $N_{ij}^k \times N_{ij}^k$ matrix $\Lambda = \{\Lambda^{\alpha\beta}\}_{\alpha,\beta \in \Theta_{ij}^k}$, such that for any $w_1^{(i)}, w_2^{(i)} \in W_i$ we have the following **transport formula** (version 1):

$$\begin{aligned}
&Y_j\big(\mathcal{Y}_{\bar{i}i}^0(\overline{w_2^{(i)}}, z_2 - z_1)w_1^{(i)}, z_1\big)\\
=&\sum_{k \in \mathcal{E}} \sum_{\alpha,\beta \in \Theta_{ij}^k} \Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w_2^{(i)}}, z_2)\mathcal{Y}_\alpha(w_1^{(i)}, z_1)\\
=&\sum_{\alpha,\beta \in \Theta_{ij}^*} \Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w_2^{(i)}}, z_2)\mathcal{Y}_\alpha(w_1^{(i)}, z_1).
\end{aligned} \tag{6.16}$$

The matrix Λ is called a **transport matrix** of V. Let $\{\check{\mathcal{Y}}^\alpha : \alpha \in \Theta_{ij}^k\}$ be the dual basis of Θ_{ij}^k. We then define a sesquilinear form $\Lambda(\cdot|\cdot)$ on $\mathcal{V}\binom{k}{i\ j}^*$ by setting

$$\Lambda(\check{\mathcal{Y}}^\alpha|\check{\mathcal{Y}}^\beta) = \Lambda^{\alpha\beta}. \tag{6.17}$$

It is easy to see that this definition does not depend on the basis chosen. These sesquilinear forms induce one on the vector space $W_i \boxtimes W_j = \bigoplus_{k \in \mathcal{E}} \mathcal{V}\binom{k}{i\ j}^* \otimes W_k$: if $k_1, k_2 \in \mathcal{E} \cap \mathcal{F}^\boxtimes$, $\check{\mathcal{Y}}_1 \in \mathcal{V}\binom{k_1}{i\ j}^*$, $\check{\mathcal{Y}}_2 \in \mathcal{V}\binom{k_2}{i\ j}^*$, $w^{(k_1)} \in W_{k_1}$, $w^{(k_2)} \in W_{k_2}$, then

$$\Lambda\big(\check{\mathcal{Y}}_1 \otimes w^{(k_1)}|\check{\mathcal{Y}}_2 \otimes w^{(k_2)}\big) = \begin{cases} \Lambda(\check{\mathcal{Y}}_1|\check{\mathcal{Y}}_2)\langle w^{(k_1)}|w^{(k_2)}\rangle & \text{if } k_1 = k_2, \\ 0 & \text{if } k_1 \neq k_2. \end{cases} \tag{6.18}$$

In the next section, we will prove that Λ is an inner product.

Remark 6.4. Our definition of transport formulas is motivated by the landmark paper [Was98] of A.Wassermann. In that paper, Wassermann used smeared intertwining operators to define transport formulas when the fusion rules are at most 1 (see section 31). In those cases transport matrices become transport coefficients. Proving the strict positivity of these coefficients is one of the key steps to compute the *Connes* fusion rules for type A_n unitary WZW models in [Was98]. However, to prove similar results for other examples, one has to combine Wassermann's methods with Huang-Lepowsky's tensor product theory on VOA modules, as we now briefly explain.

The non-zeroness of the transport coefficients in [Was98] was proved by computing explicitly the monodromy coefficients of the solutions of differential equations (2.5) (in the case of WZW models, the Knizhnik-Zamolodchikov (KZ) equations). In the case of [Was98], these equations reduce to certain generalized hypergeometric equations, the manipulation of which is still possible. But for other examples, say type G_2 WZW models, these differential equations are more complicated and the precise values of transport coefficients are therefore much harder to calculate.

Fortunately, Y.Huang's remarkable works on the rigidity and the modularity of VOA tensor categories (cf. [Hua05b, Hua08a, Hua08b]) provide a solution to this issue. Indeed, the non-zeroness of transport coefficients (or the non-degeneracy of transport formulas) is closely related to the (weak) *rigidity* of the braided tensor category $\mathrm{Rep}(V)$ (see step 3 of the proof of theorem 6.7). To prove this rigidity, Huang made the following contributions:
(a) In [Hua08a] and [Hua08b], he showed that the rigidity of $\mathrm{Rep}(V)$, which is a genus-0 property of CFT, follows from the genus-1 property of modular invariance. This important observation complements the Verlinde formula, which says that the fusion rules (which are also genus-0 data) can be computed using S-matrices (genus-1 data).
(b) The result of Y.Zhu [Zhu96] on modular invariance of VOA characters (as a special case of the modular invariance of genus-1 conformal blocks) is insufficient to prove the rigidity property. In [Hua05b], Huang proved the most general form of modular invariance by deriving certain differential equations on the moduli spaces of punctured complex tori, hence proving the rigidity of $\mathrm{Rep}(V)$. We refer the reader to [HL13] for a detailed discussion of this issue.

Generalizing Wassermann's positivity result is also not easy. Indeed, Wassermann's key idea for proving the positivity of Λ is to construct enough bounded intertwiners of conformal nets. Smeared intertwining operators (or rather their phases) provide such examples of bounded intertwiners, but they are not enough even in the case of type A WZW models. Therefore, in [Was98] Wassermann also considered products of smeared intertwining operators, which are essentially the same as the generalized smeared intertwining operators considered in the section 5.3. Moreover, to prove the positivity of all Λ, one needs to calculate certain braid relations between generalized smeared intertwining operators and their adjoints. However, Wassermann's calculation in [Was98] (especially section 32) is model dependent and hence not easy to generalize. This is perhaps due to the fact that our understanding of the corresponding VOA structures was not mature enough by the time [Was98] was written. But now, with the help of Huang-Lepowsky's important works on vertex tensor categories [HL95a, HL95b, HL95c, Hua95, Hua05a], we are able to calculate the braid and adjoint relations for generalized (smeared) intertwining operators in a general setting, as we have already seen in the previous chapter.

The sesquilinear form Λ is also closely related to the non-degenerate bilinear form constructed in [HK07]. This will be explained in section 8.3.

For any $k \in \mathcal{E} \cap \mathcal{F}^{\boxtimes}$, since W_k is irreducible, we have $N_{0k}^k = N_{k0}^k = 1$. That the sesquilinear forms Λ on $\mathcal{V}\binom{k}{0\ k}^*$ and on $\mathcal{V}\binom{k}{k\ 0}^*$ are positive definite can be seen from the following two fusion relations:

$$Y_k\big(Y(u, z_2 - z_1)v, z_1\big) = Y_k(u, z_2)Y_k(v, z_1), \tag{6.19}$$

$$Y\big(\mathcal{Y}^0_{\bar{k}k}(\overline{w_2^{(k)}}, z_2 - z_1)w_1^{(k)}, z_1\big) = \mathcal{Y}^0_{\bar{k}k}(\overline{w_2^{(k)}}, z_2)\mathcal{Y}^k_{k0}(w_1^{(k)}, z_1), \tag{6.20}$$

where $u, v \in V$, and $w_1^{(k)}, w_2^{(k)} \in W_k$. The first equation follows from proposition 2.13, and the second one follows from proposition 2.17. (Note that these two fusion relations hold for any V-module W_k.) Moreover, the dual element of Y_k is an orthonormal basis of $\mathcal{V}\binom{k}{0\ k}^*$, and the dual element of $\mathcal{Y}^k_{k0}$ is an orthonormal basis of $\mathcal{V}\binom{k}{k\ 0}^*$.

We derive now some variants of transport formulas.

Proposition 6.5. *Let $I \in \mathcal{J}$. Choose distinct complex numbers $z_1, z_2 \in I$. Choose $z_0 \in I^c$ with argument $\arg z_0$. Define a continuous argument function $\arg_I$ on I, and let $\arg z_1 = \arg_I(z_1), \arg z_2 = \arg_I(z_2)$. Let W_i, W_j be unitary V-modules in $\mathcal{F}^{\boxtimes}$.*

(1) Let W_s, W_r be unitary V-modules in $\mathcal{F}^{\boxtimes}$, and choose $\mathcal{Y}_\gamma \in \mathcal{V}\binom{r}{j\ s}$. Then for any $w_1^{(i)}, w_2^{(i)} \in W_i, w^{(j)} \in W_j$, we have the braid relation

$$\begin{aligned}&\mathcal{Y}_\gamma(w^{(j)}, z_0)\Bigg(\sum_{\alpha,\beta\in\Theta^*_{is}} \Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w_2^{(i)}}, z_2)\mathcal{Y}_\alpha(w_1^{(i)}, z_1)\Bigg)\\ =&\Bigg(\sum_{\alpha,\beta\in\Theta^*_{ir}} \Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w_2^{(i)}}, z_2)\mathcal{Y}_\alpha(w_1^{(i)}, z_1)\Bigg)\mathcal{Y}_\gamma(w^{(j)}, z_0).\end{aligned} \tag{6.21}$$

(2) For any $w_1^{(i)}, w_2^{(i)} \in W_i$ and $w^{(j)} \in W_j$, we have the transport formula (version 2)

$$\begin{aligned}&\mathcal{Y}^j_{j0}(w^{(j)}, z_0)\mathcal{Y}^0_{\bar{i}i}(\overline{w_2^{(i)}}, z_2)\mathcal{Y}^i_{i0}(w_1^{(i)}, z_1)\\ =&\Bigg(\sum_{\alpha,\beta\in\Theta^*_{ij}} \Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w_2^{(i)}}, z_2)\mathcal{Y}_\alpha(w_1^{(i)}, z_1)\Bigg)\mathcal{Y}^j_{j0}(w^{(j)}, z_0).\end{aligned} \tag{6.22}$$

(3) If $\arg z_0 < \arg z_2 < \arg z_0 + 2\pi$, then for any $w_2^{(i)} \in W_i, w^{(j)} \in W_j$, we have the transport formula (version 3)

$$\mathcal{Y}^j_{j0}(w^{(j)}, z_0)\mathcal{Y}^0_{\bar{i}i}(\overline{w_2^{(i)}}, z_2) = \sum_{\alpha,\beta\in\Theta^*_{ij}} \Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w_2^{(i)}}, z_2)\mathcal{Y}_{B_+\alpha}(w^{(j)}, z_0). \tag{6.23}$$

If $\arg z_2 < \arg z_0 < \arg z_2 + 2\pi$, then equation (6.23) still holds, with $B_+\alpha$ replaced by $B_-\alpha$.

Proof. (1) By rotating z_1, z_2 along I and changing their arguments continuously, we can assume that $0 < |z_1 - z_2| < 1$. Then clearly $\arg z_1$ is close to $\arg z_2$ as $z_2 - z_1 \to 0$. We also let $\arg(z_2 - z_1)$ be close to $\arg z_2$ as $z_1 \to 0$. Then by equation (6.16), proposition 2.13, and theorem 5.2, we have

$$\begin{aligned}
&\mathcal{Y}_\gamma(w^{(j)}, z_0)\Big(\sum_{\alpha,\beta\in\Theta^*_{is}} \Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w_2^{(i)}}, z_2)\mathcal{Y}_\alpha(w_1^{(i)}, z_1)\Big)\\
=&\mathcal{Y}_\gamma(w^{(j)}, z_0)Y_s\big(\mathcal{Y}^0_{\bar{i}i}(\overline{w_2^{(i)}}, z_2 - z_1)w_1^{(i)}, z_1\big) \qquad (6.24)\\
=&Y_r\big(\mathcal{Y}^0_{\bar{i}i}(\overline{w_2^{(i)}}, z_2 - z_1)w_1^{(i)}, z_1\big)\mathcal{Y}_\gamma(w^{(j)}, z_0) \qquad (6.25)\\
=&\Big(\sum_{\alpha,\beta\in\Theta^*_{ir}} \Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w_2^{(i)}}, z_2)\mathcal{Y}_\alpha(w_1^{(i)}, z_1)\Big)\mathcal{Y}_\gamma(w^{(j)}, z_0),
\end{aligned}$$

where (6.24) and (6.25) are understood as products of two generalized intertwining operators (see the beginning of chapter 5). This proves equation (6.21).

(2) Equation (6.22) is a special case of equation (6.21).

(3) If $\arg z_0 < \arg z_2 < \arg z_0 + 2\pi$, we choose $z_1 \in S^1\backslash\{-1\}$ close to z_2 and let $\arg z_1$ be close to $\arg z_2$ as $z_1 \to z_2$. Then by equation (6.22), corollary 2.18, and proposition 2.11, we have

$$\begin{aligned}
&\mathcal{Y}^j_{j0}(w^{(j)}, z_0)\mathcal{Y}^0_{\bar{i}i}(\overline{w_2^{(i)}}, z_2)\mathcal{Y}^i_{i0}(w_1^{(i)}, z_1)\\
=&\sum_{\alpha,\beta\in\Theta^*_{ij}} \Lambda^{\alpha\beta}\mathcal{Y}_{\beta*}(\overline{w_2^{(i)}}, z_2)\mathcal{Y}_{B_+\alpha}(w^{(j)}, z_0)\mathcal{Y}^i_{i0}(w_1^{(i)}, z_1).
\end{aligned}$$

By proposition 2.3, we obtain equation (6.23). The other case is proved in a similar way. □

6.3 Positive definiteness of Λ

Let W_i, W_j be unitary V-modules in $\mathcal{F}^\boxtimes$, and let W_k be in $\mathcal{E} \cap \mathcal{F}^\boxtimes$ as before. We prove in this section that the sesquilinear form Λ on $\mathcal{V}\binom{k}{i\,j}^*$ is positive definite. One suffices to prove this when W_i, W_j are irreducible. Indeed, if W_i, W_j are not necessarily irreducible, and have orthogonal decompositions $W_i = W_{i_1} \oplus W_{i_2} \oplus \cdots \oplus W_{i_m}, W_j = W_{j_1} \oplus W_{j_2} \oplus \cdots \oplus W_{j_n}$, then clearly the unitary V-modules $W_{i_1}, \dots, W_{i_m}, W_{j_1}, \dots, W_{j_n}$ are in $\mathcal{F}^\boxtimes$. It is easy to see that the transport matrix for $\mathcal{V}\binom{k}{i\,j}^*$ can be diagonalized into the mn blocks of the transport matrices for $\mathcal{V}\binom{k}{i_a\,j_b}^*$ $(1 \leqslant a \leqslant m, 1 \leqslant b \leqslant n)$. Therefore, if we choose $W_{i_1}, \dots, W_{i_m}, W_{j_1}, \dots, W_{j_n}$ to be irreducible, and if we can prove that the transport matrix for every $\mathcal{V}\binom{k}{i_a\,j_b}^*$ is positive definite, then the one for $\mathcal{V}\binom{k}{i\,j}^*$ is also positive definite.

So let us assume that W_i, W_j are irreducible. We let $\mathcal{Y}_{\kappa(i)} = \mathcal{Y}^i_{i0}$ and $\mathcal{Y}_{\kappa(j)} = \mathcal{Y}^j_{j0}$. Then $\mathcal{Y}_{\kappa(i)*} = \mathcal{Y}^0_{\bar{i}i}, \mathcal{Y}_{\kappa(j)*} = \mathcal{Y}^0_{\bar{j}j}$. Since W_i (resp. W_j) is in $\mathcal{F}^\boxtimes$, there exits unitary V-modules $W_{i_1}, \dots, W_{i_m}$ (resp. $W_{j_1}, \dots, W_{j_n}$) in $\mathcal{F} \cup \overline{\mathcal{F}}$, such that W_i (resp. W_j) is equivalent to a submodule of $W_{i_m\cdots i_1} = W_{i_m} \boxtimes \cdots \boxtimes W_{i_1}$ (resp. $W_{j_n\cdots j_1}$). Therefore, we can choose a chain of non-zero irreducible unitary intertwining operators $\mathcal{Y}_{\sigma_2}, \dots, \mathcal{Y}_{\sigma_m}$ (resp. $\mathcal{Y}_{\rho_2}, \dots, \mathcal{Y}_{\rho_n}$) with charge spaces $W_{i_2}, \dots, W_{i_m}$ (resp. $W_{j_2}, \dots, W_{j_n}$) respectively, such that W_{i_1} (resp. W_{j_1}) is the source space of $\mathcal{Y}_{\sigma_2}$ (resp. $\mathcal{Y}_{\rho_2}$), and that W_i (resp. W_j) is the target space of $\mathcal{Y}_{\sigma_m}$ (resp. $\mathcal{Y}_{\rho_n}$).

Fix non-zero quasi-primary vectors $w^{(i_1)} \in W_{i_1}, \dots, w^{(i_m)} \in W_{i_m}, w^{(j_1)} \in W_{j_1}, \dots, w^{(j_n)} \in W_{j_n}$. If $\mathcal{F}$ satisfies condition B in section 5.3, we assume moreover that $w^{(i_1)} \in E^1(W_{i_1}), \dots, w^{(i_m)} \in E^1(W_{i_m}), w^{(j_1)} \in E^1(W_{j_1}), \dots, w^{(j_n)} \in E^1(W_{j_n})$. Choose disjoint open intervals $I, J \in \mathcal{J}(S^1\backslash\{-1\})$, and choose $(I_1, \dots, I_m) \in \mathfrak{O}_m(I), (J_1, \dots, J_n) \in \mathfrak{O}_n(J)$. We define two sets $\mathcal{A} = \mathcal{M}_V(I)_\infty \times C_c^\infty(I_1) \times \cdots \times C_c^\infty(I_m)$ and $\mathcal{B} = \mathcal{M}_V(J)_\infty \times C_c^\infty(J_1) \times \cdots \times C_c^\infty(J_n)$. For any $a = (x, f_1, \dots, f_m) \in \mathcal{A}$ and $b = (y, g_1, \dots, g_n) \in \mathcal{B}$, we define two linear operators $A(a) : \mathcal{H}_0^\infty \to \mathcal{H}_i^\infty$ and $B(b) : \mathcal{H}_0^\infty \to \mathcal{H}_j^\infty$ as follows: if $\xi^{(0)} \in \mathcal{H}_0^\infty$ then

$$A(a)\xi^{(0)} = \pi_i(x)\mathcal{Y}_{\sigma_m\cdots\sigma_2,\kappa(i)}(w^{(i_m)}, f_m; \dots, w^{(i_1)}, f_1)\xi^{(0)}, \tag{6.26}$$
$$B(b)\xi^{(0)} = \pi_j(y)\mathcal{Y}_{\rho_n\cdots\rho_2,\kappa(j)}(w^{(j_n)}, g_n; \dots, w^{(j_1)}, g_1)\xi^{(0)}. \tag{6.27}$$

By proposition 5.9, the formal adjoints of these two linear operators exist.

Lemma 6.6. *For any $N \in \mathbb{Z}_{>0}$, $a_1, \dots, a_N \in \mathcal{A}$, $b_1, \dots, b_N \in \mathcal{B}$ and $\xi_1^{(0)}, \dots, \xi_N^{(0)} \in \mathcal{H}_0^\infty$, we have*

$$\sum_{s,t=1,\dots,N} \langle B(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)} | B(b_t)\xi_t^{(0)}\rangle \geqslant 0. \tag{6.28}$$

Proof. Suppose that

$$\sum_{s,t=1,\dots,N} \langle B(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)} | B(b_t)\xi_t^{(0)}\rangle \notin [0, +\infty). \tag{6.29}$$

Then we can find $\varepsilon > 0$, such that for any $\tau \in [0, +\infty)$,

$$\Bigg| \sum_{s,t=1,\dots,N} \langle B(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)} | B(b_t)\xi_t^{(0)}\rangle - \tau \Bigg| \geqslant \varepsilon. \tag{6.30}$$

By proposition 5.10, for any $x \in \mathcal{M}_V(J^c)$ and $r = 1, \dots, N$, we have $\pi_j(x)\overline{B(b_r)} \subset \overline{B(b_r)}\pi_0(x)$. We also regard $\overline{B(b_r)}$ as an unbounded operator on $\mathcal{H}_0 \oplus \mathcal{H}_j$, being the original operator when restrict to $\mathcal{H}_0$, and the zero map when restricted to $\mathcal{H}_j$. We let x act on $\mathcal{H}_0 \bigoplus \mathcal{H}_j$ diagonally (i.e., $x = \mathrm{diag}(\pi_0(x), \pi_I(x))$). Then $x\overline{B(b_r)} \subset \overline{B(b_r)}x$. Since x^* also satisfies this relation, elements in $\mathcal{M}_V(J^c)$ commute strongly with $\overline{B(b_r)}$. Therefore, if we take the right polar decomposition $\overline{B(b_r)} = K_rV_r$ (where K_r is self-adjoint and V_r is a partial isometry), then $\mathcal{M}_V(J^c)$ commutes strongly with V_r and K_r. We let $K_r = \int_{-\infty}^{+\infty} \lambda dQ_r(\lambda)$ be the spectral decomposition of K_r. Then for each $\lambda \geqslant 0$, $Q_r(\lambda) = \int_{-\infty}^{\lambda} dQ_r(\mu)$ commutes with $\mathcal{M}_V(J^c)$. Therefore, the *bounded* operator $\overline{Q_r(\lambda)\overline{B(b_r)}}$ commutes with $\mathcal{M}_V(J^c)$, i.e., $\overline{Q_r(\lambda)\overline{B(b_r)}} \in \mathrm{Hom}_{\mathcal{M}_V(J^c)}(\mathcal{H}_0, \mathcal{H}_j)$.

Now we choose a real number $M > 0$, such that for any $s, t = 1, \ldots, N$,

$$\|B(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)}\| \leqslant M, \quad \|B(b_t)\xi_t^{(0)}\| \leqslant M. \tag{6.31}$$

For each $r = 1, \ldots, N$, since the projection $Q_r(\lambda)$ converges strongly to 1 as $\lambda \to +\infty$, there exists $\lambda_r > 0$, such that for any $t = 1, \ldots, N$,

$$\|B(b_r)A(a_t)^\dagger A(a_r)\xi_r^{(0)} - Q_r(\lambda_r)B(b_r)A(a_t)^\dagger A(a_r)\xi_r^{(0)}\| < \frac{\varepsilon}{4MN^2}, \tag{6.32}$$

$$\|B(b_r)\xi_r^{(0)} - Q_r(\lambda_r)B(b_r)\xi_r^{(0)}\| < \frac{\varepsilon}{4MN^2}. \tag{6.33}$$

We let $\mathfrak{B}(b_r) = \overline{Q_r(\lambda_r)\overline{B(b_r)}} \in \mathrm{Hom}_{\mathcal{M}_V(J^c)}(\mathcal{H}_0, \mathcal{H}_j)$, then the above inequalities imply that

$$\left|\sum_{s,t}\langle\mathfrak{B}(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)}|\mathfrak{B}(b_t)\xi_t^{(0)}\rangle - \sum_{s,t}\langle B(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)}|B(b_t)\xi_t^{(0)}\rangle\right| < \frac{\varepsilon}{2}. \tag{6.34}$$

Now, for any $1 \leqslant r \leqslant N$, since $\mathfrak{B}(b_r) \in \mathrm{Hom}_{\mathcal{M}_V(J^c)}(\mathcal{H}_0, \mathcal{H}_j)$, we also have $\mathfrak{B}(b_r)^* \in \mathrm{Hom}_{\mathcal{M}_V(J^c)}(\mathcal{H}_j, \mathcal{H}_0)$. Thus, for any $1 \leqslant s, t \leqslant N$, we have $\mathfrak{B}(b_s)^*\mathfrak{B}(b_t) \in \mathrm{End}_{\mathcal{M}_V(J^c)}(\mathcal{H}_0) = \mathcal{M}_V(J^c)'$. By Haag duality, $\mathfrak{B}(b_s)^*\mathfrak{B}(b_t) \in \mathcal{M}_V(J)$. By proposition 5.10, $\pi_i\big(\mathfrak{B}(b_s)^*\mathfrak{B}(b_t)\big)\overline{A(a_t)} \subset \overline{A(a_t)}\mathfrak{B}(b_s)^*\mathfrak{B}(b_t)$. In particular, $\mathfrak{B}(b_s)^*\mathfrak{B}(b_t)\mathscr{D}(\overline{A(a_t)}) \subset \mathscr{D}(\overline{A(a_t)})$. Since $\xi_t^{(0)} \in \mathcal{H}_0^\infty \subset \mathscr{D}(\overline{A(a_t)})$,

$$\mathfrak{B}(b_s)^*\mathfrak{B}(b_t)\xi_t^{(0)} \in \mathfrak{B}(b_s)^*\mathfrak{B}(b_t)\mathscr{D}(\overline{A(a_t)}) \subset \mathscr{D}(\overline{A(a_t)}). \tag{6.35}$$

Therefore,

$$\begin{aligned}
&\langle\mathfrak{B}(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)}|\mathfrak{B}(b_t)\xi_t^{(0)}\rangle\\
=&\langle A(a_t)^\dagger A(a_s)\xi_s^{(0)}|\mathfrak{B}(b_s)^*\mathfrak{B}(b_t)\xi_t^{(0)}\rangle\\
=&\langle\overline{A(a_t)}^* \cdot \overline{A(a_s)}\xi_s^{(0)}|\mathfrak{B}(b_s)^*\mathfrak{B}(b_t)\xi_t^{(0)}\rangle\\
=&\langle\overline{A(a_s)}\xi_s^{(0)}|\overline{A(a_t)}\mathfrak{B}(b_s)^*\mathfrak{B}(b_t)\xi_t^{(0)}\rangle.
\end{aligned} \tag{6.36}$$

Let $\overline{A(a_s)} = H_sU_s$ be the right polar decomposition of $\overline{A(a_s)}$, and take the spectral decomposition $H_s = \int_{-\infty}^{+\infty}\kappa dP_s(\kappa)$. Then for each s, we can find $\kappa_s > 0$ such that

$$\left|\sum_{s,t}\langle\overline{A(a_s)}\xi_s^{(0)}|\overline{A(a_t)}\mathfrak{B}(b_s)^*\mathfrak{B}(b_t)\xi_t^{(0)}\rangle - \sum_{s,t}\langle\mathfrak{A}(a_s)\xi_s^{(0)}|\mathfrak{A}(a_t)\mathfrak{B}(b_s)^*\mathfrak{B}(b_t)\xi_t^{(0)}\rangle\right| < \frac{\varepsilon}{2}, \tag{6.37}$$

where $\mathfrak{A}(a_s) = \overline{P_s(\kappa_s)\overline{A(a_s)}} \in \mathrm{Hom}_{\mathcal{M}(I^c)}(\mathcal{H}_0, \mathcal{H}_i)$. Note that $\mathfrak{A}(a_s)$ and $\mathfrak{B}(b_t)$ are bounded operators. Set

$$\tau = \sum_{s,t}\langle\mathfrak{A}(a_s)\xi_s^{(0)}|\mathfrak{A}(a_t)\mathfrak{B}(b_s)^*\mathfrak{B}(b_t)\xi_t^{(0)}\rangle = \sum_{s,t}\langle\mathfrak{B}(b_s)\mathfrak{A}(a_t)^*\mathfrak{A}(a_s)\xi_s^{(0)}|\mathfrak{B}(b_t)\xi_t^{(0)}\rangle. \tag{6.38}$$

Then by inequalities (6.34), (6.37), and equation (6.36),

$$\left|\sum_{s,t}\langle B(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)}|B(b_t)\xi_t^{(0)}\rangle-\tau\right|<\varepsilon. \tag{6.39}$$

We now show that $\tau\geqslant 0$, which will contradict condition (6.30) and thus prove inequality (6.28). Let $M(N,\mathbb{C})$ be the complex valued $N\times N$ matrix algebra. By evaluating between vectors in $\mathcal{H}_0^{\oplus N}$, we find that the $\mathcal{M}_V(I)$-valued matrix $[\mathfrak{A}(a_t)^*\mathfrak{A}(a_s)]_{N\times N}$ is a positive element in the von Neumann algebra $\mathcal{M}_V(I)\otimes M(N,\mathbb{C})$. So $[\pi_{j,I}(\mathfrak{A}(a_t)^*\mathfrak{A}(a_s))]_{N\times N}\in\pi_{j,I}(\mathcal{M}_V(I))\otimes M(N,\mathbb{C})$ is also positive. Therefore, if for each s we define a vector $\eta_s=\mathfrak{B}(b_s)\xi_s^{(0)}$, then

$$\sum_{s,t}(\pi_{j,I}(\mathfrak{A}(a_t)^*\mathfrak{A}(a_s))\eta_s|\eta_t)\geqslant 0. \tag{6.40}$$

Since $\mathfrak{B}(b_s)\in\mathrm{Hom}_{\mathcal{M}_V(J^c)}(\mathcal{H}_0,\mathcal{H}_j)\subset\mathrm{Hom}_{\mathcal{M}_V(I)}(\mathcal{H}_0,\mathcal{H}_j)$, we have

$$\mathfrak{B}(b_s)\mathfrak{A}(a_t)^*\mathfrak{A}(a_s)\xi_s^{(0)}=\pi_{j,I}\big(\mathfrak{A}(a_t)^*\mathfrak{A}(a_s)\big)\mathfrak{B}(b_s)\xi_s^{(0)}=\pi_{j,I}(\mathfrak{A}(a_t)^*\mathfrak{A}(a_s))\eta_s. \tag{6.41}$$

Hence

$$\tau=\sum_{s,t}\langle\mathfrak{B}(b_s)\mathfrak{A}(a_t)^*\mathfrak{A}(a_s)\xi_s^{(0)}|\mathfrak{B}(b_t)\xi_t^{(0)}\rangle=\sum_{s,t}(\pi_{j,I}(\mathfrak{A}(a_t)^*\mathfrak{A}(a_s))\eta_s|\eta_t)\geqslant 0. \tag{6.42}$$

□

Theorem 6.7. *Suppose that V is unitary, energy bounded, and strongly local, and $\mathcal{F}$ is a non-empty set of non-zero irreducible unitary V-modules satisfying condition A or B in section 5.3. Let W_i,W_j be unitary V-modules in $\mathcal{F}^\boxtimes$. Then the sesquilinear form Λ on $W_i\boxtimes W_j$ is an inner product. Equivalently, for any irreducible unitary V-module W_k in $\mathcal{E}\cap\mathcal{F}^\boxtimes$, the sesquilinear form Λ on $\mathcal{V}{k\choose i\,j}^*$ is positive definite.*

Proof. As argued at the beginning of this section, we can assume, without loss of generality, that W_i,W_j are irreducible.

Step 1. We first show that Λ is positive. For each $k\in\mathcal{E}\cap\mathcal{F}^\boxtimes$, we choose a basis $\{\mathcal{Y}_\alpha:\alpha\in\Theta_{ij}^k\}$ of $\mathcal{V}{k\choose i\,j}$, let $\{\check{\mathcal{Y}}^\alpha:\alpha\in\Theta_{ij}^k\}$ be its dual basis in $\mathcal{V}{k\choose i\,j}^*$, and define an inner product on $\mathcal{V}{k\choose i\,j}^*$ under which $\{\check{\mathcal{Y}}^\alpha:\alpha\in\Theta_{ij}^k\}$ becomes orthonormal. We extend these inner products to a unitary structure on $W_{ij}=\bigoplus_k\mathcal{V}{k\choose i\,j}^*\otimes W_k$, just as we extend Λ using (6.18). As usual, we let $\mathcal{H}_{ij}$ be the corresponding $\mathcal{M}_V$-module. The sesquilinear form Λ on W_{ij} defined by (6.18) can be extended uniquely to a *continuous* sesquilinear form Λ on the Hilbert space $\mathcal{H}_{ij}$.

Choose intertwining operators $\mathcal{Y}_{\sigma_2},\dots,\mathcal{Y}_{\sigma_m},\mathcal{Y}_{\rho_2},\dots,\mathcal{Y}_{\rho_n}$, disjoint open intervals $I,J,(I_1,\dots,I_m)\in\mathfrak{O}_m(I),(J_1,\dots,J_n)\in\mathfrak{O}_n(J)$, and non-zero quasi-primary vectors

$w^{(i_1)}, \ldots, w^{(i_m)}, w^{(j_1)}, \ldots, w^{(j_n)}$ as at the beginning of this section. By proposition 6.2, for each $l \in \mathbb{Z}_{\geqslant 0}$, vectors of the form

$$B(b)\xi^{(0)} = \pi_j(y)\mathcal{Y}_{\rho_n\cdots\rho_2,\kappa(j)}(w^{(j_n)}, g_n; \ldots; w^{(j_1)}, g_1)\xi^{(0)} \tag{6.43}$$

span a core for $\overline{L_0}^l$ in $\mathcal{H}_j^\infty$, where $b = (y, g_1, \ldots, g_n) \in \mathcal{B}$, and $\xi^{(0)} \in \mathcal{H}_0^\infty$. For any $a = (x, f_1, \ldots, f_m) \in \mathcal{A}$, we define an unbounded operator $\widetilde{A}(a) : \mathcal{H}_j \to \mathcal{H}_{ij}$ with domain $\mathcal{H}_j^\infty$ to satisfy

$$\widetilde{A}(a) = \pi_{ij}(x)\mathcal{Y}_{\sigma_m\cdots\sigma_2,i\boxtimes j}(w^{(i_m)}, f_m; \ldots; w^{(i_1)}, f_1). \tag{6.44}$$

Then, by inequality (5.43), vectors of the form (6.43) span a core for $\widetilde{A}(a)$. Therefore, by proposition 6.2, vectors of the form

$$\xi^{(ij)} = \sum_{s=1,\ldots,N} \widetilde{A}(a_s)B(b_s)\xi_s^{(0)} \tag{6.45}$$

form a dense subspace of $\mathcal{H}_{ij}$, where $N = 1, 2, \ldots$, and for each s, $a_s = (x_s, f_{s,1}, \ldots, f_{s,m}) \in \mathcal{A}$, $b_s = (y_s, g_{s,1}, \ldots, g_{s,n}) \in \mathcal{B}$, and $\xi_s^{(0)} \in \mathcal{H}_0$. If we can prove, for any $\xi^{(ij)} \in \mathcal{H}_{ij}$ of the form (6.45), that $\Lambda(\xi^{(ij)}|\xi^{(ij)}) \geqslant 0$, then Λ is positive on $W_i \boxtimes W_j$.

Step 2. We show that $\Lambda(\xi^{(ij)}|\xi^{(ij)}) \geqslant 0$. Let us simplify the notations a little bit. Let $\vec{w}^{(\vec{i})} = (w^{(i_1)}, \ldots, w^{(i_m)}), \vec{\sigma} = (\sigma_2, \ldots, \sigma_m), \vec{f_s} = (f_{s,1}, \ldots, f_{s,m})$. If $\mathcal{Y}_\alpha$ is an intertwining operator whose charge space, source space, and target space are inside $\mathcal{F}^\boxtimes$, then we set

$$\mathcal{Y}_{\vec{\sigma},\alpha}(\vec{w}^{(\vec{i})}, \vec{f_s}) = \mathcal{Y}_{\sigma_m\cdots\sigma_2,\alpha}(w^{(i_m)}, f_{s,m}; \ldots; w^{(i_1)}, f_{s,1}). \tag{6.46}$$

Similarly, we let $\vec{w}^{(\vec{j})} = (w^{(j_1)}, \ldots, w^{(j_n)}), \vec{\rho} = (\rho_2, \ldots, \rho_n), \vec{g}_s = (g_{s,1}, \ldots, g_{s,n})$. $\mathcal{Y}_{\vec{\rho},\kappa(j)}(w^{(\vec{j})}, \vec{g}_s)$ is defined in a similar way.

Assume, without loss of generality, that I is anti-clockwise to J, i.e., for any $z \in I, \zeta \in J$, we have $-\pi < \arg\zeta < \arg z < \pi$. By proposition 5.10, for any $s = 1, \ldots, N$,

$$\begin{aligned}&\widetilde{A}(a_s)B(b_s) = x_s\mathcal{Y}_{\vec{\sigma},i\boxtimes j}(\vec{w}^{(\vec{i})}, \vec{f_s})y_s\mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})}, \vec{g}_s)\\ =&\sum_{\alpha\in\Theta_{ij}^*} \check{\mathcal{Y}}^\alpha \otimes x_s y_s \mathcal{Y}_{\vec{\sigma},\alpha}(\vec{w}^{(\vec{i})}, \vec{f_s})\mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})}, \vec{g}_s).\end{aligned} \tag{6.47}$$

So for any $s, t = 1, \ldots, N$,

$$\begin{aligned}&\Lambda\big(\widetilde{A}(a_s)B(b_s)\xi_s^{(0)}\big|\widetilde{A}(a_t)B(b_t)\xi_t^{(0)}\big)\\ =&\sum_{\alpha,\beta\in\Theta_{ij}^*} \Lambda^{\alpha\beta}\big\langle x_s y_s\mathcal{Y}_{\vec{\sigma},\alpha}(\vec{w}^{(\vec{i})}, \vec{f_s})\mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})}, \vec{g}_s)\xi_s^{(0)}\big|x_t y_t\mathcal{Y}_{\vec{\sigma},\beta}(\vec{w}^{(\vec{i})}, \vec{f_t})\mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})}, \vec{g}_t)\xi_t^{(0)}\big\rangle\\ =&\sum_{\alpha,\beta\in\Theta_{ij}^*} \Lambda^{\alpha\beta}\big\langle \mathcal{Y}_{\vec{\sigma},\beta}(\vec{w}^{(\vec{i})}, \vec{f_t})^\dagger x_t^* x_s y_s\mathcal{Y}_{\vec{\sigma},\alpha}(\vec{w}^{(\vec{i})}, \vec{f_s})\mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})}, \vec{g}_s)\xi_s^{(0)}\big|y_t\mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})}, \vec{g}_t)\xi_t^{(0)}\big\rangle\end{aligned}$$

$$= \sum_{\alpha,\beta\in\Theta^*_{ij}} \Lambda^{\alpha\beta}\big\langle \mathcal{Y}_{\vec{\sigma},\beta}(\vec{w}^{(\vec{i})},\vec{f}_t)^\dagger x_t^* x_s y_s \mathcal{Y}_{\vec{\sigma},\alpha}(\vec{w}^{(\vec{i})},\vec{f}_s)\mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})},\vec{g}_s)\xi_s^{(0)}\big|B(b_t)\xi_t^{(0)}\big\rangle. \tag{6.48}$$

By corollary 2.18 and theorem 5.12,

$$\begin{aligned}
&\sum_{\alpha,\beta\in\Theta^*_{ij}} \Lambda^{\alpha\beta}\mathcal{Y}_{\vec{\sigma},\beta}(\vec{w}^{(\vec{i})},\vec{f}_t)^\dagger x_t^* x_s y_s \mathcal{Y}_{\vec{\sigma},\alpha}(\vec{w}^{(\vec{i})},\vec{f}_s)\mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})},\vec{g}_s)\\
=&\sum_{\alpha,\beta\in\Theta^*_{ij}} \Lambda^{\alpha\beta}\mathcal{Y}_{\vec{\sigma},\beta}(\vec{w}^{(\vec{i})},\vec{f}_t)^\dagger x_t^* x_s y_s \mathcal{Y}_{\vec{\rho},B_+\alpha}(\vec{w}^{(\vec{j})},\vec{g}_s)\mathcal{Y}_{\vec{\sigma},\kappa(i)}(\vec{w}^{(\vec{i})},\vec{f}_s)\\
=&\sum_{\alpha,\beta\in\Theta^*_{ij}} \Lambda^{\alpha\beta} y_s\mathcal{Y}_{\vec{\sigma},\beta}(\vec{w}^{(\vec{i})},\vec{f}_t)^\dagger \mathcal{Y}_{\vec{\rho},B_+\alpha}(\vec{w}^{(\vec{j})},\vec{g}_s)x_t^* x_s\mathcal{Y}_{\vec{\sigma},\kappa(i)}(\vec{w}^{(\vec{i})},\vec{f}_s).
\end{aligned}\tag{6.49}$$

By theorem 5.13, for each $l = 2,\dots,m$, there exists an intertwining operators $\widetilde{\sigma}_l$ having the same type as that of σ_l, such that (5.59) holds for all $\mathcal{Y}_\alpha$ whose charge space, source space, and target space are unitary V-modules in $\mathcal{F}^{\boxtimes}$. Let $h_{t,1} = e^{i\pi\Delta_{w^{(i_1)}}}(e_{2-2\Delta_{w^{(i_1)}}}f_{t,1}),\dots,h_{t,m} = e^{i\pi\Delta_{w^{(i_m)}}}(e_{2-2\Delta_{w^{(i_m)}}}f_{t,m})$. Set $\vec{h}_t = (h_{t,1},\dots,h_{t,m}), \overline{\vec{h}_t} = (\overline{h_{t,1}},\dots,\overline{h_{t,m}}), \overline{\vec{w}^{(\vec{i})}} = (\overline{w^{(i_1)}},\dots,\overline{w^{(i_m)}})$. Then (6.49) equals

$$\sum_{\alpha,\beta\in\Theta^*_{ij}} \Lambda^{\alpha\beta} y_s \mathcal{Y}_{\vec{\widetilde{\sigma}},\beta^*}(\overline{\vec{w}^{(\vec{i})}},\overline{\vec{h}_t})\mathcal{Y}_{\vec{\rho},B_+\alpha}(\vec{w}^{(\vec{j})},\vec{g}_s)x_t^* x_s\mathcal{Y}_{\vec{\sigma},\kappa(i)}(\vec{w}^{(\vec{i})},\vec{f}_s). \tag{6.50}$$

By equation (6.23) and theorem 5.12, (6.50) equals

$$y_s\mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})},\vec{g}_s)\mathcal{Y}_{\vec{\widetilde{\sigma}},\kappa(i)^*}(\overline{\vec{w}^{(\vec{i})}},\overline{\vec{h}_t})x_t^* x_s\mathcal{Y}_{\vec{\sigma},\kappa(i)}(\vec{w}^{(\vec{i})},\vec{f}_s), \tag{6.51}$$

which, due to equation (5.59), also equals

$$\begin{aligned}
&y_s\mathcal{Y}_{\vec{\rho},\kappa(j)}(\vec{w}^{(\vec{j})},\vec{g}_s)\mathcal{Y}_{\vec{\sigma},\kappa(i)}(\vec{w}^{(\vec{i})},\vec{f}_t)^\dagger x_t^* x_s\mathcal{Y}_{\vec{\sigma},\kappa(i)}(\vec{w}^{(\vec{i})},\vec{f}_s)\\
=&B(b_s)A(a_t)^\dagger A(a_s).
\end{aligned}\tag{6.52}$$

Substitute this expression into equation (6.48), we see that

$$\Lambda\big(\widetilde{A}(a_s)B(b_s)\xi_s^{(0)}\big|\widetilde{A}(a_t)B(b_t)\xi_t^{(0)}\big) = \langle B(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)}|B(b_t)\xi_t^{(0)}\rangle. \tag{6.53}$$

Therefore, by lemma 6.6,

$$\begin{aligned}
\Lambda(\xi^{(ij)}|\xi^{(ij)}) &= \sum_{s,t=1,\dots,N}\Lambda\big(\widetilde{A}(a_s)B(b_s)\xi_s^{(0)}\big|\widetilde{A}(a_t)B(b_t)\xi_t^{(0)}\big)\\
&= \sum_{s,t=1,\dots,N}\langle B(b_s)A(a_t)^\dagger A(a_s)\xi_s^{(0)}|B(b_t)\xi_t^{(0)}\rangle \geqslant 0.
\end{aligned}\tag{6.54}$$

Step 3 (See also [HK07] theorem 3.4). We prove the non-degeneracy of Λ using the rigidity of $\mathrm{Rep}(V)$. Since Λ is positive, for each $k\in\mathcal{E}$, we can choose a basis Θ^k_{ij}, such that the transport matrix Λ is a diagonal, and that the entries are either 1 or 0. Thus, we have the transport formula

$$Y_j(\mathcal{Y}^0_{\bar{i}i}(\overline{w^{(i)}_2}, z_2 - z_1)w^{(i)}_1, z_1) = \sum_{\alpha\in\Theta^*_{ij}} \lambda_\alpha \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}_2}, z_2)\mathcal{Y}_\alpha(w^{(i)}_1, z_1), \tag{6.55}$$

where each λ_α is either 1 or 0. For each $k \in \mathcal{E}$, we let n^k_{ij} be the number of $\alpha \in \Theta^k_{ij}$ satisfying $\lambda_\alpha = 1$. Then clearly $n^k_{ij} \leqslant N^k_{ij}$. If we can show that $n^k_{ij} = N^k_{ij}$, then the non-degeneracy of Λ follows.

Since W_i is irreducible, we have $N^0_{\bar{i}i} = N^i_{0i} = 1$. So there exists a complex number $\mu_i \neq 0$ such that $\mathcal{Y}^0_{\bar{i}i}$ represents the morphism $\mu_i \mathrm{ev}_i : W_{\bar{i}} \boxtimes W_i \to V$. We also regard $\mathcal{Y}_\alpha$ as a morphism $W_i \boxtimes W_j \to W_k$, and $\mathcal{Y}_{\alpha^*}$ a morphism $W_{\bar{i}} \boxtimes W_k \to W_j$ (see section 2.4). Then equation (6.55) is equivalent to the following relation for morphisms:

$$\mu_i(\mathrm{ev}_i \otimes \mathrm{id}_j) = \sum_{k\in\mathcal{E}}\sum_{\alpha\in\Theta^k_{ij}} \lambda_\alpha \mathcal{Y}_{\alpha^*} \circ (\mathrm{id}_{\bar{i}} \otimes \mathcal{Y}_\alpha). \tag{6.56}$$

By equation (2.64),

$$\begin{aligned}
\mu_i(\mathrm{id}_i \otimes \mathrm{id}_j) =&\mu_i[(\mathrm{id}_i \otimes \mathrm{ev}_i) \circ (\mathrm{coev}_i \otimes \mathrm{id}_i)] \otimes \mathrm{id}_j\\
=&\mu_i(\mathrm{id}_i \otimes \mathrm{ev}_i \otimes \mathrm{id}_j) \circ (\mathrm{coev}_i \otimes \mathrm{id}_i \otimes \mathrm{id}_j)\\
=&\sum_{k\in\mathcal{E}}\sum_{\alpha\in\Theta^k_{ij}} \lambda_\alpha\big(\mathrm{id}_i \otimes (\mathcal{Y}_{\alpha^*} \circ (\mathrm{id}_{\bar{i}} \otimes \mathcal{Y}_\alpha))\big) \circ (\mathrm{coev}_i \otimes \mathrm{id}_i \otimes \mathrm{id}_j)\\
=&\sum_{k\in\mathcal{E}}\sum_{\alpha\in\Theta^k_{ij}} \lambda_\alpha(\mathrm{id}_i \otimes \mathcal{Y}_{\alpha^*}) \circ (\mathrm{id}_i \otimes \mathrm{id}_{\bar{i}} \otimes \mathcal{Y}_\alpha) \circ (\mathrm{coev}_i \otimes \mathrm{id}_i \otimes \mathrm{id}_j)\\
=&\sum_{k\in\mathcal{E}}\sum_{\alpha\in\Theta^k_{ij}} \lambda_\alpha(\mathrm{id}_i \otimes \mathcal{Y}_{\alpha^*}) \circ (\mathrm{coev}_i \otimes \mathcal{Y}_\alpha)\\
=&\sum_{k\in\mathcal{E}}\sum_{\alpha\in\Theta^k_{ij}} \lambda_\alpha(\mathrm{id}_i \otimes \mathcal{Y}_{\alpha^*}) \circ (\mathrm{coev}_i \otimes \mathrm{id}_k) \circ (\mathrm{id}_0 \otimes \mathcal{Y}_\alpha).
\end{aligned}$$

This equation implies that the isomorphism $\mu_i(\mathrm{id}_i \otimes \mathrm{id}_j) : W_i \boxtimes W_j \to W_i \boxtimes W_j$ factors through the homomorphism

$$\Phi : \sum^{\oplus}_{k\in\mathcal{E}} \sum^{\oplus}_{\alpha\in\Theta^k_{ij},\lambda_\alpha\neq 0} \mathrm{id}_0 \otimes \mathcal{Y}_\alpha : W_i \boxtimes W_j \to W = \bigoplus_{k\in\mathcal{E}} \bigoplus_{\alpha\in\Theta^k_{ij},\lambda_\alpha\neq 0} W_k.$$

So Φ must be injective, which implies that $W_i \boxtimes W_j$ can be embedded as a submodule of W. Note that $W_i \boxtimes W_j \simeq \bigoplus_{k\in\mathcal{E}} W_k^{\oplus N^k_{ij}}$ and $W \simeq \bigoplus_{k\in\mathcal{E}} W_k^{\oplus n^k_{ij}}$. So we must have $n^k_{ij} \geqslant N^k_{ij}$. $\square$

7 Unitarity of the ribbon fusion categories

In this chapter, we still assume that V is unitary, energy bounded, and strongly local, and that $\mathcal{F}$ is a non-empty set of non-zero irreducible unitary V-modules satisfying condition A or B in section 5.3. If W_i, W_j are unitary V-modules in $\mathcal{F}^\boxtimes$, then by theorem 6.7, for each $k \in \mathcal{E}$, the sesquilinear form Λ on $\mathcal{V}{k \choose i\ j}^*$ defined by the transport matrix is an inner product. Therefore, we have a unitary structure on $\mathcal{F}^\boxtimes$ defined by Λ (see section 2.4). We fix this unitary structure, and show that the ribbon fusion category $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{F}^\boxtimes}(V)$ is unitary.

We first note that the inner product Λ on $\mathcal{V}{k \choose i\ j}^*$ induces naturally an antilinear isomorphism map $\mathcal{V}{k \choose i\ j} \to \mathcal{V}{k \choose i\ j}^*$. We then define the inner product Λ on $\mathcal{V}{k \choose i\ j}$ so that this map becomes anti-unitary. Then a basis $\Theta^k_{ij} \subset \mathcal{V}{k \choose i\ j}$ is orthonormal if and only if its dual basis is an orthonormal basis of $\mathcal{V}{k \choose i\ j}^*$. Therefore, if for each $k \in \mathcal{E} \cap \mathcal{F}^\boxtimes$, Θ^k_{ij} is an orthonormal basis of $\mathcal{V}{k \choose i\ j}$, then the transport formulas (6.16), (6.21) and (6.23) become

$$Y_j\big(\mathcal{Y}^0_{\bar{i}i}(\overline{w^{(i)}_2}, z_2 - z_1)w^{(i)}_1, z_1\big) = \sum_{\alpha \in \Theta^*_{ij}} \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}_2}, z_2)\mathcal{Y}_\alpha(w^{(i)}_1, z_1), \tag{7.1}$$

$$\mathcal{Y}_\gamma(w^{(j)}, z_0)\Bigg(\sum_{\alpha \in \Theta^*_{is}} \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}_2}, z_2)\mathcal{Y}_\alpha(w^{(i)}_1, z_1)\Bigg) = \Bigg(\sum_{\alpha \in \Theta^*_{ir}} \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}_2}, z_2)\mathcal{Y}_\alpha(w^{(i)}_1, z_1)\Bigg)\mathcal{Y}_\gamma(w^{(j)}, z_0), \tag{7.2}$$

$$\mathcal{Y}^j_{j0}(w^{(j)}, z_0)\mathcal{Y}^0_{\bar{i}i}(\overline{w^{(i)}_2}, z_2) = \sum_{\alpha \in \Theta^*_{ij}} \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}_2}, z_2)\mathcal{Y}_{B_+\alpha}(w^{(j)}, z_0). \tag{7.3}$$

7.1 Unitarity of braid matrices

For any unitary V-modules W_i, W_j in $\mathcal{F}^\boxtimes$, and any $s, t \in \mathcal{E} \cap \mathcal{F}^\boxtimes$, we choose bases $\Theta^t_{is}, \Theta^t_{sj}$ of $\mathcal{V}{t \choose i\ s}, \mathcal{V}{t \choose s\ j}$ respectively. Now fix $i, j \in \mathcal{F}^\boxtimes$, we also define

$$\Theta^*_{i*} = \coprod_{s,t \in \mathcal{E} \cap \mathcal{F}^\boxtimes} \Theta^t_{is}, \Theta^*_{*j} = \coprod_{s,t \in \mathcal{E} \cap \mathcal{F}^\boxtimes} \Theta^t_{sj}.$$

Choose distinct $z_i, z_j \in S^1$, and let $\arg z_j < \arg z_i < \arg z_j + 2\pi$. For any $\alpha, \alpha' \in \Theta^*_{i*}, \beta, \beta' \in \Theta^*_{j*}$, if either the source space of $\mathcal{Y}_\alpha$ does not equal the target space of $\mathcal{Y}_\beta$, or the target space of $\mathcal{Y}_{\alpha'}$ does not equal the source space of $\mathcal{Y}_{\beta'}$, or the target space of $\mathcal{Y}_\alpha$ does not equal the target space of $\mathcal{Y}_{\beta'}$, or the source space of $\mathcal{Y}_\beta$ does not equal the source space of $\mathcal{Y}_{\alpha'}$, then we set $(B_+)^{\beta'\alpha'}_{\alpha\beta} = 0$; otherwise the values $(B_+)^{\beta'\alpha'}_{\alpha\beta}$ are determined by the following braid relation: for any $w^{(i)} \in W_i, w^{(j)} \in W_j$,

$$\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}_\beta(w^{(j)}, z_j) = \sum_{\alpha' \in \Theta^*_{i*}, \beta' \in \Theta^*_{j*}} (B_+)^{\beta'\alpha'}_{\alpha\beta}\mathcal{Y}_{\beta'}(w^{(j)}, z_j)\mathcal{Y}_{\alpha'}(w^{(i)}, z_i). \tag{7.4}$$

The matrix $(B_+)_{ij} = \{(B_+)^{\beta'\alpha'}_{\alpha\beta}\}^{\alpha'\in\Theta^*_{i*},\beta'\in\Theta^*_{j*}}_{\alpha\in\Theta^*_{i*},\beta\in\Theta^*_{j*}}$ is called a **braid matrix**. The matrix $(B_-)_{ij} = \{(B_-)^{\beta'\alpha'}_{\alpha\beta}\}^{\alpha'\in\Theta^*_{i*},\beta'\in\Theta^*_{j*}}_{\alpha\in\Theta^*_{i*},\beta\in\Theta^*_{j*}}$ is defined in a similar way by assuming $\arg z_i < \arg z_j < \arg z_i + 2\pi$. Clearly $(B_\pm)_{ij}$ is the inverse matrix of $(B_\mp)_{ji}$.

Proposition 7.1. *For any $\alpha, \alpha' \in \Theta^*_{i*}, \beta, \beta' \in \Theta^*_{j*}$, we have*

$$\overline{(B_\pm)^{\beta'\alpha'}_{\alpha\beta}} = (B_\mp)^{\alpha'^*\beta'^*}_{\beta^*\alpha^*}. \tag{7.5}$$

Proof. Choose distinct $z_i, z_j \in S^1$, and let $\arg z_j < \arg z_i < \arg z_j + 2\pi$. Then for any $w^{(i)} \in W_i, w^{(j)} \in W_j$, the braid relation (7.4) holds. Taking the formal adjoint of (7.4), we have

$$\mathcal{Y}_\beta(w^{(j)}, z_j)^\dagger \mathcal{Y}_\alpha(w^{(i)}, z_i)^\dagger = \sum_{\alpha',\beta'} \overline{(B_+)^{\beta'\alpha'}_{\alpha\beta}} \mathcal{Y}_{\alpha'}(w^{(i)}, z_i)^\dagger \mathcal{Y}_{\beta'}(w^{(j)}, z_j)^\dagger. \tag{7.6}$$

By equation (1.34), for any $w^{(i)} \in W_i, w^{(j)} \in W_j$ we have

$$\mathcal{Y}_{\beta^*}(\overline{w^{(j)}}, z_j)\mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, z_i) = \sum_{\alpha',\beta'} \overline{(B_+)^{\beta'\alpha'}_{\alpha\beta}} \mathcal{Y}_{\alpha'^*}(\overline{w^{(i)}}, z_i)\mathcal{Y}_{\beta'^*}(\overline{w^{(j)}}, z_j). \tag{7.7}$$

But $\{(B_-)^{\alpha'^*\beta'^*}_{\beta^*\alpha^*}\}$ is also the braid matrix for the braid relation (7.7). So we must have $\overline{(B_+)^{\beta'\alpha'}_{\alpha\beta}} = (B_-)^{\alpha'^*\beta'^*}_{\beta^*\alpha^*}$. If we let $\arg z_i < \arg z_j < \arg z_i + 2\pi$, then we obtain $\overline{(B_-)^{\beta'\alpha'}_{\alpha\beta}} = (B_+)^{\alpha'^*\beta'^*}_{\beta^*\alpha^*}$. □

Proposition 7.2. *If the bases $\Theta^*_{i*}, \Theta^*_{j*}$ are orthonormal under the inner product Λ, then for any $\alpha, \alpha' \in \Theta^*_{i*}, \beta, \beta' \in \Theta^*_{j*}$, we have*

$$(B_\pm)^{\beta'\alpha'}_{\alpha\beta} = (B_\mp)^{\alpha^*\beta'}_{\beta\alpha'^*} = (B_\pm)^{\beta^*\alpha^*}_{\alpha'^*\beta'^*}. \tag{7.8}$$

Proof. Choose distinct $z_1, z_2, z_3, z_4 \in S^1$ with arguments $\arg z_1 < \arg z_2 < \arg z_3 < \arg z_4 < \arg z_1 + 2\pi$. By relation (7.2), for any $k \in \mathcal{E} \cap \mathcal{F}^\boxtimes$, $w_0, w_5 \in W_k, w_1, w_2 \in W_i, w_3, w_4 \in W_j$, we have, following convention 2.19,

$$\begin{aligned}
&\sum_{\substack{\alpha'\in\Theta^*_{i*}\\ \beta\in\Theta^*_{j*}}} \Big\langle \mathcal{Y}_{\beta^*}(\overline{w_4}, z_4)\mathcal{Y}_\beta(w_3, z_3)\mathcal{Y}_{\alpha'^*}(\overline{w_2}, z_2)\mathcal{Y}_{\alpha'}(w_1, z_1)w_0 \Big| w_5 \Big\rangle \\
=&\sum_{\substack{\alpha\in\Theta^*_{i*}\\ \beta\in\Theta^*_{j*}}} \Big\langle \mathcal{Y}_{\beta^*}(\overline{w_4}, z_4)\mathcal{Y}_{\alpha^*}(\overline{w_2}, z_2)\mathcal{Y}_\alpha(w_1, z_1)\mathcal{Y}_\beta(w_3, z_3)w_0 \Big| w_5 \Big\rangle.
\end{aligned} \tag{7.9}$$

By exchanging $\mathcal{Y}_\alpha$ and $\mathcal{Y}_\beta$, (7.9) equals

$$\sum_{\substack{\alpha,\alpha'\in\Theta^*_{i*}\\ \beta,\beta'\in\Theta^*_{j*}}} (B_-)^{\beta'\alpha'}_{\alpha\beta}\Big\langle \mathcal{Y}_{\beta*}(\overline{w_4},z_4)\mathcal{Y}_{\alpha*}(\overline{w_2},z_2)\mathcal{Y}_{\beta'}(w_3,z_3)\mathcal{Y}_{\alpha'}(w_1,z_1)w_0\Big|w_5\Big\rangle. \tag{7.10}$$

By proposition 2.3, we have

$$\mathcal{Y}_{\beta}(w_3,z_3)\mathcal{Y}_{\alpha'*}(\overline{w_2},z_2) = \sum_{\substack{\alpha,\alpha'\in\Theta^*_{i*}\\ \beta,\beta'\in\Theta^*_{j*}}} (B_-)^{\beta'\alpha'}_{\alpha\beta}\mathcal{Y}_{\alpha*}(\overline{w_2},z_2)\mathcal{Y}_{\beta'}(w_3,z_3). \tag{7.11}$$

This proves that $(B_+)^{\alpha*\beta'}_{\beta\alpha'*} = (B_-)^{\beta'\alpha'}_{\alpha\beta}$.

Similarly, we also have

$$\begin{aligned}
&\sum_{\substack{\alpha'\in\Theta^*_{i*}\\ \beta\in\Theta^*_{j*}}} \Big\langle \mathcal{Y}_{\beta*}(\overline{w_4},z_4)\mathcal{Y}_{\beta}(w_3,z_3)\mathcal{Y}_{\alpha'*}(\overline{w_2},z_2)\mathcal{Y}_{\alpha'}(w_1,z_1)w_0\Big|w_5\Big\rangle\\
=&\sum_{\substack{\alpha'\in\Theta^*_{i*}\\ \beta'\in\Theta^*_{j*}}} \Big\langle \mathcal{Y}_{\alpha'*}(\overline{w_2},z_2)\mathcal{Y}_{\beta'*}(\overline{w_4},z_4)\mathcal{Y}_{\beta'}(w_3,z_3)\mathcal{Y}_{\alpha'}(w_1,z_1)w_0\Big|w_5\Big\rangle \qquad (7.12)\\
=&\sum_{\substack{\alpha,\alpha'\in\Theta^*_{i*}\\ \beta,\beta'\in\Theta^*_{j*}}} (B_-)^{\beta*\alpha*}_{\alpha'*\beta'*}\Big\langle \mathcal{Y}_{\beta*}(\overline{w_4},z_4)\mathcal{Y}_{\alpha*}(\overline{w_2},z_2)\mathcal{Y}_{\beta'}(w_3,z_3)\mathcal{Y}_{\alpha'}(w_1,z_1)w_0\Big|w_5\Big\rangle, \qquad (7.13)
\end{aligned}$$

which implies that $(B_+)^{\alpha*\beta'}_{\beta\alpha'*} = (B_-)^{\beta*\alpha*}_{\alpha'*\beta'*}$.

If $z_1,z_2,z_3,z_4\in S^1$ and their arguments are chosen such that $\arg z_4<\arg z_3<\arg z_2<\arg z_1<\arg z_4+2\pi$, then the same argument implies that $(B_+)^{\beta'\alpha'}_{\alpha\beta} = (B_-)^{\alpha*\beta'}_{\beta\alpha'*} = (B_+)^{\beta*\alpha*}_{\alpha'*\beta'*}$. □

Corollary 7.3. *If the bases $\Theta^*_{i*},\Theta^*_{j*}$ are orthonormal under the inner product Λ, then the braid matrix $(B_\pm)_{ij}$ is unitary.*

Proof. If we apply propositions 7.1 and 7.2, then for any $\alpha,\alpha'\in\Theta^*_{i*},\beta,\beta'\in\Theta^*_{j*}$, we have

$$(B_\pm)^{\beta'\alpha'}_{\alpha\beta} = (B_\pm)^{\beta*\alpha*}_{\alpha'*\beta'*} = \overline{(B_\mp)^{\alpha\beta}_{\beta'\alpha'}}, \tag{7.14}$$

which shows that $(B_\pm)_{ij}$ is the adjoint of $(B_\mp)_{ji}$. But we know that $(B_\pm)_{ij}$ is also the inverse matrix of $(B_\mp)_{ji}$. So $(B_\pm)_{ij}$ is unitary. □

7.2 Unitarity of fusion matrices

Recall from section 2.4 that for any W_i,W_j,W_k,W_t in $\mathcal{F}^\boxtimes$, we have a fusion matrix $\{F^{\beta'\alpha'}_{\alpha\beta}\}^{\alpha'\in\Theta^*_{ij},\beta'\in\Theta^t_{*k}}_{\alpha\in\Theta^t_{i*},\beta\in\Theta^*_{jk}}$ defined by the fusion relation

$$\mathcal{Y}_\alpha(w^{(i)},z_i)\mathcal{Y}_\beta(w^{(j)},z_j) = \sum_{\alpha'\in\Theta^*_{ij},\beta'\in\Theta^t_{*k}} F^{\beta'\alpha'}_{\alpha\beta}\mathcal{Y}_{\beta'}(\mathcal{Y}_{\alpha'}(w^{(i)},z_i-z_j)w^{(j)},z_j), \tag{7.15}$$

where $z_i, z_j \in \mathbb{C}^\times, 0 < |z_i - z_j| < |z_j| < |z_i|$, $\arg z_j$ is close to $\arg z_i$ as $z_j \to z_i$, and $\arg(z_i - z_j)$ is close to $\arg z_i$ as $z_j \to 0$. We let $F_{\alpha\beta}^{\beta'\alpha'} = 0$ if the source space of $\mathcal{Y}_\alpha$ does not equal the target space of $\mathcal{Y}_\beta$, or if the target space of $\mathcal{Y}_{\alpha'}$ does not equal the charge space of $\mathcal{Y}_{\beta'}$. In this section, we show that fusion matrices are unitary.

Proposition 7.4. *Choose unitary V-modules W_i, W_k in $\mathcal{F}^\boxtimes$, W_j, W_t in $\mathcal{E} \cap \mathcal{F}^\boxtimes$. Then for any for any $\alpha \in \Theta_{i*}^t, \beta \in \Theta_{jk}^*, \alpha' \in \Theta_{ij}^*, \beta' \in \Theta_{*k}^t$, we have*

$$F_{\alpha\beta}^{\beta'\alpha'} = (B_+)_{\alpha, B_+\beta}^{B_+\beta',\alpha'} = (B_-)_{\alpha, B_-\beta}^{B_-\beta',\alpha'}. \tag{7.16}$$

Proof. Choose distinct $z_i, z_j, z_k \in S^1$ with arguments $\arg z_k < \arg z_j < \arg z_i < \arg z_k + 2\pi$, and assume that $0 < |z_i - z_j| < 1$. Choose $w^{(i)} \in W_i, w^{(j)} \in W_j, w^{(k)} \in W_k$. By corollary 2.18, we have

$$\begin{aligned}
&\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}_\beta(w^{(j)}, z_j)\mathcal{Y}_{k0}^k(w^{(k)}, z_k)\\
=&\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}_{B_+\beta}(w^{(k)}, z_k)\mathcal{Y}_{j0}^j(w^{(j)}, z_j)\\
=&\sum_{\substack{\alpha' \in \Theta_{i*}^* \\ \beta' \in \Theta_{*k}^*}} (B_+)_{\alpha, B_+\beta}^{B_+\beta',\alpha'} \mathcal{Y}_{B_+\beta'}(w^{(k)}, z_k)\mathcal{Y}_{\alpha'}(w^{(i)}, z_i)\mathcal{Y}_{j0}^j(w^{(j)}, z_j).
\end{aligned} \tag{7.17}$$

On the other hand, by corollary 2.18 and theorem 5.2,

$$\begin{aligned}
&\mathcal{Y}_\alpha(w^{(i)}, z_i)\mathcal{Y}_\beta(w^{(j)}, z_j)\mathcal{Y}_{k0}^k(w^{(k)}, z_k)\\
=&\sum_{s \in \mathcal{E}} \sum_{\substack{\alpha' \in \Theta_{ij}^s \\ \beta' \in \Theta_{sk}^t}} F_{\alpha\beta}^{\beta'\alpha'} \mathcal{Y}_{\beta'}\big(\mathcal{Y}_{\alpha'}(w^{(i)}, z_i - z_j)w^{(j)}, z_j\big)\mathcal{Y}_{k0}^k(w^{(k)}, z_k) \qquad (7.18)\\
=&\sum_{s \in \mathcal{E}} \sum_{\substack{\alpha' \in \Theta_{ij}^s \\ \beta' \in \Theta_{sk}^t}} F_{\alpha\beta}^{\beta'\alpha'} \mathcal{Y}_{B_+\beta'}(w^{(k)}, z_k)\mathcal{Y}_{s0}^s\big(\mathcal{Y}_{\alpha'}(w^{(i)}, z_i - z_j)w^{(j)}, z_j\big), \qquad (7.19)
\end{aligned}$$

where (7.18) and (7.19) are understood as products of two generalized intertwining operators (see the beginning of chapter 5). By proposition 2.17, (7.19) equals

$$\sum_{s \in \mathcal{E}} \sum_{\substack{\alpha' \in \Theta_{ij}^s \\ \beta' \in \Theta_{sk}^t}} F_{\alpha\beta}^{\beta'\alpha'} \mathcal{Y}_{B_+\beta'}(w^{(k)}, z_k)\mathcal{Y}_{\alpha'}(w^{(i)}, z_i)\mathcal{Y}_{j0}^j(w^{(j)}, z_j). \tag{7.20}$$

Comparing this result with (7.17), we see immediately that $F_{\alpha\beta}^{\beta'\alpha'} = (B_+)_{\alpha, B_+\beta}^{B_+\beta',\alpha'}$. If we assume at the beginning that $\arg z_i < \arg z_j < \arg z_k < \arg z_i + 2\pi$, then we obtain $F_{\alpha\beta}^{\beta'\alpha'} = (B_-)_{\alpha, B_-\beta}^{B_-\beta',\alpha'}$. □

Proposition 7.5. *Let W_i, W_j be unitary V-modules in $\mathcal{F}^\boxtimes$. For each $k \in \mathcal{E} \cap \mathcal{F}^\boxtimes$, we let $\{\mathcal{Y}_\alpha : \alpha \in \Theta_{ij}^k\}$ be a set of orthonormal basis of $\mathcal{V}\binom{k}{i\ j}$ under the inner product Λ. Then $B_+\Theta_{ij}^k = \{\mathcal{Y}_{B_+\alpha} : \alpha \in \Theta_{ij}^k\}$ and $B_-\Theta_{ij}^k = \{\mathcal{Y}_{B_-\alpha} : \alpha \in \Theta_{ij}^k\}$ are orthonormal bases of $\mathcal{V}\binom{k}{j\ i}$.*

Proof. Choose distinct $z_i, z_j \in S^1$ with arguments satisfying $\arg z_i < \arg z_j < \arg z_i + 2\pi$. By proposition 6.5-(3), for any $w^{(i)} \in W_i, w^{(j)} \in W_j$, we have

$$\mathcal{Y}^j_{j0}(w^{(j)}, z_j)\mathcal{Y}^0_{\bar{i}i}(\overline{w^{(i)}}, z_i) = \sum_{\alpha \in \Theta^*_{ij}} \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, z_i)\mathcal{Y}_{B_-\alpha}(w^{(j)}, z_j). \tag{7.21}$$

Take the formal adjoint of both sides, we obtain

$$\mathcal{Y}^0_{\bar{i}i}(\overline{w^{(i)}}, z_i)^\dagger \mathcal{Y}^j_{j0}(w^{(j)}, z_j)^\dagger = \sum_{\alpha \in \Theta^*_{ij}} \mathcal{Y}_{B_-\alpha}(w^{(j)}, z_j)^\dagger \mathcal{Y}_{\alpha^*}(\overline{w^{(i)}}, z_i)^\dagger. \tag{7.22}$$

Recall that $(\mathcal{Y}^j_{j0})^\dagger = \mathcal{Y}^0_{\bar{j}j}$ and $(\mathcal{Y}^0_{\bar{i}i})^\dagger = \mathcal{Y}^i_{i0}$. Thus, by equation (1.34), equation (7.22) shows that

$$\begin{aligned}\mathcal{Y}^i_{i0}(w^{(i)}, z_i)\mathcal{Y}^0_{\bar{j}j}(\overline{w^{(j)}}, z_j) &= \sum_{\alpha \in \Theta^*_{ij}} \mathcal{Y}_{(B_-\alpha)^*}(\overline{w^{(j)}}, z_j)\mathcal{Y}_{\alpha}(w^{(i)}, z_i)\\ &= \sum_{\beta \in B_-\Theta^*_{ij}} \mathcal{Y}_{\beta^*}(\overline{w^{(j)}}, z_j)\mathcal{Y}_{B_+\beta}(w^{(i)}, z_i),\end{aligned} \tag{7.23}$$

which, by proposition 6.5-(3), shows that $B_-\Theta^k_{ij}$ is an orthonormal basis of $\mathcal{V}\binom{k}{i\ j}$ for any $k \in \mathcal{E}$. The other case is treated in a similar way. $\square$

Corollary 7.6. *For any W_i, W_j, W_k in $\mathcal{F}^\boxtimes$ and W_t in $\mathcal{E}$, the fusion matrix $\{F^{\beta'\alpha'}_{\alpha\beta}\}^{\alpha' \in \Theta^*_{ij}, \beta' \in \Theta^t_{*k}}_{\alpha \in \Theta^t_{i*}, \beta \in \Theta^*_{jk}}$ is unitary.*

Proof. If W_j is irreducible, then W_j is unitarily equivalent to a unitary V-module in $\mathcal{E} \cap \mathcal{F}^\boxtimes$. The unitarity of the fusion matrix follows then from propositions 7.4, 7.5, and the unitarity of braid matrices proved in the last section. In general, the fusion matrix is diagonalized according to the orthogonal decomposition of W_j into irreducible submodules. Thus the unitarity can be proved easily. $\square$

7.3 Unitarity of the ribbon fusion categories

In this section, we prove that $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{F}^\boxtimes}(V)$ is unitary when the unitary structure on $\mathcal{F}^\boxtimes$ is defined by Λ. By corollary 7.6, the associators are unitary. By proposition 7.5, the braid operators are unitary. That $\lambda_i : V \boxtimes W_i \to W_i$ and $\rho_i : W_i \boxtimes V \to W_i$ are unitary follows from equations (6.19) and (6.20).

Choose $W_{i_1}, W_{i_2}, W_{j_1}, W_{j_2}$ in $\mathcal{F}^\boxtimes$. We show, for any $F \in \mathrm{Hom}_V(W_{i_1}, W_{i_2}), G \in \mathrm{Hom}_V(W_{j_1}, W_{j_2})$, that

$$(F \otimes G)^* = F^* \otimes G^*. \tag{7.24}$$

Consider direct sum modules $W_i = W_{i_1} \oplus^\perp W_{i_2}, W_j = W_{j_1} \oplus^\perp W_{j_2}$. For each $k \in \mathcal{E}$, it is easy to see that $\mathcal{V}\binom{k}{i\ j}$ has the natural orthogonal decomposition

$$\mathcal{V}\binom{k}{i\ j} = \bigoplus_{a,b=1,2}^{\perp} \mathcal{V}\binom{k}{i_a\ j_b}, \tag{7.25}$$

which induces the natural decomposition

$$W_i \boxtimes W_j = \bigoplus_{a,b=1,2}^{\perp} W_{i_a} \boxtimes W_{j_b}. \tag{7.26}$$

Therefore, if we regard F, G as endomorphisms of the modules W_i, W_j respectively, then $F \otimes G$ and $F^* \otimes G^*$ can be regarded as endomorphisms of $W_i \boxtimes W_j$. Thus, it suffices to prove equation (7.24) for any $F \in \mathrm{End}_V(W_i), G \in \mathrm{End}_V(W_j)$.

Since $\mathrm{End}_V(W_i)$ and $\mathrm{End}_V(W_j)$ are C^*-algebras (see theorem 2.21), they are spanned by unitary elements inside them. Therefore, by linearity, it suffices to prove (7.24) when $F \in \mathrm{End}_V(W_i), G \in \mathrm{End}_V(W_j)$ are unitary operators. By equation (2.56), it is easy to see that $F \otimes G$ is unitary. Hence we have

$$(F^* \otimes G^*)(F \otimes G) = F^*F \otimes G^*G = \mathrm{id}_i \otimes \mathrm{id}_j = \mathrm{id}_{ij}, \tag{7.27}$$

which implies that $F^* \otimes G^* = (F \otimes G)^{-1} = (F \otimes G)^*$. This proves relation (7.24).

For each W_i in $\mathcal{F}^\boxtimes$, the twist $\vartheta_i = e^{2i\pi L_0}$ is clearly unitary. Hence, in order to prove the unitarity of $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{F}^\boxtimes}(V)$, it remains to find $\mathrm{ev}_i, \mathrm{coev}_i$, such that equations (2.69) and (2.70) hold.

To prove this, we let $\mathrm{ev}_{i,\bar{i}} \in \mathrm{Hom}_V(W_i \boxtimes W_{\bar{i}}, V)$ be the homomorphism represented by the intertwining operator $\mathcal{Y}^0_{i\bar{i}}$, and let $\mathrm{coev}_{i,\bar{i}} = \mathrm{ev}^*_{\bar{i}i}$. Since i and $\bar{\bar{i}}$ are identified, we can define $\mathrm{ev}_{\bar{i},i}$ and $\mathrm{coev}_{\bar{i},i}$ in a similar way. Set $\mathrm{ev}_i = \mathrm{ev}_{\bar{i},i}, \mathrm{coev}_i = \mathrm{coev}_{i,\bar{i}}$. If we can verify, for all W_i in $\mathcal{F}^\boxtimes$, the following relations:

$$(\mathrm{id}_i \otimes \mathrm{ev}_{\bar{i},i}) \circ (\mathrm{coev}_{i,\bar{i}} \otimes \mathrm{id}_i) = \mathrm{id}_i, \tag{7.28}$$

$$(\mathrm{ev}_{i,\bar{i}} \otimes \mathrm{id}_i) \circ (\mathrm{id}_i \otimes \mathrm{coev}_{\bar{i},i}) = \mathrm{id}_i, \tag{7.29}$$

$$\mathrm{ev}_{i,\bar{i}} = \mathrm{ev}_{\bar{i},i} \circ \sigma_{i,\bar{i}} \circ (\vartheta_i \otimes \mathrm{id}_{\bar{i}}), \tag{7.30}$$

$$\mathrm{coev}_{i,\bar{i}} = (\mathrm{id}_i \otimes \vartheta_{\bar{i}}^{-1}) \circ \sigma^{-1}_{i,\bar{i}} \circ \mathrm{coev}_{\bar{i},i}, \tag{7.31}$$

then equations (2.64), (2.65), (2.69), and (2.70) are true for all W_i, and our modular tensor category is unitary.

To begin with, we define the positive number d_i to be the norm square of the vector $\mathcal{Y}^0_{i\bar{i}}$ inside $\mathcal{V}\binom{0}{i\ \bar{i}}$, i.e.,

$$d_i = \|\mathcal{Y}^0_{i\bar{i}}\|^2. \tag{7.32}$$

By propositions 1.14 and 7.5, $d_{\bar{i}} = d_i$. The following property will indicate that d_i is the quantum dimension of W_i.

Proposition 7.7.

$$\mathrm{ev}_{i,\bar{i}} \circ \mathrm{coev}_{i,\bar{i}} = d_i. \tag{7.33}$$

Proof. First we assume that W_i is irreducible. Then $\{\mathcal{Y}^0_{i\bar{i}}\}$ is a basis of $\mathcal{V}\binom{0}{i\ \bar{i}}$. Let $\{\check{\mathcal{Y}}^0_{i\bar{i}}\}$ be its dual basis. Then $\check{\mathcal{Y}}^\alpha = d_i^{\frac{1}{2}}\check{\mathcal{Y}}^0_{i\bar{i}}$ has unit length. Now, for any $v \in V$, $\mathrm{ev}_{i,\bar{i}}$ maps $\check{\mathcal{Y}}^\alpha \otimes v \in W_i \boxtimes W_{\bar{i}}$ to $\langle \check{\mathcal{Y}}^\alpha, \mathcal{Y}^0_{i\bar{i}}\rangle v = d_i^{\frac{1}{2}}\langle \check{\mathcal{Y}}^0_{i\bar{i}}, \mathcal{Y}^0_{i\bar{i}}\rangle v = d_i^{\frac{1}{2}} v$. It follows that its adjoint $\mathrm{coev}_{i,\bar{i}}$ maps each $v \in V$ to $d_i^{\frac{1}{2}}\check{\mathcal{Y}}^\alpha \otimes v$. Hence $\mathrm{ev}_{i,\bar{i}} \circ \mathrm{coev}_{i,\bar{i}}(v) = d_i v$.

In general, W_i has decomposition $W_i = \bigoplus_a^\perp W_{i_a}$, where each W_{i_a} is irreducible. Let p_a be the projection of W_i on W_{i_a}. Then the projection $\overline{p_a}$ of $W_{\bar{i}}$ on $W_{\overline{i_a}}$ satisfies $\overline{p_a}\,\overline{w^{(i)}} = \overline{p_a w^{(i)}}$ ($w^{(i)} \in W_i$). It is easy to check that

$$\mathrm{ev}_{i,\bar{i}} = \sum_a \mathrm{ev}_{i,\bar{i}} \circ (p_a \otimes \overline{p_a}) = \sum_a \mathrm{ev}_{i_a,\overline{i_a}}, \tag{7.34}$$

$$\mathrm{coev}_{i,\bar{i}} = \sum_a (p_a \otimes \overline{p_a}) \circ \mathrm{coev}_{i,\bar{i}} = \sum_a \mathrm{coev}_{i_a,\overline{i_a}}, \tag{7.35}$$

and $d_i = \sum_a d_{i_a}$. The general case can be proved using these relations. □

Now we are ready to prove equations (7.28)-(7.31).

Proof of equation (7.29). By equations (7.34) and (7.35), it suffices to prove (7.29) when W_i is irreducible. Choose $w_1^{(i)}, w_2^{(i)} \in W_i$. Choose $z_1, z_2 \in \mathbb{C}^\times$ satisfying $0 < |z_2 - z_1| < |z_1| < |z_2|$. Choose $\arg z_2$, let $\arg z_1$ be close to $\arg z_2$ as $z_2 - z_1 \to 0$, and let $\arg(z_2 - z_1)$ be close to $\arg z_2$ as $z_1 \to 0$. Since $\|\check{\mathcal{Y}}^0_{\bar{i}i}\|^2 = d_i^{-1}$, by transport formula we have

$$\begin{aligned}
&Y_i\big(\mathcal{Y}^0_{i\bar{i}}(w_2^{(i)}, z_2 - z_1)\overline{w_1^{(i)}}, z_1\big)\\
=&d_i^{-1}(\mathcal{Y}^0_{\bar{i}i})^\dagger(w_2^{(i)}, z_2)\mathcal{Y}^0_{\bar{i}i}(\overline{w_1^{(i)}}, z_1) + \mathcal{Y}_\gamma(w_2^{(i)}, z_2)\mathcal{Y}_\beta(\overline{w_1^{(i)}}, z_1)\\
=&d_i^{-1}\mathcal{Y}^i_{i0}(w_2^{(i)}, z_2)\mathcal{Y}^0_{\bar{i}i}(\overline{w_1^{(i)}}, z_1) + \mathcal{Y}_\gamma(w_2^{(i)}, z_2)\mathcal{Y}_\beta(\overline{w_1^{(i)}}, z_1)
\end{aligned} \tag{7.36}$$

where $\mathcal{Y}_\beta, \mathcal{Y}_\gamma$ are a chain of intertwining operators, and the target space of $\mathcal{Y}_\beta$ does not contain any submodule equivalent to the vacuum module V. Equation (7.36) is equivalent to the relation

$$(\mathrm{ev}_{i,\bar{i}} \otimes \mathrm{id}_i) = d_i^{-1}(\mathrm{id}_i \otimes \mathrm{ev}_{\bar{i},i}) + \mathcal{Y}_\gamma \circ (\mathrm{id}_i \otimes \mathcal{Y}_\beta), \tag{7.37}$$

where $\mathcal{Y}_\gamma$ and $\mathcal{Y}_\beta$ denote the corresponding morphisms. By proposition 7.7,

$$\begin{aligned}
&(\mathrm{ev}_{i,\bar{i}} \otimes \mathrm{id}_i) \circ (\mathrm{id}_i \otimes \mathrm{coev}_{\bar{i},i})\\
=&d_i^{-1}(\mathrm{id}_i \otimes \mathrm{ev}_{\bar{i},i}) \circ (\mathrm{id}_i \otimes \mathrm{coev}_{\bar{i},i}) + \mathcal{Y}_\gamma \circ (\mathrm{id}_i \otimes \mathcal{Y}_\beta) \circ (\mathrm{id}_i \otimes \mathrm{coev}_{\bar{i},i})\\
=&\mathrm{id}_i + \mathcal{Y}_\gamma \circ \big(\mathrm{id}_i \otimes (\mathcal{Y}_\beta \circ \mathrm{coev}_{\bar{i},i})\big).
\end{aligned} \tag{7.38}$$

Since $\mathcal{Y}_\beta \circ \mathrm{coev}_{\bar{i},i}$ is a morphism from the vacuum module V to a V-module with no irreducible submodule equivalent to V, $\mathcal{Y}_\beta \circ \mathrm{coev}_{\bar{i},i}$ must be zero. So (7.38) equals id_i, and equation (7.29) is proved. □

Proof of equations (7.28), (7.30), *and* (7.31). Take the adjoint of equation (7.29), we immediately obtain equation (7.28). Equation (7.30) follows from equation (1.41). Equation (1.42) indicates that

$$\mathrm{ev}_{i,\bar{i}} = \mathrm{ev}_{\bar{i},i} \circ \sigma_{i,\bar{i}} \circ (\mathrm{id}_i \otimes \vartheta_{\bar{i}}), \tag{7.39}$$

the adjoint of which is (7.31). □

Thus we've proved the unitarity of our ribbon fusion category.

Theorem 7.8. *Let V be unitary, energy bounded, and strongly local, and let $\mathcal{F}$ be a non-empty set of non-zero irreducible unitary V-modules satisfying condition A or B in section 5.3. If we define a unitary structure on $\mathcal{F}^{\boxtimes}$ using Λ, then the ribbon fusion category $\mathrm{Rep}^{\mathrm{u}}_{\mathcal{F}^{\boxtimes}}(V)$ is unitary.*

Note that our proof of the unitarity of the tensor categories uses only the positive definiteness of Λ. Thus our results in this chapter can also be summarized in the following way.

Theorem 7.9. *Let V be unitary. If all irreducible V-modules are unitarizable, and the sesquilinear form Λ defined on any tensor product of unitary V-modules is positive (definite), then $\mathrm{Rep}^{\mathrm{u}}(V)$ is a unitary modular tensor category.*

8 Epilogue

8.1 Application to unitary Virasoro VOAs $(c < 1)$

Let $\mathrm{Vir} = \mathrm{Span}_{\mathbb{C}}\{C, L_n : n \in \mathbb{Z}\}$ be the Virasoro Lie algebra satisfying the relation

$$\begin{gathered}[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m,-n}C \qquad (m, n \in \mathbb{Z}),\\ [C, L_n] = 0 \qquad (n \in \mathbb{Z}).\end{gathered}$$

If W is a Vir-module, and the vector space W is equipped with an inner product $\langle\cdot|\cdot\rangle$, we say that W is a unitary Vir-module, if $L_n^\dagger = L_{-n}$ holds for any $n \in \mathbb{Z}$. More precisely, this means that for any $w_1, w_2 \in W$, we have

$$\langle L_n w_1 | w_2\rangle = \langle w_1 | L_{-n} w_2\rangle. \tag{8.1}$$

Choose Lie subalgebras $\mathrm{Vir}_+ = \mathrm{Span}_{\mathbb{C}}\{L_n : n \in \mathbb{Z}_{>0}\}$ and $\mathrm{Vir}_- = \mathrm{Span}_{\mathbb{C}}\{L_n : n \in \mathbb{Z}_{<0}\}$ of Vir, and let $U(\mathrm{Vir})$ be the universal enveloping algebra of Vir. For each $c, h \in \mathbb{C}$, the Verma module $M(c,h)$ for Vir is the free $U(\mathrm{Vir}_-)$-module generated by a distinguished vector (the highest weight vector) $v_{c,h}$, subject to the relation

$$U(\mathrm{Vir}_+)v_{c,h} = 0, \qquad Cv_{c,h} = cv_{c,h}, \qquad L_0 v_{c,h} = hv_{c,h}. \tag{8.2}$$

Then there exists a unique maximal proper submodule $J(c,h)$ of $M(c,h)$. We let $L(c,h) = M(c,h)/J(c,h)$. It was proved in [FQS84] and [GKO86] that when $0 \leqslant c < 1$, the Vir-module $L(c,h)$ is unitarizable if and only if there exist $m, r, s \in \mathbb{Z}$ satisfying $2 \leqslant m, 1 \leqslant r \leqslant m-1, 1 \leqslant s \leqslant m$, such that

$$c = 1 - \frac{6}{m(m+1)}, \tag{8.3}$$

$$h = h_{r,s} = \frac{\big((m+1)r - ms\big)^2 - 1}{4m(m+1)}. \tag{8.4}$$

For such a module $L(c,h)$, we fix a unitary structure such that $\langle v_{c,h}|v_{c,h}\rangle$=1.

Let $\Omega = v_{c,0}, \nu = L_{-2}\Omega$. Then there exists a unique VOA structure on $L(c,0)$, such that Ω is the vacuum vector, and $Y(\nu,x) = \sum_{n\in\mathbb{Z}} L_n x^{-n-2}$ (cf.[FZ92]). Let $E = \{\Omega, \nu\}$, then E is a set of quasi-primary vectors generating $L(c,0)$.

We now assume that c satisfies relation (8.3). Then by [DL14] theorem 4.2 or [CKLW18] proposition 5.17, $L(c,0)$ is a unitary VOA. The PCT operator θ is determined by the fact that θ fixes vectors in E. $L(c,0)$ satisfies conditions $(\alpha),(\beta)$, and (γ) in the introduction. (See the introduction of [Hua08b], and the reference therein.)

Since $Y(\nu,n) = L_{n-1}$, representations of $L(c,0)$ are determined by their restrictions to Vir. By [Wang93] theorem 4.2, irreducible representations of $L(c,0)$ are precisely those that can be restricted to irreducible Vir-modules of the form $L(c,h_{r,s})$, where the highest weight $h_{r,s}$ satisfies relation (8.4). By proposition 1.10, $L(c,h_{r,s})$ is a unitary $L(c,0)$-module. It follows that any $L(c,0)$-module is unitarizable. Clearly the conformal dimension of $L(c,h_{r,s})$ is $h_{r,s}$.

Let $\mathcal{F} = \{L(c,h_{1,2}), L(c,h_{2,2})\}$. The fusion rules of $L(c,0)$ (see [Wang93] theorem 4.3) indicate that $\mathcal{F}$ is **generating**, i.e., any unitary $L(c,0)$-module is in $\mathcal{F}^{\boxtimes}$. We check that $\mathcal{F}$ satisfies condition A in section 5.3:

Condition A-(a): Since we know that any $L(c,0)$-module is unitarizable, condition A-(a) is obvious.

Condition A-(b): Since $E \subset E^1(L(c,0))$, $E^1(L(c,0))$ is generating.

Condition A-(c): If $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$ is unitary and irreducible (hence W_i, W_j, W_k restrict to irreducible highest weight Vir-modules), we choose a non-zero highest weight vector $v^{(i)} \in W_i$. We then define a linear map

$$\phi_\alpha : W_j \to W_k\{x\},$$
$$w^{(j)} \mapsto \phi_\alpha(x)w^{(j)} = \mathcal{Y}_\alpha(v^{(i)},x)w^{(j)}.$$

Then ϕ_α is a primary field in the sense of [Loke94] chapter II. By [Loke94] proposition IV.1.3, if $W_i \in \mathcal{F}$, then ϕ_α satisfies 0-th order energy bounds. This proves condition A-(c). Theorem 7.8 now implies the following:

Theorem 8.1. *Let $c = 1 - \frac{6}{m(m+1)}$ where $m = 2,3,4,\ldots$, and let $L(c,0)$ be the unitary Virasoro VOA with central charge c. Then any $L(c,0)$-module is unitarizable, Λ is positive definite on the tensor product of any two $L(c,0)$-modules, and the modular tensor category $\mathrm{Rep}^{\mathrm{u}}(L(c,0))$ of the unitary representations of $L(c,0)$ is unitary.*

8.2 Application to unitary affine VOAs

Let $\mathfrak{g}$ be a complex simple Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\lambda \in \mathfrak{h}^*$ be a highest weight, and let $L(\lambda)$ be the irreducible highest weight module of $\mathfrak{g}$ with highest weight λ and a distinguished highest (non-zero) vector $v_\lambda \in L(\lambda)$.

Choose the normalized invariant bilinear form $(\cdot,\cdot)$ satisfying $(\theta,\theta) = 2$, where θ is the highest root of $\mathfrak{g}$. Let $\widehat{\mathfrak{g}} = \mathrm{Span}_{\mathbb{C}}\{K, X(n) : X \in \mathfrak{g}, n \in \mathbb{Z}\}$ be the affine Lie algebra satisfying

$$[X(m), Y(n)] = [X,Y](m+n) + m(X,Y)\delta_{m,-n}K \qquad (X, Y \in \mathfrak{g}, m, n \in \mathbb{Z}),$$
$$[K, X(n)] = 0 \qquad (X \in \mathfrak{g}, n \in \mathbb{Z}).$$

Let $\mathfrak{g}_{\mathbb{R}}$ be a compact real form of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \oplus_{\mathbb{R}} i\mathfrak{g}_{\mathbb{R}}$. If W is a $\widehat{\mathfrak{g}}$-module, and the vector space W is equipped with an inner product $\langle\cdot|\cdot\rangle$, we say that W is a unitary $\widehat{\mathfrak{g}}$-module, if for any $X \in \mathfrak{g}_{\mathbb{R}}$ and $n \in \mathbb{Z}$, we have

$$X(n)^\dagger = -X(-n), \qquad K^\dagger = K. \tag{8.5}$$

Let $U(\widehat{\mathfrak{g}})$ be the universal enveloping algebra of $\widehat{\mathfrak{g}}$. Choose Lie subalgebras $\widehat{\mathfrak{g}}_+ = \mathrm{Span}_{\mathbb{C}}\{X(n) : X \in \mathfrak{g}, n > 0\}, \widehat{\mathfrak{g}}_- = \mathrm{Span}_{\mathbb{C}}\{X(n) : X \in \mathfrak{g}, n < 0\}$ of $\widehat{\mathfrak{g}}$. We regard $\mathfrak{g}$ as a Lie subalgebra of $\widehat{\mathfrak{g}}$ by identifying $X \in \mathfrak{g}$ with $X(0) \in \widehat{\mathfrak{g}}$. For any $k \in \mathbb{C}$, highest weight $\lambda \in \mathfrak{h}^*$, the Verma module $M(k,\lambda)$ for $\widehat{\mathfrak{g}}$ is the free $U(\widehat{\mathfrak{g}}_-)$-module generated by $L(\lambda)$ and subject to the conditions

$$U(\widehat{\mathfrak{g}}_+)L(\lambda) = 0, \qquad K|_{L(\lambda)} = k \cdot \mathrm{id}|_{L(\lambda)}. \tag{8.6}$$

We let $M(k,\lambda)$ be graded by $\mathbb{Z}_{\geqslant 0}$: For any $X_1, \dots, X_m \in \mathfrak{g}, n_1, \dots, n_m > 0, v \in L(\lambda)$, the weight of $X_1(-n_1)\cdots X_m(-n_m)v$ equals $n_1 + \cdots + n_m$. There exits a unique maximal proper graded submodule $J(k,\lambda)$ of $M(k,\lambda)$. We let $L(k,\lambda) = M(k,\lambda)/J(k,\lambda)$. Then by [Kac94] theorem 11.7, the $\widehat{\mathfrak{g}}$-module $L(k,\lambda)$ is unitarizable if and only if

$$k = 0, 1, 2, \dots, \tag{8.7}$$
$$\lambda \text{ is a dominant integral weight of } \mathfrak{g}\text{, and } (\lambda,\theta) \leqslant k. \tag{8.8}$$

For such a $\widehat{\mathfrak{g}}$-module $L(k,\lambda)$, we fix a unitary structure.

Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$. Let Ω be a highest weight vector of $L(k,0)$. It was proved in [FZ92] that when $k \neq -h^\vee$, there exists a unique VOA structure on $L(k,0)$, such that Ω is the vacuum vector, that for any $X \in \mathfrak{g}$ we have

$$Y\big(X(-1)\Omega, x\big) = \sum_{n\in\mathbb{Z}} X(n)x^{-n-1}, \tag{8.9}$$

and that the conformal vector ν is defined by

$$\nu = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{\dim\mathfrak{g}} X_i(-1)^2\Omega, \tag{8.10}$$

where $\{X_i\}$ is an orthonormal basis of $i\mathfrak{g}_{\mathbb{R}}$ under the inner product $(\cdot,\cdot)$. The set $E = \{\Omega, X(-1)\Omega : X \in \mathfrak{g}_{\mathbb{R}}\}$ generates $L(k,0)$. Writing the operator $L_1 = Y(\nu, 2)$ in terms of $X(n)$'s using Jacobi identity, one can show that the vectors in E are quasi-primary.

We now assume that $k \in \mathbb{Z}_{\geqslant 0}$. Then $L(k,0)$ satisfies conditions $(\alpha), (\beta)$, and (γ) in the introduction. (See the introduction of [Hua08b], and the reference therein.) By [DL14] theorem 4.7 or [CKLW18] proposition 5.17, $L(k,0)$ is a unitary VOA, and the PCT operator θ is determined by the fact that it fixes the vectors in E.

Representations of $L(k,0)$ are determined by their restrictions to $\widehat{\mathfrak{g}}$. By [FZ92] theorem 3.1.3, irreducible $L(k,0)$-modules are precisely those which can be restricted to the $\widehat{\mathfrak{g}}$-modules of the form $L(k,\lambda)$, where $\lambda \in \mathfrak{h}^*$ satisfies condition (8.8). By proposition 1.10, these $L(k,0)$-modules are unitary. Hence all $L(k,0)$-modules are unitarizable, and any set $\mathcal{F}$ of irreducible unitary $L(k,0)$-module satisfies condition A-(a).

By proposition 3.6, $E \subset E^1(L(k,0))$. Since E generates $L(k,0)$, any $\mathcal{F}$ also satisfies condition A-(b). Checking condition A-(c) is much harder, and requires case by case studies. Note that given the set $\mathcal{F}$, finding out which irreducible modules are inside $\mathcal{F}^{\boxtimes}$ requires the knowledge of fusion rules. A very practical way of calculating fusion rules for a unitary affine VOA is to calculate the dimensions of the spaces of primary fields.

Primary fields

Fix $k \in \mathbb{Z}_{>0}$. For each $\lambda \in \mathfrak{h}^*$ satisfying condition (8.8), we write $U_\lambda = L(\lambda), W_\lambda = L(k,\lambda)$. Let Δ_λ be the conformal dimension of the $L(k,0)$-module W_λ. We define the normalized energy operator on W_λ to be $D = L_0 - \Delta_\lambda$.

Assume that $\lambda, \mu, \nu \in \mathfrak{h}^*$ satisfy condition (8.8). We let $\Delta^\nu_{\lambda\mu} = \Delta_\lambda + \Delta_\mu - \Delta_\nu$. A **type** $\binom{\nu}{\lambda\ \mu}$ **primary field** ϕ_α is a linear map

$$\phi_\alpha : U_\lambda \otimes W_\mu \to W_\nu[[x^{\pm 1}]]x^{-\Delta^\nu_{\lambda\mu}},$$
$$u^{(\lambda)} \otimes w^{(\mu)} \mapsto \phi_\alpha(u^{(\lambda)}, x)w^{(\mu)} = \sum_{n\in\mathbb{Z}} \phi_\alpha(u^{(\lambda)}, n)w^{(\mu)}x^{-\Delta^\nu_{\lambda\mu}-n}$$
$$(\text{where } \phi_\alpha(u^\lambda, n) \in \mathrm{Hom}(W_\mu, W_\nu)),$$

such that for any $u^{(\lambda)} \in U_\lambda, X \in \mathfrak{g}, m \in \mathbb{Z}$, we have

$$[X(m), \phi_\alpha(u^{(\lambda)}, x)] = \phi_\alpha(Xu^{(\lambda)}, x)x^m, \tag{8.11}$$
$$[L_0, \phi_\alpha(u^{(\lambda)}, x)] = \Big(x\frac{d}{dx} + \Delta_\lambda\Big)\phi_\alpha(u^{(\lambda)}, x). \tag{8.12}$$

We say that U_λ is the **charge space** of ϕ_α.

Note that the above two conditions are equivalent to that for any $m, n \in \mathbb{Z}, u^{(\lambda)} \in U_\lambda, X \in \mathfrak{g}$,

$$[X(m), \phi_\alpha(u^{(\lambda)}, n)] = \phi_\alpha(Xu^{(\lambda)}, n+m), \tag{8.13}$$
$$[D, \phi_\alpha(u^{(\lambda)}, n)] = -n\phi_\alpha(u^{(\lambda)}, n). \tag{8.14}$$

Primary fields and intertwining operators are related in the following way: Let $\mathcal{V}_{\mathrm{p}}\binom{\nu}{\lambda\ \mu}$ be the vector space of type $\binom{\nu}{\lambda\ \mu}$ primary fields. If $\mathcal{Y}_\alpha \in \mathcal{V}\binom{\nu}{\lambda\ \mu}$ is a type $\binom{\nu}{\lambda\ \mu}$ intertwining operator of $L(k,0)$, then by relation (1.26), for any $w^{(\lambda)} \in W_\lambda$ we have,

$$\mathcal{Y}_\alpha(w^{(\lambda)}, x) = x^{L_0}\mathcal{Y}_\alpha(x^{-L_0}w^{(\lambda)}, 1)x^{-L_0} \in \mathrm{End}(W_\mu, W_\nu)[[x^{\pm 1}]]x^{-\Delta^\nu_{\lambda\mu}} \tag{8.15}$$

where $\mathcal{Y}_\alpha(\cdot, 1) = \mathcal{Y}_\alpha(\cdot, x)\big|_{x=1}$. We define a linear map $\phi_\alpha : U_\lambda \otimes W_\mu \to W_\nu[[x^{\pm 1}]]x^{-\Delta^\nu_{\lambda\mu}}$ to be the restriction of $\mathcal{Y}_\alpha$ to $U_\lambda \otimes W_\mu$. Then the Jacobi identity and the translation property for $\mathcal{Y}_\alpha$ implies that $\mathcal{Y}_\alpha$ satisfies equations (8.11) and (8.12). Therefore, we have a linear map

$$\Phi : \mathcal{V}\begin{pmatrix} \nu \\ \lambda\ \mu \end{pmatrix} \to \mathcal{V}_{\mathrm{p}}\begin{pmatrix} \nu \\ \lambda\ \mu \end{pmatrix}, \qquad \mathcal{Y}_\alpha \mapsto \phi_\alpha. \tag{8.16}$$

The injectivity of Φ follows immediately from relation (1.22) or from corollary 2.15. Φ is also surjective. Indeed, if we fix any $z \in \mathbb{C}^\times$ and define another linear map

$$\Psi_z : \mathcal{V}_{\mathrm{p}}\begin{pmatrix} \nu \\ \lambda\ \mu \end{pmatrix} \to (W_\lambda \otimes W_\mu \otimes W_{\overline{\nu}})^*,$$
$$\phi_\alpha \mapsto \phi_\alpha(\cdot, z) = \phi_\alpha(\cdot, x)\big|_{x=z},$$

then by equation (8.12), Ψ_z is injective. By equation (8.11) and [Ueno08] theorem 3.18, the dimension of the image of Ψ_z is no greater than that of "the space of vacua" $\mathcal{V}^\dagger_{\mu\lambda\overline{\nu}}(\mathbb{P}^1; 0, z, \infty)$ defined in [TUY89] and [Ueno08]. The later can be calculated using the Verlinde formula proved in [Bea94], [Fal94], and [Tel95]. The same Verlinde formula for $N^\nu_{\lambda\mu}$ proved in [Hua08a] shows that the dimension of the vector space $\mathcal{V}\binom{\nu}{\lambda\ \mu}$ (which is the fusion rule $N^\nu_{\lambda\mu}$) equals that of $\mathcal{V}^\dagger_{\mu\lambda\overline{\nu}}(\mathbb{P}^1; 0, z, \infty)$. So $\dim \mathcal{V}_{\mathrm{p}}\binom{\nu}{\lambda\ \mu} \leqslant N^\nu_{\lambda\mu}$, and hence Φ must be surjective. We conclude the following:

Proposition 8.2. *The linear map Φ defined in (8.16) is an isomorphism. In particular, the fusion rule $N^\nu_{\lambda\mu}$ of $L(k,0)$ equals the dimension of the vector space of type $\binom{\nu}{\lambda\ \mu}$ primary fields of $L(k,0)$.*

Theorem 8.3. *Let $k = 0, 1, 2, \ldots$, and let $L(k,0)$ be the level k unitary affine VOA associated to $\mathfrak{g}$. Then any $L(k,0)$-module is unitarizable. Suppose that $\mathcal{F}$ is a* generating *set of irreducible unitary $L(k,0)$-modules (i.e., $\mathcal{F}^{\boxtimes}$ contains any unitary $L(k,0)$-module), and that for any $\lambda \in \mathcal{F}$, all primary fields of $L(k,0)$ with charge spaces U_λ are energy-bounded. Then Λ is positive definite on the tensor product of any two $L(k,0)$-modules, and the modular tensor category $\mathrm{Rep}^{\mathrm{u}}(L(k,0))$ is unitary.*

We now show that theorem 8.3 can be applied to the unitary affine VOAs of type A_n and D_n.

The case $\mathfrak{g} = \mathfrak{sl}_n$ $(n \geqslant 2)$

Let $L(\square)$ be the (n-dimensional) vector representation of $\mathfrak{sl}_n$, and let

$$\mathcal{F} = \{L(k, \square)\}.$$

In [Was98], especially in section 25, it was proved that if $\lambda = \Box$ and the weights μ, ν of $\mathfrak{sl}_n$ satisfy condition (8.8), then

$$\dim \mathcal{V}_{\mathrm{p}}\binom{\nu}{\lambda\ \mu} = \dim \Big(\mathrm{Hom}_{\mathfrak{g}}\big(L(\lambda) \otimes L(\mu), L(\nu)\big)\Big). \tag{8.17}$$

(Note that this relation is not true for general $L(\lambda)$.) Using this relation, one can show that $\mathcal{F}$ is generating. In the same section, it was proved that any $\phi_\alpha \in \mathcal{V}_{\mathrm{p}}\binom{\nu}{\Box\ \mu}$ satisfies 0-th order energy bounds.

The case $\mathfrak{g} = \mathfrak{so}_{2n}$ $(n \geqslant 3)$

Let $L(\Box)$ be the vector representation of $\mathfrak{so}_{2n}$, and let $L(s_+)$ and $L(s_-)$ be the two half-spin representations of $\mathfrak{so}_{2n}$. In [TL04] chapter IV, it was proved that if λ equals $\Box$ or $s_\pm$, and the weights μ, ν of $\mathfrak{so}_{2n}$ satisfy condition (8.8), then relation (8.17) holds. This shows that the set

$$\mathcal{F} = \{L(k, \Box), L(k, s_+), L(k, s_-)\}$$

is generating. By [TL04] theorem VI.3.1, any primary field whose charge space is $L(k, \Box), L(k, s_+)$, or $L(k, s_-)$ is energy-bounded.

We conclude the following.

Theorem 8.4. *Let $\mathfrak{g}$ be $\mathfrak{sl}_n$ $(n \geqslant 2)$ or $\mathfrak{so}_{2n}$ $(n \geqslant 3)$, let $k = 0, 1, 2, \ldots$, and let $L(k, 0)$ be the unitary affine VOA associated to $\mathfrak{g}$. Then Λ is positive definite on the tensor product of any two $L(k, 0)$-modules, and the modular tensor category $\mathrm{Rep}^{\mathrm{u}}(L(k, 0))$ of the unitary representations of $L(k, 0)$ is unitary.*

Other examples

As we see in theorem 8.3, to finish proving the unitarity of the modular tensor categories associated to unitary affine VOAs, one has to show, for the remaining types, that a "generating" set of primary fields are energy-bounded. The success in type A_n and D_n unitary WZW models, as well as in unitary minimal models, shows that achieving this goal is promising. Indeed, the main idea of proving the energy-boundedness of a primary field in [Was98], [Loke94], and [TL04] is to embedd the original VOA V in a larger (super) VOA $\widetilde{V}$, the energy-boundedness of the field operators of which is easy to show, and realize the primary field as the compression of the vertex operator or an energy-bounded intertwining operator of $\widetilde{V}$. This strategy can in fact be successfully carried out for the other classical Lie types (B and C) and for type G_2 (cf. [Gui19b]). We hope that it will also work for the remaining exceptional types E and F_4.

8.3 Full conformal field theory with reflection positivity

In this section, we give an interpretation of our unitarity results from the perspective of full conformal field theory. In [HK07], Y.Z.Huang and L.Kong constructed a (genus

0) full conformal field theory for V called "diagonal model". This construction relies on the non-degeneracy of a bilinear form on each pair $\mathcal{V}\binom{k}{i\,j}\otimes\mathcal{V}\binom{\overline{k}}{\overline{i}\,\overline{j}}$, which follows from the rigidity of $\mathrm{Rep}(V)$. These bilinear forms $(\cdot,\cdot)$ are directly related to our sesquilinear forms $\Lambda(\cdot|\cdot)$ on each $\mathcal{V}\binom{k}{i\,j}$:

$$(\mathcal{Y}_\alpha,\mathcal{Y}_{\overline{\beta}})=\Lambda(\mathcal{Y}_\alpha|\mathcal{Y}_\beta)\qquad(\alpha,\beta\in\mathcal{V}\binom{k}{i\,j}).\tag{8.18}$$

In light of this relation, we sketch the construction of diagonal model in [HK07] from a unitary point of view.

Let us assume that V is unitary, all V-modules are unitarizable, and all transport matrices are positive definite. (This last condition holds for V if there exists a generating set $\mathcal{F}$ of irreducible unitary V-modules satisfying condition A of B in section 5.3.) We define a vector space

$$F=\bigoplus_{i\in\mathcal{E}}W_i\otimes W_{\overline{i}}.\tag{8.19}$$

Its algebraic completion is $\widehat{F}=\bigoplus_{i\in\mathcal{E}}\widehat{W_i}\otimes\widehat{W_{\overline{i}}}$.

For each $i,j,k\in\mathcal{E}$, we choose an orthonormal basis Θ^k_{ij} of $\mathcal{V}\binom{k}{i\,j}$ under the inner product Λ. The full field operator $\mathbb{Y}$ is defined to be an $\mathrm{End}(F\otimes F,\widehat{F})$-valued continuous function on $\mathbb{C}^\times$, such that for any $w^{(i)}_L\otimes\overline{w^{(i)}_R}\in W_i\otimes W_{\overline{i}}\subset F$, $w^{(j)}_L\otimes\overline{w^{(j)}_R}\in W_j\otimes W_{\overline{j}}\subset F$,

$$\mathbb{Y}(w^{(i)}_L\otimes\overline{w^{(i)}_R};z,\overline{z})(w^{(j)}_L\otimes\overline{w^{(j)}_R})=\sum_{k\in\mathcal{E}}\sum_{\alpha\in\Theta^k_{ij}}\mathcal{Y}_\alpha(w^{(i)}_L,z)w^{(j)}_L\otimes\mathcal{Y}_{\overline{\alpha}}(\overline{w^{(i)}_R},\overline{z})\overline{w^{(j)}_R}.\tag{8.20}$$

Then $(F,\mathbb{Y})$ is a full field algebra of V satisfying certain important properties, including commutativity ([HK07] proposition 1.5) and associativity ([HK07] proposition 1.4). In fact, in our unitarity context, it is not hard for the reader to check that these two properties are equivalent to the unitarity of braid matrices and fusion matrices respectively. $(F,\mathbb{Y})$ also satisfies modular invariance ([HK10] proposition 5.1), which is indeed equivalent ([HK10] theorem 3.8) to the unitarity of the projective representation of $SL_2(\mathbb{Z})$ on the vector space of the traces of the intertwining operators. This in turn is equivalent ([HK10] theorem 4.11) to the unitarity of projective representation of $SL_2(\mathbb{Z})$ in the unitary modular tensor category $\mathrm{Rep}^{\mathrm{u}}(V)$ proved by [Kir96] theorem 2.5.

Let us equip the vector space F with an *inner product* $\langle\cdot|\cdot\rangle$, such that the decomposition (8.19) is orthogonal, and for any $i\in\mathcal{E}$, $w^{(i)}_{L,1},w^{(i)}_{R,1},w^{(i)}_{L,2},w^{(i)}_{R,2}\in W_i$,

$$\langle w^{(i)}_{L,1}\otimes\overline{w^{(i)}_{R,1}}|w^{(i)}_{L,2}\otimes\overline{w^{(i)}_{R,2}}\rangle=d_i^{-1}\langle w^{(i)}_{L,1}|w^{(i)}_{L,2}\rangle\langle w^{(i)}_{R,2}|w^{(i)}_{R,1}\rangle.\tag{8.21}$$

We also define an antilinear operator $\theta:F\to F$ sending each $w^{(i)}_L\otimes\overline{w^{(i)}_R}$ to $\overline{w^{(i)}_L}\otimes w^{(i)}_R$, which is easily checked to be an anti-automorphism:

$$\theta Y(w;z,\overline{z})=Y(\theta w;\overline{z},z)\theta\qquad(w\in F).\tag{8.22}$$

We call θ the **PCT** operator of $(F, \mathbb{Y})$.

Note that when V is non-unitary, we can only define a non-degenerate bilinear form on F, and show that under this bilinear form, the full field algebra $(F, \mathbb{Y})$ satisfies the *invariance property* ([HK07] definition 3.9). But in our case, this invariance property should be replaced by the **reflection positivity**:

$$\mathbb{Y}(w; z, \overline{z})^\dagger = \mathbb{Y}\big(\theta \cdot e^{zL_1^L + \overline{z}L_1^R}(e^{-i\pi}z^{-2})^{L_0^L}(\overline{e^{-i\pi}z^{-2}})^{L_0^R}w; \overline{z^{-1}}, z^{-1}\big) \qquad (w \in F), \tag{8.23}$$

where for each $n \in \mathbb{Z}$,, the linear operators $L_n^L = L_n \otimes 1$, $L_n^R = 1 \otimes L_n$ are defined on F. The factor $e^{-i\pi}$ in equation (8.23) can be replaced by any $e^{i(2n+1)\pi}$, where $n \in \mathbb{Z}$. The reflection positivity is equivalent to the fact that for any $i, j, k \in \mathcal{E}, \mathcal{Y}_\alpha, \mathcal{Y}_\beta \in \mathcal{V}\binom{k}{i\ j}$,

$$\langle \mathcal{Y}_{C\alpha} | \mathcal{Y}_{C\beta} \rangle = \frac{d_k}{d_j} \langle \mathcal{Y}_\alpha | \mathcal{Y}_\beta \rangle. \tag{8.24}$$

This relation is essentially proved in [HK07] using properties of the fusion matrices of intertwining operators. We remark that it can also be proved using graphical calculations for ribbon fusion categories.

A final remark: The positivity of Λ is not used in full power to prove the reflection positivity of F. One only uses the positivity of quantum dimensions d_i's and the fact that Λ is Hermitian (i.e., $\Lambda(\mathcal{Y}_\alpha | \mathcal{Y}_\beta) = \overline{\Lambda(\mathcal{Y}_\beta | \mathcal{Y}_\alpha)}$), which can be checked more directly without doing long and tedious analysis as in our papers. So unlike the non-degeneracy of Λ, which is of significant importance in the construction of the full field algebras of diagonal models, the positivity of Λ only plays a marginal role. However, we expect that a systematic treatment of all full rational CFTs (but not just diagonal models) with reflection positivity will rely heavily on the unitarity of the MTC of V, and hence on the positivity of Λ.

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