Categorical extensions of Conformal Nets are Haag-Kastler nets of charged fields for chiral CFT.

- Motivation: Relate VOA and Conformal net tensor categories. This can be done by relating charged fields on both sides (Motivated by A. Wassermann 98' computing fusion rules)
- Categorial extensions give a unified description of VOA and Conformal net charged fields.

- Wightman axioms $\longleftrightarrow$ Haag-Kastler Nets

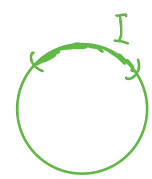
  $\varphi(x)$ $\qquad\qquad$ $O \mapsto \mathcal{A}(O)$

- 2d Chiral CFT : (Carpi-Kawahigashi-Longo-Weiner 18')

Vertex operator algebra $\longleftrightarrow$ Conformal Net $\mathcal{A}$

$$\varphi(z) = \sum_{n\in\mathbb{Z}} \varphi_n z^n \qquad\qquad I \subset S^1 \longmapsto \mathcal{A}(I)$$

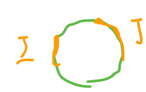
open connected non-dense interval of S^1



$$\varphi(f) = \int_I \varphi(z) f(z)\, dz \qquad f \in C_c^\infty(I)$$

then $\mathcal{A}(I) = vN\{\varphi(f) : \varphi \in V,\ f \in C_c^\infty(I)\}$

- Locality: If $f \in C_c^\infty(I)$, $g \in C_c^\infty(J)$, $I \cap J = \phi$, $\varphi, \psi \in V$, then $[\varphi(f), \psi(g)] = 0$.



  Also, $\varphi(f)^* = \varphi^*(\bar{f})$ for some $\varphi^* \in V$. So $[\varphi(f)^*, \psi(g)] = 0$.

This is the adjoint commutativity of $\varphi(f), \psi(g)$.

- (Strong) Locality: $[\mathcal{A}(I), \mathcal{A}(J)] = 0$ if $I \cap J = \phi$

  or $\quad [vN(\varphi(f)), vN(\psi(g))] = 0$

  $\varphi(f)^*$

- Categorical extension of $\mathcal{A}$ is the Haag-Kastler net of charged fields.

  A charged field is $\varphi(z) : W_i \longrightarrow W_j$

  where W_i, W_j are representations of V.

  If W_i, W_j are irreducible, $\varphi(z) = \sum_{n \in \mathbb{Z}} \varphi_n z^{n+\Delta}$

  for some $\Delta \in \mathbb{R}$.

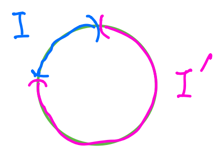
- $\mathcal{A}$ acts on the vacuum rep. $\mathcal{H}_0$.

  $\Omega \in \mathcal{H}_0$ is the vacuum vector.

  Cat. Ext. is modeled on Connes fusion of $\mathcal{A}$-rep. (Connes, Wassermann)

- Let $\mathcal{H}_i, \mathcal{H}_j, \mathcal{H}_k, \cdots$ be rep. of $\mathcal{A}$.

  $I' =$ interior $(S^1 - I)$



$$\mathrm{Hom}_{\mathcal{A}(I')}(\mathcal{H}_0, \mathcal{H}_i)$$

$= \{$ Bounded linear operators from $\mathcal{H}_0 \to \mathcal{H}_i$ intertwing the actions of $\mathcal{A}(I')\}$

$$\mathcal{H}_i(I) = \mathrm{Hom}_{\mathcal{A}(I')}(\mathcal{H}_0, \mathcal{H}_i) \cdot \Omega \overset{\text{dense}}{\subset} \mathcal{H}_i$$

$$\begin{matrix} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \xi & = & Z(\xi, I)\,\Omega \end{matrix}$$

$Z(\xi, I) : \mathcal{H}_0 \to \mathcal{H}_i$ is uniquely determined by ξ

(State - field correspondence)

We say vectors in $\mathcal{H}_i(I)$ to be <u>I-bounded</u>

- If $\xi_1, \xi_2 \in \mathcal{H}_i(I)$, then

$$\mathcal{H}_0 \xrightarrow{Z(\xi_1, I)} \mathcal{H}_i(I) \xrightarrow{Z(\xi_2, I)^*} \mathcal{H}_0$$

commutes with $\mathcal{A}(I')$. So by Haag-duality, it is in $\mathcal{A}(I)$.

So $\underbrace{Z(\xi_2, I)^* Z(\xi_1, I)}$ acts on any rep. $\mathcal{H}_j$.

- Assume $\xi \in \mathcal{H}_i(I)$

We let $Z(\xi, I)$ act not only on $\mathcal{H}_0$ but on any $\mathcal{H}_j$:

$$Z(\xi, I) : \mathcal{H}_j \longrightarrow \mathcal{H}_i(I) \boxtimes \mathcal{H}_j = \mathcal{H}_j \boxtimes \mathcal{H}_i(I)$$

described as follows

- $\mathcal{H}_i(I) \boxtimes \mathcal{H}_j$: Hilbert space generated by $\underset{\xi}{\cup\!\!\!\mid} \boxtimes \underset{\eta}{\cup\!\!\!\mid}$

$\xi \boxtimes \eta$ Inner product:

$$\langle \xi_1 \boxtimes \eta_1 \mid \xi_2 \boxtimes \eta_2 \rangle$$

$$= \langle Z(\xi_2, I)^* Z(\xi_1, I) \eta_1 \mid \eta_2 \rangle$$

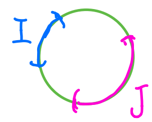
Then $Z(\xi, I) : \mathcal{H}_j \to \mathcal{H}_i(I) \boxtimes \mathcal{H}_j = \mathcal{H}_j \boxtimes \mathcal{H}_i(I)$,

$$\eta \longmapsto \xi \boxtimes \eta$$

$\mathcal{H}_i(I) \boxtimes \mathcal{H}_j$ is naturally a rep. of $\mathcal{A}$

(Bartels – Douglas – Henriques 17', G)

- Locality: Assume $I \cap J = \phi$



$$(\mathcal{H}_i(I) \boxtimes \mathcal{H}_k) \boxtimes \mathcal{H}_j(J) = \mathcal{H}_i(I) \boxtimes (\mathcal{H}_k \boxtimes \mathcal{H}_j(J))$$

written as $\underset{\substack{\rotatebox{90}{$\in$} \\ \xi}}{\mathcal{H}_i(I)} \boxtimes \underset{\substack{\rotatebox{90}{$\in$} \\ \psi}}{\mathcal{H}_k} \boxtimes \underset{\substack{\rotatebox{90}{$\in$} \\ \eta}}{\mathcal{H}_j(J)}$

$$(\xi \boxtimes \psi) \boxtimes \eta = \xi \boxtimes (\psi \boxtimes \eta)$$

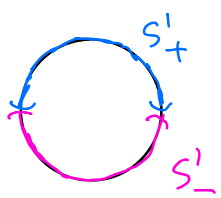
$$\begin{tikzcd}
\mathcal{H}_k \arrow[r, "Z(\eta, J)"] \arrow[d, "Z(\xi, I)"'] & \mathcal{H}_k \boxtimes \mathcal{H}_j(J) \arrow[d, "Z(\xi, I)"] \\
\mathcal{H}_i(I) \boxtimes \mathcal{H}_k \arrow[r, "Z(\eta, J)"] & \mathcal{H}_i(I) \boxtimes \mathcal{H}_k \boxtimes \mathcal{H}_j(J)
\end{tikzcd}$$

Moreover, this diagram commutes adjointly, i.e., this diagram and the following both commute:

$$\begin{tikzcd}
\mathcal{H}_K \arrow[r, "Z(\eta, J)"] & \mathcal{H}_K \boxtimes \mathcal{H}_j(J) \\
\mathcal{H}_i(I) \boxtimes \mathcal{H}_K \arrow[u, "Z(\xi, I)^*"] \arrow[r, "Z(\eta, J)"] & \mathcal{H}_i(I) \boxtimes \mathcal{H}_K \boxtimes \mathcal{H}_j(J) \arrow[u, "Z(\xi, I)^*"']
\end{tikzcd}$$

We call this the adjoint commutativity of $Z(\xi, I)$, $Z(\eta, J)$. (Recall $I \cap J = \phi$)

We now get rid of I, J in Connes fusion.

- 

$\mathcal{H}_i \boxtimes \mathcal{H}_j \neq \mathcal{H}_j \boxtimes \mathcal{H}_i$

$= \mathcal{H}_j \boxtimes \mathcal{H}_i(S^1_+)$

Define $\mathcal{H}_i \boxtimes \mathcal{H}_j = \underline{\mathcal{H}_i(S^1_+) \boxtimes \mathcal{H}_j}$.

identified with $\underline{\mathcal{H}_i \boxtimes \mathcal{H}_j(S^1_-)}$

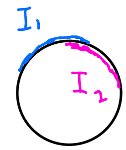
via $Z(\xi, S^1_+)\eta = Z(\eta, S^1_-)\xi$

if $\xi \in \mathcal{H}_i(S^1_+)$, $\eta \in \mathcal{H}_j(S^1_-)$

$\mathcal{H}_i(I_1) \boxtimes \mathcal{H}_j \qquad \mathcal{H}_i(I_2) \boxtimes \mathcal{H}_j$

- How to relate $\mathcal{H}_i(I) \boxtimes \mathcal{H}_j$ with $\mathcal{H}_i(S^1_+) \boxtimes \mathcal{H}_j$?

- Notice : If I_1, I_2 are "close"



in the sense that $I_1 \cap I_2$ is a non-empty connected open interval

$\mathcal{H}_i(I_1 \cap I_2) \boxtimes \mathcal{H}_j$

These two example are not close.

We have $\mathcal{H}_i(I_1) \boxtimes \mathcal{H}_j \xrightarrow{\cong} \mathcal{H}_i(I_2) \boxtimes \mathcal{H}_j$.

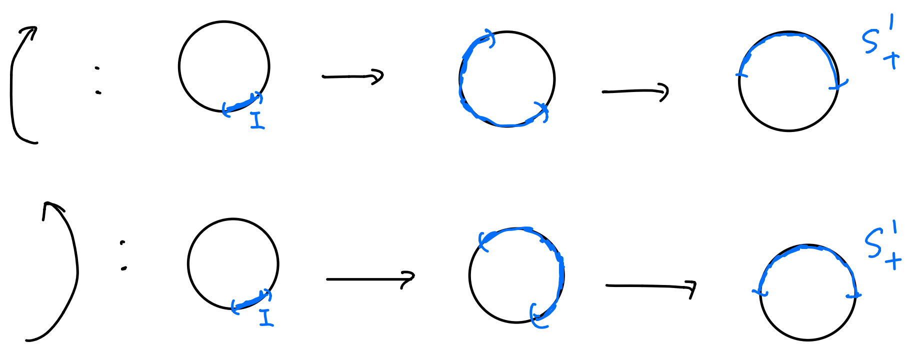
$$\xi \boxtimes \eta \longmapsto \xi \boxtimes \eta$$

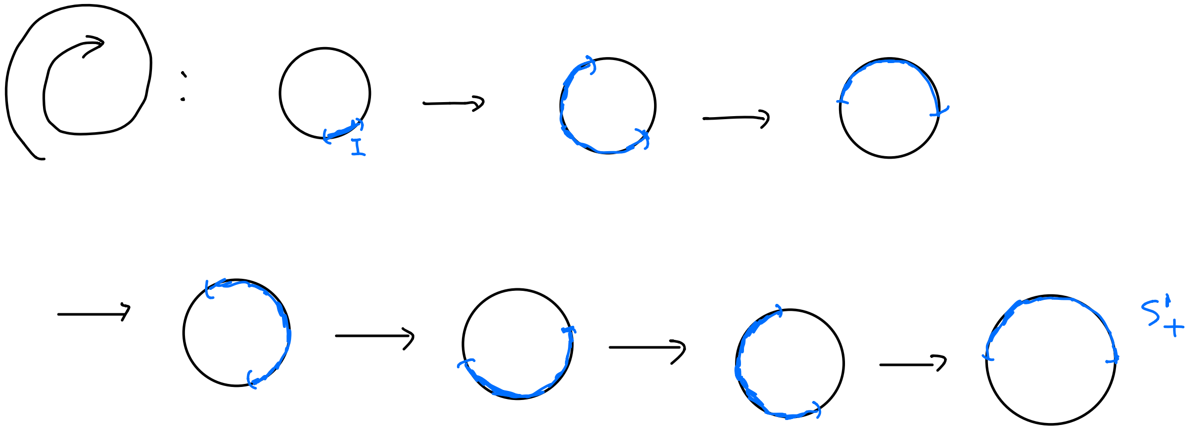
if $\xi \in \mathcal{H}_i(I_1 \cap I_2)$

- We have isomorphisms $\mathcal{H}_i(I) \boxtimes \mathcal{H}_j \xrightarrow{\cong} \mathcal{H}_i(S^1_+) \boxtimes \mathcal{H}_j$

$$\| $$

$$\mathcal{H}_i \boxtimes \mathcal{H}_j$$

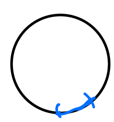


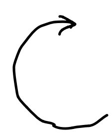


(These three might be different isomorphisms)

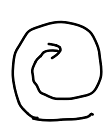
- $\tilde{I} = (I, \arg_I)$ where $\arg_I$ is a continuous function on I satisfying $\arg_I(z) = \arg(z)$

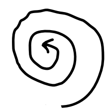
$\tilde{I}$ uniquely determines a path $\alpha_{\tilde{I}}$ from I to S^1_+ s.t. $\arg_I$ changes continuously to $(0, \pi)$

Ex: Let I =  $\tilde{I} = (I, \arg_I)$

$\arg_I : \left(\frac{3}{2}\pi, \frac{7}{4}\pi\right)$ 

$\arg_I : \left(-\frac{1}{2}\pi, -\frac{1}{4}\pi\right)$ 

$\arg_I : \left(\frac{3}{2}\pi + 2\pi, \frac{7}{4}\pi + 2\pi\right)$ 

$\arg_I : \left(-\frac{1}{2}\pi - 4\pi, -\frac{1}{4}\pi - 4\pi\right)$ 

- $\dot{\alpha}_{\tilde{I}} : \mathcal{H}_i(I) \boxtimes \mathcal{H}_j \xrightarrow{\cong} \mathcal{H}_i(S^1_+) \boxtimes \mathcal{H}_j$

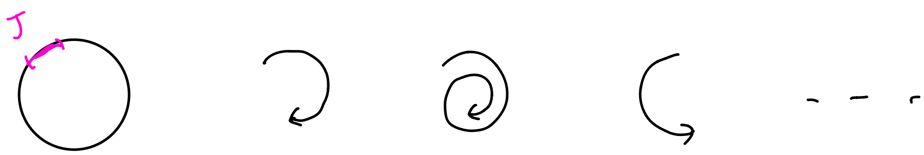
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad \mathcal{H}_i \boxtimes \mathcal{H}_j$

determined by $\alpha_{\tilde{I}}$.

Similary, $\beta_{\tilde{J}}$ the path from $\tilde{J}$ to S^1_-

s.t. $\arg_J$ changes continuously to $(-\pi, 0)$



$\dot{\beta}_{\tilde{J}} : \mathcal{H}_i \boxtimes \mathcal{H}_j(J) \xrightarrow{\cong} \mathcal{H}_i \boxtimes \mathcal{H}_j(S^1_-)$

$\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad \mathcal{H}_i \boxtimes \mathcal{H}_j.$

- Define, for each $\mathcal{H}_k$, $\xi \in \mathcal{H}_i(I)$, $\eta \in \mathcal{H}_j(J)$,

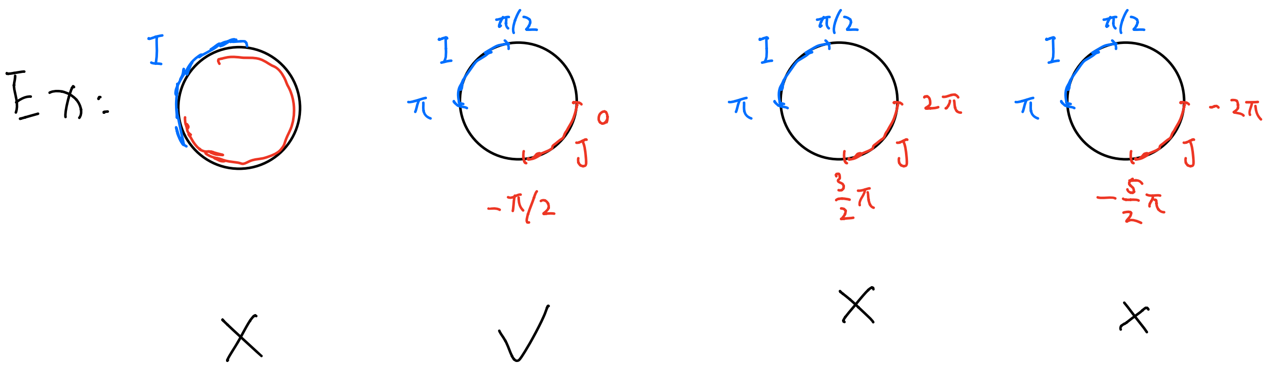
$$L(\xi, \tilde{I}) : \mathcal{H}_k \xrightarrow{Z(\xi, I)} \mathcal{H}_i(I) \boxtimes \mathcal{H}_k \xrightarrow{\dot{\alpha}_{\tilde{I}}} \mathcal{H}_i \boxtimes \mathcal{H}_k$$

$$R(\eta, \tilde{J}) : \mathcal{H}_k \xrightarrow{Z(\eta, J)} \mathcal{H}_k \boxtimes \mathcal{H}_j(J) \xrightarrow{\dot{\beta}_{\tilde{J}}} \mathcal{H}_k \boxtimes \mathcal{H}_j.$$

We say $\tilde{J}$ is clockwise to $\tilde{I}$

if $I \cap J = \phi$, $\quad \arg_I - 2\pi < \arg_J < \arg_I$

Ex:

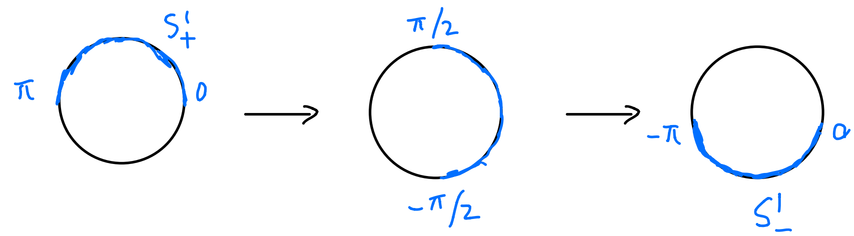


• <u>Locality axiom:</u>

$$\begin{tikzcd}
\mathcal{H}_k \arrow[r, "R(\eta, \tilde{J})"] \arrow[d, "L(\xi, \tilde{I})"'] & \mathcal{H}_k \boxtimes \mathcal{H}_j \arrow[d, "L(\xi, \tilde{I})"] \\
\mathcal{H}_i \boxtimes \mathcal{H}_k \arrow[r, "R(\eta, \tilde{J})"] & \mathcal{H}_i \boxtimes \mathcal{H}_k \boxtimes \mathcal{H}_j \arrow[d, equal] \\
 & \mathcal{H}_i(S^1_+) \boxtimes \mathcal{H}_k \boxtimes \mathcal{H}_j(S^1_-)
\end{tikzcd}$$

<u>Commutes adjointly</u> if $\xi \in \mathcal{H}_i(I)$ $\eta \in \mathcal{H}_j(J)$

and $\tilde{J}$ is clockwise to $\tilde{I}$

- The path ρ: 

induces $\rho^\bullet : \mathcal{H}_i(S^1_+) \boxtimes \mathcal{H}_k \xrightarrow{\cong} \mathcal{H}_i(S^1_-) \boxtimes \mathcal{H}_k$

$$\mathcal{H}_i(S^1_+) \boxtimes \mathcal{H}_k = \mathcal{H}_i \boxtimes \mathcal{H}_k$$

$$\mathcal{H}_i(S^1_-) \boxtimes \mathcal{H}_k = \mathcal{H}_k \boxtimes \mathcal{H}_i(S^1_-) = \mathcal{H}_k \boxtimes \mathcal{H}_i$$

<u>Braiding axiom:</u>

$$\begin{tikzcd}
& \mathcal{H}_k \arrow[dl, "L(\xi, \widetilde{I})"'] \arrow[dr, "R(\xi, \widetilde{I})"] & \\
\mathcal{H}_i \boxtimes \mathcal{H}_k \arrow[rr, "\rho^\bullet", "\cong"'] & & \mathcal{H}_k \boxtimes \mathcal{H}_i
\end{tikzcd}$$

Theorem: $\rho^\bullet$ satisfies Hexagon axioms

It is the braiding

- Categorical extensions of $\mathcal{A}$:

Axioms of $L(\xi, \widetilde{I})$, $R(\eta, \widetilde{J})$.