Twisted/untwisted correspondence in permutation orbifold conformal field theory

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Untwisted chiral CFT

- A Vertex Operator Algebra (VOA) $\mathbb{V}$, or a conformal net.
- $\mathrm{Rep}(\mathbb{V})$, the category of $\mathbb{V}$-modules (satisfying...)
- $\mathrm{Rep}(\mathbb{V})$ is an abelian category. To make $\mathrm{Rep}(\mathbb{V})$ a tensor category, we need to study conformal blocks associated to $\mathbb{V}$-modules. These are certain data associated to $\mathbb{V}$-modules and pointed Riemann surfaces (assuming compactness throughout the talk).
- Genus-0 conformal blocks contain all the information of $\mathrm{Rep}(\mathbb{V})$. (Folklore: Higher genus data are determined by genus-0 ones.) But to prove deeper properties (e.g. rigidity, modularity, Verlinde formula, modular invariance, etc.), it is also necessary to study genus-1 conformal blocks.

Twisted chiral CFT (i.e. orbifold theory)

- A (say) finite group G of automorphisms of a VOA $\mathbb{U}$.
- We want to study $\mathrm{Rep}(\mathbb{U}^G)$ where $\mathbb{U}^G = \{u \in \mathbb{U} : gu = u \ \forall g \in G\}$. Since any $\mathbb{U}$-module automatically restricts to a $\mathbb{U}^G$-module, $\mathrm{Rep}(\mathbb{U}^G)$ contains more information than $\mathrm{Rep}(\mathbb{U})$, and one cannot recover $\mathrm{Rep}(\mathbb{U}^G)$ from $\mathrm{Rep}(\mathbb{U})$.
- In good cases (e.g. $\mathbb{U}^G$ is C_2-cofinite), the study of $\mathrm{Rep}(\mathbb{U}^G)$ is more or less equivalent to the study of $\mathrm{Rep}^G(\mathbb{U})$, which is the tensor category of G-twisted $\mathbb{U}$-modules. (Kirillov, Müger, McRae, etc.)

Permutation orbifold CFT

- For permutation orbifold CFTs, twisted modules and their conformal blocks can be reduced to the study of untwisted ones. We call this the permutation-twisted/untwisted correspondence.
- More precisely: Let E be a finite set ($\simeq \{1, 2, \ldots, |E|\}$), let G be $\mathrm{Aut}(E) = \{$permutations of $E\}$ or its subgroup, acting by permutation on $\mathbb{U} = \mathbb{V}^{\otimes E}$.
- The correspondence says that G-twisted $\mathbb{V}^{\otimes E}$-modules and their conformal blocks can be constructed from (untwisted) $\mathbb{V}$-modules and conformal blocks. Moreover, the Riemann surface C on the untwisted side should be a (possibly) branched covering of the one C_0 on the twisted side.

What are twisted modules?

- Recall that $\mathbb{V}$ is a vector space together with a vertex operation Y associating linearly to each u an operator-valued holomorhic function $Y(u,z)$ over $z \in \mathbb{C}^\times = \mathbb{C}\backslash\{0\}$. $Y(u,z)$ is a linear operator from $\mathbb{V}$ to the "algebraic closure" $\mathbb{V}^{\mathrm{cl}}$. But each $Y(u)_n = \mathrm{Res}_{z=0} Y(u,z) z^n \frac{dz}{2\mathbf{i}\pi}$ is a genuine linear operator on $\mathbb{V}$.
- A crucial property for Y is the Jacobi identity $Y(u,z)Y(v,\zeta) \sim Y(v,\zeta)Y(u,z) \sim Y(Y(u,z-\zeta)v,\zeta)$.
- For an untwisted $\mathbb{V}$-module $\mathbb{W}$, we also have a similar vertex operation $Y_{\mathbb{W}}$. This time, for each $u \in \mathbb{V}$, $Y_{\mathbb{W}}(u,z) : \mathbb{W} \to \mathbb{W}^{\mathrm{cl}}$, and for each $u, v \in \mathbb{V}$, $Y_{\mathbb{W}}(u,z)Y_{\mathbb{W}}(v,\zeta) \sim Y_{\mathbb{W}}(v,\zeta)Y_{\mathbb{W}}(u,z) \sim Y_{\mathbb{W}}(Y(u,z-\zeta)v,\zeta)$.

What are twisted modules?

- Now, let $G \curvearrowright \mathbb{U}$. If $g \in G$ (with order $|g|$) and $\mathcal{W}$ is a g-twisted module, then $Y(u,z) : \mathcal{W} \to \mathcal{W}^{\mathrm{cl}}$ is single-valued over $z^{1/|g|} \in \mathbb{C}^\times$, and hence $Y(u,z)$ depends on $\arg z$. And
$$Y_{\mathcal{W}}(gu,z) = Y_{\mathcal{W}}(u, e^{-2\mathbf{i}\pi} z)$$
Here $\arg(e^{-2\mathbf{i}\pi}z) = -2\pi + \arg z$.

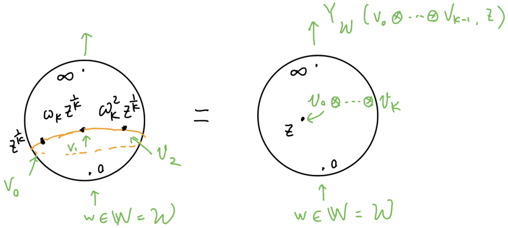
Construction of permutation-twisted modules

- Due to Barron-Dong-Mason ('02)
- First, assume $g \in G$ has only one orbit, i.e. $E = \{0, 1, \ldots, k-1\}$, $g = (012 \cdots k-1)$. Let $\omega_k = e^{-2\mathbf{i}\pi/k}$. Then for each $\mathbb{V}$-module $\mathbb{W}$, we can construct a g-twisted $\mathbb{V}^{\otimes E}$-module $\mathcal{W}$ which equals $\mathbb{W}$ as vector spaces, and

$$Y_{\mathcal{W}}\big(v_0 \otimes v_1 \otimes \cdots \otimes v_{k-1}, z\big) = \prod_{j=0}^{k-1} Y_{\mathbb{W}}\big(\star\, v_j, (\omega_k)^j z^{1/k}\big)$$

Here $\star$ are linear operators which account for the change of coordinate. And $\arg((\omega_k)^j z^{1/k}) = -\frac{2j\pi}{k} + \frac{1}{k}\arg z$.

Construction of permutation-twisted modules



- Note that the left hand side is the branched covering of the right hand side by $\mathbb{P}^1 \to \mathbb{P}^1, \gamma \mapsto \gamma^k$ with $0, \infty$ the branched points.
- General case: Let $\mathrm{Orb}(g)$ be the set of g-orbits in E. For each g-orbit o we choose a $\mathbb{V}$-module $\mathbb{W}_o$. Then $\mathcal{W} = \bigotimes_{o \in \mathrm{Orb}(g)} \mathbb{W}_o$ can be equipped with a g-twisted module such that $Y_{\mathcal{W}}$ is the tensor product of all twisted vertex opertors on $\mathbb{W}_o$ defined above.

A brief (and incomplete) history of the correspondence

Physics:

- Bantay ('98, '02) pointed out this twisted/untwisted correspondence. But he mainly considered *unbranced* coverings (of elliptic curves).

Conformal nets:

- Kawahigashi-Longo-Müger ('01) used (genus-0) $\mathbb{Z}_2$ permutation CFT (i.e. (12)-twisted modules of $\mathcal{A} \otimes \mathcal{A}$ where $\mathcal{A}$ is a conformal net) to study the relation between 1. the "complete rationality" of $\mathcal{A}$ and 2. the modularity of $\mathrm{Rep}(\mathcal{A})$.

A brief (and incomplete) history of the correspondence

Conformal nets (continued):

- The methods of conformal nets (a family of operator algebras indexed by open intervals on $\mathbb{S}^1$) are seemingly only applicable to genus-0 CFT. But modularity, S-matrices, etc. are genus-1 phenomena. So KLW's work shows that genus-0 $\mathbb{Z}_2$-permutation twisted CFT contains useful data of genus-1 (or higher genus) untwisted CFT!
- A retrospective explanation: Conformal blocks for permutation-twisted $\mathbb{V}^{\otimes E}$-modules on $\mathbb{P}^1$ correspond to conformal blocks for untwisted $\mathbb{V}$-modules of a branched covering C of $\mathbb{P}^1$, which is possibly of higher genus.

A brief (and incomplete) history of the correspondence

- Later works on conformal nets: Longo-Xu ('04), Kac-Longo-Xu ('05), Liu-Xu ('19), etc. Note that KLX computed many fusion rules (i.e. dimensions of the space of conformal blocks associated to 3-pointed $\mathbb{P}^1$) for **cyclic** (i.e. $\mathbb{Z}_n$) permutation-twisted conformal net modules. Their result suggests a relation between genus-0 permutation orbifold CFT and higher genus untwisted CFT. But no (geometric) explanation was given there.

VOAs:

- Constructing twisted modules (no conformal blocks or fusion rules): Barron-Dong-Mason ('02), Dong-Xu-Yu ('21).
- Computing (certain) fusion rules among **cyclic** permutation-twisted modules: Dong-Li-Xu-Yu ('19)

A brief (and incomplete) history of the correspondence

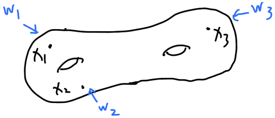
Tensor categories and modular functors:

- Barmeier-Schweigert ('11): constructing **topological** branched coverings, i.e., explains the twisted/untwisted correspondence in the topological setting.
- Bischoff-C.Jones ('19) and Delaney ('19): developed algorithms of computing twisted-permutation fusion rules for any G, but explicit yet uniform results of computation are restricted to the **cyclic** case. Did *not* explain the correspondence.
- When applying these results to the explicit VOA or conformal net contexts, it is not clear how to identify the explicitly constructed twisted modules (e.g. via Barron-Dong-Mason) with the abstract objects in the categorical approaches.

Conformal blocks

- Let $\mathfrak{X} = (C; x_1, \ldots, x_N; \eta_1, \ldots, \eta_N)$ be a (possibly disconnected) N-pointed Riemann surface with local coordinates, where $x_1, \ldots, x_N \in C$ are distinct points, each component of C contains some x_i, and each η_i is a local coordinate at x_i. (Namely, $\eta_i : U_i \to \mathbb{C}$ is injective and holomorphic for some neighborhood $U_i \ni x_i$, and $\eta_i(x_i) = 0$.)
- To each x_i we associate a $\mathbb{V}$-module $\mathbb{W}_i$. Then a conformal block is a linear functional $\phi : \mathbb{W}_\bullet = \mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N \to \mathbb{C}$ satisfying certain covariance property.

Conformal blocks



- Covariance property (E.Frenkel-BenZvi '04): For all i, the expression $f_i = \phi(w_1 \otimes \cdots \otimes Y(v, \eta_i) w_i \otimes \cdots \otimes w_N)$ (as a (formal) meromorphic function on $U_i \ni x_i$ with (finite) poles at x_i) can be analytically continued to the same meromorphic section on C with possible poles only at $x_1, \ldots, x_N$.
- If $\mathbb{W}_i$ is twisted, this definition no longer makes sense since f_i is not a single-valued function of η_i.

Basic facts about (untwisted) conformal blocks

(Beilinson-Feigin-Mazur, Tsuchiya-Ueno-Yamada, Zhu, Huang, Nagatomo-Tsuchiya, Damiolini-Gibney-Tarasca, G., etc.)

Assume $\mathbb{V}$ is $\mathbb{N}$-graded and C_2-cofinite, each $\mathbb{W}_i$ is finitely-generated.

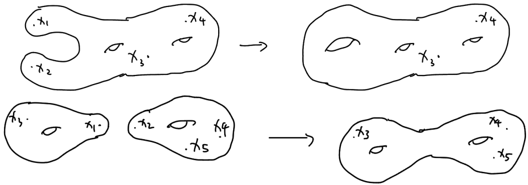
- Let $CB(\mathfrak{X}, \mathbb{W}_\bullet)$ be the space of conformal blocks associated to $\mathfrak{X}$ and the assigned modules. Then $\dim CB(\mathfrak{X}, \mathbb{W}_\bullet) < +\infty$ and is independent of $\eta_\bullet$, the complex structure of C, and the locations of $x_\bullet$ (if C is connected).
- If $\mathbb{W}_1$ is dual to $\mathbb{W}_2$, then the contraction

$$\phi(\underbrace{\cdot \otimes \cdot}_{\text{contraction}} \otimes w_3 \otimes \cdots \otimes w_N)$$

  is convergent and is a conformal block associated to $\mathscr{S}\mathfrak{X}$ (the sewing of $\mathfrak{X}$ along $\eta_1^{-1}(S^1), \eta_2^{-1}(S^1)$) when the sewing makes sense geometrically.

Basic facts about (untwisted) conformal blocks

- Self-sewing and disjoint sewing:

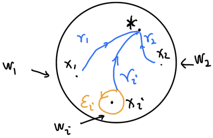


- Factorization: Assume $\mathbb{V}$ is also rational. Fix $\mathbb{W}_3, \dots, \mathbb{W}_N$. Then any element of $CB(\mathscr{S}\mathfrak{X}, \mathbb{W}_3 \otimes \cdots \otimes \mathbb{W}_N)$ is a linear combination of sewing (contraction) of elements of $CB(\mathfrak{X}, \mathbb{W}_1 \otimes \mathbb{W}_1^\vee \otimes \mathbb{W}_3 \otimes \cdots \otimes \mathbb{W}_N)$ (for possibly several $\mathbb{W}_1$ and its dual module $\mathbb{W}_1^\vee$).

Basic facts about (untwisted) conformal blocks

- Special cases of this sewing/factorization include: the associativity isomorphisms of $\mathrm{Rep}(\mathbb{V})$, modular invariance, etc.
- A variant of factorization: $\dim CB(\mathscr{S}\mathfrak{X}, \mathbb{W}_3 \otimes \cdots \otimes \mathbb{W}_N) = \sum_{[\mathbb{W}_1]} \dim CB(\mathfrak{X}, \mathbb{W}_1 \otimes \mathbb{W}_1^\vee \otimes \mathbb{W}_3 \otimes \cdots \otimes \mathbb{W}_N)$ where the sum is over all equivalence classes of irreducible $\mathbb{V}$-modules $[\mathbb{W}_1]$.
- This factorization gives us an explicit algorithm of expressing the dimensions of spaces of conformal blocks in terms of those with lower genera or fewer marked points, and ultimately in terms of fusion rules (i.e. those of genus-0 and 3 marked points).

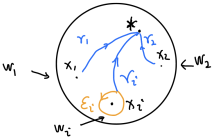
Twisted conformal blocks



Consider for simplicity the genus-0 cases.

- Consider N-pathed sphere $\mathfrak{P} = (\mathbb{P}^1; x_\bullet; \eta_\bullet; \gamma_\bullet)$ where each γ_i $(1 \leqslant i \leqslant N)$ is a path in $\mathbb{P}^1 \backslash x_\bullet$ from $\gamma_i(0)$ (near but not equal to x_i) to $\gamma_i(1) = \star$ (independent of i) such that $\eta_i(\gamma_i(1)) > 0$.
- Let ϵ_i be a small anticlockwise circle around x_i from and to $\gamma_i(0)$, and let $\alpha_i = \gamma_i^{-1} * \epsilon_i * \gamma_i$. Then the homotopy class $[\alpha_i]$ is an element of $\Gamma := \pi_1(\mathbb{P}^1 \backslash x_\bullet, \star)$. We assume that Γ is generated by all $[\alpha_i]$.

Twisted conformal blocks



- Let $\mathcal{W}_i$ be a g_i-twisted $\mathbb{U}$-module. Then a conformal block associated to $\mathcal{W}_\bullet$ and $\mathfrak{P}$ is a linear functional $\psi : \mathcal{W}_\bullet = \mathcal{W}_1 \otimes \cdots \otimes \mathcal{W}_N \to \mathbb{C}$ such that the expression
$$\psi(\mathbf{w}_1 \otimes \cdots Y(\mathbf{u}, \eta_i)\mathbf{w}_i \otimes \cdots \otimes \mathbf{w}_N)$$
(where we take $\arg \eta_i(\gamma_i(0))$ to be 0 recalling the assumption $\eta_i(\gamma_i(0)) > 0$) for all i can be extended to a common holomorhic section on a neighborhood of $\bigcup_i \mathrm{Rng}\gamma_i$, and furthermore to a multivalued holomorphic section on $\mathbb{P}^1 \backslash x_\bullet$.

Permutation-twisted/untwisted correspondence

We return to the setting $\mathbb{U} = \mathbb{V}^{\otimes E}$, $G \leqslant \mathrm{Aut}(E)$, $g_i \in G$, and consider $\mathcal{W}_i$ a g_i-twisted $\mathbb{V}^{\otimes E}$-module from the BDM-construction, which equals $\bigotimes_{o \in \mathrm{Orb}(g_i)} \mathbb{W}_{i,o}$ as a vector space, and each $\mathbb{W}_{i,o}$ is a $\mathbb{V}$-module. (We assume: $\mathbb{V}$ and $\mathbb{W}_{i,o}$ are $\mathbb{N}$-graded with finite-dimensional graded subspaces).

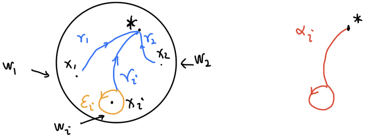
Theorem (G. '21)

A linear functional on $\bigotimes_{i=1}^{N} \mathcal{W}_i = \bigotimes_{i=1}^{N} \bigotimes_{o \in \mathrm{Orb}(g_i)} \mathbb{W}_{i,o}$ *is a conformal block associated to* $\mathcal{W}_1, \dots, \mathcal{W}_N$ *and* $\mathfrak{P}$ *iff* ϕ *is a conformal block associated to all* $\mathbb{W}_{i,o}$ *and a pointed compact Riemann surface* $\mathfrak{X}$ *described below. (Note that all* $\mathbb{W}_{i,o}$ *correspond bijectively to the marked points of* $\mathfrak{X}$.)

In particular, the two spaces of conformal blocks in the theorem have the same dimension.

- Recall in general that if Y is a connected manifold, $\star \in Y$, then the finite *connected* covering spaces $X \to Y$ correspond bijectively to the (conjugacy classes of) cofinite subgroups of $\pi_1(Y, \star)$, i.e., a transitive action of $\pi_1(Y, \star)$. In general, finite covering spaces $X \to Y$ correspond bijectively to actions of $\pi_1(Y, \star)$ on some finite sets E. The components of X correspond bijectively to the $\pi_1(Y, \star)$-orbits of E.

Permutation-twisted/untwisted correspondence



- $\mathfrak{X} = (C; \text{marked points}; \text{local coordinates})$ where we have a holomorphic branched covering $\varphi : C \to \mathbb{P}^1$ which is unbranched outside $\varphi^{-1}(x_\bullet)$. $\varphi^{-1}(x_\bullet)$ is the set of marked points of $\mathfrak{X}$.
- Let $\Gamma = \pi_1(\mathbb{P}^1 \backslash x_\bullet, \star)$ act on E such that $[\alpha_i]$ acts as g_i. (The existence of Γ adds constraints on g_i. If such action does not exist, we understand that there are no non-zero conformal blocks.) Then the unbranched covering $C \backslash \varphi^{-1}(x_\bullet) \to \mathbb{P}^1 \backslash x_\bullet$ is the one corresponding to the action $\Gamma \curvearrowright E$.

Permutation-twisted/untwisted correspondence

- The way we associate each $\mathbb{W}_{i,o}$ to a marked point of $\mathfrak{X}$ is more complicated. (See the intro of my article.)
- But note that the genus of each connected component of C can be computed via the Riemann-Hurwitz formula: Let Ω be any $\Gamma = \pi_1(\mathbb{P}^1 \backslash x_\bullet, \star)$ orbit (equivalently, G-orbit) in E, which corresponds to a component C^Ω. Then the genus is

$$g(C^\Omega) = 1 - |\Omega| + \frac{1}{2} \sum_{i=1}^{N} \sum_{o \in \mathrm{Orb}^\Omega(g_i)} (|o| - 1)$$

where $\mathrm{Orb}^\Omega(g_i)$ is the set of g_i-orbits of Ω.

Permutation-twisted/untwisted correspondence

Example

Let $\mathfrak{P} = (\mathbb{P}^1; 0, a, b, \infty)$ where $a \neq b \in \mathbb{C}\backslash\{0\}$. Associate $\sigma = (12)$-twisted modules of $\mathbb{V} \otimes \mathbb{V}$ to the four marked points. Then the associated branched covering is the elliptic curve $\zeta^2 = z(z-a)(z-b)$ (with 4 marked points) projected onto the z-coordinate.

- Thus, when $\mathbb{V}$ is C_2-cofinite and rational, the fusion rules among permutation-twisted $\mathbb{V}^{\otimes E}$-modules can be expressed by those among (untwisted) $\mathbb{V}$-modules thanks to the factorization of (untwisted) conformal blocks.