Qiuzhen Lectures on Analysis

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Last page before index   Last page of TOC

Sections on history include but are not limited to: 2.1 (point-set topology), 10.4 (Hilbert spaces, Riesz-Fischer theorem), 13.1 (integral theory, Fourier series), 17.5 (Banach-Alaoglu, Hahn-Banach), 17.9 (quotient Banach spaces, Hahn-Banach), 21.1 and most part of Ch. 22 (Hilbert spaces), 23.6 (measurable sets), 27.7 (functional calculus, spectral theory)

Contents

0 Preface

0.1 Notations

Note: Topics marked with $\star\star$ are technical and/or their methods are rarely used in later studies. Topics marked with $\star$ are interesting, but not necessarily technical or difficult. They are not essential for understanding the rest of the notes. You can skim or skip the starred topics on first reading. When a chapter/section/subsection is starred, it means that all of the material in that chapter/section/subsection is starred.

We use frequently the abbreviations:

iff=if and only if

LHS=left hand side   RHS=reft hand side

$\exists$=there exists $\forall$=for all

i.e.=id est=that is=namely   e.g.=for example

cf.=compare/check/see/you are referred to

resp.=respectively   WLOG=without loss of generality

If P, Q are properties, then

$$P \wedge Q = P \text{ and } Q \qquad P \vee Q = P \text{ or } Q \qquad \neg P = \text{ Not } P$$

When we write $A := B$ or $A \xlongequal{\text{def}} B$, we mean that A is defined by the expression B. When we write $A \equiv B$, we mean that A are B are different symbols of the same object.

If $\mathbb{F}$ is any field (e.g. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$), we let $\mathbb{F}^\times = \mathbb{F}\backslash\{0\}$. If α is a complex number and $n \in \mathbb{N}$, we define the **binomial coefficient**

$$\binom{\alpha}{n} = \begin{cases} \dfrac{\alpha \cdot (\alpha-1)\cdots(\alpha-n+1)}{n!} & \text{if } n \geqslant 1 \\ 1 & \text{if } n = 0 \end{cases} \tag{0.1}$$

where $n! = n(n-1)(n-2)\cdots 2\cdot 1$ and $0! = 1$. The bold letter $\mathbf{i}$ means

$$\mathbf{i} = \sqrt{-1}$$

If $z = a + b\mathbf{i}$ where $a, b \in \mathbb{R}$, we let

$$\mathrm{Re}(z) = a \qquad \mathrm{Im}(z) = b$$

If x, y are two elements, we let

$$\delta_{x,y} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \tag{0.2}$$

When studying manifolds, we also write $\delta_{x,y}$ as δ_x^y.

We let $\overline{\mathbb{R}} = [-\infty, +\infty]$, $\overline{\mathbb{R}}_{\geqslant 0} = [0, +\infty]$, and $\mathbb{R}_{\geqslant 0} = [0, +\infty)$. Additions and multiplications in $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}_{\geqslant 0}$ are described in Def. 1.36.

If $f, g : X \to Y$ where X is a set and Y is a preordered set, we write

$$f \leqslant g$$

whenever $f(x) \leqslant g(x)$ for all $x \in X$. If Y is a totally ordered set where $a < b$ in Y means $a \leqslant b$ and $a \neq b$, we write

$$f < g$$

whenever $f(x) < g(x)$ for all $x \in X$.

If X is a topological space, then $\mathcal{T}_X$ denotes the topology of X, i.e., the set of open subsets of X.

If $f : X \to V$ where V is a normed vector space, $|f| : X \to \mathbb{R}_{\geqslant 0}$ denotes its **absolute value function**, i.e., the one sending each $x \in X$ to $\|f(x)\|$.

If A is a precompact subset of B (cf. Def. 15.21), we write

$$A \Subset B$$

1 Basic set theory and numbers

In this chapter, we discuss informally some of the basic notions in set theory and basic properties about numbers. A more thorough treatment can be found in [Mun, Ch. 1] (for set theory) and [Rud-P, Ch. 1] (for numbers).

1.1 Basic operations and axioms

Intuitively, a set denotes a collection of elements. For instance:

$$\mathbb{Z} = \{\text{all integers}\} \qquad \mathbb{N} = \mathbb{Z}_{\geqslant 0} = \{n \in \mathbb{Z} : n \geqslant 0\} \qquad \mathbb{Z}_+ = \{n \in \mathbb{Z} : n > 0\}$$

have infinitely many elements. (In this course, we will not be concerned with the rigorous construction of natural numbers and integers from Peano axioms.) We also let

$$\mathbb{Q} = \{\text{all rational numbers}\} \qquad \mathbb{R} = \{\text{all real numbers}\}$$

if we that rational and real numbers exist and satisfy the properties we are familiar with in high school mathematics. (We will construct $\mathbb{Q}$ and $\mathbb{R}$ rigorously, by the way.)

Set theory is the foundation of modern mathematics. It consists of several Axioms telling us what we can do about the sets. For example, the following way of describing sets

$$\{x : x \text{ satisfies property...}\} \tag{1.1}$$

is illegal, since it gives **Russell's paradox**: Consider

$$S = \{A : A \text{ is a set and } A \notin A\} \tag{1.2}$$

If S were a set, then $S \in S \Rightarrow S \notin S$ and $S \notin S \Rightarrow S \in S$. This is something every mathematician doesn't want to happen.

Instead, the following way of defining sets is legitimate:

$$\{x \in X : x \text{ satisfies property} \dots\} \tag{1.3}$$

where X *is a given set*. For instance, we can define the **difference** of two sets:

$$A \backslash B = A - B = \{x \in A : x \notin B\}$$

So let us figure out the legal way of defining unions and intersections of sets. The crucial point is that we assume the following axiom:

Axiom. If $\mathscr{A}$ is a set of sets, then there exists a set X such that $A \subset X$ for all $A \in \mathscr{A}$.

Thus, if $\mathscr{A}$ is a set of sets, let X satisfy $A \subset X$ for all $A \in \mathscr{A}$, then we can define the **union** and the **intersection**

$$\bigcup_{A\in\mathscr{A}} A = \{x \in X : \text{there exists } A \in \mathscr{A} \text{ such that } x \in A\} \tag{1.4a}$$

$$\bigcap_{A\in\mathscr{A}} A = \{x \in X : \text{for all } A \in \mathscr{A} \text{ we have } x \in A\} \tag{1.4b}$$

It is clear that this definition does not rely on the particular choice of X.

Remark 1.1. In many textbooks, it is not uncommon that sets are defined as in (1.1). You should interpret such definition as (1.3), where the set X is omitted because it is clear from the context. For instance, if the context is clear, the set $\{x \in \mathbb{R} : x \geqslant 0\}$ could be simply written as $\{x : x \geqslant 0\}$ or even $\{x \geqslant 0\}$. By the same token, the phrase "$\in X$" in (1.4) could be omitted. So we can also write

$$A \cup B = \{x : x \in A \text{ or } x \in B\} \qquad A \cap B = \{x : x \in A \text{ and } x \in B\}$$

which are special cases of (1.4).

Remark 1.2. In the same spirit, when discussing subsets of a given "large" set X, and if X is clear from the context, we shall write $X\backslash A$ (where $A \subset X$) as A^c and call it the **complement** of A.

Example 1.3. We have

$$\bigcup_{x\in(1,+\infty)} [0,x) = [0,+\infty) \qquad \bigcap_{n\in\mathbb{Z}_+} (0,1+1/n) = (0,1] \qquad \bigcup_{n\in\mathbb{N}} (0,1-1/n] = (0,1)$$

The readers may notice that these examples are not exactly in the form (1.4). They are actually unions and intersections of indexed families of sets. (See Def. 1.10.) We need some preparation before discussing this notion.

Axiom. If $A_1, \dots, A_n$ are sets, their **Cartesian product** exists:

$$A_1 \times \cdots \times A_n = \{(a_1, \dots, a_n) : a_i \in A_i \text{ for all } 1 \leqslant i \leqslant n\}$$

where two elements $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ of the Cartesian product are regarded equal iff $a_1 = b_1, \dots, a_n = b_n$. We also write

$$(a_1, \dots, a_n) = a_1 \times \cdots \times a_n$$

especially when a, b are real numbers and (a, b) can mean an open interval. We understand $A_1 \times \cdots \times A_n$ as $\varnothing$ if some A_i is $\varnothing$.

If $A_1 = \cdots = A_n = A$, we write the Cartesian product as A^n. $\square$

Example 1.4. Assume that the set of real numbers $\mathbb{R}$ exists. Then the set of complex numbers $\mathbb{C}$ is defined to be $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ as a set. We write $(a, b) \in \mathbb{C}$ as $a + b\mathbf{i}$ where $a, b \in \mathbb{R}$. Define

$$(a + b\mathbf{i}) + (c + d\mathbf{i}) = (a + c) + (b + d)\mathbf{i}$$
$$(a + b\mathbf{i}) \cdot (c + d\mathbf{i}) = (ac - bd) + (ad + bc)\mathbf{i}$$

Define the zero element 0 of $\mathbb{C}$ to be $0 + 0\mathbf{i}$. More generally, we consider $\mathbb{R}$ as a subset of $\mathbb{C}$ by viewing $a \in \mathbb{R}$ as $a + 0\mathbf{i} \in \mathbb{C}$. This defines the usual arithmetic of complex numbers.

If $z = a + b\mathbf{i}$, we define its **absolute value** $|z| = \sqrt{a^2 + b^2}$. Then $z = 0$ iff $|z| = 0$. We define the **(complex) conjugate** of z to be $\overline{z} = a - b\mathbf{i}$. Then $|z|^2 = z\overline{z}$.

If $z \neq 0$, then there clearly exists a unique $z^{-1} \in \mathbb{C}$ such that $zz^{-1} = z^{-1}z = 1$: $z^{-1} = |z|^{-2} \cdot \overline{z}$. Thus, using the language of modern algebra, $\mathbb{C}$ is a **field**.[1] $\square$

The axiom of Cartesian product allows us to define relations and functions:

Definition 1.5. If A, B are sets, a subset R of $A \times B$ is called a **relation**. For $(a, b) \in A \times B$, we write aRb iff $(x, y) \in R$. We understand "aRb" as "a is related to b through the relation R".

Definition 1.6. A relation f of A, B is called a **function** or a **map** (or a **mapping**), if for every $a \in A$ there is a unique $b \in B$ such that afb. In this case, we write $b = f(a)$.

When we write $f : A \to B$, we always mean that A, B are sets and f is a function from A to B. A and B are called respectively the **domain** and the **codomain** of f. (Sometimes people also use the words "source" and "target" to denote A and B.)

If $E \subset A$ and $F \subset B$, we define the **image under** f of E and the **preimage under** f of F to be

$$f(E) = \{b \in B : \exists a \in E \text{ such that } b = f(a)\}$$
$$f^{-1}(F) = \{a \in A : f(a) \in F\}.$$

$f(A)$ is simply called the **image** of f, or the **range** of f. If $b \in B$, $f^{-1}(\{b\})$ is often abbreviated to $f^{-1}(b)$. The function

$$f|_E : E \to B \qquad x \mapsto f(x)$$

is called the **restriction** of f to E. $\square$

[1] See Rem. 6.7 or [Rud-P, Def. 1.12] for the definition of fields. Rather than memorizing the full definition of fields, it is more important to keep in mind some typical (counter)examples: $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields. $\mathbb{Z}$ is not a field, because not every non-zero element of $\mathbb{Z}$ has an inverse. The set of quaternions $\{a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k} : a, b, c, d \in \mathbb{R}\}$ is not a field because it is not commutative ($\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$). The set of rational functions $P(x)/Q(x)$, where P, Q are polynomials with coefficients in $\mathbb{R}$ and $Q \neq 0$, is a field.

The intuition behind the definition of functions is clear: we understand functions as the same as their graphs. So a subset f of the "coordinate plane" $A \times B$ is the graph of a function iff it "intersects every vertical line precisely once".

Remark 1.7. According to our definition, $\varnothing$ (as a subset of $\varnothing \times B$) is the only function from $\varnothing$ to B. (A false assumption implies any statement.) If $A \neq \varnothing$, there are no functions $A \to \varnothing$.

Definition 1.8. A function $x : \mathbb{Z}_+ \to A$ is called a **sequence in** A. We write $x(n)$ as x_n, and write this sequence as $(x_n)_{n\in\mathbb{Z}_+}$ (or simply $(x_n)_n$ or (x_n)).

Many people write such a sequence as $\{x_n\}_{n\in\mathbb{Z}_+}$. We do not use this notation, since it can be confused with $\{x_n : n \in \mathbb{Z}_+\}$ (the range of the function x).

Axiom. If X is a set, then the **power set** 2^X exists, where

$$2^X = \{\text{Subsets of } X\}$$

Example 1.9. The set $2^{\{1,2,3\}}$ has 8 elements: $\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$. Surprisingly, $8 = 2^3$. As we shall see in Exp. 1.54 and Cor. 1.57, this relationship holds more generally, which explains the terminology 2^X.

Now we are ready to define indexed families of sets.

Definition 1.10. An **indexed family of sets** $(S_i)_{i\in I}$ is defined to be a function $S : I \to 2^X$ for some sets I, X. We write $S(i)$ as S_i. (So S_i is a subset of X.) I is called the **index set**. Define

$$\bigcup_{i\in I} S_i = \bigcup_{T\in S(I)} T \qquad \bigcap_{i\in I} S_i = \bigcap_{T\in S(I)} T$$

Note that $S(I)$ is the image of the function S.

Example 1.11. In the union $\bigcup_{x\in(1,+\infty)}[0,x)$, the index set is $I = (1,+\infty)$, and X can be the set of real numbers $\mathbb{R}$. Then $S : I \to 2^X$ is defined to be $S_i = S(i) = [0,i)$.

Exercise 1.12. Let $f : A \to B$ be a function. We say that f is **injective** if for all $a_1, a_2 \in A$ satisfying $a_1 \neq a_2$ we have $f(a_1) \neq f(a_2)$. We say that f is **surjective** if for each $b \in B$ we have $f^{-1}(b) \neq \varnothing$. f is called **bijective** if it is both surjective and bijective. Define the **identity maps** $\mathrm{id}_A : A \to A, a \mapsto a$ and id_B in a similar way. Prove that

$$f \text{ is injective} \Longleftrightarrow \text{there is } g : B \to A \text{ such that } g \circ f = \mathrm{id}_A \tag{1.5a}$$

$$f \text{ is surjective} \Longleftrightarrow \text{there is } g : B \to A \text{ such that } f \circ g = \mathrm{id}_B \tag{1.5b}$$

$$f \text{ is bijective} \Longleftrightarrow \text{there is } g : B \to A \text{ such that } g \circ f = \mathrm{id}_A \text{ and } f \circ g = \mathrm{id}_B \tag{1.5c}$$

Show that the g in (1.5a) (resp. (1.5b), (1.5c)) is surjective (resp. injective, bijective).

The equivalence (1.5b) is subtler, since its proof requires Axiom of Choice.

Axiom. Let $(S_i)_{i\in I}$ be an indexed family of sets. The **Axiom of Choice** asserts that if $S_i \neq \varnothing$ for all $i \in I$, then there exists a function (the **choice function**)

$$f : I \to \bigcup_{i\in I} S_i$$

such that $f(i) \in S_i$ for each $i \in I$.

Intuitively, the axiom of choice says that for each $i \in I$ we can choose an element $f(i)$ of S_i. And such choice gives a function f.

Example 1.13. Let $f : A \to B$ be surjective. Then each member of the family $(f^{-1}(b))_{b\in B}$ is nonempty. Thus, by axiom of choice, there is a choice function g defined on the index set B such that $g(b) \in f^{-1}(b)$ for each b. Clearly $f \circ g = \mathrm{id}_B$.

Remark 1.14. Suppose that each member S_i of the family $(S_i)_{i\in I}$ has exactly one element. Then the existence of a choice function does not require Axiom of Choice: Let $X = \bigcup_{i\in I} S_i$ and define relation

$$f = \{(i, x) \in I \times X : x \in S_i\}$$

Then one checks easily that this relation between I and X is a function, and that it is the (necessarily unique) choice function of $(S_i)_{i\in I}$.

According to the above remark, one does not need Axiom of Choice to prove (1.5a) and (1.5c). Can you see why?

1.2 Partial and total orders, equivalence relations

Definition 1.15. Let A be a set. A **partial order** (or simply an **order**) $\leqslant$ on A is a relation on $A \times A$ satisfying for all $a, b, c \in A$ that:

- (Reflexivity) $a \leqslant a$.
- (Antisymmetry) If $a \leqslant b$ and $b \leqslant a$ then $a = b$.
- (Transitivity) If $a \leqslant b$ and $b \leqslant c$ then $a \leqslant c$.

We write $b \geqslant a$ iff $a \leqslant b$. Write $a > b$ iff $a \geqslant b$ and $a \neq b$. Write $a < b$ iff $b > a$. So $\geqslant$ is also an order on A. The pair $(A, \leqslant)$ is called a **partially ordered set**, or simply a **poset**. A partial order $\leqslant$ on A is called a **total order**, if for every $a, b \in A$ we have either $a \leqslant b$ or $b \leqslant a$.

Example 1.16. The following are examples of orders.

- Assume that $\mathbb{R}$ exists. Then $\mathbb{R}$ has the canonical total order, which restricts to the total order of $\mathbb{Z}$. This is the total order that everyone is familiar with.
- Let X be a set. Then $(2^X, \subset)$ is a poset.
- $\mathbb{R}^2$ is a poset, if we define $(a,b) \leqslant (c,d)$ to be $a \leqslant c$ and $b \leqslant d$.

Definition 1.17. A relation $\sim$ on a set A is called an **equivalence relation**, if for all $a, b, c \in A$ we have

- (Reflexivity) $a \sim a$.
- (Symmetry) $a \sim b$ iff $b \sim a$.
- (Transitivity) If $a \sim b$ and $b \sim c$ then $a \sim c$.

Later, we will use the notions of partial orders and equivalence relation not just for a set, but for a collection of objects "larger" than a set. See Sec. 1.4.

Definition 1.18. Let A be a set, together with an equivalence relation $\sim$. Define a new set

$$A/\sim\, = \{[a] : a \in A\}$$

where the notion $[a]$ can be understood in the following two equivalent ways (we leave it to the readers to check the equivalence):

(1) $[a]$ is a new symbol. We understand $[a]$ and $[b]$ as equal iff $a \sim b$.

(2) $[a] = \{x \in A : x \sim a\}$

We call $[a]$ the **equivalence class** (or the **residue class**) of a, and call $A/\sim$ the **quotient set** of A under $\sim$. The surjective map $\pi : a \in A \mapsto [a] \in A/\sim$ is called the **quotient map**.

Exercise 1.19. Prove that every surjective map is equivalent to a quotient map. More precisely, prove that for every surjection $f : A \to B$, there is an equivalence relation $\sim$ on A and a bijective map $\Phi : A/\sim\, \to B$ such that the following diagram commutes (i.e. $f = \Phi \circ \pi$):

$$\begin{tikzcd} & A \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\ A/\sim \arrow[rr, "\Phi"] & & B \end{tikzcd} \tag{1.6}$$

This is the first time we see commutative diagrams. Commutative diagrams are very useful for indicating that certain maps between sets are "equivalent" or are satisfying some more general relations. For example, (1.6) shows that the maps f and π are equivalent, and that this equivalence is implemented by the bijection Φ. The formal definition of commutative diagrams is the following:

Definition 1.20. A diagram (i.e. some sets denoted by symbols, and some maps denoted by arrows) is called a **commutative diagram**, if all directed paths in the diagram with the same start and endpoints lead to the same result.

Here is an example of commutative diagram in linear algebra. This example assumes some familiarity with the basic properties of vector spaces and linear maps.[2]

Example 1.21. Let V, W be vector spaces over a field $\mathbb{F}$ with finite dimensions m, n respectively. Let $e_1, \dots, e_m$ be a basis of V, and let $\varepsilon_1, \dots, \varepsilon_n$ be a basis of W. We know that there are unique linear isomorphisms $\Phi : \mathbb{F}^m \xrightarrow{\simeq} V$ and $\Psi : \mathbb{F}^n \xrightarrow{\simeq} W$ such that

$$\Phi(a_1, \dots, a_m) = a_1 e_1 + \cdots + a_m e_m \qquad \Psi(b_1, \dots, b_n) = b_1 \varepsilon_1 + \cdots + b_n \varepsilon_n$$

Let $T : V \to W$ be a **linear map**, i.e., a map satisfying $T(a\xi + b\eta) = aT\xi + bT\eta$ for all $a, b \in \mathbb{F}, \xi, \eta \in V$. Then there is a unique $n \times m$ matrix $A \in \mathbb{F}^{n \times m}$ (viewed as a linear map $\mathbb{F}^m \to \mathbb{F}^n$ defined by matrix multiplication) such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{F}^m \arrow[r, "\Phi", "\simeq"'] \arrow[d, "A"'] & V \arrow[d, "T"] \\ \mathbb{F}^n \arrow[r, "\Psi", "\simeq"'] & W \end{tikzcd} \tag{1.7}$$

namely, $T\Phi = \Psi A$. This commutative diagram tells us that T is equivalent to its **matrix representation** A under the bases $e_\bullet, \varepsilon_\star$, and that this equivalence is implemented by the linear isomorphisms Φ (on the sources) and Ψ (on the targets).

Commutative diagrams are ubiquitous in mathematics. You should learn how to read commutative diagrams and understand their intuitive meanings. We will see more examples in the future of this course.

1.3 $\mathbb{Q}$, $\mathbb{R}$, and $\overline{\mathbb{R}} = [-\infty, +\infty]$

Using equivalence classes, one can construct rational numbers from integers, and real numbers from rationals. We leave the latter construction to the future, and discuss the construction of rationals here.

Example 1.22 (Construction of $\mathbb{Q}$ from $\mathbb{Z}$)**.** Define a relation on $\mathbb{Z} \times \mathbb{Z}^\times$ (where $\mathbb{Z}^\times = \mathbb{Z} \backslash \{0\}$) as follows. If $(a, b), (a', b') \in \mathbb{Z} \times \mathbb{Z}^\times$, we say $(a, b) \sim (a', b')$ iff $ab' = a'b$. It is a routine check that $\sim$ is an equivalence relation. Let $\mathbb{Q} = (\mathbb{Z} \times \mathbb{Z}^\times)/\sim$, and write

[2] Again, we refer the readers to Internet or any Linear Algebra textbook (e.g. [Axl]) for the definition of vector spaces and linear maps.

the equivalence class of (a,b) as a/b or $\frac{a}{b}$. Define additions and multiplications in $\mathbb{Q}$ to be

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}\qquad\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

We leave it to the readers to check that this definition is **well-defined**: If $(a,b)\sim(a',b')$ and $(c,d)\sim(c',d')$ then $(ad+bc,bd)\sim(a'd'+b'c',b'd')$ and $(ac,bd)\sim(a'c',b'd')$.

We regard $\mathbb{Z}$ as a subset of $\mathbb{Q}$ by identifying $n\in\mathbb{Z}$ with $\frac{n}{1}$. (This is possible since the map $n\in\mathbb{Z}\mapsto\frac{n}{1}\in\mathbb{Q}$ is injective.) Each $a/b\in\mathbb{Q}$ has additive inverse $\frac{-a}{b}$. If $a/b\in\mathbb{Q}$ is not zero (i.e. $(a,b)\nsim(0,1)$), then a/b has multiplicative inverse b/a. This makes $\mathbb{Q}$ a field: the field of **rational numbers**.

If $a/b\in\mathbb{Q}$, we say $a/b\geqslant 0$ if $ab\geqslant 0$. Check that this is well-defined (i.e., if $(a,b)\sim(a',b')$, then $ab\geqslant 0$ iff $a'b'\geqslant 0$). More generally, if $a/b,c/d\in\mathbb{Q}$, we say $\frac{a}{b}\geqslant\frac{c}{d}$ if $\frac{a}{b}-\frac{c}{d}\geqslant 0$. Check that $\geqslant$ is a total order on $\mathbb{Q}$. Check that $\mathbb{Q}$ is an Archimedean ordered field, defined below. $\square$

Definition 1.23. A field $\mathbb{F}$, together with a total order $\leqslant$, is called an **ordered field**, if for every $a,b,c\in\mathbb{F}$ we have

- (Addition preserves $\leqslant$) If $a\leqslant b$ then $a+c\leqslant b+c$.
- (Multiplication by $\mathbb{F}_{\geqslant 0}$ preserves $\geqslant 0$) If $a,b\geqslant 0$ then $ab\geqslant 0$.

These two properties relate $\leqslant$ to $+$ and $\cdot$ respectively.

Remark 1.24. Many familiar properties about inequalities in $\mathbb{Q}$ hold for an ordered field. For instance:

$$\begin{gathered}
a\geqslant b\ \wedge\ c\geqslant d\qquad\Longrightarrow\qquad a+c\geqslant b+d\\
a\geqslant 0\qquad\Longleftrightarrow\qquad -a\leqslant 0\\
a\geqslant 0\ \wedge\ b\geqslant c\qquad\Longleftrightarrow\qquad ab\geqslant ac\\
a\leqslant 0\ \wedge\ b\geqslant c\qquad\Longleftrightarrow\qquad ab\leqslant a\\
a^2\geqslant 0\\
0<a\leqslant b\qquad\Longrightarrow\qquad 0<b^{-1}\leqslant a^{-1}
\end{gathered}$$

Check them yourself, or see [Rud-P, Prop. 1.18].

Definition 1.25. We say that an ordered field $\mathbb{F}$ satisfies **Archimedean property** if for each $a,b\in\mathbb{F}$ we have

$$a>0\qquad\Longrightarrow\qquad\exists n\in\mathbb{N}\text{ such that }na>b$$

where na means $\underbrace{a+\cdots+a}_{n}$.

Example 1.26. $\mathbb{Q}$ satisfies Archimedean property. Indeed, let $a, b \in \mathbb{Q}$ and $a > 0$. Then $a = p/q$ and $b = r/s$ where $p, q, s \in \mathbb{Z}_+$ and $r \in \mathbb{Z}$. So $na > b$ where $n = q|r| + q$.

Prop. 1.29 gives an important application of Archimedian property. We will use this in the construction of $\mathbb{R}$ from $\mathbb{Q}$, and in the proof that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 1.27. Let $\mathbb{F}$ be a field. A subset $\mathbb{E} \subset \mathbb{F}$ is called a **subfield** of $\mathbb{F}$, if $\mathbb{E}$ contains the 1 of $\mathbb{F}$, and if $\mathbb{E}$ is closed under the operations of addition, multiplication, taking negative, and taking inverse in $\mathbb{F}$ (i.e. if $a, b \in \mathbb{E}$ then $a + b, ab, -a \in \mathbb{E}$, and $a^{-1} \in \mathbb{E}$ whenever $a \neq 0$). We also call $\mathbb{F}$ a **field extension** of $\mathbb{E}$, since $\mathbb{E}$ is clearly a field.

Note that if $\mathbb{E}$ is a subfield of $\mathbb{F}$, the 0 of $\mathbb{F}$ is in $\mathbb{E}$ since $0 = 1 + (-1) \in \mathbb{E}$.

Definition 1.28. Let $\mathbb{E}$ be an ordered field. A field extension $\mathbb{F}$ of $\mathbb{E}$ is called an **ordered field extension**, if $\mathbb{F}$ is equipped with a total order $\leqslant$ such that $\mathbb{F}$ is an ordered field, and if the order $\leqslant$ of $\mathbb{F}$ restricts to that of $\mathbb{E}$. We also call $\mathbb{E}$ an **ordered subfield** of $\mathbb{F}$.

Our typical example of ordered field extension will be $\mathbb{Q} \subset \mathbb{R}$.

Proposition 1.29. *Let $\mathbb{F}$ be an ordered field extension of $\mathbb{Q}$. Assume that $\mathbb{F}$ is Archimedean. Then for every $x, y \in \mathbb{F}$ satisfying $x < y$, there exists $p \in \mathbb{Q}$ such that $x < p < y$.*

Proof. Assume $x, y \in \mathbb{F}$ and $x < y$. Then $y - x > 0$ (since $y - x \neq 0$ and $-x + x \leqslant -x + y$). By Archimedean property, there exists $n \in \mathbb{Z}_+$ such that $n(y - x) > 1$. So $y - x > \dfrac{1}{n}$ and hence $x + \dfrac{1}{n} < y$.

Let us prove that the subset

$$A = \left\{k \in \mathbb{Z} : \frac{k}{n} \leqslant x\right\}$$

is nonempty and bounded from above in $\mathbb{Z}$. By Archimedean property, there is $m \in \mathbb{Z}_+$ such that $m > nx$, i.e. $\dfrac{m}{n} > x$. So for each $k \in \mathbb{Z}_+$ satisfying $k \geqslant m$, we have $\dfrac{k}{n} = \dfrac{m}{n} + \dfrac{k - m}{n} > x$. Therefore, for each $k \in A$ we have $k < m$. So A is bounded above. By Archimedean property again, there is $l \in \mathbb{Z}_+$ such that $\dfrac{l}{n} > -x$. So $-\dfrac{l}{n} < x$, and hence A is nonempty.

We now use the fact that *every nonempty subset of $\mathbb{Z}$ bounded above has a maximal element*. Let $k = \max A$. Since $k + 1 \notin A$, we have $x < \dfrac{k+1}{n}$. Since $\dfrac{k}{n} \leqslant x$, we have

$$\frac{k+1}{n} = \frac{k}{n} + \frac{1}{n} \leqslant x + \frac{1}{n} < y$$

This proves $x < p < y$ with $p = \dfrac{k+1}{n}$. $\square$

To introduce $\mathbb{R}$ formally, we need more definitions:

Definition 1.30. Let $(X, \leqslant)$ be a poset and $E \subset X$. An **upper bound of E in X** is an element $x \in X$ satisfying $e \leqslant x$ for all $e \in E$. An upper bound $x \in X$ of E is called a **least upper bound** or a **supremum** if $x \leqslant y$ for every upper bound $y \in Y$ of E. In this case, we write the supremum as $\sup E$. It is not hard to check that supremums are unique if they exist.

We leave it to the readers to define **lower bounds** and the **greatest lower bound** (if exists) of E, also called the **infinimum** and is denoted by $\inf E$. □

Definition 1.31. Let $(X, \leqslant)$ be a poset. We say that X satisfies the **least-upper-bound property**, if every every nonempty subset $E \subset X$ which is bounded above (i.e. E has an upper bound) has a supremum in X. The **greatest-lower-bound property** is defined in the opposite way.

Example 1.32. $\mathbb{Z}$ satisfies the least-upper-bound and the greatest-lower-bound property: Let $A \subset \mathbb{Z}$. If A is bounded above (resp. below), then the maximum $\max A$ (resp. minimum $\min A$) exists and is the supremum (resp. infinimum) of A.

Example 1.33. Let X be a set. Then $(2^X, \subset)$ satisfies the least-upper-bound and the greatest-lower-bound property: Let $\mathscr{A} \subset 2^X$, i.e., $\mathscr{A}$ is a set of subsets of X. Then $\mathscr{A}$ is bounded from above by X, and is bounded from below by $\varnothing$. Moreover,

$$\sup \mathscr{A} = \bigcup_{A \in \mathscr{A}} A \qquad \inf \mathscr{A} = \bigcap_{A \in \mathscr{A}} A$$

Theorem 1.34. *There is an ordered field extension of $\mathbb{Q}$ which is Archimedian and satisfies the least-upper-bound property. This field is denoted by $\mathbb{R}$. Its elements are called **real numbers**.*

Thm. 1.34 will be proved in Ch. 6. Note that by taking negative, we see that $\mathbb{R}$ also satisfies the greatest-lower-bound property.

Remark 1.35. The ordered field extensions satisfying the conditions in Thm. 1.34 are unique "up to isomorphisms". (The words "**isomorphism**" and "equivalence" are often interchangeable, though "isomorphism" is more often used in the algebraic setting, whereas "equivalence" can be used in almost every context. For example, in point-set topology, "equivalence" means "homeomorphism".) We leave it to the readers to give the precise statement. We will not use this uniqueness in this course.

Note that to compare two extensions $\mathbb{F}, \mathbb{R}$ of $\mathbb{Q}$, it is very confusing to regard $\mathbb{Q}$ as a subset of both $\mathbb{F}$ and $\mathbb{R}$. You'd better consider two different injective maps $\tau : \mathbb{Q} \to \mathbb{F}$ and $\iota : \mathbb{Q} \to \mathbb{R}$ preserving the algebraic operations and the order of $\mathbb{Q}$, and use a commutative diagram to indicate that τ and ι are equivalent. (Thus, what's happening here is that we have an equivalence of maps, not just an equivalence of the fields $\mathbb{F}$ and $\mathbb{R}$.) □

Definition 1.36. Let $-\infty, +\infty$ be two different symbols, and extend the total order $\leqslant$ of $\mathbb{R}$ to the **extended real line**

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$$

by letting $-\infty < x < +\infty$ for all $x \in \mathbb{R}$. Define for each $x \in \mathbb{R}$ that

$$x \pm \infty = \pm\infty + x = \pm\infty \qquad +\infty - (-\infty) = +\infty$$

$$x \cdot (\pm\infty) = \pm\infty \cdot x = \begin{cases} \pm\infty & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \mp\infty & \text{if } x < 0 \end{cases}$$

$$\frac{x}{\pm\infty} = 0$$

$$\frac{\pm\infty}{x} = x^{-1} \cdot (\pm\infty) \qquad \text{if } x \neq 0$$

If $a, b \in \overline{\mathbb{R}}$ and $a \leqslant b$, we define **intervals** with endpoints a, b:

$$\begin{aligned} [a,b] &= \{x \in \overline{\mathbb{R}} : a \leqslant x \leqslant b\} \qquad & (a,b) &= \{x \in \overline{\mathbb{R}} : a < x < b\} \\ (a,b] &= \{x \in \overline{\mathbb{R}} : a < x \leqslant b\} \qquad & [a,b) &= \{x \in \overline{\mathbb{R}} : a \leqslant x < b\} \end{aligned} \tag{1.8}$$

So $\mathbb{R} = (-\infty, +\infty)$ and $\overline{\mathbb{R}} = [-\infty, +\infty]$. If a, b are in $\mathbb{R}$, we say that the corresponding interval is **bounded**.

In this course, unless otherwise stated, an interval always means one of the four sets in (1.8). The first two intervals are called respectively a **closed interval** and an **open interval**.

Remark 1.37. Clearly, every subset E of $\overline{\mathbb{R}}$ is bounded and has a supremum and an infinimum. We have that $\sup E = +\infty$ iff E is not bounded above in $\mathbb{R}$, and that $\inf E = -\infty$ iff E is not bounded below in $\mathbb{R}$.

1.4 Cardinalities, countable sets, and product spaces Y^X

Definition 1.38. Let A and B be sets. We say that A and B have the same **cardinality** and write $\mathrm{card}(A) = \mathrm{card}(B)$ (or simply $A \approx B$), if there is a bijection $f : A \to B$. We write $\mathrm{card}(A) \leqslant \mathrm{card}(B)$ (or simply $A \precsim B$) if A and a subset of B have the same cardinality.

Exercise 1.39. Show that $\mathrm{card}(A) \leqslant \mathrm{card}(B)$ iff there is an injection $f : A \to B$, iff there is a surjection $g : B \to A$. (You need either Axiom of Choice or its consequence (1.5b) to prove the last equivalence.)

It is clear that $\approx$ is an equivalence relation on the collection of sets. It is also true that $\precsim$ is a partial order: Reflexivity and transitivity are easy to show. The proof of antisymmetry is more involved:

Theorem 1.40 (Schröder-Bernstein)**.** *Let A, B be two sets. If $A \precsim B$ and $B \precsim A$, then $A \approx B$.*

$\star\star$ *Proof.* Assume WLOG that $A \subset B$. Let $f : B \to A$ be an injection. Let $A_n = f^n(A)$ defined inductively by $f^0(A) = A$, $f^n(A) = f(f^{n-1}(A))$. Let $B_n = f^n(B)$. Then

$$B_0 \supset A_0 \supset \cdots \supset B_n \supset A_n \supset B_{n+1} \supset \cdots$$

In particular, $C = \bigcap_{n\in\mathbb{N}} A_n$ equals $\bigcap_{n\in\mathbb{N}} B_n$. Note that f gives a bijection $B_n\backslash A_n \to B_{n+1}\backslash A_{n+1}$ (since f gives bijections $B_n \to B_{n+1}$ and $A_n \to A_{n+1}$). Therefore, we have a bijection $g : B \to A$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in B_n\backslash A_n \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

where "otherwise" means either $x \in C$ or $x \in A_n\backslash B_{n+1}$ for some n. □

Intuition about the above proof: View B as an onion. The layers of B are $B_n\backslash A_n$ (the odd layers) and $A_n\backslash B_{n+1}$ (the even layers). The bijection g maps each odder layer to the subsequent odd one, and fixes the even layers and the core C.

Example 1.41. If $-\infty < a < b < +\infty$, then $(0,1) \approx (a,b)$.

Proof. $f : (0,1) \to (a,b)$ sending x to $(b-a)x + a$ is a bijection. □

Example 1.42. If $-\infty < a < b < +\infty$, then $\mathbb{R} \approx (a,b)$

Proof. By the previous example, it suffices to prove $\mathbb{R} \approx (-1,1)$. The function

$$f : \mathbb{R} \to (-1,1) \qquad f(x) = \begin{cases} \frac{x}{1+x} & \text{if } x \geqslant 0 \\ -f(-x) & \text{if } x < 0 \end{cases} \tag{1.9}$$

is bijective. □

Alternatively, one may use the tangent function to give a bijection between $(-\pi/2, \pi/2)$ and $\mathbb{R}$. I have avoided this method, since (1.9) is more elementary than trigonometric functions. The mathematically rigorous definition of trigonometric functions and the verification of their well-known properties are far from easy tasks.

Proposition 1.43. *Let I be an interval with endpoints $a < b$ in $\overline{\mathbb{R}}$. Then $I \approx \mathbb{R}$.*

Proof. Let $A = (0,1) \cup \{-\infty, +\infty\}$. By Exp. 1.42, we have

$$(a,b) \subset I \precsim \overline{\mathbb{R}} \approx A \approx [0,1] \subset (-2,2) \approx (a,b)$$

So $I \approx \overline{\mathbb{R}}$ by Schröder-Bernstein Thm. 1.40. In particular, $\mathbb{R} = (-\infty, +\infty) \approx \overline{\mathbb{R}}$. □

Definition 1.44. A set A is called **finite** if $A \precsim \{1, \ldots, n\}$ for some $n \in \mathbb{Z}_+$. A is called **countable** if $A \precsim \mathbb{N}$.

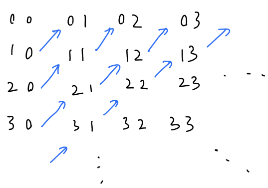
Clearly, a set A is finite iff either $A \approx \emptyset$ or $A \approx \{1, \ldots, n\}$ for some $n \in \mathbb{Z}_+$.

Remark 1.45. Let $A \subset \mathbb{N}$. If A is bounded above, then $A \subset \{0, \ldots, n\}$ and hence A is finite. If A is not bounded above, then we can construct a strictly increasing sequence $(x_n)_{n\in\mathbb{N}}$ in A. (Pick any $x_0 \in A$. Suppose we have $x_n \in A$. Since x_n is not an upper bounded of A, there is $x_{n+1} \in A$ larger than x_n. So $(x_n)_{n\in\mathbb{N}}$ can be constructed inductively.) This gives an injection $\mathbb{N} \to A$. Therefore $A \succsim \mathbb{N}$, and hence $A \approx \mathbb{N}$ by Schröder-Bernstein.

It follows that if $B \precsim \mathbb{N}$, then either B is a finite set, or $B \approx \mathbb{N}$. Therefore, "a set B is **countably infinite**" means the same as "$B \approx \mathbb{N}$". $\square$

Theorem 1.46. *A countable union of countable sets is countable. In particular,* $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$.

Proof. Recall Exe. 1.39. Let $A_1, A_2, \ldots$ be countable sets. Since each A_i is countable, there is a surjection $f_i : \mathbb{N} \to A_i$. Thus, the map $f : \mathbb{N} \times \mathbb{N} \to \bigcup_i A_i$ defined by $f(i,j) = f_i(j)$ is surjective. Therefore, it suffices to show that there is a surjection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$. This is true, since we have a bijection $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ where $g(0), g(1), g(2), \ldots$ are $(0,0)$, $(1,0)$, $(0,1)$, $(2,0)$, $(1,1)$, $(0,2)$, $(3,0)$, $(2,1)$, $(1,2)$, $(0,3)$, etc., as shown by the figure



$\square$

As an application, we prove the extremely important fact that $\mathbb{Q}$ is countable.

Corollary 1.47. *We have* $\mathbb{N} \approx \mathbb{Z}_+ \approx \mathbb{Z} \approx \mathbb{Q}$.

Proof. Clearly $\mathbb{Z}_{<0} \approx \mathbb{N} \approx \mathbb{Z}_+$. By Thm. 1.46, $\mathbb{Z} = \mathbb{Z}_{<0} \cup \mathbb{N}$ is countably infinite, and hence $\mathbb{Z} \approx \mathbb{N}$. It remains to prove $\mathbb{Z} \approx \mathbb{Q}$. By Schröder-Bernstein, it suffices to prove $\mathbb{Q} \precsim \mathbb{Z}$. By Thm. 1.46 again, $\mathbb{Z} \times \mathbb{Z} \approx \mathbb{Z}$. By Exp. 1.22, there is a surjection from a subset of $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Q}$. So $\mathbb{Q} \precsim \mathbb{Z} \times \mathbb{Z} \approx \mathbb{Z}$. $\square$

Later, when we have learned Zorn's Lemma (an equivalent form of Axiom of Choice), we will be able to prove the following generalization of $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$. So we defer the proof of the following theorem to a later section.

Theorem 1.48. *Let X be a infinite set. Then* $X \times \mathbb{N} \approx X$.

Proof. See Thm. 16.7. $\square$

Our next goal is to prove an exponential law $a^{b+c} = a^b \cdot a^c$ for cardinalities. For that purpose, we first need to define the set-theoretic operations that correspond to the summation $b + c$ and the exponential a^b.

Definition 1.49. We write $X = \bigsqcup_{\alpha\in\mathscr{A}} A_\alpha$ and call X the **disjoint union** of $(A_\alpha)_{\alpha\in\mathscr{A}}$, if $X = \bigcup_{\alpha\in\mathscr{A}} A_\alpha$ and $(A_\alpha)_{\alpha\in\mathscr{A}}$ is a family of pairwise disjoint sets (i.e. $A_\alpha \cap A_\beta = \varnothing$ if $\alpha \neq \beta$). If moreover $\mathscr{A} = \{1, \dots, n\}$, we write $X = A_1 \sqcup \cdots \sqcup A_n$.

Definition 1.50. Let X, Y be sets. Then

$$Y^X = \{\text{functions } f : X \to Y\} \tag{1.10}$$

A more precise definition of Y^X (in the spirit of (1.3)) is $\{f \in X \times Y \mid f : X \to Y \text{ is a function}\}$. Note that by Rem. 1.7,

$$Y^\varnothing = \{\varnothing\} \tag{1.11}$$

This new notation is compatible with the old one $Y^n = Y \times \cdots \times Y$:

Example 1.51. Let $n \in \mathbb{Z}_+$. We have $Y^{\{1,\dots,n\}} \approx Y^n$ due to the bijection

$$Y^{\{1,\dots,n\}} \to Y^n \qquad f \mapsto (f(1), \dots, f(n))$$

Remark 1.52. The above example suggests that in the general case that X is not necessarily finite, we can view each function $f : X \to Y$ as $(f(x))_{x\in X}$, an **indexed family of elements** of Y with index set X. Thus, intuitively and hence not quite rigorously,

$$Y^X = \underbrace{Y \times Y \times \cdots}_{\text{card}(X) \text{ pieces}} \tag{1.12}$$

This generalizes the intuition in Def. 1.8 that a function $f : \mathbb{Z}_+ \to Y$ is equivalently a sequence $(f(1), f(2), f(3), \dots)$.

The viewpoint that Y^X is a **product space** with index set X is very important and will be adopted frequently in this course. More generally, we can define: $\square$

Definition 1.53. Let $(X_i)_{i\in I}$ be a family of sets with index set I. Their **product space** is defined by

$$\prod_{i\in I} X_i = \{f \in \mathfrak{X}^I : f(i) \in X_i \text{ for all } i \in I\}$$

where $\mathfrak{X} = \bigcup_{i\in I} X_i$. If each X_i is nonempty, then $\prod_{i\in I} X_i$ is nonempty by Axiom of Choice. An element of $\prod_{i\in I} X_i$ is also written as $(f_i)_{i\in I}$ when the i-th component of it is $f_i \in X_i$.

In particular, if all X_i are equal to X, then $X^I = \prod_{i\in I} X$.

Example 1.54. Let X be a set. For each $A \subset X$, define the **characteristic function** $\chi_A : X \to \{0,1\}$ to be

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Then we have

$$2^X \approx \{0,1\}^X$$

since the following map is bijective:

$$2^X \to \{0,1\}^X \qquad A \mapsto \chi_A$$

Its inverse is $f \in \{0,1\}^X \mapsto f^{-1}(1) \in 2^X$.

Proposition 1.55 (Exponential Law). *Suppose that $X = A_1 \sqcup \cdots \sqcup A_n$. Then*

$$Y^X \approx Y^{A_1} \times \cdots \times Y^{A_n}$$

Proof. We have a bijection

$$\begin{aligned} \Phi : Y^X &\to Y^{A_1} \times \cdots \times Y^{A_n} \\ f &\mapsto (f|_{A_1}, \dots, f|_{A_n}) \end{aligned} \tag{1.13}$$

where we recall that $f|_{A_i}$ is the restriction of f to A_i. □

Exercise 1.56. Assume that $A_1, \dots, A_n$ are subsets of X. Define Φ by (1.13). Prove that Φ is injective iff $X = A_1 \cup \cdots \cup A_n$. Prove that Φ is surjective iff $A_1, \dots, A_n$ are pairwise disjoint.

Corollary 1.57. *Let X, Y be finite sets with cardinalities $m, n \in \mathbb{N}$ respectively. Assume that $Y \neq \emptyset$. Then Y^X is a finite set with cardinality n^m.*

Proof. The special case that $m = 0$ (i.e. $X = \emptyset$, cf. (1.11)) and $m = 1$ is clear. When $m > 1$, assume WLOG that $X = \{1, \dots, m\}$. Then $X = \{1\} \sqcup \cdots \sqcup \{m\}$. Apply Prop. 1.55 to this disjoint union. We see that $Y^X \simeq Y \times \cdots \times Y \simeq \{1, \dots, n\}^m$ has n^m elements. □

We end this section with some (in)equalities about the cardinalities of product spaces. To begin with, we write $X \precnsim Y$ (or $\mathrm{card}(X) < \mathrm{card}(Y)$) if $X \precsim Y$ and $X \not\approx Y$.

Proposition 1.58. *Let X, Y be sets with $\mathrm{card}(Y) \geqslant 2$ (i.e. Y has at least two elements). Then $X \precnsim Y^X$. In particular, $X \precnsim 2^X$.*

Proof. The case $X = \varnothing$ is obvious since $0 < 1$. So we assume $Y \neq \varnothing$. Clearly $2^X \simeq \{0,1\}^X$ is $\precsim Y^X$. So it suffices to prove $X \precnsim 2^X$. Since the map $X \to 2^X$ sending x to $\{x\}$ is injective, $X \precsim 2^X$. Let us prove $X \not\approx 2^X$.

Assume that $X \approx 2^X$. So there is a bijection $\Phi : X \to 2^X$ sending each $x \in X$ to a subset $\Phi(x)$ of X. Motivated by Russell's Paradox (1.2), we define

$$S = \{x \in X : x \notin \Phi(x)\}$$

Since Φ is surjective, there exists $y \in X$ such that $S = \Phi(y)$. If $y \in \Phi(y)$, then $y \in S$, and hence $y \notin \Phi(y)$ by the definition of S. If $y \notin \Phi(y)$, then $y \notin S$, and hence $y \in \Phi(y)$ by the definition of S. This gives a contradiction. $\square$

Remark 1.59. Write $\{1, \dots, n\}^X$ as n^X for short. Assuming that real numbers have decimal, binary, or (more generally) base-n representations where $n \in \mathbb{Z}_{\geqslant 2}$, then $\mathbb{R} \approx n^{\mathbb{N}}$. So by Prop. 1.58, $\mathbb{N} \precnsim \mathbb{R}$, i.e. $\mathbb{R}$ *is uncountable*. The base-n representations of real numbers suggest that $\mathrm{card}(n^{\mathbb{N}})$ is independent of n. This fact can be proved by elementary methods without resorting to the analysis of real numbers:

Theorem 1.60. *Let X be an infinite set. Then*

$$2^X \approx 3^X \approx 4^X \approx \cdots \approx \mathbb{N}^X$$

Proof. First, we assume that $X = \mathbb{N}$. Clearly, for each $n \in \mathbb{Z}_{\geqslant 2}$ we have $2^X \precsim n^X \precsim \mathbb{N}^X$. Since elements of $\mathbb{N}^X$ are subsets of $X \times \mathbb{N}$ (i.e. elements of $2^{X \times \mathbb{N}}$), we have

$$\mathbb{N}^X \subset 2^{X \times \mathbb{N}} \simeq 2^X$$

since $X \times \mathbb{N} \approx X$ by Thm. 1.46. So $2^X \approx n^X \approx \mathbb{N}^X$ by Schröder-Bernstein.

As pointed out earlier (cf. Thm. 1.48), it can be proved by Zorn's Lemma that $X \times \mathbb{N} \approx X$ for every infinite set X. So the same conclusion holds for such X. $\square$

2 Metric spaces

We first give an informal introduction to metric spaces, hoping to motivate the readers from a (relatively) historical perspective. It is okay if you do not understand all of the concepts mentioned in the introduction on the first read. Simply return to this section when you feel unmotivated while formally studying these concepts in later sections. (The same suggestion applies to all the introductory sections and historical comments in our notes.)

2.1 Introduction: what is point-set topology?

> The method which has been used with success by Volterra and Hilbert consists in observing that a function (for instance a continuous one) can be replaced by a countable infinity of parameters. One treats the problem first as though one had only a finite number of parameters and then one goes to the limit... We believe that this method has played a useful role because it followed intuition, but that its time has passed... The most fruitful method in functional analysis seems to us to treat the element of which the functional depends directly as a variable and in the form in which it presents itself naturally.
>
> —- Fréchet, 1925 (cf. [Jah, Sec. 13.8])

In this chapter, we begin the study of point-set topology by learning one of its most important notions: metric spaces. Similar to [Rud-P], we prefer to introduce metric spaces and basic point-set topology at the early stage of our study. An obvious reason for doing so is that metric spaces provide a uniform language for the study of basic analysis problems in $\mathbb{R}, \mathbb{R}^n, \mathbb{C}^n$, and more generally in function spaces such as the space of continuous functions $C([a,b])$ on the interval $[a,b] \subset \mathbb{R}$. With the help of such a language, for example, many useful criteria for the convergence of series in $\mathbb{R}$ and $\mathbb{C}$ (e.g. root test, ratio test) are generalized straightforwardly to criteria for the *uniform* convergence of series of functions in $C([a,b])$.

Point-set topology was born in 1906 when Fréchet defined metric spaces, motivated mainly by the study of function spaces in analysis (i.e. *functional analysis*). Indeed, point-set topology and functional analysis are the two faces of the same coin: they both originated from the study of **functionals**, i.e., functions of functions. See for example (2.1). The core ideas of point-set topology are as follows:

(1) Take X to be a set of functions defined on a "classical space" (e.g. the set of all continuous functions $f : [a,b] \to \mathbb{C}$). Then a functional is a function $S : X \to \mathbb{C}$. This is a generalization of functions on $\mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n$ or on their subsets.

(2) Unlike $\mathbb{R}^n$, a function space X is usually "infinite dimensional". Thus, one may think that a functional S is a function with infinite variables. In point-set topology, this viewpoint is abandoned; the philosophy of point-set topology is diametrically opposed to that of multivariable calculus.[1] Instead, one should view a functional S as a function with one variable x, where x denotes a general point of the function space X.

(3) Rather than looking at each variable/component and doing explicit mutivariable calculations, one uses geometric intuitions to study the analytic properties of functionals.[2] These geometric intuitions (e.g. distances, open balls, convergence) are borrowed from $\mathbb{R}$ and $\mathbb{R}^n$ and are mostly irrelevant to dimensions or numbers of variables.

(Sequential) **compactness**, **completeness**, and **separability** are prominent geometric properties that are useful in the study of the analytic properties of functionals. The importance of these three notions was already recognized by Fréchet by the time he defined metric spaces. The study of these three properties will be a main theme of our course.

Consider sequential compactness for example. The application of compactness to function spaces originated from the problems in calculus of variations. For instance, let $L(x,y,z)$ be a polynomial or (more generally) a continuous function in 3 variables. We want to find a "good" (e.g. differentiable) function $f:[0,1]\rightarrow\mathbb{R}$ minimizing or maximizing the expression

$$S(f)=\int_0^1 L(t,f(t),f'(t))dt \tag{2.1a}$$

This is the general setting of Lagrangian mechanics. In the theory of integral equations, one considers the extreme values and points of the functional

$$S(f)=\int_0^1\int_0^1 f(x)K(x,y)\overline{f(y)}dxdy \tag{2.1b}$$

where $K:[0,1]^2\rightarrow\mathbb{R}$ is continuous and $f:[0,1]\rightarrow\mathbb{C}$ is subject to the condition $\int_0^1|f(x)|^2dx\leqslant 1$. Any f maximizing (resp. minimizing) $S(f)$ is an eigenvector of the linear operator $g\mapsto\int_0^1 K(x,y)g(y)dy$ with maximal (resp. minimal) eigenvalue.

As we shall learn, (sequential) compactness is closely related to the problem of finding (or proving the existence of) maximal/minimal values of a continuous

[1] Very often, the formula of $S(f)$ involves an integral. See e.g. (2.1). Mathematicians (e.g. Volterra, Lévy, Fredholm, and early Hilbert) used to study $S(f)$ by discretizing $S(f)$, i.e., by approximating integrals by finite sums. Thus, S is approximated by a sequence of functions with n variables where $n\rightarrow\infty$. This viewpoint is abandoned in point-set topology.

[2] This is similar to linear algebra where one prefers vectors, linear subspaces, and linear operators to n-tuples, sets of solutions, and matrices.

function and the points at which the function attains its maximum/minimum. So, in 19th century, when people were already familiar with sequential compactness in $\mathbb{R}^n$ (e.g. Bolzano-Weierstrass theorem, Heine-Borel theorem), they applied compactness to function spaces and functionals. The idea is simple: Suppose we are given X, a set of functions (say continuous and differentiable) from $[a,b]$ to $\mathbb{R}$. We want to find $f \in X$ maximizing $S(f)$. Here is an explicit process (see also the proof of Lem. 3.2):

(A) Find a sequence $(f_n)_{n\in\mathbb{Z}_+}$ in X such that $S(f_n)$ increases to $M = \sup S(X)$.

(B) Define convergence in X in a suitable way, and verify that $S : X \to \mathbb{R}$ is continuous (i.e. if f_n converges to f in the way we define, then $S(f_n) \to S(f)$).

(C) Suppose we can find a subsequence $(f_{n_k})_{k\in\mathbb{Z}_+}$ converging to some $f \in X$, then S attains its maximum at f. In particular, $S(f) = M$ and hence $M < +\infty$.

To carry out step (B), we need to define suitable geometric structures for a function space X so that the convergence of sequences in X and the continuity of functions $S : X \to \mathbb{R}$ can be defined and studied in a similar pattern as that for $\mathbb{R}^n$. **Metric** (of a metric space) and **topology** (of a topological space) are such geometric structures. As we shall learn, the topology of a metric space is uniquely determined by the convergence of sequences in this space.

Step (C) can be carried out if every sequence in X has a convergent subsequent, i.e., if X is sequentially compact. Thus, we need a good criterion for sequential compactness for subsets of a function space. Arzelà-Ascoli theorem, the $C([a,b])$-version of Heine-Borel theorem, is such a criterion. This famous theorem was proved in late 19th century (and hence before the birth of point-set topology), and it gave an important motivation for Fréchet to consider metric spaces in general. We will learn this theorem at the end of the first semester.

To summarize: Metric spaces are defined not just for fun. We introduce such geometric objects because we want to study the convergence of sequences and the analytic properties of continuous functions using geometric intuitions. And moreover, the examples we are interested in are not just subsets of $\mathbb{R}^n$, but also subsets of function spaces. With this in mind, we now begin our journey into point-set topology.

2.2 Basic definitions and examples

Definition 2.1. Let X be a set. A function $d : X \times X \to \mathbb{R}_{\geqslant 0}$ is called a **metric** if for all $x, y, z \in X$ we have

(1) $d(x,y) = d(y,x)$.

(2) $d(x, y) = 0$ iff $x = y$.

(3) (Triangle inequality) $d(x, z) \leqslant d(x, y) + d(y, z)$.

The pair (X, d), or simply X, is called a **metric space**. If $x \in X$ and $r \in (0, +\infty]$,[3] the set

$$B_X(x, r) = \{y \in X : d(x, y) < r\}$$

often abbreviated to $B(x, r)$, is called the **open ball** with center x and radius r. If $r \in [0, +\infty)$,

$$\overline{B}_X(x, r) = \{y \in X : d(x, y) \leqslant r\}$$

also abbreviated to $\overline{B}(x, r)$, is called the **closed ball** with center x and radius r.

We make some comments on this definition.

Remark 2.2. That "$d(x, y) = 0$ iff $x = y$" is very useful. Think about X as a set of functions $[0, 1] \to \mathbb{R}$ and d is a metric on X. To show that $f, g \in X$ are equal, instead of checking that infinitely many values are equal (i.e. $f(t) = g(t)$ for all $t \in \mathbb{R}$), it suffices to check that one value (i.e. $d(f, g)$) is zero.

Remark 2.3. Triangle inequality clearly implies "polygon inequality":

$$d(x_0, x_n) \leqslant \sum_{j=1}^{n} d(x_{j-1}, x_j) \tag{2.2}$$

Remark 2.4. Choose distinct points $x, y \in X$. Then x, y are separated by two open balls centered at them: there exists $r, \rho > 0$ such that $B(x, r) \cap B(y, \rho) = \varnothing$. This is called the **Hausdorff property**.

To see this fact, note that $d(x, y) > 0$. Choose r, ρ such that $r + \rho \leqslant d(x, y)$. If $z \in B(x, r) \cap B(y, \rho)$, then $d(x, z) + d(y, z) < r + \rho \leqslant d(x, y)$, contradicting triangle inequality.

We will see (cf. Prop. 2.19) that Hausdorff property guarantees that any sequence in a metric space cannot converge to two different points. Intuition: one cannot find a point which is very close to x and y at the same time. □

We give some examples, and leave it to the readers to check that they satisfy the definition of metric spaces. We assume that square roots of positive real numbers can be defined. (We will rigorously define square roots after we define e^x using the series $\sum_{n \in \mathbb{N}} x^n/n!$.)

Example 2.5. $\mathbb{R}$ is a metric space if we define $d(x, y) = |x - y|$

[3]We want open and closed balls to be nonempty. So we assume $r \neq 0$ only for open balls.

Example 2.6. On $\mathbb{R}^n$, we can define **Euclidean metric**

$$d(x,y)=\sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}$$

if $x_\bullet, y_\bullet$ are the components of x,y. The following are also metrics:

$$\begin{aligned}d_1(x,y)&=|x_1-y_1|+\cdot+|x_n-y_n|\\ d_\infty(x,y)&=\max\{|x_1-y_1|,\dots,|x_n-y_n|\}\end{aligned}$$

Example 2.7. The **Euclidean metric** on $\mathbb{C}^n$ is

$$d(z,w)=\sqrt{|z_1-w_1|^2+\cdots+|z_n-w_n|^2}$$

which agrees with the Euclidean metric on $\mathbb{R}^{2n}$. The following are also metrics:

$$\begin{aligned}d_1(z,w)&=|z_1-w_1|+\cdot+|z_n-w_n|\\ d_\infty(z,w)&=\max\{|z_1-w_1|,\dots,|z_n-w_n|\}\end{aligned}$$

Convention 2.8. Unless otherwise stated, the metrics on $\mathbb{R}^n$ and $\mathbb{C}^n$ (and their subsets) are assumed to be the Euclidean metrics.

Remark 2.9. One may wonder what the subscripts $1,\infty$ mean. This notation is actually due to the general fact that

$$d_p(z,w)=\sqrt[p]{|z_1-w_1|^p+\cdots+|z_n-w_n|^p}$$

is a metric where $1\leqslant p<+\infty$, and $d_\infty=\lim_{q\to+\infty}d_q$. It is not easy to prove that d_p satisfies triangle inequality: one needs Minkowski inequality. For now, we will not use such general d_p. And we will discuss Minkowski inequality in later sections.

Example 2.10. Let $X=X_1\times\cdots\times X_N$ where each X_i is a metric space with metric d_i. Write $x=(x_1,\dots,x_N)\in X$ and $y=(y_1,\dots,y_N)\in Y$. Then the following are metrics on X:

$$\begin{aligned}d(x,y)&=d_1(x_1,y_1)+\cdots+d_N(x_N,y_N)\\ \delta(x,y)&=\max\{d_1(x_1,y_1),\dots,d_N(x_N,y_N)\}\\ \rho(x,y)&=\sqrt{d_1(x_1,y_1)^2+\cdots+d_N(x_N,y_N)^2}\end{aligned}$$

With respect to the metric δ, the open balls of X are "polydisks"

$$B_X(x,r)=B_{X_1}(x_1,r)\times\cdots\times B_{X_N}(x_N,r)$$

There is no standard choice of metric on the product of metric spaces. d,δ,ρ are all good, and they are equivalent in the following sense:

Definition 2.11. We say that two metrics d_1, d_2 on a set X are **equivalent** and write $d_1 \approx d_2$, if there exist $\alpha, \beta > 0$ such that for any $x, y \in X$ we have

$$d_1(x,y) \leqslant \alpha d_2(x,y) \qquad d_2(x,y) \leqslant \beta d_1(x,y)$$

This is an equivalence relation. More generally, we may write $d_1 \lesssim d_2$ if $d_1 \leqslant \alpha d_2$ for some $\alpha > 0$. Then $d_1 \approx d_2$ iff $d_1 \lesssim d_2$ and $d_2 \lesssim d_1$.

Example 2.12. In Exp. 2.10, we have $\delta \leqslant \rho \leqslant d \leqslant N\delta$. So $\delta \approx \rho \approx d$.

Convention 2.13. Given finitely many metric spaces $X_1, \dots, X_N$, the metric on their product space $X = X_1 \times \cdots \times X_N$ is chosen to be any one that is equivalent to the ones defined in Exp. 2.10. In the case that each X_i is a subset of $\mathbb{R}$ or $\mathbb{C}$, we follow Convention 2.8 and choose the metric on X to be the Euclidean metric (unless otherwise stated).

Definition 2.14. Let (X, d) be a metric space. Then a **metric subspace** denotes an object $(Y, d|_Y)$ where $Y \subset X$ and $d|_Y$ is the restriction of d to Y, namely, for all $y_1, y_2 \in Y$ we set

$$d|_Y(y_1, y_2) = d(y_1, y_2)$$

Convention 2.15. Suppose Y is a subset of a given metric space (X, d). Unless otherwise stated, the metric of Y is chosen to be $d|_Y$ whenever Y is viewed as a metric space. For example, the metric of any subset of $\mathbb{R}^n$ is assumed to be the Euclidean metric, unless otherwise stated.

2.3 Convergence of sequences

Definition 2.16. Let $(x_n)_{n\in\mathbb{Z}_+}$ be a sequence in a metric space X. Let $x \in X$. We say that x is the **limit** of x_n and write $\lim\limits_{n\to\infty} x_n = x$ (or $x_n \to x$), if:

- For every real number $\varepsilon > 0$ there exists $N \in \mathbb{Z}_+$ such that for every $n \geqslant N$ we have $d(x_n, x) < \varepsilon$.

Equivalently, this means that every (nonempty) open ball centered at x contains **all but finitely many** x_n.[4]

Remark 2.17. The negation of $x_n \to x$ is clear:

- There exists $\varepsilon > 0$ such that for all $N \in \mathbb{Z}_+$ there exists $n \geqslant N$ such that $d(x_n, x) \geqslant \varepsilon$.

[4]"All but finitely many x_n satisfies..." means "for all but finitely many n, x_n satisfies...". It does NOT mean that "all but finitely many elements of the set $\{x_n : n \in \mathbb{Z}_+\}$ satisfies...".

Namely, one changes each "for all" to "there exists", changes each "there exists" to "for all", and negate the last sentence.

Exercise 2.18. Show that in the above definition of limits, it suffices to consider rational numbers $\varepsilon > 0$. (Note: You need to use Prop. 1.29.)

This exercise implies that the definition of limits does not require the existence of real numbers, i.e., does not assume Thm. 1.34. Indeed, we will use limits of sequences (and "double sequences") to prove Thm. 1.34.

In many textbooks and research papers, you will see phrases such as

$$x_n \text{ satisfies property } P \text{ for } \textbf{sufficiently large } n \tag{2.3}$$

This means that "there exists $N \in \mathbb{Z}_+$ such that P holds for all $n \geqslant N$". (We also say that x_n **eventually** satisfies P.) Then $\lim_{n\to\infty} x_n = x$ means that "for every $\varepsilon > 0$, we have $d(x_n, x) < \varepsilon$ for sufficiently large n".

Proposition 2.19. *Any sequence $(x_n)_{n\in\mathbb{Z}_+}$ in a metric space X has at most one limit.*

Proof. Suppose $(x_n)_{n\in\mathbb{Z}_+}$ converges to $x, y \in X$ where $x \neq y$. By Hausdorff property (Rem. 2.4), there exist $r, \rho > 0$ such that $B(x,r) \cap B(y,\rho) = \emptyset$. By the definition of $x_n \to x$, there exists $N_1 \in \mathbb{Z}_+$ such that $x_n \in B(x,r)$ for all $n \geqslant N_1$. Similarly, $x_n \to y$ means that there is $N_2 \in \mathbb{Z}_+$ such that $x_n \in B(y,\rho)$ for all $n \geqslant N_2$. Choose any $n \geqslant N_1, N_2$ (e.g. $n = \max\{N_1, N_2\}$). Then $x_n \in B(x,r) \cap B(y,\rho) = \emptyset$, impossible. □

2.3.1 Methods for proving convergence and computing limits

Example 2.20. $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0.$

Proof. Choose any $\varepsilon \in \mathbb{Q}_{>0}$. By Archimedean property, there exists $N \in \mathbb{Z}_+$ such that $N\varepsilon > 1$, i.e. $1/N < \varepsilon$. Thus, for all $n \geqslant N$ we have $1/n < \varepsilon$. □

Proposition 2.21. *Let $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}\}$ and $(x_n), (y_n)$ be sequences in $\mathbb{F}$ converging to $x, y \in \mathbb{R}$. If $x_n \leqslant y_n$ for all n, then $x \leqslant y$.*

Proof. If $y < x$, let $\varepsilon = x - y$. Then all but finitely many members of (x_n) are in $(x - \varepsilon/2, x + \varepsilon/2)$, and all but finitely many members of (y_n) are in $(y - \varepsilon/2, y + \varepsilon/2)$. Since $y + \varepsilon/2 < x - \varepsilon/2$, there must exist n such that $y_n < x_n$. □

Definition 2.22. If A and B are posets (or more generally, preordered sets, see Def. 5.1), we say a function $f : A \to B$ is **increasing** (resp. **strictly increasing**), if for each $x, y \in A$ we have

$$x \leqslant y \qquad \Longrightarrow \qquad f(x) \leqslant f(y)$$

resp.

$$x < y \qquad \Longrightarrow \qquad f(x) < f(y)$$

We leave the definitions of **decreasing** and **strictly decreasing** to the readers. We say that f is **monotonic** (resp. **strictly monotonic**), if f is either increasing or decreasing (resp. either strictly increasing or strictly decreasing).

Proposition 2.23. *Let $(x_n)_{n\in\mathbb{Z}}$ be a sequence in $[a,b] \subset \mathbb{R}$. If (x_n) is increasing (resp. decreasing), then $\lim\limits_{n\to\infty} x_n$ equals $\sup\{x_n : n \in \mathbb{Z}_+\}$ (resp. $\inf\{x_n : n \in \mathbb{Z}_+\}$).*

Proof. Assume (x_n) increases. (The case of decreasing is similar and hence its proof is omitted.) Let $A = \sup\{x_n : n \in \mathbb{Z}_+\} < +\infty$. Then for each $\varepsilon > 0$ there is N such that $x_N > A - \varepsilon$ (since $A - \varepsilon$ is not an upper bound). Since (x_n) is increasing, for all $n \in \mathbb{N}$ we have $A - \varepsilon < x_n \leqslant A$ and so $|x_n - A| < \varepsilon$. $\square$

Example 2.24. Let $(x_n)_{n\in\mathbb{Z}_+}$ be a sequence in a metric space X, and let $x \in X$. It is easy to see that

$$\lim_{n\to\infty} x_n = x \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} d(x_n, x) = 0$$

Example 2.25. Suppose that (a_n) and (b_n) are sequences in $\mathbb{R}_{\geqslant 0}$, that $a_n \leqslant b_n$ for all n, and that $b_n \to 0$. Then $a_n \to 0$.

Proof. For each $\varepsilon > 0$, $[0, \varepsilon)$ contains all but finitely many b_n, and hence all but finitely many a_n. $\square$

More generally, we have:

Proposition 2.26 (Squeeze theorem). *Suppose that (x_n) is a sequence in a metric space X. Let $x \in X$. Suppose that there is a sequence (a_n) in $\mathbb{R}_{\geqslant 0}$ such that $\lim\limits_{n\to\infty} a_n = 0$ and that $d(x_n, x) \leqslant a_n$ for all n. Then $\lim\limits_{n\to\infty} x_n = x$.*

Proof. This follows immediately from Exp. 2.24 and 2.25. $\square$

The above proposition explains why many people say that "analysis is the art of inequalities": It transforms the problem of convergence to the problem of finding a sequence $(a_n) \in \mathbb{R}_{\geqslant 0}$ converging to 0 such that the inequality $d(x_n, x) \leqslant a_n$ holds. And very often, a good (hard) analyst is one who knows how to find such good sequences!

Proposition 2.27. *Let $X = X_1 \times \cdots \times X_N$ be a product of metric spaces (X_i, d_i). Let d be any of the three metrics of X in Exp. 2.10. Let $\mathbf{x}_n = (x_{1,n}, \dots, x_{N,n})$ be a sequence in X. Let $\mathbf{y} = (y_1, \dots, y_N)$. Then*

$$\lim_{n\to\infty} \mathbf{x}_n = \mathbf{y} \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} x_{i,n} = y_i \ \ (\forall 1 \leqslant i \leqslant N)$$

Proof. We let d be the metric δ in Exp. 2.10, i.e. defined by $\max_j d_j(x_j, y_j)$. Now choose a sequence $(\mathbf{x}_n)$ and an element $\mathbf{y}$ in X. Then

$$\mathbf{x}_n \to \mathbf{y} \iff d_X(\mathbf{x}_n, \mathbf{y}) \to 0 \iff \max_{1 \leqslant j \leqslant N} d_j(x_{j,n}, y_j) \to 0 \tag{2.4}$$

Suppose that the RHS of (2.4) is true. Fix any $1 \leqslant i \leqslant N$. Then $d_i(x_{i,n}, y_i) \leqslant \max_j d_j(x_{j,n}, y_j)$. So $x_{i,n} \to y_i$ by Prop. 2.26.

Conversely, assume that for every i we have $x_{i,n} \to y_i$. Then for every $\varepsilon > 0$ there is $K_i \in \mathbb{Z}_+$ such that $d_i(x_{i,n}, y_i) < \varepsilon$ for every $n \geqslant K_i$. Let $K = \max\{K_1, \ldots, K_N\}$. Then for all $n \geqslant K$ we have $\max_j d_j(x_{j,n}, y_j) < \varepsilon$. This proves the RHS of (2.4).

If d is one of the other two metrics in Exp. 2.10, one can either use a similar argument, or use the following important (but easy) fact. □

Proposition 2.28. *Let d, δ be two equivalent metrics on a set X. Let $(x_n)_{n \in \mathbb{Z}_+}$ and x be in X. Then (x_n) converges to x under the metric d iff (x_n) converges to x under δ. Namely, $d(x_n, x) \to 0$ iff $\delta(x_n, x) \to 0$.*

Proof. Prove it yourself. (Or see Prop. 2.62.) □

More useful formulas about limit will be given in Exp. 4.24.

2.3.2 Criteria for divergence

Definition 2.29. A subset E of a metric space (X, d) is called **bounded** if either $E = \emptyset$ or there exist $p \in X$ and $R \in \mathbb{R}_{>0}$ such that $E \subset B_X(p, R)$. If X is bounded, we also say that d is a **bounded metric**.

Remark 2.30. Note that if E is bounded, then for *any* $q \in X$ there exists $\widetilde{R} \in \mathbb{R}_{>0}$ such that $E \subset B_X(q, \widetilde{R})$. (Indeed, choose $\widetilde{R} = R + d(p, q)$, then by triangle inequality, $B(p, R) \subset B(q, \widetilde{R})$.)

Some easy examples are as follows.

Example 2.31. In a metric space X, if $x \in X$ and $0 < r < +\infty$, then $B(x, r)$ is bounded. Hence $\overline{B}(x, r)$ is bounded (since it is inside $B(x, 2r)$).

Also, it is easy to see:

Example 2.32. A finite union of bounded subsets is bounded.

Proposition 2.33. *Let $(x_n)_{n \in \mathbb{Z}_+}$ be a convergent sequence in a metric space X. Then $(x_n)_{n \in \mathbb{Z}_+}$ is bounded.*

By saying that a sequence $(x_n)_{n \in \mathbb{Z}_+}$ in X is **bounded**, we mean that its range in X (namely $\{x_n : n \in \mathbb{Z}_+\}$) is bounded.

Proof. Suppose that $x_n \to x$. Then for each $\varepsilon > 0$, say $\varepsilon = 1$, all but finitely many elements of x_n (say $x_1, \dots, x_N$) are in $B(x,1)$. So this whole sequence is in $A = \{x_1\} \cup \cdots \{x_N\} \cup B(x,1)$. Since each $\{x_i\}$ is bounded, and since $B(x,1)$ is bounded, A is bounded by Exp. 2.32. $\square$

Remark 2.34. Prop. 2.33 gives our first criterion on divergence: If a sequence is unbounded (e.g. $x_n = n^2$), then it does not converge. But there are many bounded and divergent sequences. (See Exp. 2.37.) In this case, we need the second criterion: If a sequence has two subsequences converging to two different points, then this sequence diverge. (See Prop. 2.36)

Definition 2.35. Let $(x_n)_{n\in\mathbb{Z}_+}$ be a sequence in a set X. If $(n_k)_{k\in\mathbb{Z}_+}$ is a strictly increasing sequence in $\mathbb{Z}_+$, we say that $(x_{n_k})_{k\in\mathbb{Z}_+}$ is a **subsequence** of $(x_n)_{n\in\mathbb{Z}_+}$.

Thus, a subsequence of $(x_n)_{n\in\mathbb{Z}_+}$ is equivalently the restriction of the function $x : \mathbb{Z}_+ \to X$ to an infinite subset of $\mathbb{Z}_+$.

Proposition 2.36. *Let $(x_n)_{n\in\mathbb{Z}_+}$ be a sequence in a metric space X converging to $x \in X$. Then every subsequence $(x_{n_k})_{k\in\mathbb{Z}_+}$ converges to x.*

Proof. For every $\varepsilon > 0$, $B(x,\varepsilon)$ contains all but finitely many $\{x_n : n \in \mathbb{Z}_+\}$, and hence all but finitely many $\{x_{n_k} : k \in \mathbb{Z}_+\}$. $\square$

Example 2.37. The sequence $x_n = (-1)^n$ in $\mathbb{R}$ is divergent, because the subsequence $(x_{2k})_{k\in\mathbb{Z}_+}$ converges to 1, whereas $(x_{2k-1})_{k\in\mathbb{Z}_+}$ converges to -1.

One may wonder if the two criteria in Rem. 2.34 are complete in order to determine whether a sequence diverges. This is true for sequences in $\mathbb{R}^n$. We will discuss this topic later. (See Cor. 3.17.)

2.4 Continuous maps of metric spaces

Continuous maps are a powerful tool for showing that a sequence converges.

Definition 2.38. Let $f : X \to Y$ be a map of metric spaces. Let $x \in X$. We say that f is **continuous at** x if one of the following equivalent conditions hold:

(1) For every sequence $(x_n)_{n\in\mathbb{Z}_+}$ in X, we have

$$\lim_{n\to\infty} x_n = x \qquad \Longrightarrow \qquad \lim_{n\to\infty} f(x_n) = f(x)$$

(2) For every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $p \in X$ satisfying $d(p,x) < \delta$, we have $d(f(p), f(x)) < \varepsilon$.

(2') For every $\varepsilon > 0$, there exists $\delta > 0$ such that $B_X(x,\delta) \subset f^{-1}(B_Y(f(x),\varepsilon))$.

We say that f is a **continuous map/function**, if f is continuous at every point of X.

Proof of the equivalence. (2)⇔(2′): Obvious.

(2)⇒(1): Assume (2). Assume $x_n \to x$. For every $\varepsilon > 0$, let $\delta > 0$ be as in (2). Then since $x_n \to x$, there is $N \in \mathbb{Z}_+$ such that for all $n \geqslant N$ we have $d(x_n, x) < \delta$. By (2), we have $d(f(x_n), f(x)) < \varepsilon$ for all $n \geqslant N$. This proves $f(x_n) \to f(x)$.

¬(2)⇒ ¬(1): Assume that (2) is not true. Then there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exists $p \in X$ such that $d(p, x) < \delta$ and $d(f(p), f(x)) \geqslant \varepsilon$. Thus, for every $n \in \mathbb{Z}_+$, by taking $\delta = 1/n$, we see that there exists $x_n \in X$ such that $d(x_n, x) < 1/n$ and $d(f(x_n), f(x)) \geqslant \varepsilon$. By Squeeze theorem (Prop. 2.26), $x_n \to x$. But $f(x_n) \nrightarrow f(x)$ (i.e. $d(f(x_n), f(x)) \nrightarrow 0$). So (1) is not true. □

Remark 2.39. One can compare Def. 2.38-(1) to the definition of linear maps. A map is continuous iff it *preserves the convergence of sequences*, i.e., iff it maps convergent sequences to convergent ones. A map (between vector spaces) is linear iff it perserves the addition and the scalar multiplication of vectors. In general, a good map between two sets with "structures" is a map which preserves the structures. (Thus, linear combinations encode the linear structures of vector spaces. Similarly, the convergence of sequences remembers the "topological" structures of metric spaces.) As another example, we will define an isometry of metric spaces to be one that preserves the metrics (the structures of metric spaces), see Exe. 2.63.

Remark 2.40. In this section, we mainly use Def. 2.38-(1) to study continuity. But in later sections we will also use Def. 2.38-(2′). An advantage of (2′) is that it is more geometric. Indeed, if X is a metric space and $E \subset X$, we say that $x \in E$ is an **interior point of E in X** if there exists $\delta > 0$ such that $B_X(x, \delta) \subset E$. For example, a point $z \in \mathbb{C}$ is an interior point of the closed unit disk $\overline{B}_{\mathbb{C}}(0,1) = \{w \in \mathbb{C} : |w| \leqslant 1\}$ iff $|z| < 1$.

Thus, Def. 2.38-(2′) says that for any map of metric spaces $f : X \to Y$ and $x \in X$, the following are equivalent:

(a) f is continuous at x.

(b) For each $\varepsilon > 0$, every $x \in X$ is an interior point of $f^{-1}\big(B_Y(f(x), \varepsilon)\big)$.

We say that a subset $U \subset X$ is **open** if each point of U is an interior point. For example, by triangle inequality, every open ball in a metric space is an open set. Thus, we have:

- A map of metric spaces $f : X \to Y$ is continuous iff the preimage under f of every open ball of Y is an open subset of X.

In the study of point-set topology, we will see that many properties can be studied in two approaches: using sequences (or using nets, the natural generalizations of sequences) and their convergence, and using open sets. The first example of such property is continuity, as we have seen in Def. 2.38. Another prominent example is sequential compactness vs. compactness. These two approaches represent two (very) different intuitions: one is dynamic, while the other is static and more geometric. (So it is surprising that these two very things are actually equivalent!) Sometimes both approaches work for a problem, but sometimes only one of them works, or one of them is much simpler. If you are a beginner in analysis and point-set topology, I suggest that whenever you see one approach applied to a problem, try to think about whether the other approach also works and which one is better. □

2.4.1 Methods for proving continuity

Lemma 2.41. *Let $f : X \to Y$ be a map of metric spaces. Let $(B_i)_{i\in I}$ be a collection of open balls in X such that $X = \bigcup_{i\in I} B_i$. Suppose that for each i, the restriction $f|_{B_i} : B_i \to Y$ is continuous. Then f is continuous.*

This lemma shows that if we can prove that f is "locally" continuous, then f is globally continuous.

Proof. Choose (x_n) in X converging to $x \in X$. We shall show $f(x_n) \to f(x)$. Choose i such that $x \in B_i$. Then one can find $\delta > 0$ such that $B(x, \delta) \subset B_i$. (In the language of point-set topology: x is an interior point of B_i.) To see this, write $B_i = B(y, r)$. Since $x \in B(y, r)$, we have $r - d(x, y) > 0$. Choose $0 < \delta \leqslant r - d(x, y)$. Then triangle inequality implies $B(x, \delta) \subset B(y, r)$.

Since $x_n \to x$, there is $N \in \mathbb{Z}_+$ such that $x_n \in B(x, \delta)$ for all $n \geqslant N$. Thus, $(x_{k+N})_{k\in\mathbb{Z}_+}$ converges in B_i to x. Since $f|_{B_i}$ is continuous, $\lim_{k\to\infty} f(x_{k+N}) = f(x)$. So $f(x_n) \to f(x)$. □

Definition 2.42. A map of metric spaces $f : X \to Y$ is called **Lipschitz continuous** if there exists $L \in \mathbb{R}_{>0}$ (called **Lipschitz constant**) such that for all $x_1, x_2 \in X$,

$$d_Y\big(f(x_1), f(x_2)\big) \leqslant L \cdot d_X(x_1, x_2) \tag{2.5}$$

Lemma 2.43. *Lipschitz continuous maps are continuous.*

Proof. Suppose that $f : X \to Y$ is Lipschitz continuous with Lipschitz constant L. Suppose $x_n \to x$ in X. Then $L \cdot d(x_n, x) \to 0$. By (2.5) and Squeeze theorem (Prop. 2.26), $f(x_n) \to f(x)$. (You can also use Def. 2.38-(2) to prove this lemma.) □

Example 2.44. Let $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. The map $z \in \mathbb{F}\backslash\{0\} \mapsto z^{-1} \in \mathbb{F}$ is continuous.

Proof. Let this map be f. Since $\mathbb{F}$ is covered by open balls $B(z, \delta)$ where $z \in \mathbb{F}\backslash\{0\}$ and $0 < \delta < |z|$, by Lem. 2.41, it suffices to prove that f is continuous when restricted to every such $B(z, \delta)$. Let $\varepsilon = |z| - \delta > 0$. Choose $x, y \in B(z, \delta)$. Then $|x| = |x - z + z| \geqslant |z| - |z - x| > \varepsilon$ by triangle inequality. Similarly, $|y| > \varepsilon$. So

$$|f(x) - f(y)| = |x^{-1} - y^{-1}| = |x - y|/|xy| \leqslant \varepsilon^{-2}|x - y|$$

So $f|_{B(z,\delta)}$ has Lipschitz constant ε^{-2}, and hence is continuous. □

(Question: in the above proof, is the map $f : \mathbb{F}\backslash\{0\} \to \mathbb{F}$ Lipschitz continuous?)

We have given a fancy way of proving that if (z_n) is a sequence in $\mathbb{F}\backslash\{0\}$ converging to $z \in \mathbb{F}\backslash\{0\}$, then $z_n^{-1} \to z^{-1}$. You should think about how to prove this fact directly using $\varepsilon - N$ language, and compare your proof with the above proof to find the similarities!

Proposition 2.45. *Let $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. Then the following maps are continuous:*

$$\begin{aligned} + : \mathbb{F} \times \mathbb{F} \to \mathbb{F} \qquad & (x, y) \mapsto x + y \\ - : \mathbb{F} \times \mathbb{F} \to \mathbb{F} \qquad & (x, y) \mapsto x - y \\ \times : \mathbb{F} \times \mathbb{F} \to \mathbb{F} \qquad & (x, y) \mapsto xy \\ \div : \mathbb{F} \times \mathbb{F}^\times \to \mathbb{F} \qquad & (x, y) \mapsto x/y \end{aligned}$$

Recall our Convention 2.13 on the metrics of finite product spaces.

Proof. We only prove that the last two are continuous: the first two can be treated in a similar (and easier) way.

Denote the multiplication map by μ. We choose the metric on $\mathbb{F}^2$ to be $d(\mathbf{x}, \mathbf{x}') = \max\{|x_1 - x_1'|, |x_2 - x_2'|\}$. Since $\mathbb{F} \times \mathbb{F}$ is covered by open balls of the form $B(0, r) = \{(x, y) \in \mathbb{F}^2 : |x| < r, |y| < r\}$ where $0 < r < +\infty$, by Lem. 2.41 and 2.43, it suffices to show that $\mu|_{B(0,r)}$ is Lipschitz continuous. This is true, since for each $(x, y), (x', y') \in B(0, r)$, we have

$$\begin{aligned} &|\mu(x, y) - \mu(x', y')| = |xy - x'y'| \leqslant |(x - x')y| + |x'(y - y')| \\ <&2r \cdot \max\{|x - x'|, |y - y'|\} = 2r \cdot d((x, y), (x', y')) \end{aligned} \tag{2.6}$$

By Exp. 2.44 and Prop. 2.47, the map $(x, y) \in \mathbb{F} \times \mathbb{F}^\times \mapsto (x, y^{-1}) \in \mathbb{F} \times \mathbb{F}$ is continuous. So its composition with the continuous map μ is continuous, thanks to Prop. 2.46. So $\div$ is continuous. $\square$

Proposition 2.46. *Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous maps of metric spaces. Then $g \circ f : X \to Z$ is continuous.*

Proposition 2.47. *Suppose that $f_i : X_i \to Y_i$ is a map of metric spaces, where $1 \leqslant i \leqslant N$. Then the product map*

$$\begin{gathered} f_1 \times \cdots \times f_N : X_1 \times \cdots \times X_N \to Y_1 \times \cdots \times Y_N \\ (x_1, \ldots, x_N) \mapsto (f_1(x_1), \ldots, f_N(x_N)) \end{gathered}$$

is continuous if and only if $f_1, \ldots, f_N$ are continuous.

Proof of Prop. 2.46 and 2.47. Immediate from Def. 2.38-(1) and Prop. 2.27. $\square$

Corollary 2.48 (Squeeze theorem). *Let $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}\}$ and $(x_n), (y_n), (z_n)$ be sequences in $\mathbb{R}$. Assume that $x_n \leqslant y_n \leqslant z_n$ for all n. Assume that x_n and z_n both converge to $A \in \mathbb{R}$. Then $\lim_{n\to\infty} y_n = A$.*

Proof. Let $a_n = y_n - x_n$ and $b_n = z_n - x_n$. Then $0 \leqslant a_n \leqslant b_n$, and $\lim_n b_n = \lim_n z_n - \lim_n x_n = 0$ because the subtraction map is continuous (Prop. 2.45). By Exp. 2.25, $a_n \to 0$. Since $x_n \to A$, $y_n = x_n + a_n$ converges to A, since the addition map is continuous by Prop. 2.45. □

Again, this is a fancy way of proving Squeeze theorem. The readers should know how to prove it directly from the definition of limits of sequences.

We give some more examples of continuous maps.

Example 2.49. By Prop. 2.45, $f(x,y,z) = x^2y + 5y^4z^7 - 3xyz^2$ is a continuous function on $\mathbb{C}^3$. Clearly $z \in \mathbb{C} \mapsto \overline{z} \in \mathbb{C}$ is continuous. So $g(x,y,z) = f(\overline{x},\overline{y},z) + 2\overline{f(z,x^2,xy^{-9})} - 5xy^{-2}\overline{z^{-3}}$ is a continuous function on $\mathbb{C} \times \mathbb{C}^\times \times \mathbb{C}^\times$.

Example 2.50. Let $f,g : X \to \mathbb{F}$ be continuous functions where $\mathbb{F} \in \{\mathbb{Q},\mathbb{R},\mathbb{C}\}$. Then by Prop. 2.45 and 2.46, $f \pm g$ and fg are continuous, and f/g is continuous when $0 \notin g(X)$. Here

$$(f \pm g)(x) = f(x) \pm g(x) \qquad (fg)(x) = f(x)g(x) \qquad (f/g)(x) = f(x)/g(x) \tag{2.7}$$

Example 2.51. Let $f : X \to Y$ be a map of metric spaces. Let E, F be subsets of X, Y respectively. (Recall that the metrics of subsets are chosen as in Def. 2.14.)

- The inclusion map $\iota_E : E \to X, x \mapsto x$ is clearly continuous. Thus, if f is continuous, then $f|_E : E \to Y$ is continuous, since $f|_E = f \circ \iota_E$.
- If $f(X) \subset F$, then we can restrict the codomain of f from Y to F: let $\widetilde{f} : X \to F$ be $\widetilde{f}(x) = f(x)$. Then it is clear that f is continuous iff $\widetilde{f}$ is continuous.

Proposition 2.52. *Let $X_1, \dots, X_N$ be metric spaces. The following projection is clearly continuous:*

$$\pi_{X_i} : X_1 \times \cdots \times X_N \to X_i \qquad (x_1, \dots, x_N) \mapsto x_i$$

Proposition 2.53. *Let $f_i : X \to Y_i$ be maps where $X, Y_1, \dots, Y_N$ are continuous. Then*

$$f_1 \vee \cdots \vee f_N : X \to Y_1 \times \cdots \times Y_N \qquad x \mapsto (f_1(x), \dots, f_N(x))$$

is continuous iff $f_1, \dots, f_N$ are continuous.

Example 2.54. Let X be a metric space. Then $d : X \times X \to \mathbb{R}, (x,y) \mapsto d(x,y)$ is Lipschiz continuous with Lipschitz constant 1 (by triangle inequality). So d is continuous.

Proposition 2.55. *Let X be a metric space and $E \subset X$ is nonempty. Define **distance function***

$$d(\cdot, E) : X \to \mathbb{R}_{\geqslant 0} \qquad d(x,E) = \inf_{e \in E} d(x,e)$$

Then $d(\cdot, E)$ is has Lipschitz constant 1. So $d(\cdot, E)$ is continuous.

Proof. Choose any $x, y \in X$. By triangle inequality, for each $e \in E$ we have $d(x,e) \leqslant d(x,y) + d(y,e)$. Since $d(x,E) \leqslant d(x,e)$, we get $d(x,E) \leqslant d(x,y) + d(y,e)$. Applying $\inf_{e\in E}$ to the RHS gives $d(x,E) \leqslant d(x,y) + d(y,E)$. Hence $d(x,E) - d(y,E) \leqslant d(x,y)$. Exchanging x and y gives

$$\big|d(x,E) - d(y,E)\big| \leqslant d(x,y) \tag{2.8}$$

This proves that $d(\cdot, E)$ has Lipschitz constant 1. $\square$

Definition 2.56. More generally, if E, F are subsets of a metric space E, we can define

$$d(E,F) = \inf_{e\in E, f\in F} d(e,f) \tag{2.9}$$

to be the **distance between** E **and** F.

Exercise 2.57. Let $E, F \subset X$. Prove that

$$d(E,F) = \inf_{f\in F} d(E,f) \tag{2.10}$$

Example 2.58. If X is a metric space and $p \in X$, then by Prop. 2.55 (or simply by triangle inequality), the function $d_p : x \in X \mapsto d(x,p) \in \mathbb{R}$ has Lipschitz constant 1 and hence is continuous. In particular, if $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, the function $z \in \mathbb{F} \mapsto |z|$ is continuous (since $|z| = d_{\mathbb{F}}(z, 0)$). Thus, if $f : X \to \mathbb{F}$ is continuous, then $|f| : X \to \mathbb{R}_{\geqslant 0}$ is continuous where

$$|f|(x) = |f(x)| \tag{2.11}$$

Example 2.59. Let $N \in \mathbb{Z}_+$. Then the following function is continuous:

$$\max : \mathbb{R}^N \to \mathbb{R} \qquad (x_1, \dots, x_N) \mapsto \max\{x_1, \dots, x_N\} \in \mathbb{R}$$

Similarly, the minimum function is continuous.

Proof. To avoid confusion, we write $\max$ as $\max_N$. The case $N = 1$ is obvious. When $N = 2$, we have

$$\max(x_1, x_2) = \frac{x_1 + x_2 + |x_1 - x_2|}{2} \tag{2.12}$$

So $\max_2$ is continuous by Exp. 2.50 and 2.58.

We use induction. Suppose we have proved that $\max_N$ is continuous. Then $\max_N \times \mathrm{id}_{\mathbb{R}} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is continuous. So $\max_{N+1} = \max_2 \circ (\max_N \times \mathrm{id}_{\mathbb{R}})$ is continuous. $\square$

2.5 Homeomorphisms and isometric isomorphisms; convergence in $\overline{\mathbb{R}}$

2.5.1 General theory

Definition 2.60. A bijection of metric spaces $f : X \to Y$ is called a **homeomorphism** if one of the following equivalent (cf. Def. 2.38) statements holds:

(1) $f : X \to Y$ and its inverse map $f^{-1} : Y \to X$ are continuous.

(2) For each sequence (x_n) in X and each $x \in X$, we have $\lim_{n\to\infty} x_n = x$ iff $\lim_{n\to\infty} f(x_n) = f(x)$.

If such f exists, we say that X, Y are **homeomorphic**.

A special case of the above definition is:

Definition 2.61. Let X be a set with metrics d, δ. We say that d and δ induce the **same topology** on X (or that d, δ are **topologically equivalent**) if one of the following clearly equivalent statements holds:

(1) The map $(X, d) \to (X, \delta), x \mapsto x$ is a homeomorphism.[5]

(2) For each sequence (x_n) in X and each $x \in X$, (x_n) converges to x under the metric d iff (x_n) converges to x under δ.

Proposition 2.62. *Suppose that d, δ are equivalent metrics on a set X. Then d, δ are topologically equivalent.*

Proof. Suppose $\delta \leqslant \alpha d$ and $d \leqslant \beta\delta$ for some $\alpha, \beta > 0$. Then the map $f : (X, d) \to (X, \delta), x \mapsto x$ and its inverse f^{-1} have Lipschitz constants α and β respectively. So f, f^{-1} are continuous. □

Exercise 2.63. Let $f : X \to Y$ be a map of metric spaces. We say that $f : X \to Y$ is an **isometry** (or is **isometric**) if for all $x_1, x_2 \in X$ we have

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2) \tag{2.13}$$

Show that an isometry is injective and continuous.

We say that f is an **isometric isomorphism** if f is a surjective isometry. If an isometric isomorphism between two metric spaces X, Y exists, we say that X and Y are **isometric metric spaces**. Show that an isometric isomorphism is a homeomorphism. □

[5]We prefer not to call this map the identity map, because the metrics on the source and on the target are different.

Remark 2.64. Isometric isomorphisms are important examples of homeomorphisms. That $f : X \to Y$ is an isometric isomorphism means that X and Y are equivalent as metric spaces, and that this equivalence can be implemented by the bijection f.

We now look at isometric isomorphisms in a different direction. Suppose that $f : X \to Y$ is a bijection of sets. Suppose that Y is a metric space. Then there is unique metric d_X on X such that f is an isometric isomorphism: one defines d_X using (2.13). We write such d_X as f^*d_Y, i.e.,

$$f^*d_Y(x_1, x_2) = d_Y(f(x_1), f(x_2))$$

and call f^*d_Y the **pullback metric** of d_Y by f. □

Pullback metrics are a very useful way of constructing metrics on a set. We consider some examples below.

Exercise 2.65. Two metrics inducing the same topology are not necessarily equivalent metrics. For example, let $f : [0,1] \to [0,1]$ be $f(x) = x^2$. Let $X = [0,1]$, and let d_X be the Euclidean metric: $d_X(x,y) = |x - y|$. So

$$f^*d_X(x,y) = |x^2 - y^2|$$

is a metric on X. It is not hard to check that $f : (X, d_X) \to (X, d_X)$ is a homeomorphism. So d_X and f^*d_X give the same topology on $[0,1]$ (cf. Exe. 2.66). Show that f^*d_X and d_X are not equivalent metrics.

Exercise 2.66. Let $f : X \to Y$ be a bijection of sets with metrics d_X, d_Y. Show that d_X and f^*d_Y give the same topology on X iff $f : (X, d_X) \to (Y, d_Y)$ is a homeomorphism.

In particular, if $f : X \to X$ is a bijection, and d_X is a metric on X. Then d_X and f^*d_X give the same topology on X iff $f : (X, d_X) \to (X, d_X)$ is a homeomorphism.

2.5.2 Convergence in $\overline{\mathbb{R}}$

Our second application of pullback metrics is the convergence in $\overline{\mathbb{R}}$.

Definition 2.67. We say that a sequence (x_n) in $\overline{\mathbb{R}}$ **converges to** $+\infty$ (resp. $-\infty$), if for every $A \in \mathbb{R}$ there is $N \in \mathbb{Z}_+$ such that for all $n \geqslant N$ we have $x_n > A$ (resp. $x_n < A$).

Suppose $x \in \mathbb{R}$. We say that a sequence (x_n) in $\overline{\mathbb{R}}$ **converges to** x, if there is $N \in \mathbb{Z}_+$ such that $x_n \in \mathbb{R}$ for all $n \geqslant N$, and that the subsequence $(x_{k+N})_{k \in \mathbb{Z}_+}$ converges in $\mathbb{R}$ to x. □

This notion of convergence is weird: it is not defined by a metric. So one wonders if there is a metric d on $\overline{\mathbb{R}}$ such that convergence of sequences under d agrees with that in Def. 2.67. We shall now find such a metric.

Lemma 2.68. *Let $-\infty \leqslant a < b \leqslant +\infty$ and $-\infty \leqslant c < d \leqslant +\infty$. Then there is a strictly increasing bijective map $[a,b] \to [c,d]$.*

Note that this map clearly sends a to c and b to d. So it restricts to strictly increasing bijections $(a,b) \to (c,d)$, $(a,b] \to (c,d]$, $[a,b) \to [c,d)$.

Proof. We have a strictly increasing bijection $f : \mathbb{R} \to (-1,1)$ defined by (1.9). f can be extended to a strictly increasing bijective map $\overline{\mathbb{R}} \to [-1,1]$ if we set $f(\pm\infty) = \pm 1$. Thus, f restricts to a strictly increasing bijection $[a,b] \to [f(a), f(b)]$. Choose a linear function $g(x) = \alpha x + \beta$ (where $\alpha > 0$) giving an increasing bijection $[f(a), f(b)] \to [0,1]$. Then $h = g \circ f : [a,b] \to [0,1]$ is a strictly increasing bijection. Similarly, we have a strictly increasing bijection $k : [c,d] \to [0,1]$. Then $k^{-1} \circ h : [a,b] \to [c,d]$ is a strictly increasing bijection. □

Theorem 2.69. *Let $\varphi : \overline{\mathbb{R}} \to [a,b]$ be a strictly increasing bijective map where $[a,b] \subset \mathbb{R}$ is equipped with the Euclidean metric $d_{[a,b]}$. Then a sequence (x_n) in $\overline{\mathbb{R}}$ converges to $x \in \overline{\mathbb{R}}$ in the sense of Def. 2.67 iff $\varphi(x_n)$ converges to $\varphi(x)$ under the metric $d_{[a,b]}$. In other words, the convergence in $\overline{\mathbb{R}}$ is given by the metric $\varphi^* d_{[a,b]}$.*

Proof. Let $y = \varphi(x)$ and $y_n = \varphi(x_n)$. We need to prove that $x_n \to x$ (in the sense of Def. 2.67) iff $y_n \to y$ (under the Euclidean metric). Write $\psi = \varphi^{-1}$, which is a strictly increasing map $[a,b] \to \overline{\mathbb{R}}$. Note that $\varphi(+\infty) = b$ and $\varphi(-\infty) = a$.

Case 1: $x \in \mathbb{R}$. By discarding the first several terms, we may assume that (x_n) is always in $\mathbb{R}$. If $x_n \to x$, then for every $\varepsilon > 0$, all but finitely many x_n are inside the open interval $(\psi(y-\varepsilon), \psi(y+\varepsilon))$. So all but finitely many y_n are inside $(y-\varepsilon, y+\varepsilon)$. So $y_n \to y$. That $y_n \to y$ implies $x_n \to x$ is proved in a similar way.

Case 2: $x = \pm\infty$. We consider $x = +\infty$ only; the other case is similar. Note that if $0 < \varepsilon < b - a$, then $B_{[a,b]}(b, \varepsilon) = (b - \varepsilon, b]$. If $x_n \to +\infty$, then for each $0 < \varepsilon < b - a$, all but finitely many x_n are $> \psi(b - \varepsilon)$. So all but finitely many y_n are inside $(b - \varepsilon, b]$. This proves $y_n \to b$. Conversely, if $y_n \to b$, then for each $A \in \mathbb{R}$, all but finitely many y_n are inside $(\varphi(A), b]$ and hence $> \varphi(A)$. So all but finitely many x_n are $> A$. □

Convention 2.70. Unless otherwise stated, a metric on $\overline{\mathbb{R}}$ is one that makes Def. 2.67 true, for instance $\varphi^* d_{[a,b]}$ in Thm. 2.69. Unless otherwise stated, we do NOT view $\mathbb{R}$ (or any subset of $\mathbb{R}$) as a metric subspace of $\overline{\mathbb{R}}$. Namely, we do not follow Convention 2.15 for $\mathbb{R} \subset \overline{\mathbb{R}}$, or more generally for $\mathbb{R}^N \subset \overline{\mathbb{R}}^N$. Instead, we choose Euclidean metrics on $\mathbb{R}^N$, following Convention 2.8.

The main reason for not following Convention 2.15 here is that metrics on $\overline{\mathbb{R}}$ are all bounded (by Prop. 3.7). Thus, every subset of $\mathbb{R}$ is bounded if we view $\mathbb{R}$ as a metric subspace of $\overline{\mathbb{R}}$. However, we want a subset of $\mathbb{R}$ to be bounded precisely when it is contained in $[a,b]$ for some $-\infty < a < b < +\infty$. (Recall also Def. 1.36.)

After learning topological spaces, we shall forget about the metrics on $\overline{\mathbb{R}}$ and only care about its topology. (See Conv. 7.18.)

Remark 2.71. By Thm. 2.69, the properties of $[a, b]$ about convergence of sequences and inequalities can be transported to $\overline{\mathbb{R}}$, for example:

1. If $(x_n), (y_n)$ are sequences in $\overline{\mathbb{R}}$ converging to $A, B \in \overline{\mathbb{R}}$, and if $x_n \leqslant y_n$ for all n, then $A \leqslant B$.

2. **Squeeze theorem**: Suppose that $(x_n), (y_n), (z_n)$ are sequences in $\overline{\mathbb{R}}$, $x_n \leqslant y_n \leqslant z_n$ for all n, and x_n and z_n both converge to $A \in \overline{\mathbb{R}}$. Then $y_n \to A$.

3. Prop. 2.23 also holds for $[-\infty, +\infty] = \overline{\mathbb{R}}$: if (x_n) is an increasing resp. decreasing sequence in $\overline{\mathbb{R}}$, then $\lim_n x_n$ exists in $\overline{\mathbb{R}}$ and equals $\sup_n x_n$ resp. $\inf_n x_n$.

We will see more examples when studying $\limsup$ and $\liminf$ in the future.

We have shown that there is a metric on $\overline{\mathbb{R}}$ which defines the convergence in Def. 2.67. However, there is no standard choice of such a metric on $\overline{\mathbb{R}}$. Even worse, two possible choices of metrics might not be equivalent: Let $\varphi, \psi : \overline{\mathbb{R}} \to [0, 1]$ be a strictly increasing bijections where $\psi \circ \varphi^{-1} : [0, 1] \to [0, 1]$ is $x \mapsto x^2$. Then by Exe. 2.65, $\varphi^* d_{[0,1]}$ and $\psi^* d_{[0,1]}$ are non-equivalent but topologically equivalent metrics on $\overline{\mathbb{R}}$. This is the first example that metrics are not convenient for the description of convergence. When studying the convergence in $\overline{\mathbb{R}}$, thinking about metrics is distracting. In the future, we will see a better notion for the study of convergence: the notion of topological spaces.

We end this section with a generalization of Thm. 2.69.

Theorem 2.72. *Let φ be a strictly increasing bijection in one of the following forms*

$$\begin{array}{ll} [a, b] \to [c, d] & \quad (a, b) \to (c, d) \\ (a, b] \to (c, d] & \quad [a, b) \to [c, d) \end{array}$$

where $-\infty \leqslant a \leqslant b \leqslant +\infty$ and $-\infty \leqslant c \leqslant d \leqslant +\infty$. Then φ is a homeomorphism, i.e., if (x_n) and x are in the domain, then $x_n \to x$ iff $\varphi(x_n) \to \varphi(x)$ (in the sense of Def. 2.67).

Proof. The case $a = b$ is obvious. So we consider $a < b$, and hence $c < d$. We consider the left-open-right-closed case for example. The other cases are treated in a similar way. If the theorem can be proved for $(-\infty, +\infty] \to (c, d]$, then it can also be proved $(-\infty, +\infty] \to (a, b]$. By composing the inverse of the second map with the first map, we see that the theorem holds for $(a, b] \to (c, d]$.

Let us consider $\varphi : (-\infty, +\infty] \to (c, d]$. φ can be extended to a strictly increasing bijection $\varphi : \overline{\mathbb{R}} \to [c, d]$ by letting $\varphi(-\infty) = c$. It suffices to prove that this φ is a homeomorphism. When $-\infty < c < d < +\infty$, then the theorem holds by Thm. 2.69. If one of c, d is $\pm\infty$, the same argument as in the proof of Thm. 2.69 proves that φ is a homeomorphism. We leave it to the readers to fill in the details. $\square$

2.6 Problems and supplementary material

Definition 2.73. Let A be a subset of $\mathbb{R}$ satisfying $x + y \in A$ for all $x, y \in A$. (Or more generally, let A be an abelian semigroup.) We say that a function $f : A \to \mathbb{R}$ is **subadditive** if for every $x, y \in A$ we have $f(x + y) \leqslant f(x) + f(y)$.

Problem 2.1. Consider the following increasing functions:

$$f_1 : \mathbb{R}_{\geqslant 0} \to [0, 1) \qquad f_1(x) = \frac{x}{1+x}$$
$$f_2 : \mathbb{R}_{\geqslant 0} \to [0, 1] \qquad f_2(x) = \min\{x, 1\}$$

Prove that they are subadditive functions.

Problem 2.2. Let $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be an increasing subadditive function satisfying the following conditions:

(1) $f^{-1}(0) = \{0\}$.

(2) For any $(x_n)_{n\in\mathbb{Z}_+}$ in $\mathbb{R}_{\geqslant 0}$ we have $x_n \to 0$ iff $f(x_n) \to 0$.

Let (X, d) be a metric space. Define

$$\delta : X \times X \to [0, A) \qquad \delta(x, y) = f \circ d(x, y)$$

Prove that δ is a metric, and that δ and d are topologically equivalent.

Proposition 2.74. *Let (X, d) be a metric space. Then there is a bounded metric δ on X such that d and δ are topologically equivalent.*

Proof. Let f be either f_1 or f_2 defined in Pb. 2.1. Then f satisfies the assumptions in Pb. 2.2. So $\delta = f \circ d$ is a desired metric due to Pb. 2.2. We write down the formulas explicitly:

$$\delta_1(x, y) = \frac{d(x, y)}{1 + d(x, y)} \qquad \delta_2(x, y) = \min\{d(x, y), 1\}$$

□

Problem 2.3. Let $(X_i, d_i)_{i\in\mathbb{Z}_+}$ be a sequence of metric spaces. Assume that $d_i \leqslant 1$ for each i. Let $S = \prod_{i\in\mathbb{Z}_+} X_i$. For each elements $f = (f(i))_{i\in\mathbb{Z}_+}$ and $g = (g(i))_{i\in\mathbb{Z}_+}$ of S, define

$$d(f, g) = \sup_{i\in\mathbb{Z}_+} \frac{d_i(f(i), g(i))}{i} \tag{2.14}$$

Prove that d is a metric on S. Let $f_n = (f_n(i))_{i\in\mathbb{Z}_+}$ be a sequence in S. Let $g \in S$. Prove that the following are equivalent:

(a) $\lim_{n\to\infty} f_n = g$ under the metric d.

(b) f_n **converges pointwise** to g, namely, $\lim_{n\to\infty} f_n(i) = g(i)$ for every $i \in \mathbb{Z}_+$.

Remark 2.75. The above problem gives our first non-trivial example of function spaces as metric spaces, where the domain of functions is a countable set. After learning series, the readers can check that

$$\delta(f,g) = \sum_{i\in\mathbb{Z}_+} 2^{-i} d_i(f(i), g(i)) \tag{2.15}$$

also defines a metric, and that (a) (with d replaced by δ) and (b) are equivalent. So δ and d (defined by (2.14)) induce the same topology on X, called the **pointwise convergence topology** or simply **product topology**. Unfortunately, if the index set $\mathbb{Z}_+$ is replaced by an uncountable set, there is in general no metric inducing the product topology. We will prove this in Pb. 7.9.

$\star$ **Problem 2.4.** Let $X = \bigsqcup_{\alpha\in\mathscr{A}} X_\alpha$ be a disjoint union of metric spaces (X_α, d_α). Assume that $d_\alpha \leqslant 1$ for all i. For each $x, y \in X$, define

$$d(x,y) = \begin{cases} d_\alpha(x,y) & \text{if } x,y \in X_\alpha \text{ for some } \alpha \in \mathscr{A} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

1. Prove that d defines a metric on X.
2. Choose $(x_n)_{n\in\mathbb{Z}_+}$ in X and $x \in X$. What does $\lim_{n\to\infty} x_n = x$ mean in terms of the convergence in each X_α?

Think about the question: Let X be a set. For each $x, y \in X$ define $d(x,y) = 0$ if $x = y$, and $d(x,y) = 1$ if $x \neq y$. What does convergence in (X,d) mean?

3 Sequential compactness and completeness

3.1 Sequential compactness

3.1.1 Basic properties of sequentially compact spaces

Definition 3.1. Let X be a metric space. We say that X is **sequentially compact** if every sequence in X has a subsequence converging to some point of X.

The notion of sequential compactness is extremely useful for finding solutions in an analysis problem. In general, suppose we want to find a point $x \in X$ which makes a property $P(x)$ to be true. Suppose that we can find an "approximate solution", i.e. an $y \in X$ such that $P(y)$ is close to being true. Thus, we can find a sequence (x_n) in X such that $P(x_n)$ is closer and closer to being true when $n \to \infty$. Now, if X is sequentially compact, then (x_n) has a subsequence (x_{n_k}) converging to $x \in X$. Then $P(x)$ is true, and hence x is a solution for the problem. (See also Sec. 2.1.) Let us see an explicit example:

Lemma 3.2 (Extreme value theorem). *Let X be a sequentially compact metric space. Let $f : X \to \mathbb{R}$ be a continuous function. Then f attains its maximum and minimum at points of X. In particular, $f(X)$ is a bounded subset of $\mathbb{R}$.*

This extremely important result is the main reason for introducing sequentially compact spaces. We call this a lemma, since we will substantially generalize this result later. (See Exe. 3.6.)

Note that the **boundedness** of subsets of $\mathbb{R}$ (or more generally, of $\mathbb{R}^N$) is always understood under the Euclidean metric of $\mathbb{R}$, not under any metric of $\overline{\mathbb{R}}$ or $\overline{\mathbb{R}}^N$. (Recall Convention 2.70.)

Proof. We show that f attains its maximum on X. The proof for minimum is similar. Let $A = \sup f(X)$. Then $A \in (-\infty, +\infty]$. If $A < +\infty$, then for each $n \in \mathbb{Z}_+$ there is $x_n \in X$ such that $A - 1/n < f(x_n) \leqslant A$ (since $A - 1/n$ is not an upper bound of $f(X)$). If $A = +\infty$, then for each n there is $x_n \in X$ such that $f(x_n) > n$. In either case, we have a sequence (x_n) in X such that $f(x_n) \to A$ in $\overline{\mathbb{R}}$.

Since X is sequentially compact, (x_n) has a subsequence $(x_{n_k})_{k \in \mathbb{Z}_+}$ converging to some $x \in X$. Now, consider f as a map $f : X \to \overline{\mathbb{R}}$, which is continuous (cf. Exp. 2.51). Since $f(x_n) \to A$, its subsequence $f(x_{n_k})$ also converges to A. But since $x_{n_k} \to x$ and f is continuous at x, we have $A = f(x)$. So f attains its maximum at x. Since $f(X) \subset \mathbb{R}$, we have $A \in \mathbb{R}$. □

The following are some elementary examples of sequential compactness:

Exercise 3.3. Show that finite unions of sequentially compact spaces is sequentially compact. (In particular, a finite set is sequentially compact.)

More precisely, let X be a metric space. Assume $X = A_1 \cup \cdots \cup A_N$ where each metric subspace A_i is sequentially compact. Show that X is sequentially compact.

Proposition 3.4. *Let $X_1, \dots, X_N$ be sequentially compact metric spaces. Then $X = X_1 \times \cdots \times X_N$ is sequentially compact.*

Proof. Since $X = (X_1 \times \cdots \times X_{N-1}) \times X_N$, by induction, it suffices to assume $N = 2$. So we write $X = A \times B$ where A, B are sequentially compact. Let (a_n, b_n) be a sequence in X. Since A is sequentially compact, (a_n) has a convergent subsequence (a_{n_k}). Since B is sequentially compact, (b_{n_k}) has a convergent subsequence $(b_{n_{k_l}})$. So $(a_{n_{k_l}}, b_{n_{k_l}})$ is a convergent subsequence of (a_n, b_n). □

Proposition 3.5. *Let $f : X \to Y$ be a continuous map of metric spaces. Assume that X is sequentially compact. Then $f(X)$, as a metric subspace of Y, is sequentially compact.*

Proof. Choose any sequence (y_n) in $f(X)$. We can write $y_n = f(x_n)$ where $x_n \in X$. Since X is sequentially compact, (x_n) has a subsequence (x_{n_k}) converging to some $x \in X$. Since f is continuous, $y_{n_k} = f(x_{n_k})$ converges to $f(x)$. □

Exercise 3.6. Prove that if Y is a sequentially compact subset of $\mathbb{R}$, then $\sup Y \in Y$ and $\inf Y \in Y$. Therefore, Prop. 3.5 generalizes Lem. 3.2.

Proposition 3.7. *Let X be a sequentially compact metric space. Then X is bounded under its metric d.*

Proof. Choose any $p \in X$. The function $d_p : x \in X \mapsto d(x,p) \in \mathbb{R}_{\geqslant 0}$ is continuous by Exp. 2.58. So, by Lem. 3.2, d_p is bounded by some $0 < R < +\infty$. So $X = \overline{B}_X(p, R) \subset B_X(p, 2R)$. □

3.1.2 Limits inferior and superior, and Bolzano-Weierstrass

The goal of this subsection is to prove that closed intervals are sequentially compact.

Definition 3.8. Let (x_n) be a sequence in a metric space X. We say that $x \in X$ is a **cluster point** of (x_n), if (x_n) has a subsequence (x_{n_k}) converging to x.

Warning: In a general topological space, the cluster points of a sequence will be defined in a different way. (See Pb. 7.2 and Rem. 7.117.)

Definition 3.9. Let (x_n) be a sequence in $\overline{\mathbb{R}}$. Define

$$\alpha_n = \inf\{x_k : k \geqslant n\} \qquad \beta_n = \sup\{x_k : k \geqslant n\} \tag{3.1}$$

It is clear that $\alpha_n \leqslant x_n \leqslant \beta_n$, that (α_n) is increasing and (β_n) is decreasing. Define

$$\liminf_{n\to\infty} x_n = \sup\{\alpha_n : n \in \mathbb{Z}_+\} = \lim_{n\to\infty} \alpha_n \tag{3.2a}$$

$$\limsup_{n\to\infty} x_n = \inf\{\beta_n : n \in \mathbb{Z}_+\} = \lim_{n\to\infty} \beta_n \tag{3.2b}$$

(cf. Rem. 2.71), called respectively the **limit inferior** and the **limit superior** of (x_n).

Remark 3.10. Let $(x_n), (y_n)$ be sequences in $\overline{\mathbb{R}}$. Suppose that $x_n \leqslant y_n$ for every n. It is clear that

$$\liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n \qquad \liminf_{n\to\infty} x_n \leqslant \liminf_{n\to\infty} y_n \qquad \limsup_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} y_n$$

Theorem 3.11. *Let (x_n) be a sequence in $\overline{\mathbb{R}}$, and let S be the set of cluster points of (x_n). Then $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$ belong to S. They are respectively the minimum and the maximum of S.*

In particular, every sequence in $\overline{\mathbb{R}}$ has at least one cluster point.

Proof. We use the notations in Def. 3.9. Let $A = $ (3.2a) and $B = $ (3.2b). If $x \in S$, pick a subsequence (x_{n_k}) converging to x. Since $\alpha_{n_k} \leqslant x_{n_k} \leqslant \beta_{n_k}$, we have $A \leqslant x \leqslant B$ by Rem. 2.71. It remains to show that $A, B \in S$. We prove $B \in S$ by constructing a subsequence (x_{n_k}) converging to B; the proof of $A \in S$ is similar.

Consider first of all the special case that (x_n) is bounded, i.e., is inside $[a, b] \subset \mathbb{R}$. Choose an arbitrary $n_1 \in \mathbb{Z}_+$. Suppose $n_1 < \cdots < n_k$ have been constructed. By the definition of β_{1+n_k}, there is $n_{k+1} \geqslant 1 + n_k$ such that $x_{n_{k+1}}$ is close to β_{1+n_k}, say

$$\beta_{1+n_k} - \frac{1}{k} < x_{n_{k+1}} \leqslant \beta_{1+n_k} \tag{3.3}$$

Since the left most and the right most of (3.3) both converge to B as $k \to \infty$, by Squeeze theorem (Cor. 2.48) we conclude $\lim_k x_{n_k} = B$.

In general, by Lem. 2.68 and Thm. 2.72, there is an increasing (i.e. order-preserving) homeomorphism (i.e. topopogy-preserving map) $\varphi : \overline{\mathbb{R}} \to [0, 1]$. Then $\varphi(\beta_n) = \sup\{\varphi(x_k) : k \geqslant n\}$ (cf. Exe. 3.13) and $\varphi(B) = \lim_n \varphi(\beta_n)$. So $\varphi(B) = \limsup_n \varphi(x_n)$. By the above special case, $(\varphi(x_n))$ has a subsequence $(\varphi(x_{n_k}))$ converging to $\varphi(B)$. So (x_{n_k}) converges to B. □

Remark 3.12. One can also prove the above general case directly using a similar idea as in the special case. And you are encouraged to do so! (Pay attention to the case $B = \pm\infty$.)

The proof given above belongs to a classical proof pattern: To prove that a space X satisfies some property, one first prove it in a convenient case. Then, in the general case, one finds an "isomorphism" (i.e. "equivalence" in a suitable sense) $\varphi : X \to Y$ where Y is in the convenient case. Then the result on Y can be translated via φ^{-1} to X, finishing the proof.

For example, to solve a linear algebra problem about linear maps between finite-dimensional vector spaces V, W, one first proves it in the special case that $V = \mathbb{F}^m$ and $W = \mathbb{F}^n$. Then, the general case can be translated to the special case via an equivalence as in Exp. 1.21. □

Exercise 3.13. Let X, Y be posets. Let $\varphi : X \to Y$ be an increasing bijection whose inverse is also increasing. (Namely, φ induces an equivalence of posets). Suppose $E \subset X$ has supremum $\sup E$. Explain why $\varphi(E)$ has supremum $\varphi(\sup E)$.

It is now fairly easy to prove the famous

Theorem 3.14 (Bolzano-Weierstrass). *Let $[a_1, b_1], \ldots, [a_N, b_N]$ be closed intervals in $\overline{\mathbb{R}}$. Then $[a_1, b_1] \times \cdots \times [a_N, b_N]$ is sequentially compact.*

Proof. By Prop. 3.4, it suffices to assume $N = 1$. Write $a_1 = a, b_1 = b$. Let (x_n) be a sequence in $[a, b]$. By Thm. 3.11, (x_n) has a subsequence (x_{n_k}) converging to some $x \in \overline{\mathbb{R}}$. (E.g. $x = \limsup_n x_n$.) Since $a \leqslant x_{n_k} \leqslant b$, we have $a \leqslant x \leqslant b$ by Rem. 2.71. □

Bolzano-Weierstrass theorem illustrates why we sometimes prefer to work with $\overline{\mathbb{R}}$ instead of $\mathbb{R}$: $\overline{\mathbb{R}}$ is sequentially compact, while $\mathbb{R}$ is not. That every sequence has limits superior and inferior in $\overline{\mathbb{R}}$ but not necessarily in $\mathbb{R}$ is closely related to this fact. In the language of point-set topology, $\overline{\mathbb{R}}$ is a **compactification** of $\mathbb{R}$.

Bolzano-Weierstrass theorem (restricted to $\mathbb{R}^N$) will be generalized to **Heine-Borel theorem**, which says that a subset of $\mathbb{R}^N$ is sequentially compact iff it is bounded and closed (cf. Def. 3.26 for the definition of closed subsets). See Thm. 3.55.

3.1.3 A criterion for convergence in sequentially compact spaces

At the end of Sec. 2.3, we have raised the following question: Suppose that (x_n) is a bounded sequence in a metric space X such that any two convergent subsequences converge to the same point. Does (x_n) converge?

When X is sequentially compact, (x_n) is automatically bounded due to Prop. 3.7. The answer to the above question is yes:

Theorem 3.15. *Let X be a sequentially compact metric space. Let (x_n) be a sequence in X. Then the following are equivalent.*

(1) The sequence (x_n) converges in X.

(2) Any two convergent subsequences of (x_n) converge to the same point. In other words, (x_n) has only one cluster point.

Proof. (1)⇒(2): By Prop. 2.36.

(2)⇒(1): Assume that (x_n) has at most one cluster point. Since X is sequentially compact, (x_n) has at least one cluster point $x \in X$. We want to prove $\lim_{n\to\infty} x_n = x$. Suppose not. Then there exists $\varepsilon > 0$ such that for every $N \in \mathbb{Z}_+$ there is $n \geqslant N$ such that $d(x_n, x) \geqslant \varepsilon$. Thus, one can inductively construct a subsequence (x_{n_k}) of (x_n) such that $d(x_{n_k}, x) \geqslant \varepsilon$ for all k. Since X is sequentially compact, (x_{n_k}) has a subsequence x'_n converging to $x' \in X$. So $d(x'_n, x) \geqslant \varepsilon$ for all n. Since the function $y \in X \mapsto d(y, x)$ is continuous (Exp. 2.58), we have $\lim_{n\to\infty} d(x'_n, x) = d(x', x)$. This proves that $d(x', x) \geqslant \varepsilon > 0$. However, x', x are both cluster points of (x_n), and so $x = x'$. This gives a contradiction. □

Remark 3.16. Thm. 3.15 can be used in the following way. Suppose that we want to prove that a given sequence (x_n) in a sequentially compact space X converges to x. Then it suffices to prove that if (x'_n) is a subsequence of (x_n) converging to some $y \in X$, then $y = x$. This is sometimes easier to prove than directly proving the convergence of (x_n). We will use this strategy in the proof of L'Hôpital's rule, for example. (See Subsec. 12.1.2.)

Corollary 3.17. *Let (x_n) be a sequence in $\mathbb{R}^N$. The following are equivalent.*

(1) The sequence (x_n) converges in $\mathbb{R}^N$.

(2) The sequence (x_n) is bounded. Moreover, any two convergent subsequences of (x_n) converge to the same point of $\mathbb{R}^N$.

Proof. (1)$\Rightarrow$(2): By Prop. 2.33 and 2.36.

(2)$\Rightarrow$(1): Assume (2). Since (x_n) is bounded, it can be contained in $X = I_1 \times \cdots \times I_N$ where each I_i is a closed interval in $\mathbb{R}$. Clearly, any two cluster points of (x_n) are inside X, and are equal by (2). By Bolzano-Weierstrass, X is sequentially compact. Thus, by Thm. 3.15, (x_n) converges in X and hence in $\mathbb{R}^N$. $\square$

Corollary 3.18. *The following are true.*

1. *Let (x_n) be a sequence in $\overline{\mathbb{R}}$. Then (x_n) converges in $\overline{\mathbb{R}}$ iff $\limsup_{n\to\infty} x_n$ equals $\liminf_{n\to\infty} x_n$.*

2. *Let (x_n) be a sequence in $\mathbb{R}$. Then (x_n) converges in $\mathbb{R}$ iff $\limsup_{n\to\infty} x_n$ equals $\liminf_{n\to\infty} x_n$ and (x_n) is bounded.*

Note that if (x_n) converges in $\overline{\mathbb{R}}$, we must have $\lim x_n = \limsup x_n = \liminf x_n$ by Thm. 3.11.

Proof. 1. Let $A = \liminf x_n$ and $B = \limsup x_n$. Let S be the set of cluster points of (x_n). By Thm. 3.11, $A = \min S, B = \max S$. So $A = B$ iff S has only one element. This is equivalent to the convergence of (x_n) in $\overline{\mathbb{R}}$ due to Thm. 3.15 (since $\overline{\mathbb{R}}$ is sequentially compact by Bolzano-Weierstrass.)

2. If (x_n) converges, then $A = B$ by part 1. And (x_n) is bounded due to Prop. 2.33. Conversely, if $A = B$ and if (x_n) is bounded, say $\alpha \leqslant x_n \leqslant \beta$ for all n where $-\infty < \alpha < \beta < +\infty$. Then $\alpha \leqslant A \leqslant B \leqslant \beta$. So $A, B \in \mathbb{R}$. By part 1, (x_n) converges to $A \in \mathbb{R}$. $\square$

3.2 Outlook: sequentially compact function spaces

In Sec. 2.1, we mentioned that metric spaces and (more generally) point-set topology were introduced by mathematicians in order to study (typically infinite dimensional) function spaces with the help of the geometric intuition of $\mathbb{R}^N$. Now we have learned a couple of important results about sequentially compact spaces. But we have not met any example arising from function spaces. So let me show one example to the curious readers: The product space $[0,1]^{\mathbb{Z}_+}$, equipped with the metric defined in Pb. 2.3, is sequentially compact. We will prove this result at the end of this chapter. (Indeed, we will prove a slightly more general version. See Thm. 3.54.) This is a famous result, not only because it has many important applications (some of which will be hinted at in this section), but also because its proof uses the clever "diagonal method".

Moreover, we will later prove an even more surprising fact: every sequentially compact metric space is homeomorphic to a closed subset of $[0,1]^{\mathbb{Z}_+}$. (See Thm. 8.45.) Thus, all sequentially compact metric spaces can be constructed explicitly, in some sense.

The readers may still complain that functions on $\mathbb{Z}_+$ are very different from those we often see and use in analysis and (especially) in differential equations: We are ultimately interested in functions on $\mathbb{R}$ or on $[a,b]$, but not on countable sets. This is correct. But $[0,1]^{\mathbb{Z}_+}$ (and its closed subsets) are in fact very helpful for the study of spaces of functions on $\mathbb{R}$ and on $[a,b]$. In this course, we shall learn two major examples that the sequential compactness of $[0,1]^{\mathbb{Z}_+}$ helps with:

1. $A = \mathbb{Q} \cap [a,b]$ is a countable dense subset of $[a,b]$. Thus, if we let $C([a,b])$ denote the set of continuous $\mathbb{R}$-functions on $[a,b]$, then the restriction map $f \in C([a,b]) \mapsto f|_A \in \mathbb{R}^A$ is injective. In many applications, we are interested in a subset $\mathcal{X} \subset C([a,b])$ of uniformly bounded functions, say all $f \in \mathcal{X}$ take values in $[-1,1]$. Then we have an injective map

$$\Phi : \mathcal{X} \to [-1,1]^A \qquad f \mapsto f|_A$$

If $\mathcal{X}$ satisfies a condition called "**equicontinuous**", then a sequence f_n in $\mathcal{X}$ converges *uniformly* to $f \in C([a,b])$ iff $f_n|_A$ converges *pointwise* to $f|_A$. (See Rem. 3.62.) Thus, from the sequential compactness of $[-1,1]^A$ under pointwise convergence topology, one concludes that every sequence in $\mathcal{X}$ has a subsequence converging uniformly in $C([a,b])$. This remarkable sequential compactness result on (the closure of) $\mathcal{X}$ is called **Arzelà-Ascoli theorem**, and will be used to prove the fundamental Peano existence theorem in ordinary differential equations. We also see that the fact that $[a,b]$ has a countable dense subset A plays a crucial role. This property of metric spaces is called "**separable**" and will be studied later.

2. **Fourier series** are powerful for the study of partial differential equations. A continuous function $f : [-\pi,\pi] \to \mathbb{C}$ satisfying $f(-\pi) = f(\pi)$ has Fourier

series expansion $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{\mathbf{i}nx}$ where $a_n \in \mathbb{C}$. However, for the sake of studying differential equations, one needs to consider series $\sum_{n\in\mathbb{Z}} a_n e^{\mathbf{i}nx}$ converging to a function much worse than a continuous function. For example, in the study of integral equations (which are closely related to certain partial differential equations), Hilbert and Schmidt discovered that one has to consider all $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{\mathbf{i}nx}$ satisfying $\sum_n |a_n|^2 \leqslant 1$. Therefore, one lets $\overline{B} = \{z \in \mathbb{C} : |z| \leqslant 1\}$ and considers $\widehat{f} : n \in \mathbb{Z} \mapsto a_n \in \mathbb{C}$ as an element of $\overline{B}^{\mathbb{Z}}$. The sequential compactness of $\overline{B}^{\mathbb{Z}}$ helps one find the $\widehat{f}$ such that the corresponding $f(x) = \sum_n \widehat{f}(n) \cdot e^{\mathbf{i}nx}$ is a desired solution of the integral equation.

3.3 Complete metric spaces and Banach spaces

In this section, we let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and assume that all vector spaces are over $\mathbb{F}$.

3.3.1 Cauchy sequences and complete metric spaces

Definition 3.19. A sequence (x_n) in a metric space X is called a **Cauchy sequence**, if:

- For every $\varepsilon > 0$ there exists $N \in \mathbb{Z}_+$ such that for all $m, n \geqslant N$ we have $d(x_m, x_n) < \varepsilon$.

Here, "$\varepsilon > 0$" can mean either "$\varepsilon \in \mathbb{R}_{>0}$" or "$\varepsilon \in \mathbb{Q}_{>0}$". The choice of this meaning does not affect the definition. The above definition can be abbreviated to "for every $\varepsilon > 0$, we have $d(x_m, x_n) < \varepsilon$ for sufficiently large m, n".

Remark 3.20. It is an easy consequence of triangle inequality that (x_n) is a Cauchy sequence iff

- For every $\varepsilon > 0$ there exists $N \in \mathbb{Z}_+$ such that for all $n \geqslant N$ we have $d(x_n, x_N) < \varepsilon$.

Also, it is clear that every Cauchy sequence is bounded.

Proposition 3.21. *Every convergent sequence in a metric space X is a Cauchy sequence.*

Proof. Assume (x_n) converges to x in X. Then for every $\varepsilon > 0$ there is $N \in \mathbb{Z}_+$ such that $d(x, x_n) < \varepsilon/2$ for all $n \geqslant N$. Since this is true for every $m \geqslant N$, we have $d(x_m, x_n) \leqslant d(x, x_n) + d(x, x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

Definition 3.22. A metric space X is called **complete** if every Cauchy sequence in X converges.

We have many examples of complete metric spaces:

Theorem 3.23. *If (x_n) is a Cauchy sequence in a metric space X with at least one cluster point x, then (x_n) converges in X to x. Consequently, every sequentially compact metric space is complete.*

Proof. Let (x_n) be a Cauchy sequence in X with subsequence (x_{n_k}) converging to $x \in X$. Let us show that $x_n \to x$.

Since (x_n) is Cauchy, for every $\varepsilon > 0$ there is $N \in \mathbb{Z}_+$ such that $d(x_n, x_m) < \varepsilon/2$ for all $m, n \geqslant N$. Since $x_{n_k} \to x$, there is $k \geqslant N$ such that $d(x_{n_k}, x) < \varepsilon/2$. Since n_k is strictly increasing over k, we have $n_k \geqslant k$. So $n_k \geqslant N$. So we can let $m = n_k$. This gives $d(x_n, x_{n_k}) < \varepsilon/2$. Therefore $d(x_n, x) \leqslant d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon$ for all $n \geqslant N$. □

Example 3.24. Let $X = [a_1, b_1] \times \cdots \times [a_N, b_N]$ where each $[a_i, b_i]$ is a closed interval in $\mathbb{R}$, then X is sequentially compact by Bolzano-Weierstrass. Thus, by Thm. 3.23. X is complete.

Corollary 3.25. $\mathbb{R}^N$ *and* $\mathbb{C}^N$ *are complete (under the Euclidean metrics).*

Proof. Since $\mathbb{C}^N$ is isometrically isomorphic to $\mathbb{R}^{2N}$, it suffices to prove that $\mathbb{R}^N$ is complete. Choose a Cauchy sequence (x_n) in $\mathbb{R}^N$. Since (x_n) is bounded, (x_n) is contained inside $X = I_1 \times \cdots \times I_N$ where each $I_i = [a, b]$ is in $\mathbb{R}$. By Exp. 3.24, X is complete. So (x_n) converges to some $x \in X$. □

Definition 3.26. We say that a subset A of a metric space X is **closed** if the following condition is true: For every sequence (x_n) in A converging to a point $x \in X$, we have $x \in A$.

Thus, the word "closed" here means "closed under taking limits".

Proposition 3.27. *Let A be a metric subspace of a metric space X. Recall that the metric of A inherits from that of X (cf. Def. 2.14). Consider the statements:*

(1) A is complete.

(2) A is a closed subset of X.

Then (1)⇒(2). If X is complete, then (2)⇒(1).

Proof. First, assume that X is complete and (2) is true. Let (x_n) be a Cauchy sequence in A. Then it is a Cauchy sequence in X. So $x_n \to x \in X$ because X is complete. So $x \in A$ by the definition of closedness. This proves (1).

Next, we assume (1). Choose a sequence (x_n) in A converging to a point $x \in X$. By Prop. 3.21, (x_n) is a Cauchy sequence in X, and hence a Cauchy sequence in A. Since A is complete, there is $a \in A$ such that $x_n \to a$. So we must have $x = a$ because any sequence has at most one limit in a metric space. This proves $x \in A$. So (2) is proved. □

A similar result holds for sequential compactness. See Pb. 3.4.

Example 3.28. Let $-\infty < a < b < +\infty$. Then (a,b) is not complete (under the Euclidean metric), because (a,b) is not closed in the metric space $\mathbb{R}$. (For sufficiently large n, $b-1/n$ is in (a,b), but $\lim_{n\to\infty}(b-1/n) = b$ is not in b.)

Example 3.29. By Prop. 1.29, for each $x \in \mathbb{R}\backslash\mathbb{Q}$, we can choose an increasing sequence in $\mathbb{Q}$ converging to x. So $\mathbb{Q}$ is not closed in $\mathbb{R}$. So $\mathbb{Q}$ is not complete under the Euclidean topology.

Example 3.30. Let X be a metric space. Let $p \in X$ and $0 \leqslant R < +\infty$. Then $\overline{B}_X(x,R)$ is a closed subset of X. Therefore, if X is complete, then $\overline{B}_X(p,R)$ is complete by Prop. 3.27.

Proof of closedness. Let (x_n) be a sequence in $\overline{B}(p,R)$ converging to $x \in X$. Then $d(p,x_n) \leqslant R$. Since the function $y \in X \mapsto d(p,y) \in \mathbb{R}$ is continuous (Exp. 2.58), we have $d(p,x) = \lim_{n\to\infty} d(p,x_n) \leqslant R$. So $x \in \overline{B}(p,R)$. $\square$

Exercise 3.31. Let d,δ be two equivalent metrics on a set X. Show that a sequence (x_n) in X is Cauchy under d iff (x_n) is Cauchy under δ.

Note that if, instead of assuming d,δ are equivalent, we only assume that d,δ are topologically equivalent. Then the above conclusion is not necessarily true:

Exercise 3.32. Find a non-complete metric δ on $\mathbb{R}$ topologically equivalent to the Euclidean metric.

3.3.2 Normed vector spaces and Banach spaces

A major application of complete metric spaces is to show that many series converge without knowing to what exact values these series converge. A typical example is the convergence of $\sum_{n\in\mathbb{Z}_+}\sin(\sqrt{2}n)/n^2$ in $\mathbb{R}$. We are also interested in the convergence of series in function spaces, for instance: the uniform convergence of $f(x) = \sum_{n\in\mathbb{Z}_+}\sin(\sqrt{2}nx^3)/n^2$ on $\mathbb{R}$; a suitable convergence of the Fourier series $\sum_{n\in\mathbb{Z}} a_n e^{\mathbf{i}nx}$. But we cannot take sum in a general metric space since it has no vector space structures. Therefore, we need a notion which combines complete metric spaces with vector spaces. Banach spaces are such a notion.

Definition 3.33. Let V be a vector space over $\mathbb{F}$ with zero vector 0_V. A function $\|\cdot\| : V \to \mathbb{R}_{\geqslant 0}$ is called a **norm** if for every $u,v \in V$ and $\lambda \in \mathbb{F}$, the following hold:

- (Subadditivity) $\|u+v\| \leqslant \|u\| + \|v\|$.
- (Absolute homogeneity) $\|\lambda v\| = |\lambda|\cdot\|v\|$. In particular, (by taking $\lambda = 0$) we have $\|0_V\| = 0$.
- If $\|v\| = 0$ then $v = 0_V$.

We call $(V, \|\cdot\|)$ (often abbreviated to V) a **normed vector space**.

Remark 3.34. Assuming $\|0_V\| = 0$, to check the absolute homogeneity, it suffices to check

$$\|\lambda v\| \leqslant |\lambda| \cdot \|v\|$$

for all λ and v. Then clearly $\|\lambda v\| = |\lambda| \cdot \|v\|$ when $\lambda = 0$. Suppose $\lambda \neq 0$. Then

$$\|v\| = \|\lambda^{-1}\lambda v\| \leqslant |\lambda|^{-1}\|\lambda v\|$$

which implies $\|\lambda v\| = |\lambda| \cdot \|v\|$.

Remark 3.35. Let V be a vector space. If V is a normed vector space, then

$$d(u, v) = \|u - v\| \tag{3.4}$$

clearly defines a metric. (Note that triangle inequality follows from subadditivity.) Unless otherwise stated, we always assume that the metric of a normed vector space is defined by (3.4).

Definition 3.36. Let V be a normed vector space. We say that V is a **Banach space** if V is a complete metric space where the metric is the canonical one (3.4). If V is over the field $\mathbb{C}$ (resp. $\mathbb{R}$), we call V a **complex** (resp. **real**) **Banach space**.

Example 3.37. We always assume that the norm on $\mathbb{F}^N$ is the **Euclidean norm**

$$\|(a_1, \dots, a_N)\| = \sqrt{|a_1|^2 + \cdots + |a_N|^2} \tag{3.5}$$

The canonical metric it gives is the Euclidean metric. Thus, by Cor. 3.25, $\mathbb{F}^N$ is a Banach space.

If (λ_n) is a sequence in $\mathbb{F}$ converging to λ, and if (x_n) is a sequence in $\mathbb{F}^N$ converging to x, then one can show that $\lambda_n x_n$ converges to λx by checking that each component of $\lambda_n x_n$ converges to the corresponding component of λx. This is due to Prop. 2.27. However, if (x_n) is in general a sequence in a normed vector space, this method fails. So we need a different argument:

Proposition 3.38. *Let V be a normed vector space. The following maps are continuous*

$$\begin{aligned} + : V \times V \to V \qquad & (u, v) \mapsto u + v \\ - : V \times V \to V \qquad & (u, v) \mapsto u - v \\ \times_{\mathbb{F}} : \mathbb{F} \times V \to V \qquad & (\lambda, v) \mapsto \lambda v \\ \|\cdot\| : V \to \mathbb{R}_{\geqslant 0} \qquad & v \mapsto \|v\| \end{aligned}$$

We didn't mention the continuity of the division map $(\lambda, v) \in \mathbb{F}^\times \times V \mapsto \lambda^{-1}v$ since it follows from that of $\times_{\mathbb{F}}$ and of the inversion map $\lambda \mapsto \lambda^{-1}$ by Exp. 2.44.

Proof. One can check that the addition map, the subtraction map, and the last map $\|\cdot\|$ are Lipschitz continuous.

Define metric $d((\lambda,v),(\lambda',v')) = \max\{|\lambda-\lambda'|, \|v-v'\|\}$ on $\mathbb{F}\times V$. Then $\mathbb{F}\times V$ is covered by open balls of the form $B(0,r) = \{(\lambda,v)\in\mathbb{F}\times V : |\lambda|<r, \|v\|<r\}$. Similar to the argument in (2.6), one uses subadditivity (i.e. triangle inequality) and absolute homogeneity to show that $\times_{\mathbb{F}}$ has Lipschitz constant $2r$ on $B(0,r)$. So $\times_{\mathbb{F}}$ is continuous by Lem. 2.41 and 2.43. $\square$

3.4 The Banach spaces $l^\infty(X,V)$ and $C(X,V)$

In this section, we let $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$ and assume that the vector spaces V are over $\mathbb{F}$. As the title suggests, in this section we shall introduce two important examples of Banach spaces: the space of bounded functions $l^\infty(X,V)$ and its subspace of continuous functions $C(X,V)$ (when X is a sequentially compact metric space). In order for these two spaces to be Banach spaces, we must assume that V is also Banach.

In application, the main examples are $V=\mathbb{R},\mathbb{C},\mathbb{R}^N,\mathbb{C}^N$. Indeed, $C([a,b],\mathbb{R}^N)$ is one of the main examples of function spaces considered by Fréchet when he defined metric spaces. Therefore, the readers can assume that V is one of such spaces if they want to make life easier. Just keep in mind that we sometimes also consider the case where V itself is a function space.

Definition 3.39. Let X be a set and let V be a vector space. The set V^X is a vector space if we define for each $f,g\in V^X$ and $\lambda\in\mathbb{F}$:

$$f+g : X\to V \qquad (f+g)(x) = f(x)+g(x)$$
$$\lambda f : X\to V \qquad (\lambda f)(x) = \lambda f(x)$$

We also define the **absolute value function**

$$|f| : X\to\mathbb{R}_{\geqslant 0} \qquad x\in X\mapsto \|f(x)\| \tag{3.6}$$

The symbol $|f|$ is sometimes also written as $\|f\|$ when it will not be confused with $\|f\|_\infty$ or other norms of f.

Definition 3.40. Let X be a set and let V be a normed vector space. For each $f\in V^X$, define the ***l*$^\infty$-norm**

$$\|f\|_{l^\infty(X,V)} \equiv \|f\|_{l^\infty} \equiv \|f\|_\infty = \sup_{x\in X}\|f(x)\| \tag{3.7}$$

where $\|f(x)\|$ is defined by the norm of V. Define the ***l*$^\infty$-space**

$$l^\infty(X,V) = \{f\in V^X : \|f\|_\infty < +\infty\} \tag{3.8}$$

which is a vector subspace of V^X. Then $l^\infty(X,V)$ is a normed vector space under the l^∞-norm. A function $f: X\to V$ is called **bounded** if $f\in l^\infty(X,V)$.

Exercise 3.41. Prove that for every $f, g \in V^X$ and $\lambda \in \mathbb{F}$ we have

$$\begin{aligned} \|f+g\|_\infty &\leqslant \|f\|_\infty + \|g\|_\infty \\ \|\lambda f\|_\infty &= |\lambda| \cdot \|f\|_\infty \end{aligned} \tag{3.9}$$

(Note that clearly we have that $\|f\|_\infty = 0$ implies $f = 0$.) Here, we understand $0 \cdot (+\infty) = 0$. Use these relations to verify that $l^\infty(X, V)$ is a linear subspace of V^X (i.e. it is closed under addition and scalar multiplication) and that $\|\cdot\|_\infty$ is a norm on $l^\infty(X, V)$. □

Definition 3.42. Let V be a normed vector space. We say that a sequence (f_n) in V^X **converges uniformly** to $f \in V^X$ if $\lim_{n\to\infty} \|f - f_n\|_\infty = 0$. In this case, we write $f_n \rightrightarrows f$.

We say that (f_n) **converges pointwise** to $f \in V^X$ if for every $x \in X$ we have $\lim_{n\to\infty} f_n(x) = f(x)$, i.e. $\lim_{n\to\infty} \|f_n(x) - f(x)\| = 0$.

The same definition will be applied to nets $(f_\alpha)_{\alpha\in I}$ in V^X after learning net convergence in Sec. 5.2. □

In more details, the uniform convergence of f_n to f means that "for every $\varepsilon > 0$ there is $N \in \mathbb{Z}_+$ such that for all $n \geqslant N$ and *for all* $x \in X$, we have $\|f_n(x) - f(x)\| < \varepsilon$". If we place the words "for all $x \in X$" at the very beginning of the sentence, we get pointwise convergence.

Uniform convergence implies pointwise convergence: If $\|f - f_n\|_\infty \to 0$, then for each $x \in X$ we have $\|f_n(x) - f(x)\| \to 0$ since $\|f(x) - f_n(x)\| \leqslant \|f - f_n\|_\infty$.

Example 3.43. Let $f_n : (0,1) \to \mathbb{R}$ be $f_n(x) = x^n$. Then f_n converges pointwise to 0 (cf. Exp. 4.10). But $\sup_{x\in(0,1)} |x^n - 0| = 1$ does not converge to 0. So f_n does not converge uniformly to 0.

Remark 3.44. The uniform convergence of sequences in $l^\infty(X, V)$ is induced by the l^∞-norm, and hence is induced by the metric $d(f, g) = \|f - g\|_\infty$. However, this formula cannot be extended to a metric on V^X, since for arbitrary $f, g \in V^X$, $\|f - g\|_\infty$ is possibly $+\infty$.

In fact, it is true that the uniform convergence of sequences in V^X is induced by a metric, see Pb. 3.6. When X is countable, we have seen in Pb. 2.3 that the pointwise convergence in V^X is also given by a metric. □

Theorem 3.45. *Let X be a set, and let V be a Banach space (over $\mathbb{F}$). Then $l^\infty(X, V)$ is a Banach space (over $\mathbb{F}$).*

Proof. Let (f_n) be a Cauchy sequence in $l^\infty(X, V)$. Then for every $\varepsilon > 0$ there is $N \in \mathbb{Z}_+$ such that for all $m, n \geqslant N$ we have that $\sup_{x\in X} \|f_n(x) - f_m(x)\| < \varepsilon$, and hence $\|f_n(x) - f_m(x)\| < \varepsilon$ for each $x \in X$. This shows that for each $x \in X$, $(f_n(x))$ is a Cauchy sequence in V, which converges to some element $f(x) \in V$ because V is complete.

We come back to the statement that for each $\varepsilon > 0$, there exists $N \in \mathbb{Z}_+$ such that for all $n \geqslant N$ and all x,

$$\|f_n(x) - f_m(x)\| < \varepsilon$$

for every $m \geqslant N$. Let $m \to \infty$. Then by the continuity of subtraction and taking norm (cf. Prop. 3.38.), we obtain $\|f_n(x) - f(x)\| \leqslant \varepsilon$ for all $n \geqslant N$ and $x \in X$. In other words, $\|f_n - f\|_\infty \leqslant \varepsilon$ for all $n \geqslant N$. In particular, $\|f\|_\infty \leqslant \|f_N\|_\infty + \|f_N - f\|_\infty < +\infty$ by (3.9). This proves $f \in l^\infty(X, V)$ and $f_n \rightrightarrows f$. $\square$

Mathematicians used to believe that "if a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ converges pointwise to a function $f : [0,1] \to \mathbb{R}$, then f is continuous". Cauchy, one of the main figures in 19th century working on putting analysis on a rigorous ground, has given a problematic proof of this wrong statement. Counterexamples were later found in the study of Fourier series: Let $f : \mathbb{R} \to \mathbb{R}$ be a function with period 2π such that $f(x) = x$ when $-\pi < x < \pi$, and $f(x) = 0$ when $x = \pm\pi$. Then the Fourier series of this noncontinuous function f converges pointwise to f, yet the partial sums of this series are clearly continuous functions. Later, it was realized that uniform convergence is needed to show the continuity of the limit function. (See Thm. 3.48.) This was the first time the importance of uniform convergence was realized.

The following discussions about (resp. sequentially compact) metric spaces also apply to general (resp. compact) topological spaces. The reader can come back and check the proofs for these more general spaces after studying them in the future.

Definition 3.46. Let X, Y be metric spaces (resp. topological spaces). Then $C(X, Y)$ denotes the set of continuous functions from X to Y.

Lemma 3.47. *Let X be a metric space (resp. a topological space), and let V be a normed vector space. Then $C(X, V)$ is a linear subspace of V^X. If X is sequentially compact (resp. compact), then $C(X, V)$ is a linear subspace of $l^\infty(X, V)$.*

Proof. Using Prop. 3.38, one checks easily that $C(X, V)$ is a linear subspace of V^X. For any $f \in C(X, V)$, the absolute value function $|f| : x \in X \mapsto \|f(x)\|$ is continuous. Thus, assuming that X is sequentially compact, then by Lem. 3.2, $|f|$ is bounded on X. This proves that $\|f\|_\infty < +\infty$. Thus $C(X, V)$ is a subset (and hence a linear subspace) of $l^\infty(X, V)$. $\square$

Theorem 3.48. *Let X be a metric space (resp. a topological space), and let V be a normed vector space. Then $C(X, V) \cap l^\infty(X, V)$ is a closed linear subspace of $l^\infty(X, V)$. In particular, if X is sequentially compact (resp. compact), then $C(X, V)$ is a closed linear subspace of $l^\infty(X, V)$.*

Proof. Choose a sequence (f_n) in $C(X,V) \cap l^\infty(X,V)$ converging in $l^\infty(X,V)$ to f. Namely, $f_n \rightrightarrows f$. We want to prove that f is continuous. We check that f satisfies Def. 2.38-(2′). (One can also use Def. 2.38-(1). The proofs using these two definitions are not substantially different.)

Fix $p \in X$. Choose any $\varepsilon > 0$. Since $f_n \rightrightarrows f$, there exists $N \in \mathbb{Z}_+$ such that for all $n \geqslant N$ and we have $\|f - f_n\|_\infty < \varepsilon$. Since f_N is continuous, there exists $r > 0$ such that for each $x \in B_X(p,r)$ we have $\|f_N(x) - f_N(p)\| < \varepsilon$. Thus, for each $x \in B_X(p,r)$ we have

$$\|f(x) - f(p)\| \leqslant \|f(x) - f_N(x)\| + \|f_N(x) - f_N(p)\| + \|f_N(p) - f(p)\| < 3\varepsilon$$

This finishes the proof. □

Convention 3.49. Unless otherwise stated, if X is sequentially compact metric space (or more generally, a compact topological space to be defined latter), and if V is a normed vector space, the norm on $C(X,V)$ is chosen to be the l^∞-norm.

Corollary 3.50. *Let X be a metric space (resp. a topological space), and let V be a Banach space. Then $C(X,V) \cap l^\infty(X,V)$ is a Banach space under the l^∞-norm. In particular, if X is sequentially compact (resp. compact), then $C(X,V)$ is a Banach space.*

Proof. This follows immediately from Prop. 3.27, Thm. 3.48, and the fact that $l^\infty(X,V)$ is complete (Thm. 3.45). □

3.5 Problems and supplementary material

Problem 3.1. Let (x_n) be a sequence in a metric space X. Let $x \in X$. Prove that the following are equivalent.

(1) x is a cluster point of (x_n), i.e., the limit of a convergent subsequence of (x_n).

(2) For each $\varepsilon > 0$ and each $N \in \mathbb{Z}_+$, there exists $n \geqslant N$ such that $d(x_n, x) < \varepsilon$.

(Note: in a general topological space, these two statements are not equivalent.)

Remark 3.51. Condition (2) is often abbreviated to "for each $\varepsilon > 0$, the sequence (x_n) is frequently in $B(x,\varepsilon)$". In general, we say "(x_n) **frequently** satisfies P" if for each $N \in \mathbb{Z}_+$ there is $n \geqslant N$ such that x_n satisfies P. We say that "(x_n) **eventually** satisfies P" if there exists $N \in \mathbb{Z}_+$ such that for every $n \geqslant N$, x_n satisfies P.

Thus "(x_n) eventually satisfies P" means the same as "all but finitely many x_n satisfies P". Its negation is "(x_n) frequently satisfies $\neg$P". □

Remark 3.52. Condition (2) of Pb. 3.1 is sometimes easier to use than (1). For example, compared to the original definition of cluster points, it is much easier to find an explicit negation of (2) by using the rule suggested in Rem. 2.17: There exist $\varepsilon > 0$ and $N \in \mathbb{Z}_+$ such that $d(x_n, x) \geqslant \varepsilon$ for all $n \geqslant N$. (Or simply: there exists $\varepsilon > 0$ such that x_n is eventually not in $B(x,\varepsilon)$.)

Problem 3.2. Use Pb. 3.1-(2) to prove that if (x_n) is a sequence in $\overline{\mathbb{R}}$, then $\limsup_{n\to\infty} x_n$ is a cluster point of (x_n).

Remark 3.53. You will notice that your proof of Pb. 3.2 is slightly simpler than the proof we gave for Thm. 3.11. This is because our construction of subsequence as in (3.3) has been incorporated into your proof of (2)⇒(1) in Pb. 3.1.

Problem 3.3. Let $f : X \to Y$ be a continuous map of metric spaces. Assume that f is bijective and X is sequentially compact. Prove that f is a homeomorphism using the following hint.

Hint. You need to prove that if (y_n) is a sequence in Y converging to $y \in Y$, then $x_n = f^{-1}(y_n)$ converges to $x = f^{-1}(y)$. Prove that (x_n) has only one cluster point, and hence converges to some point $x' \in X$ (why?). Then prove $x' = x$. (In the future, we will use the language of open sets and closed sets to prove this result again. Do not use this language in your solution.) □

Theorem 3.54 (Tychonoff theorem, countable version). *Let $(X_n)_{n\in\mathbb{Z}_+}$ be a sequence of sequentially compact metric spaces. Then the product space $S = \prod_{n\in\mathbb{Z}_+} X_n$ is sequentially compact under the metric defined as in Pb. 2.3.*

The method of choosing subsequence in the following proof is the reknowned **diagonal method**. A different method will be given in Pb. 8.7.

Proof. Let $(x_m)_{m\in\mathbb{Z}_+}$ be a sequence in S. Since $(x_m(1))_{m\in\mathbb{Z}_+}$ is a sequence in the sequentially compact space X_1, $(x_m)_{m\in\mathbb{Z}_+}$ has a subsequence $x_{1,1}, x_{1,2}, x_{1,3} \dots$ whose value at $n = 1$ converges in X_1. Since X_2 is sequentially compact, we can choose a subsequence $x_{2,1}, x_{2,2}, x_{2,3}, \dots$ of the previous subsequence such that its values at $n = 2$ converge in X_2. Then pick a subsequence from the previous one whose values at 3 converge in X_3.

By repeating this process, we get an $\infty \times \infty$ matrix $(x_{i,j})_{i,j\in\mathbb{Z}_+}$:

$$\begin{matrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ x_{3,1} & x_{3,2} & x_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix} \tag{3.10}$$

such that the following hold:

- The 1-st line is a subsequence of the original sequence $(x_m)_{m\in\mathbb{Z}_+}$.
- The $(i + 1)$-th line is a subsequence of the i-th line.
- For each n, $\lim_{j\to\infty} x_{n,j}(n)$ converges in X_n.

Then the diagonal line $(x_{i,i})_{i\in\mathbb{Z}_+}$ is a subsequence of the original sequence $(x_m)_{m\in\mathbb{Z}_+}$. Moreover, for each n, $(x_{i,i})_{i\geqslant n}$ is a subsequence of the n-th line, whose value at n therefore converges in X_n. Thus $\lim_{i\to\infty} x_{i,i}(n)$ converges in X_n. Thus, by Pb. 2.3, $(x_{i,i})_{i\in\mathbb{Z}_+}$ converges under any metric inducing the product topology. □

Problem 3.4. Let X be a sequentially compact metric space. Let $A \subset X$ be a metric subspace. Consider the statements:

(1) A is sequentially compact.

(2) A is a closed subset of X.

Prove that (1)⇒(2). Prove that if X is sequentially compact, then (2)⇒(1).

The above problem implies immediately:

Theorem 3.55 (Heine-Borel theorem). *Let A be a subset of $\mathbb{R}^N$. Then A is sequentially compact iff A is a bounded closed subset of $\mathbb{R}^N$.*

Proof. Suppose that A is sequentially compact. Then A is bounded under the Euclidean metric by Prop. 3.7. By Pb. 3.4, A is a closed subset of $\mathbb{R}^N$.

Conversely, assume that A is a bounded and closed subset of $\mathbb{R}^N$. Then $A \subset B$ where B is the product of N pieces of closed intervals in $\mathbb{R}$. Then B is sequentially compact by Bolzano-Weierstrass. Since A is closed in $\mathbb{R}^N$, it is not hard to check that A is closed in B.[1] Thus A is sequentially compact by Pb. 3.4. □

Example 3.56. Choose any $p \in \mathbb{R}^N$ and $0 \leqslant R < +\infty$. Then $\overline{B}_{\mathbb{R}^N}(p, R)$ is a bounded closed subset of $\mathbb{R}^N$ (Exp. 3.30), and hence is sequentially compact by Heine-Borel.

Remark 3.57. Think about the question: Equip $\mathbb{R}^{\mathbb{Z}_+}$ with metric

$$d(x, y) = \sup_{n\in\mathbb{Z}_+} \frac{\min\{|x(n), y(n)|, 1\}}{n}$$

What are the sequentially compact subsets of $\mathbb{R}^{\mathbb{Z}_+}$? (Namely, think about how to generalize Heine-Borel theorem to $\mathbb{R}^{\mathbb{Z}_+}$.)

Problem 3.5. Do Exercise 3.41.

Problem 3.6. Let V be a normed vector space. For every $f, g \in V^X$ define

$$d(f, g) = \min\{1, \|f - g\|_\infty\} \tag{3.11}$$

1. Show that d defines a metric on V^X.

[1] If Z is a metric space, if $X \subset Y \subset Z$, and if X is closed in Z, then it is easy to check that X is closed in Y.

2. Show that for every sequence (f_n) in V^X and every $g \in V^X$, we have $f_n \to g$ under the metric d iff $f_n \rightrightarrows g$.

Definition 3.58. Let X be a set, and let V be a normed vector space. A metric on V^X is called a **uniform convergence metric** if it is equivalent to (3.11). Thus, by Def. 2.61, a uniform convergence metric is one such that a sequence (f_n) in V^X converges to f under this metric iff $f_n \rightrightarrows f$.

Problem 3.7. Let X, Y be metric spaces, and assume that Y is sequentially compact. Let V be a normed vector space. Choose $f \in C(X \times Y, V)$, i.e., $f : X \times Y \to V$ is continuous. For each $x \in X$, let

$$f_x : Y \to V \qquad y \mapsto f(x,y)$$

Namely $f_x(y) = f(x,y)$. It is easy to check that $f_x \in C(Y,V)$. Define a new function

$$\Phi(f) : X \to C(Y,V) \qquad x \mapsto f_x \tag{3.12}$$

Recall that $C(Y,V)$ is equipped with the l^∞-norm.

1. Prove that $\Phi(f)$ is continuous. In other words, prove that if (x_n) is a sequence in X converging to $x \in X$, then $f_{x_n} \rightrightarrows f_x$ on Y, i.e.

$$\lim_{n\to\infty} \|f_{x_n} - f_x\|_{l^\infty(Y,V)} = 0$$

⋆ 2. Give an example of $f \in C(X \times Y, \mathbb{R})$ where Y is not sequentially compact, (x_n) converges to x in X, and f_{x_n} does not converge uniformly to f_x. (Note: you may consider $X = Y = \mathbb{R}$.)

Hint. In part 1, to prove that $\Phi(f)$ is continuous, one can prove the equivalent fact that for every fixed $x \in X$ the following is true:

- For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $p \in B_X(x,\delta)$, we have $\sup_{y\in Y} \|f(p,y) - f(x,y)\| < \varepsilon$.

(Cf. Def. 2.38.) Prove this by contradiction and by using the sequential compactness of Y appropriately. □

Remark 3.59. Let $X = \mathbb{Z}_+ \cup \{\infty\}$, equipped with the metric

$$d(m,n) = |m^{-1} - n^{-1}|$$

In other words, the metric on X is $\tau^* d_{\mathbb{R}}$ where $d_{\mathbb{R}}$ is the Euclidean metric on $\mathbb{R}$, and $\tau : X \to \mathbb{R}, n \mapsto n^{-1}$. It is not hard to show that X is sequentially compact: either prove it directly, or apply Heine-Borel to $\tau(X)$.

Let Y be a metric space. Let $(y_n)_{n\in\mathbb{Z}_+}$ be a sequence in Y, and let $y_\infty \in Y$. It is not hard to see that the following two statements are equivalent:

(1) The function $F : X \to Y, n \mapsto y_n$ is continuous.

(2) The sequence $(y_n)_{n\in\mathbb{Z}_+}$ converges to y_∞.

The following problem is a generalization of this equivalence. $\square$

⋆ **Problem 3.8.** Let V be a normed vector space. Let Y be a metric space. Let $X = \mathbb{Z}_+ \cup \{\infty\}$ with metric defined as in Rem. 3.59. Let $(f_n)_{n\in\mathbb{Z}_+}$ be a sequence in $C(Y, V)$. Let $f_\infty \in V^Y$. Prove that the following are equivalent:

(1) The following function is continuous:

$$F : X \times Y \to V \qquad (n, y) \mapsto f_n(y) \tag{3.13}$$

In particular, by restricting F to $\infty \times Y$, we see that $f_\infty \in C(Y, V)$.

(2) $(f_n)_{n\in\mathbb{Z}_+}$ converges pointwise to f_∞. Moreover, $(f_n)_{n\in\mathbb{Z}_+}$ is **pointwise equicontinuous**, which means the following:

- For every $y \in Y$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $p \in B_Y(y, \delta)$ we have

$$\sup_{n\in\mathbb{Z}_+} \|f_n(p) - f_n(y)\| < \varepsilon$$

Note. In part (1), the only nontrivial thing to prove is that F is continuous at (∞, y) for every $y \in Y$.

Remark 3.60. There is a concise way to define pointwise equicontinuity: a sequence $(f_n)_{n\in\mathbb{Z}_+}$ in V^Y is pointwise equicontinuous iff the function

$$Y \mapsto V^{\mathbb{Z}_+} \qquad y \mapsto (f_1(y), f_2(y), \dots) \tag{3.14}$$

is continuous, where $V^{\mathbb{Z}_+}$ is equipped with any uniform convergence metric (cf. Def. 3.58).

⋆ **Remark 3.61.** In Pb. 3.8, there is a quick and tricky way to conclude (1)⇒(2): Use Pb. 3.7 and the sequential compactness of X. (Do not use this method in your solution. Prove (1)⇒(2) directly; it is a good exercise and is not difficult.)

⋆ **Remark 3.62.** Pb. 3.7 and 3.8, together with Thm. 3.48, imply the following fact (can you see why?):

- Let Y be a sequentially compact metric space. Let V be a normed vector space. Let $(f_n)_{n\in\mathbb{Z}_+}$ be a pointwise equicontinuous sequence of functions $Y \to V$ converging pointwise to some $f : Y \to V$. Then $f_n \rightrightarrows f$ on Y.

You can also try to give a straightforward proof of this fact without using Pb. 3.7 and 3.8.

4 Series

In this chapter, we assume that vector spaces are over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ unless otherwise stated.

4.1 Definitions and basic properties

Definition 4.1. Let V be a Banach space (over $\mathbb{F}$). A **series** in V is an expression of the form

$$\sum_{i=1}^{\infty} v_i \tag{4.1}$$

where $(v_i)_{i \in \mathbb{Z}_+}$ is a sequence in V. If $s \in V$, we say that the series (4.1) **converges to** s if

$$s = \lim_{n \to \infty} \sum_{i=1}^{n} v_i$$

namely, $s_n \to s$ where s_n is the **partial sum** $s_n = \sum_{i=1}^{n} v_i$. In this case, we write

$$s = \sum_{i=1}^{\infty} v_i$$

Remark 4.2. Since V is complete, the series (4.1) converges iff the sequence of partial sum (s_n) is a Cauchy sequence: for every $\varepsilon > 0$ there exists $N \in \mathbb{Z}_+$ such that for all $n > m \geqslant N$ we have $\|s_n - s_m\| < \varepsilon$, i.e.,

$$\Big\| \sum_{i=m+1}^{n} v_i \Big\| < \varepsilon \tag{4.2}$$

Proposition 4.3. *Suppose that $\sum_{i=1}^{\infty} v_i$ is a convergent series in a Banach space V. Then $\lim\limits_{n \to \infty} v_n = 0$.*

Proof. Let $s_n = v_1 + \cdots + v_n$, which converges to $s \in V$. Then $\lim_{n \to \infty} s_{n+1} = s$. So $v_n = s_{n+1} - s_n \to s - s = 0$ since subtraction in continuous (Prop. 3.38). $\square$

Thus, for example, $\sum_{n=1}^{\infty} (-1)^n$ diverges in the Banach space $\mathbb{R}$ since $\lim_{n \to \infty} (-1)^n$ does not converge to 0.

Definition 4.4. Consider a **series** in $\overline{\mathbb{R}}_{\geqslant 0}$:

$$\sum_{i=1}^{\infty} a_i \tag{4.3}$$

namely, each a_i is in $\overline{\mathbb{R}}_{\geqslant 0}$. Note that the partial sum $s_n = \sum_{i=1}^n a_i$ is increasing. We say that $\lim_{n\rightarrow\infty} s_n$ (which exists in $\overline{\mathbb{R}}_{\geqslant 0}$ and equals $\sup\{s_n : n \in \mathbb{Z}_+\}$, cf. Rem. 2.71) is the value of the series (4.3) and write

$$\sum_{i=1}^{\infty} a_i = \lim_{n\rightarrow\infty} s_n$$

Definition 4.5. We say that a series $\sum_{i=1}^{\infty} a_i$ in $\mathbb{R}_{\geqslant 0}$ **converges** if it converges in $\mathbb{R}$ (but not just converges in $\overline{\mathbb{R}}_{\geqslant 0}$, which is always true). Clearly, $\sum_{i=1}^{\infty} a_i$ converges iff

$$\sum_{i=1}^{\infty} a_i < +\infty$$

More generally, we say that a series $\sum_{i=1}^{\infty} v_i$ in a Banach space V **converges absolutely**, if

$$\sum_{i=1}^{\infty} \|v_i\| < +\infty$$

Remark 4.6. By the Cauchy condition of convergence, $\sum_{i=1}^{\infty} v_i$ converges absolutely iff for every $\varepsilon > 0$ there exists $N \in \mathbb{Z}_+$ such that for all $n > m \geqslant N$ we have

$$\sum_{i=m+1}^{n} \|v_i\| < \varepsilon \tag{4.4}$$

By comparing (4.4) with (4.2) and using the subadditivity of the norm (recall Def. 3.33), we immediately see:

Proposition 4.7. *Let $\sum_{i=1}^{\infty} v_i$ be a series in a Banach space. The following are true.*

1. *If $\sum_{i=1}^{\infty} v_i$ converges absolutely, then it converges.*
2. *For each i we choose $a_i \in \mathbb{R}_{\geqslant 0}$ satisfying $\|v_i\| \leqslant a_i$. Suppose that $\sum_{i=1}^{\infty} a_i < +\infty$. Then $\sum_{i=1}^{\infty} v_i$ converges absolutely.*

Proof. Part 1 has been explained above. In part 2, we have $\sum \|v_i\| \leqslant \sum a_i < +\infty$. So $\sum v_i$ converges absolutely. □

Exercise 4.8. Suppose that $\sum_{i=1}^{\infty} u_i$ and $\sum_{i=1}^{\infty} v_i$ are convergent (resp. absolutely convergent) series in a Banach space V. Let $\lambda \in \mathbb{F}$. Show that the LHS of the following equations converges (resp. converges absolutely) in V, and that the following equations hold:

$$\sum_{i=1}^{\infty}(u_i + v_i) = \sum_{i=1}^{\infty} u_i + \sum_{i=1}^{\infty} v_i$$

$$\sum_{i=1}^{\infty} \lambda v_i = \lambda \cdot \sum_{i=1}^{\infty} v_i$$

Remark 4.9. We have seen that absolute convergence implies convergence. In fact, at least when $V = \mathbb{F}^N$, absolute convergence is in many ways more natural than convergence. For example, we will learn that if a series $\sum_i v_i$ in $\mathbb{F}^N$ converges absolutely, then the value of $\sum_i v_i$ is invariant under rearrangement of the series: for every bijection $\varphi : \mathbb{Z}_+ \to \mathbb{Z}_+$ we have $\sum_i v_i = \sum_i v_{\varphi(i)}$. In the next semester, we shall learn Lebesgue integral theory and, more generally, measure theory. When applying measure theory to infinite sums over the countable set $\mathbb{Z}_+$, many good results (e.g. dominated convergence theorem, Fubini's theorem) hold only for absolute convergence series, but not for arbitrary convergent series in general. In fact, there is no analog of convergent (but not absolutely convergent) series in measure theory at all!

When V is not necessarily finite-dimensional, the situation is subtler: there is a version of convergence which lies between the usual convergence and absolute convergence, and which coincides with absolute convergence when $V = \mathbb{F}^N$. This version of convergence is defined using nets instead of sequences. Moreover, many good properties (as mentioned above) hold for this convergence, and these properties can be proved in a very conceptual way (rather than using brute-force computation). We will learn this convergence in the next chapter. $\square$

4.2 Basic examples

Let us study the **geometric series** $\sum_{n=0}^{\infty} z^n$ where $z \in \mathbb{C}$. We first note the famous **binomial formula**: for each $z, w \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$(z+w)^n = \sum_{k0}^{n} \binom{n}{k} z^k w^{n-k} \tag{4.5}$$

In particular,

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2} z^2 + \frac{n(n-1)(n-2)}{6} z^3 + \cdots + nz^{n-1} + z^n \tag{4.6}$$

Example 4.10. Assume $z \in \mathbb{C}$ and $|z| < 1$. Then $\lim_{n\to\infty} z^n = 0$.

Proof. If $z = 0$ then it is obvious. Assume that $0 < |z| < 1$. Choose $\delta > 0$ such that $|z| = 1/(1+\delta)$. By (4.6), $(1+\delta)^n \geqslant 1 + n\delta$. So

$$0 \leqslant |z^n| \leqslant (1+n\delta)^{-1}$$

Since $\lim\limits_{n\to\infty} (1+n\delta)^{-1} = 0$, we have $|z^n| \to 0$ by squeeze theorem. Hence $z^n \to 0$. $\square$

Example 4.11. Let $z \in \mathbb{C}$. If $|z| < 1$, then $\sum\limits_{n=0}^{\infty} z^n$ converges absolutely, and

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \tag{4.7}$$

where 0^0 is understood as 1. If $|z| \geqslant 1$, then $\sum_{n=0}^{\infty} z_n$ diverges in $\mathbb{C}$.

Proof. The partial sum $s_n = 1 + z + z^2 + \cdots + z^n$ equals $(1 - z^{n+1})/(1 - z)$ when $z \neq 1$. Therefore, when $|z| < 1$, $s_n \to 1/(1 - z)$. When $|z| \geqslant 1$, we have $|z^n| \geqslant 1$ and hence $z^n \nrightarrow 0$. So $\sum_{n=0}^{\infty} z^n$ diverges by Prop. 4.3. $\square$

Example 4.12. The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (in $\mathbb{R}$).

Proof. We want to show that the Cauchy condition (cf. Rem. 4.2) does not hold. Thus, we want to prove that there exists $\varepsilon > 0$ such that for every $N \in \mathbb{Z}_+$ there exist $n > m \geqslant N$ such that $|(m+1)^{-1} + (m+2)^{-1} + \cdots + n^{-1}| \geqslant \varepsilon$.

To see this, for each N we choose $m = 2^N$ and $n = 2^{N+1}$. Then $n > m > N$, and

$$\Big|\sum_{i=m+1}^{n} i^{-1}\Big| = \Big|\frac{1}{2^N + 1} + \frac{1}{2^N + 2} + \cdots + \frac{1}{2^N + 2^N}\Big|$$
$$\geqslant \underbrace{\Big|\frac{1}{2^{N+1}} + \frac{1}{2^{N+1}} + \cdots + \frac{1}{2^{N+1}}\Big|}_{2^N \text{ terms}} = \varepsilon$$

where $\varepsilon = \frac{1}{2}$. $\square$

Exercise 4.13. Choose any $p \in \mathbb{Z}$. Prove that $\sum_{n=1}^{\infty} n^{-p}$ converges iff $p \geqslant 2$.

Hint. Use Prop. 4.7 and Exp. 4.12 to reduce the problem to the case $p = 2$. Prove this case by proving $\sum_{n=1}^{\infty} 1/n(n+1) = 1 < +\infty$. $\square$

Definition 4.14. Let V be a Banach space, let X be a set, and let (f_n) be a sequence in $l^\infty(X, V)$, and let $g \in l^\infty(X, V)$. We say that the series of functions $\sum_{i=1}^{\infty} f_i$ **converges uniformly to** g (on X) if it converges to g as a series in the Banach space $l^\infty(X, V)$ and under the l^∞-norm. Equivalently, this means that the partial sum function $s_n = f_1 + \cdots + f_n$ converges uniformly to g as $n \to \infty$.

Example 4.15. The series of functions $\sum_{n=1}^{\infty} \frac{\sin |nz^3|}{n^2}$ converges uniformly on $\mathbb{C}$ to a continuous function $g : \mathbb{C} \to \mathbb{R}$ which is bounded (i.e. $\sup_{z \in \mathbb{C}} |g(z)| < +\infty$).

Proof. Let $f_n(z) = \sin |nz^3|/n^2$. Then each f_n is in $\mathfrak{X} = C(\mathbb{C}, \mathbb{R}) \cap l^\infty(\mathbb{C}, \mathbb{R})$ where $\mathfrak{X}$ is a real Banach space under the l^∞-norm by Cor. 3.50. Note that $\|f_n\|_\infty \leqslant n^{-2}$. By Exe. 4.13, $\sum_{n=1}^{\infty} n^{-2} < +\infty$. Therefore, by Prop. 4.7, the series $\sum_n f_n$ converges in $\mathfrak{X}$, i.e., it converges uniformly to an element $g \in \mathfrak{X}$. (In particular, $\sum_n f_n(z) = g(z)$ for all $z \in \mathbb{C}$.) $\square$

4.3 Root test and ratio test; power series; construction of e^z

Root test and ratio test are useful criteria for proving the convergence or divergence of series, especially power series. In addition, the method of power series provides a unified and elegant proof for many useful formulas about limit (see Prop. 4.23 and Exp. 4.24). We begin our discussion with the following easy observation:

Remark 4.16. Let (x_n) be a sequence in $\overline{\mathbb{R}}$, and let $A \in \overline{\mathbb{R}}$. The following are true.

1. If $\limsup\limits_{n\to\infty} x_n < A$, then $x_n < A$ is eventually true.

2. If $\limsup\limits_{n\to\infty} x_n > A$, then $x_n > A$ is frequently true.

By taking negative, we obtain similar statements for $\liminf$.

Proof. Recall that $\limsup x_n = \inf_{n\in\mathbb{Z}_+} \alpha_n$ where where $\alpha_n = \sup\{x_n, x_{n+1}, \dots\}$.

Assume that $\inf_{n\in\mathbb{Z}_+} \alpha_n < A$. Then A is not a lower bound of $\{\alpha_n : n \in \mathbb{Z}_+\}$. Thus, there exists $N \in \mathbb{Z}_+$ such that $\alpha_N < A$. Then $x_n < A$ for all $n \geqslant N$.

Assume that $\inf_{n\in\mathbb{Z}_+} \alpha_n > A$. Then for each $N \in \mathbb{Z}_+$ we have $\alpha_N > A$. So A is not an upper bound of $\{x_n, x_{n+1}, \dots\}$. So there is $n \geqslant N$ such that $x_n > A$. □

We will heavily use $\sqrt[n]{x}$ (where $x \geqslant 0$ and $n \in \mathbb{Z}_+$) in the following discussions. $\sqrt[n]{x}$ will be rigorously constructed in Exp. 7.108, whose proof does not rely on the results of this section.

Proposition 4.17 (Root test). *Let $\sum\limits_{n=1}^{\infty} v_n$ be a series in a Banach space V. Let $\beta = \limsup\limits_{n\to\infty} \sqrt[n]{\|v_n\|}$. Then:*

1. *If $\beta < 1$, then $\sum v_n$ converges absolutely, and hence converges in V.*

2. *If $\beta > 1$, then $\sum v_n$ diverges in V.*

Proof. Suppose $\beta < 1$. Then we can choose γ such that $\beta < \gamma < 1$. So $\limsup \sqrt[n]{\|v_n\|} < \gamma$. By Rem. 3.8, there exists $N \in \mathbb{Z}_+$ such that for all $n \geqslant N$, we have $\sqrt[n]{\|v_n\|} < \gamma$, and hence $\|v_n\| < \gamma^n$. Since $\sum_{n=0}^{\infty} \gamma^n = (1-\gamma)^{-1} < +\infty$ (Exp. 4.11), the series $\sum_{n=N}^{\infty} v_n$ converges absolutely by Prop. 4.7. So the original series converges absolutely.

Assume that $\beta > 1$. Then by Rem. 4.16, for each N there is $n \geqslant N$ such that $\sqrt[n]{\|v_n\|} > 1$ and hence $\|v_n - 0\| > 1$. So $v_n \nrightarrow 0$. So $\sum v_n$ diverges by Prop. 4.3. □

Example 4.18. Let $V = \mathbb{R}$ and $v_n = 1/n$ resp. $v_n = 1/n^2$. Then $\beta = 1$, and $\sum v_n$ diverges resp. converges absolutely due to Exe. 4.13. So Root test gives no information on the convergence of series when $\beta = 1$. The same can be said about ratio test.

Proposition 4.19 (Ratio test). *Let* $\sum_{n=1}^{\infty} v_n$ *be a series in a Banach space* V *such that* $v_n \neq 0$ *for all* n. *Let* $\alpha = \liminf_{n\to\infty} \frac{\|v_{n+1}\|}{\|v_n\|}$ *and* $\beta = \limsup_{n\to\infty} \frac{\|v_{n+1}\|}{\|v_n\|}$. *Then:*

1. *If* $\beta < 1$, *then* $\sum v_n$ *converges absolutely, and hence converges in* V.

2. *If* $\alpha > 1$, *then* $\sum v_n$ *diverges in* V.

Proof. Suppose $\beta < 1$. Choose γ such that $\beta < \gamma < 1$. Then by Rem. 4.16, there is N such that for all $n \geqslant N$ we have $\|v_{n+1}\|/\|v_n\| < \gamma$. So $\|v_n\| < \gamma^{n-N}\|v_N\|$. So $\sum_{n\geqslant N} \|v_n\| \leqslant \|v_N\| \cdot \sum_{n\geqslant N} \gamma^{n-N} = \|v_N\| \cdot (1-\gamma)^{-1} < +\infty$. So $\sum v_n$ converges absolutely.

Suppose $\alpha > 1$. Then by Rem. 4.16, there is N such that for all $n \geqslant N$ we have $\|v_{n+1}\|/\|v_n\| > 1$. So $\|v_n\| \geqslant \|v_N\| > 0$ for all $n \geqslant N$. So $v_n \nrightarrow 0$ and hence $\sum v_n$ diverges, as in the proof of root test. □

Definition 4.20. A **power series** in a complex Banach space V is an expression of the form $\sum_{n=0}^{\infty} v_n z^n$ where the **coefficients** $v_0, v_1, v_2, \dots$ are elements of V, and z is a **complex variable**, i.e., a symbol which can take arbitrary values in $\mathbb{C}$. If the power series $\sum v_n z^n$ converges at $z_0 \in \mathbb{C}$, we often let $\sum v_n z_0^n$ denote this limit.

Proposition 4.21. *Let* $\sum v_n z^n$ *be a power series in a complex Banach space* V. *Then there is a unique* $0 \leqslant R \leqslant +\infty$ *satisfying the following properties:*

(a) If $z \in \mathbb{C}$ *and* $|z| < R$, *then* $\sum v_n z^n$ *converges absolutely in* V.

(b) If $z \in \mathbb{C}$ *and* $|z| > R$, *then* $\sum v_n z^n$ *diverges in* V.

Such R *is called the* ***radius of convergence*** *of* $\sum v_n z^n$. *Moreover, we have*

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{\|v_n\|}} = \liminf_{n\to\infty} \frac{1}{\sqrt[n]{\|v_n\|}} \tag{4.8}$$

Proof. Clearly, there are at most one R satisfying (a) and (b). Let us define R using (4.8) (note that the second and the third terms of (4.8) are clearly equal), and prove that R satisfies (a) and (b). Let

$$\beta(z) = \limsup_{n\to\infty} \sqrt[n]{\|v_n z^n\|}$$

Then $\beta(z) = |z|/R$. So (a) and (b) follow immediately from root test. □

Remark 4.22. Note that if one can find $0 \leqslant r \leqslant R$ such that $\sum v_n z^n$ converges whenever $|z| < r$, then $r \leqslant R$ where R is the radius of convergence: otherwise, the series diverges for any positive z satisfying $R < z < r$, impossible.

It follows that if $\sum v_n z^n$ converges for all $|z| < r$, then $\sum v_n z^n$ converges *absolutely* for all $|z| < r$. □

Prop. 4.21 provides a useful method for computing limits of a positive sequence:

Proposition 4.23. *Let (λ_n) be a sequence in $\mathbb{R}_{>0}$. Then*

$$\liminf_{n\to\infty}\frac{\lambda_{n+1}}{\lambda_n}\leqslant\liminf_{n\to\infty}\sqrt[n]{\lambda_n}\leqslant\limsup_{n\to\infty}\sqrt[n]{\lambda_n}\leqslant\limsup_{n\to\infty}\frac{\lambda_{n+1}}{\lambda_n}\tag{4.9}$$

In particular, (by Cor. 3.18) we have

$$\lim_{n\to\infty}\sqrt[n]{\lambda_n}=\lim_{n\to\infty}\frac{\lambda_{n+1}}{\lambda_n}\tag{4.10}$$

provided that the limit on the RHS of (4.10) exists in $\overline{\mathbb{R}}$.

The four numbers in (4.9) can be completely different. See [Rud-P, Exp. 3.35].

Proof. Let R be the radius of convergence of $\sum\lambda_n z^n$. Then $R=1/\limsup\sqrt[n]{\lambda_n}$ by (4.10). Thus, by Prop. 4.21, if $|z|>R$ then $\sum\lambda_n z^n$ diverges, and hence $\limsup|\lambda_{n+1}z^{n+1}|/|\lambda_n z^n|\geqslant 1$ by ratio test. Therefore,

$$|z|>\frac{1}{\limsup\sqrt[n]{\lambda_n}}\qquad\Longrightarrow\qquad|z|\cdot\limsup\frac{\lambda_{n+1}}{\lambda_n}\geqslant 1$$

This proves

$$\limsup\sqrt[n]{\lambda_n}\leqslant\limsup\frac{\lambda_{n+1}}{\lambda_n}$$

Replacing λ_n by λ_n^{-1}, we get

$$\frac{1}{\liminf\sqrt[n]{\lambda_n}}=\limsup\sqrt[n]{\lambda_n^{-1}}\leqslant\limsup\frac{\lambda_n}{\lambda_{n+1}}=\frac{1}{\liminf\frac{\lambda_{n+1}}{\lambda_n}}$$

This proves (4.9). □

Example 4.24. Let $a\in\mathbb{R}_{>0}$ and $p\in\mathbb{Z}$. The following formulas follow immediately from Prop. 4.23 (especially, from (4.10)):

$$\lim_{n\to\infty}\sqrt[n]{a}=1\tag{4.11a}$$

$$\lim_{n\to\infty}\sqrt[n]{n!}=+\infty\tag{4.11b}$$

$$\lim_{n\to\infty}\sqrt[n]{n^p}=1\tag{4.11c}$$

Note that (4.11c) follows from

$$\lim_{n\to\infty}\Big(\frac{n}{n+1}\Big)^p=1\tag{4.11d}$$

(This is clearly true when $p = \pm 1$, and hence is true for any p by induction.) By (4.11c), the radius of convergence of $\sum_n n^p z^n$ is 1. Therefore, by Prop. 4.21, $\sum n^p A^{-n}$ converges absolutely when $A > 1$. Thus, by Prop. 4.3,

$$\lim_{n\to\infty} \frac{n^p}{A^n} = 0 \qquad (\text{if } A > 1) \tag{4.11e}$$

This means that "polynomials grow slower than exponentials".

The same conclusions hold for arbitrary $p \in \mathbb{R}$ once we know how to define x^p and prove the continuity of $x \in \mathbb{R}_{>0} \mapsto x^p$. (See Sec. 7.8.) $\square$

$\star$ **Exercise 4.25.** Prove (4.11a) directly. Then use (4.11a) to give a direct proof of Prop. 4.23. Do not use root test, ratio test, or any results about power series.

Definition 4.26. By (4.11b), the power series

$$\exp(z) \equiv e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

has radius of convergence $+\infty$, and hence converges absolutely on $\mathbb{C}$. (In particular, $\lim_{n\to\infty} z^n/n! = 0$ for all $z \in \mathbb{C}$.) This gives a function $\exp : \mathbb{C} \to \mathbb{C}$, called the **exponential function**.

Part (a) of Prop. 4.21 can be strengthened in the following way.

Theorem 4.27. *Let $\sum v_n z^n$ be a power series with coefficients in a complex Banach space V. Let R be its radius of convergence, and assume that $0 < R \leqslant +\infty$. For each $z \in B_{\mathbb{C}}(0,R)$, let $f(z)$ denote the value of this series at z (which is an element of V). Then $f : B_{\mathbb{C}}(0,R) \to V$ is continuous. Moreover, for each $0 < \rho < R$, the series of functions $\sum v_n z^n$ converges uniformly on $\overline{B}_{\mathbb{C}}(0,\rho)$ to f.*

Note that by calling $\sum v_n z^n$ a series of functions, we understand each term $v_n z^n$ as a function $\mathbb{C} \to V$.

Proof. For each $0 < \rho < R$, let $X_\rho = \overline{B}_{\mathbb{C}}(0,\rho)$. Then X_ρ is clearly a bounded closed subset of $\mathbb{C}$, and hence is sequentially compact by Heine-Borel Thm. 3.55. Let $g_n = v_n z^n$, which is a continuous function $X_\rho \to V$. We view g_n as an element of the Banach space (cf. Cor. 3.50) $C(X_\rho, V)$. Then $\|g_n\|_\infty = \rho^n \|v_n\|$. Thus

$$\limsup_{n\to\infty} \sqrt[n]{\|g_n\|_\infty} = \limsup_{n\to\infty} \rho \sqrt[n]{\|v_n\|} = \rho/R < 1$$

Therefore, by root test, the series $\sum g_n$ converges in the Banach space $C(X_\rho, V)$ to some $f_\rho \in C(X_\rho, V)$.

We have proved that for each $0 < \rho < R$, the series of functions $\sum v_n z^n$ converges uniformly on X_ρ to a continuous function f_ρ. Let $f : B_{\mathbb{C}}(0,R) \to V$ whose value at each z is the value of the original series at z. Thus, if $|z| \leqslant \rho$, then

$f_\rho(z) = f(z)$. Namely, $f|_{X_\rho} = f_\rho$. This shows that $\sum v_n z^n$ converges uniformly on X_ρ to f. It also shows that $f|_{B_\mathbb{C}(0,\rho)}$ is continuous (because f_ρ is continuous). Therefore, since $B_\mathbb{C}(0, R)$ is covered by all open disks $B_\mathbb{C}(0, \rho)$ (where $0 < \rho < R$), we conclude from Lem. 2.41 that f is continuous on $B_\mathbb{C}(0, R)$. □

Example 4.28. By Thm. 4.27, the exponential function $\exp : \mathbb{C} \to \mathbb{C}$ is continuous; moreover, $\sum_{n=0}^\infty z^n/n!$ converges uniformly to e^z on $\overline{B}_\mathbb{C}(0, R)$ for every $0 < R < +\infty$, and hence on every bounded subset of $\mathbb{C}$.

4.4 Problems and supplementary material

Let V be a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Problem 4.1. Let W be a normed vector space. Prove that W is complete iff every absolutely convergent series in W is convergent (i.e. if $\sum_{n=1}^\infty \|w_n\| < +\infty$ then $\sum_{n=1}^\infty w_n$ converges).

Hint. "$\Rightarrow$" was proved in Prop. 4.7. To prove "$\Leftarrow$", for each Cauchy sequence (w_n) in W, choose a subsequence (w_{n_k}) such that $\|w_{n_k} - w_{n_{k+1}}\| \leqslant 2^{-k}$. Apply Thm. 3.23. □

Problem 4.2. Use the formula of **summation by parts**

$$\sum_{k=m+1}^{n} f_k g_k = F_n g_n - F_m g_m - \sum_{k=m}^{n-1} F_k (g_{k+1} - g_k) \tag{4.12}$$

(where $F_n = \sum_{j=0}^n f_j$) to prove the following Dirichlet's test.

Theorem 4.29 (Dirichlet's test). *Let X be a set. Assume that (f_n) is a sequence in $l^\infty(X, V)$ such that $\sup_{n \in \mathbb{Z}_+} \|F_n\|_{l^\infty} < +\infty$ (where $F_n = \sum_{j=1}^n f_j$). Assume that (g_n) is a decreasing sequence (i.e. $g_1 \geqslant g_2 \geqslant g_3 \geqslant \cdots$) in $l^\infty(X, \mathbb{R}_{\geqslant 0})$ converging uniformly to 0. Then $\sum_{n=1}^\infty f_n g_n$ converges uniformly on X.*

Problem 4.3. Let $e_n : \mathbb{R} \to \mathbb{C}$ be $e_n(x) = e^{\mathbf{i}nx} = \cos(nx) + \mathbf{i}\sin(nx)$. Show that for each $x \in \mathbb{R}$, $\sum_{n=1}^\infty e_n(x)/n$ does not converge absolutely. Use Dirichlet's test to show that the series of functions $\sum_{n=1}^\infty e_n/n$ converges pointwise on $\mathbb{R} \backslash \{2k\pi : k \in \mathbb{Z}\}$, and uniformly on $[\delta, 2\pi - \delta]$ for every $0 < \delta < \pi$.

5 Nets and discrete integrals

5.1 Introduction: why do we need nets?

Nets were introduced by Moore and Smith in 1922 as a generalization of sequences. The most well-known motivation for introducing nets is that sequences are not enough for the study of non-metrizable topological spaces (i.e. topological spaces whose topologies are not induced by metrics). Here are two examples:

- In a general topological space, the definition of continuous maps using sequential convergence (as in Def. 2.38-(1)) is weaker than the definition using interior points and open sets (as in Def. 2.38-(2′), see also Rem. 4.23). Therefore, the dynamic intuition of sequences is not equivalent to the static intuition of open sets.

- Some important topological spaces are compact (i.e. every open cover has a finite subcover) but not sequentially compact. $[0,1]^I$ (where I is uncountable), equipped with the "product topology" (i.e. "pointwise convergence topology"), is such an example.

As we shall see, nets provide a remedy for these issues: For a general topological space, the definition of continuity using net convergence is equivalent to that using open sets; compactness is equivalent to "net-compactness", where the latter means that every net has a convergent subset. Thus, by generalizing sequences to nets, the dynamic intuition and the static and geometric intuition are unified again.

Nevertheless, the most common topological spaces appearing in analysis are metrizable. This raises the question: Why should we care about nets, given that our primary interest is in metrizable topological spaces? Here is my answer: Even though we are mainly interested in metrizable spaces, we can still find nets helpful in the following aspects.

First of all, many convergence processes cannot be described by sequential convergence, but can be described by net convergence. For example, the following limits can be formulated and understood in the language of net convergence:

(1) The limit of a function $\lim_{x\to x_0} f(x)$ where $f: X \to Y$ is a map of metric spaces and $x_0 \in X$.

(2) The limit $\lim\limits_{m,n\to\infty} a_{m,n}$ where $(a_{m,n})_{m,n\in\mathbb{Z}_+}$ is a **double sequence** in a metric space X. Note that this is not the same as (but is more natural than) the **iterated limit** $\lim\limits_{n\to\infty}\lim\limits_{m\to\infty} a_{m,n}$. Moreover, the limit $\lim\limits_{m,n\to\infty} a_{m,n}$ is the key to understanding the problem of commutativity of iterated integrals:

$$\lim_{n\to\infty}\lim_{m\to\infty} a_{m,n} \overset{?}{=\!=} \lim_{m\to\infty}\lim_{n\to\infty} a_{m,n}$$

(3) The Riemann integral $\int_a^b f(x)dx$. This is the limit of the Riemann sum $\lim\sum f(\xi_i)(a_i-a_{i-1})$ as the partition of the interval $[a,b]$ is getting finer and finer.

Moreover, as for (3), we shall see that the net version of Cor. 3.18 provides a quick and conceptual proof of the following fact: If the upper and lower Darboux integrals are equal, then the Riemann integral exists and are equal to the two Darboux integrals. Indeed, the upper and lower Darboux integrals are respectively the $\limsup$ and $\liminf$ of a net in $\mathbb{R}$.

Second, nets provide a conceptual solution to many problems about **double series**. Let $(a_{m,n})_{m,n\in\mathbb{Z}_+}$ be a double sequence in $\mathbb{R}$. Think about the following questions, which arise naturally when one is trying to prove $e^ze^w=e^{z+w}$.

(a) When is it true that $\displaystyle\sum_{n=1}^\infty\sum_{m=1}^\infty a_{m,n}=\sum_{m=1}^\infty\sum_{n=1}^\infty a_{m,n}$?

(b) Since $\mathrm{card}(\mathbb{Z}_+\times\mathbb{Z}_+)=\mathrm{card}(\mathbb{Z}_+)$, why not use an ordinary series to study a double series? So let us **parametrize** $\mathbb{Z}_+\times\mathbb{Z}_+$ by $\mathbb{Z}_+$: choose a bijection $\varphi:\mathbb{Z}_+\to\mathbb{Z}_+\times\mathbb{Z}_+$. When is it true that $\displaystyle\sum_{k=1}^\infty a_{\varphi(k)}=\sum_{n=1}^\infty\sum_{m=1}^\infty a_{m,n}$?

(c) Choose another parametrization (i.e. bijection) $\psi:\mathbb{Z}_+\to\mathbb{Z}_+\times\mathbb{Z}_+$. When is it true that $\displaystyle\sum_{k=1}^\infty a_{\varphi(k)}=\sum_{k=1}^\infty a_{\psi(k)}$?

(d) More generally, let X be a countably infinite set, and let $f:X\to\mathbb{R}$. Intuitively, we can take an infinite sum $\displaystyle\sum_{x\in X}f(x)$. How to define it rigorously? One may think about choosing a parametrization, i.e., a bijection $\varphi:\mathbb{Z}_+\to X$. Then one defines the infinite sum by $\sum_{k=1}^\infty f(\varphi(k))$. Is this definition independent of the choice of parametrization?

(e) As a special case of (d), when is a series invariant under **rearrangement**? Namely, choose a bijection $\varphi:\mathbb{Z}_+\to\mathbb{Z}_+$, and choose a sequence (a_n) in $\mathbb{R}$, when is it true that $\displaystyle\sum_{n=1}^\infty a_n=\sum_{n=1}^\infty a_{\varphi(n)}$?

Modern differential geometry (whose "intrinsic" spirit stems from Gauss's Theorema Egregium) teaches us that in order to answer these questions, one should first define the infinite sum $\sum_{x\in X}f(x)$ in a parametrization-independent way. (The reason we call a bijection $\varphi:\mathbb{Z}_+\to X$ a parametrization is that we want the readers to compare it with the parametrizations of curves, surfaces, and more generally manifolds.) We will call this sum a **discrete integral**. Then, one tries to answer when this definition agrees with those that depend on parametrizations (such as the sums in (a)-(e) above). These goals can be achieved with the help of nets.

5.2 Nets

5.2.1 Directed sets and nets

Definition 5.1. A relation $\leqslant$ on a set I is called a **preorder** if for all $\alpha, \beta, \gamma \in I$, the following are satisfied:

- (Reflexivity) $\alpha \leqslant \alpha$.
- (Transitivity) If $\alpha \leqslant \beta$ and $\beta \leqslant \gamma$ then $a \leqslant \gamma$.

The pair $(I, \leqslant)$ (or simply I) is called a **preordered set**.

Therefore, a partial order is a preorder satisfying antisymmetry: $(\alpha \leqslant \beta) \wedge (\beta \leqslant \alpha) \Rightarrow (\alpha = \beta)$.

Definition 5.2. A preordered set $(I, \leqslant)$ is called a **directed set** if

$$\forall \alpha, \beta \in I \quad \exists \gamma \in I \quad \text{such that } \alpha \leqslant \gamma, \beta \leqslant \gamma \tag{5.1}$$

If I is a directed set and X is a set, then a function $x : I \to X$ is called a **net** with directed set/index set I. We often write $x(\alpha)$ as x_α if $\alpha \in I$, and write x as $(x_\alpha)_{\alpha \in I}$.

Example 5.3. $(\mathbb{Z}_+, \leqslant)$ is a directed set. A net with index set $\mathbb{Z}_+$ in a set X is precisely a sequence in X.

Definition 5.4. Suppose that $(I, \leqslant_I)$ and $(J, \leqslant_J)$ are preordered set (resp. directed set), then the **product** $(I \times J, \leqslant)$ is a preordered set (resp. directed set) if for every $\alpha, \alpha' \in I, \beta, \beta' \in J$ we define

$$(\alpha, \beta) \leqslant (\alpha', \beta') \qquad \Longleftrightarrow \qquad \alpha \leqslant_I \alpha' \text{ and } \beta \leqslant_J \beta' \tag{5.2}$$

Unless otherwise stated, the preorder on $I \times J$ is assumed to be defined by (5.2).

Example 5.5. $\mathbb{Z}_+ \times \mathbb{Z}_+$ (or similarly, $\mathbb{N} \times \mathbb{N}$) is naturally a directed set whose preorder is defined by (5.2). A net $(x_{m,n})_{(m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_+}$ with index set $\mathbb{Z}_+ \times \mathbb{Z}_+$ is called a **double sequence** and is written as $(x_{m,n})_{m,n \in \mathbb{Z}_+}$ or simply $(x_{m,n})$. (We will even write it as (x_{mn}) when no confusion arises.)

More generally, we call $(x_{\alpha,\beta})_{(\alpha,\beta) \in I \times J} = (x_{\alpha,\beta})_{\alpha \in I, \beta \in J}$ a **double net** if its index set is $I \times J$ for some directed sets I, J. $\square$

Example 5.6. If X is a set, then $(2^X, \subset)$ and $(\text{fin}(2^X), \subset)$ are directed sets where

$$\text{fin}(2^X) = \{A \subset X : A \text{ is a finite set}\} \tag{5.3}$$

We will use nets with index set $\text{fin}(2^X)$ to study infinite sums.

Example 5.7. Let X be a metric space and $x \in X$. Then $X_x = (X, \leqslant)$ is a directed set if for each $p_1, p_2 \in X$ we define

$$p_1 \leqslant p_2 \text{ in } X_x \qquad \Longleftrightarrow \qquad d(p_1, x) \geqslant d(p_2, x) \tag{5.4}$$

(Namely, a larger element of X_x is one closer to x.) Nets with this directed set can be used to study the limits of functions (cf. Rem. 7.83). Note that X_x is our first example of directed set which is not a poset! ($d(p_1, x) = d(p_2, x)$ does not imply $p_1 = p_2$.)

5.2.2 Limits of nets

If I is an preordered set and $\beta \in I$, we write

$$I_{\geqslant \beta} = \{\alpha \in I : \alpha \geqslant \beta\} \tag{5.5}$$

Definition 5.8. Let P be a property about elements of a set X, i.e., P is a function $X \to \{\text{true, false}\}$. Let $(x_\alpha)_{\alpha \in I}$ be a net in X.

We say that x_α **eventually** satisfies P (equivalently, we say that x_α satisfies P for **sufficiently large** α) if:

- There exists $\beta \in I$ such that for every $\alpha \in I_{\geqslant \beta}$, the element x_α satisfies P.

"Sufficiently large" is also called "**large enough**".

We say that x_α **frequently** satisfies P if:

- For every $\beta \in I$ there exists $\alpha \in I_{\geqslant \beta}$ such that x_α satisfies P.

□

Remark 5.9. Note that unlike sequences, for a general net, "x_α eventually satisfies P" does not imply "all but finitely many x_α satisfy P" because the complement of $I_{\geqslant \beta}$ is not necessarily a finite set.

Remark 5.10. Let P and Q be two properties about elements of X. Then

$$\neg(x_\alpha \text{ eventually satisfies } P) \quad = \quad (x_\alpha \text{ frequently satisfies } \neg P) \tag{5.6a}$$

By the crucial condition (5.1) for directed sets, we have

$$\begin{gathered}(x_\alpha \text{ eventually satisfies } P) \wedge (x_\alpha \text{ eventually satisfies } Q) \\ \Downarrow \\ x_\alpha \text{ eventually satisfies } P \wedge Q\end{gathered} \tag{5.6b}$$

By taking contraposition and replacing P, Q by $\neg P, \neg Q$, we have

$$\begin{gathered}x_\alpha \text{ frequently satisfies } P \vee Q \\ \Downarrow \\ (x_\alpha \text{ frequently satisfies } P) \vee (x_\alpha \text{ frequently satisfies } Q)\end{gathered} \tag{5.6c}$$

Definition 5.11. Let $(x_\alpha)_{\alpha\in I}$ be a net in a metric space X. Let $x \in X$. We say that (x_α) **converges to** x and write

$$\lim_{\alpha\in I} x_\alpha \equiv \lim_{\alpha} x_\alpha = x$$

or simply $x_\alpha \to x$ if the following statement holds:

- For every $\varepsilon > 0$, x_α is eventually in $B_X(x, \varepsilon)$.

Clearly, $x_\alpha \to x$ iff $d(x_\alpha, x) \to 0$.

Definition 5.12. Let $(x_{m,n})_{m,n\in\mathbb{Z}_+}$ be a double sequence in a metric space. Then we write

$$\lim_{(m,n)\in\mathbb{Z}_+\times\mathbb{Z}_+} x_{m,n} \equiv \lim_{m,n\to\infty} x_{m,n} \tag{5.7}$$

and call it the **(double) limit** of $(x_{m,n})$.

Remark 5.13. Let us spell out the meaning of $\lim_{m,n\to\infty} x_{m,n} = x$: For each $\varepsilon > 0$ there exists $M, N \in \mathbb{Z}_+$ such that $d(x_{m,n}, x) < \varepsilon$ for all $m \geqslant M$ and $n \geqslant N$. Clearly, this is equivalent to the statement:

- For each $\varepsilon > 0$ there exists $N \in \mathbb{Z}_+$ such that $d(x_{m,n}, x) < \varepsilon$ for all $m, n \geqslant N$.

Therefore, if (x_n) is a sequence in X, then

$$(x_n) \text{ is a Cauchy sequence} \qquad \Longleftrightarrow \qquad \lim_{m,n\to\infty} d(x_m, x_n) = 0 \tag{5.8}$$

Thus, the Cauchyness of sequences can be studied in terms of double limits, and hence in terms of nets.

Proposition 5.14. *Let $(x_\alpha)_{\alpha\in I}$ be a net in a metric space X converging to x, y. Then $x = y$.*

Proof. Suppose that $x \neq y$. Then there are $r, \rho > 0$ such that $B(x, r) \cap B(y, \rho) = \varnothing$, say $r = \rho = d(x, y)/2$. Since $x_\alpha \to x$, the point x_α is eventually in $B(x, r)$. Since $x_\alpha \to y$, the point x_α is eventually in $B(y, \rho)$. Therefore, by the logic (5.6b), x_α is eventually in $B(x, r) \cap B(y, \rho)$, impossible. $\square$

Theorem 5.15. *Let $f : X \to Y$ be map of metric spaces continuous at $x \in X$. Let $(x_\alpha)_{\alpha\in I}$ be a net in X converging to x. Then $\lim\limits_{\alpha} f(x_\alpha) = f(x)$.*

Proof. Choose any $\varepsilon > 0$. By Def. 2.38-(2) and the continuity of f at x, there exists $\delta > 0$ such that for all $p \in B(x, \delta)$ we have $f(p) \in B(f(x), \varepsilon)$. Since $x_\alpha \to x$, x_α is eventually in $B(x, \delta)$. Therefore $f(x_\alpha)$ is eventually in $B(f(x), \varepsilon)$. $\square$

This theorem implies, for example, that if (v_α) is a net in a complex normed vector space converging to v, and if (λ_α) is a net in $\mathbb{C}$ converging to λ, then $\lambda_\alpha v_\alpha$ converges to λv because the scalar multiplication map is continuous (Prop. 3.38).

Exercise 5.16. Prove the generalization of Rem. 2.71:

1. If $(x_\alpha)_{\alpha\in I}, (y_\alpha)_{\alpha\in I}$ are nets in $\overline{\mathbb{R}}$ converging to $A, B \in \overline{\mathbb{R}}$, and if $x_\alpha \leqslant y_\alpha$ for all α, then $A \leqslant B$.
2. **Squeeze theorem**: Suppose that $(x_\alpha)_{\alpha\in I}, (y_\alpha)_{\alpha\in I}, (z_\alpha)_{\alpha\in I}$ are nets in $\overline{\mathbb{R}}$, $x_\alpha \leqslant y_\alpha \leqslant z_\alpha$ for all α, and x_α and z_α both converge to $A \in \overline{\mathbb{R}}$. Then $y_\alpha \to A$.
3. If (x_α) is an increasing resp. decreasing net in $\overline{\mathbb{R}}$, then $\lim_\alpha x_\alpha$ exists in $\overline{\mathbb{R}}$ and equals $\sup_\alpha x_\alpha$ resp. $\inf_\alpha x_\alpha$.

5.2.3 Subnets (in the sense of Willard)

Definition 5.17. A subset E of a directed set I is called **cofinal** if:

$$\forall \alpha \in I \quad \exists \beta \in E \quad \text{such that } \alpha \leqslant \beta$$

By the transitivity in Def. 5.1 and property (5.1), we clearly have

$$\forall \alpha_1, \dots, \alpha_n \in I \quad \exists \beta \in E \quad \text{such that } \alpha_1 \leqslant \beta, \dots, \alpha_n \leqslant \beta$$

Definition 5.18. Let $(x_\alpha)_{\alpha\in I}$ be a net in a set X. A **subnet** of $(x_\alpha)_{\alpha\in I}$ is, by definition, of the form $(x_{\alpha_s})_{s\in S}$ where S is a directed set, and

$$(\alpha_s)_{s\in S} : S \to I \qquad s \mapsto \alpha_s$$

is an increasing function whose range $\{\alpha_s : s \in S\}$ is cofinal in I.

Remark 5.19. There are several different definitions of subnets that are equivalent for proving the main results in point-set topology. Unfortunately, there is no common agreement on the standard definition of subnets. The definition we gave is due to Willard [Wil], and is also the one given in the famous textbook of Munkres [Mun]. Some famous analysis and topology textbooks (e.g. [?, Kel, RS]) use a weaker definition, which does not assume that the map $S \to I$ is increasing.

Example 5.20. A subsequence of a sequence is a subnet of that sequence.

Example 5.21. Let $(x_{m,n})_{m,n\in\mathbb{Z}_+}$ be a net with index set $\mathbb{Z}_+ \times \mathbb{Z}_+$. Then $(x_{k,k})_{k\in\mathbb{Z}_+}$ and $(x_{2k,k})_{k\in\mathbb{Z}_+}$ are subnets. $(x_{k,1})_{k\in\mathbb{Z}_+}$ is not a subnet, because the cofinal condition is not satisfied. More generally, it is not hard to show that for every function $\varphi, \psi : \mathbb{Z}_+ \to \mathbb{Z}_+$, $(x_{\varphi(k),\psi(k)})_{k\in\mathbb{Z}_+}$ is a subnet iff φ, ψ are increasing and $\lim_{k\to\infty} \varphi(k) = \lim_{k\to\infty} \psi(k) = +\infty$.

Exercise 5.22. Prove the following facts:

- The cofinal subset of a cofinal subset of a directed set I is a cofinal subset of I.
- The subnet of a subnet of a net (x_α) is a subnet of (x_α).

Note that in your proof you need to use the transitivity in Def. 5.1.

The biggest difference between subnets and subsequences is that the index set of a subnet is not necessarily a subset of the index set of the original net. Indeed, subnets are defined in this way mainly because we want to have a net version of Pb. 3.1 in any topological space. (This will be achieved in Pb. 7.2.) Let us see an elementary example of subnet whose index set is larger than that of the original net. Its importance is justified by the proofs of Exp. 5.28 and Prop. 5.34.

Example 5.23. Let J be a directed set. Then every net $(x_\alpha)_{\alpha\in I}$ has subnet $(x_\alpha)_{(\alpha,\beta)\in I\times J}$. The corresponding increasing map of directed sets is the projection $I\times J\to I$ onto the first component.

To appreciate the importance of cofinalness (as well as transitivity), we prove the following generalization of Prop. 2.36. This result has a wide range of surprising applications that are unavailable when one only considers sequences. (We will see them soon in this chapter. For instance, this result explains why the values of absolutely convergent series are invariant under rearrangement.) So I call this result a theorem, even though its proof is simple.

Theorem 5.24. *Let $(x_\alpha)_{\alpha\in I}$ be a net in a metric space (or more generally, a topological space) X converging to $x\in X$. Then every subnet $(x_{\alpha_s})_{s\in S}$ converges to x.*

The following proof for metric spaces can be generalized straightforwardly to topological spaces. The readers can come back and check the details after learning topological spaces.

Proof. Choose any $\varepsilon>0$. Since $x_\alpha\to x$, there exists $\beta\in I$ such that for all $\alpha\geqslant\beta$ we have $d(x_\alpha,x)<\varepsilon$. By the cofinalness, there exists $t\in S$ such that $\alpha_t\geqslant\beta$. Thus, since $s\in S\mapsto\alpha_s\in I$ is increasing, for every $s\geqslant t$, we have $\alpha_s\geqslant\alpha_t\geqslant\beta$ and hence $\alpha_s\geqslant\beta$ by the transitivity in Def. 5.1. So $d(x_{\alpha_s},x)<\varepsilon$ for all $s\geqslant t$. This finishes the proof. □

This proposition does not hold if one does not assume cofinalness in the definition of subnets:

Example 5.25. Let (x_n) be a sequence in $\mathbb{R}$ converging to $x\in\mathbb{R}$. Since (x_n) is a Cauchy sequence, we know that $\lim_{m,n\to\infty}x_m-x_n=0$. We have seen in Exp. 5.21 that $(x_{2k}-x_k)_{k\in\mathbb{Z}_+}$ is a subnet of $(x_{m,n})$. Therefore, $\lim_{k\to\infty}x_{2k}-x_k=0$. But

$(x_k - x_1)_{k\in\mathbb{Z}_+}$ is not a subnet since the cofinal condition is not satisfied. And if $x \neq x_1$, then $\lim_k (x_k - x_1) = x - x_1 \neq 0$, i.e.,

$$\lim_{k\to\infty}(x_k - x_1) \neq \lim_{m,n\to\infty}(x_m - x_n)$$

In Subsec. 2.3.2, we have seen two criteria for the divergence of sequence: a sequence diverges if it is unbounded, or if it has two subsequences converging to different points. By Thm. 5.24, the second criterion can be generalized to nets. However, the following example shows that the first criterion does not has its net version:

Example 5.26. A convergent net $(x_\alpha)_{\alpha\in I}$ in a metric space X is not necessarily **bounded**. Namely, it is not necessarily true that $\{x_\alpha : \alpha \in I\}$ is a bounded subset of X. Let $f : \mathbb{R}_{>0} \to \mathbb{R}$ be $f(x) = 1/x$. Then f is net in $\mathbb{R}$ with directed set $(\mathbb{R}_{>0}, \leqslant)$. This net is not bounded, although $\lim f(x) = 0$.

Example 5.27. The double sequence $x_{m,n} = n/(m+n)$ in $\mathbb{R}$ has subnets $x_{n,n} = n/(n+n) = 1/2$ and $x_{2n,n} = 1/3$. Since these two subnets converge to different values, Thm. 5.24 implies that $\lim_{m,n} x_{m,n}$ does not exist. However, the **iterated limits** exist and take different values:

$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{n}{m+n} = 1 \qquad \lim_{n\to\infty}\lim_{m\to\infty}\frac{n}{m+n} = 0$$

As we shall see, this gives another criterion for the divergence of double series: If the two iterated limits exist and are different, then the double series diverge.

Finally, we do an example of convergent double sequence:

Example 5.28. Let $x_{m,n} = (m^{-2} - n^{-1})\sin\dfrac{\pi(m+\sqrt{n})}{4}$. Then $\lim\limits_{m,n\to\infty} x_{m,n} = 0$.

Proof. The sequence $(m^{-2})_{m\in\mathbb{Z}_+}$ converges to 0. By Exp. 5.23, the double sequence $(m^{-2})_{m,n\in\mathbb{Z}_+}$ is its subnet, and hence converges to 0 by Thm. 5.24. Similarly, the double sequence $(n^{-1})_{m,n\in\mathbb{Z}_+}$ converges to 0. Therefore, $m^{-2}+n^{-1}$ converges to 0 due to Thm. 5.15 and the continuity of the addition map $(x,y) \in \mathbb{R}^2 \to x+y \in \mathbb{R}$ (Prop. 2.45). Since $0 \leqslant |x_{m,n}| \leqslant m^{-2}+n^{-1}$, we conclude $|x_{m,n}| \to 0$ (and hence $x_{m,n} \to 0$) by squeeze theorem (Exe. 5.16). $\square$

5.2.4 Double limits and iterated limits

Theorem 5.29. *Let $(x_{\alpha,\beta})_{\alpha\in I,\beta\in J}$ be a double net in a metric space X. Assume that the following are true:*

(1) The limit $\lim\limits_{(\alpha,\beta)\in I\times J} x_{\alpha,\beta}$ exists in X.

(2) *For each $\alpha \in I$, the limit $\lim\limits_{\beta\in J} x_{\alpha,\beta}$ exists in X.*

Then the LHS limit in the following equation exists and equals the RHS:

$$\lim_{\alpha\in I}\lim_{\beta\in J} x_{\alpha,\beta} = \lim_{(\alpha,\beta)\in I\times J} x_{\alpha,\beta} \tag{5.9}$$

In particular, suppose that the following is also true:

(3) *For each $\beta \in J$, the limit $\lim\limits_{\alpha\in J} x_{\alpha,\beta}$ exists in X.*

Then the following limits exist and are equal:

$$\lim_{\alpha\in I}\lim_{\beta\in J} x_{\alpha,\beta} = \lim_{\beta\in J}\lim_{\alpha\in I} x_{\alpha,\beta} \tag{5.10}$$

Proof. Let $x_\alpha = \lim\limits_{\beta} x_{\alpha,\beta}$ and $x = \lim\limits_{\alpha,\beta} x_{\alpha,\beta}$. We want to show that $\lim\limits_{\alpha} x_\alpha = x$. Choose any $\varepsilon > 0$. Then there exist $A \in I, B \in J$ such that for every $\alpha \geqslant A$ and $\beta \geqslant B$ we have $d(x_{\alpha,\beta}, x) < \varepsilon/3$. In particular, $d(x_{\alpha,\beta}, x) \leqslant \varepsilon/2$. Using Thm. 5.15 and the fact that $p \in X \mapsto d(p, x) \in \mathbb{R}$ is continuous (Exp. 2.58), we see that for every $\alpha \geqslant A$ we have $d(x_\alpha, x) = \lim\limits_{\beta\in J_{\geqslant B}} d(x_{\alpha,\beta}, x) \leqslant \varepsilon/2 < \varepsilon$. □

The readers may skip the next remark and proof and come back to them when they have learned about topological spaces.

⋆ **Remark 5.30.** Thm. 5.29 can be generalized to the case that X is a regular topological space. By saying that the topological space X is **regular**, we mean that for every $x \in X$ and every open set U containing x, there is a smaller open set V containing x such that the closure $\overline{V}$ (cf. Def. 7.28) is contained in U.

⋆ *Proof.* Let $x_\alpha = \lim\limits_{\beta} x_{\alpha,\beta}$ and $x = \lim\limits_{\alpha,\beta} x_{\alpha,\beta}$. Choose any open set U containing x. We want to prove that x_α is eventually in U. Choose an open set V containing x such that $\overline{V} \subset U$. Then there are $A \in I, B \in J$ such that for all $\alpha \geqslant A$ and $\beta \geqslant B$ we have $x_{\alpha,\beta} \in V$. Thus, for each $\alpha \geqslant A$, since $x_{\alpha,\beta}$ approaches x_α, we have $x_\alpha \in \overline{V}$ and hence $x_\alpha \in U$. □

Corollary 5.31. *Let $(x_{\alpha,\beta})_{\alpha\in I,\beta\in J}$ be a double net in $\overline{\mathbb{R}}$. Assume that $x_{\bullet,\bullet}$ is increasing, i.e., $x_{\alpha,\beta} \leqslant x_{\alpha',\beta'}$ if $\alpha \leqslant \alpha'$ and $\beta \leqslant \beta'$. Then the following equation* (5.11) *hold, where all the limits* (5.11) *exist in $\overline{\mathbb{R}}$:*

$$\lim_{\alpha\in I}\lim_{\beta\in J} x_{\alpha,\beta} = \lim_{\beta\in J}\lim_{\alpha\in I} x_{\alpha,\beta} = \lim_{(\alpha,\beta)\in I\times J} x_{\alpha,\beta} = \sup\{x_{\alpha,\beta} : \alpha \in I, \beta \in J\} \tag{5.11}$$

Clearly, a similar result holds for decreasing double nets in $\overline{\mathbb{R}}$.

Proof. By Exe. 5.16, the three limits $\lim\limits_{\alpha} x_{\alpha,\beta}$, $\lim\limits_{\beta} x_{\alpha,\beta}$, and $\lim\limits_{\alpha,\beta} x_{\alpha,\beta}$ exist in $\overline{\mathbb{R}}$. Therefore, by Thm. 5.29, the three limits in (5.11) exist and are equal. The last equality in (5.11) is also due to Exe. 5.16. □

5.2.5 Cauchy nets

Definition 5.32. A net $(x_\alpha)_{\alpha\in I}$ in a metric space X is called a Cauchy net if

$$\lim_{\alpha,\beta\in I} d(x_\alpha, x_\beta) = 0$$

Equivalently, this means that

$$\forall\varepsilon > 0 \quad \exists\gamma \in I \quad \text{such that } \forall\alpha, \beta \geqslant \gamma \quad \text{we have } d(x_\alpha, x_\beta) < \varepsilon \tag{5.12}$$

Exercise 5.33. Show that the subnet of a Cauchy net is Cauchy.

Proposition 5.34. *A convergent net in a metric space is a Cauchy net.*

Proof. Let $(x_\alpha)_{\alpha\in I}$ converge to x in a metric space X. Then $\lim_\alpha d(x_\alpha, x) = 0$. Since $(d(x_\alpha, x))_{\alpha,\beta\in I}$ is a subnet (cf. Exp. 5.23), we have $\lim_{\alpha,\beta} d(x_\alpha, x) = 0$ by Thm. 5.24. Similarly, we have $\lim_{\alpha,\beta} d(x, x_\beta) = 0$. Since $0 \leqslant d(x_\alpha, x_\beta) \leqslant d(x_\alpha, x) + d(x, x_\beta)$, by Squeeze theorem (Exe. 5.16) we have $\lim_{\alpha,\beta} d(x_\alpha, x_\beta) = 0$. □

Proposition 5.35. *Let $(x_\alpha)_{\alpha\in I}$ be a Cauchy net in a metric space X. Suppose that $(x_\alpha)_{\alpha\in I}$ has a convergent subnet $(x_{\alpha_s})_{s\in S}$ converging to $x \in X$. Then $(x_\alpha)_{\alpha\in I}$ converges to x.*

Proof. Choose any $\varepsilon > 0$. Since (x_α) is a Cauchy net, there exists $\gamma \in I$ such that $d(x_\alpha, x_\beta) \leqslant \varepsilon$ for all $\alpha, \beta \geqslant \gamma$. Since $(\alpha_s)_{s\in S}$ has cofinal range, $\alpha_{s_0} \geqslant \gamma$ for some $s_0 \in S$. Thus $\alpha_s \geqslant \gamma$ for all $s \geqslant s_0$ because $(\alpha_s)_{s\in S}$ is increasing and because of the transitivity in Def. 5.1. Thus, for every $\beta \geqslant \gamma$, $d(x_{\alpha_s}, x_\beta) \leqslant \varepsilon$ for sufficiently large s. By taking limit over s and using the continuity of $y \in X \mapsto d(y, x_\beta)$ as well as Thm. 5.15, we get $d(x, x_\beta) \leqslant \varepsilon$ for all $\beta \geqslant \gamma$. □

Definition 5.36. Two nets $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ in a metric space X are called **Cauchy-equivalent** if

$$\lim_{\alpha\in I} d(x_\alpha, y_\alpha) = 0$$

Two Cauchy nets are simply called **equivalent** if they are Cauchy-equivalent. It is not hard to see that Cauchy-equivalence is an equivalence relation (recall Def. 1.17) on X^I.

Exercise 5.37. Let $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ be nets in a metric space X.

1. Assume that $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ are Cauchy-equivalent. Prove that (x_α) is a Cauchy net iff (y_α) is a Cauchy net.

2. Assume that $(x_\alpha)_{\alpha\in I}$ converges to x. Prove that $(y_\alpha)_{\alpha\in I}$ converges to x iff $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ are Cauchy-equivalent.

Theorem 5.38. *Every Cauchy net $(x_\alpha)_{\alpha\in I}$ in a complete metric space X is convergent.*

We give a hint of the proof and leave the details to the readers as an exercise.

Hint. Construct an increasing sequence $(\alpha_n)_{n\in\mathbb{Z}_+}$ in I such that for every $\beta,\gamma \geqslant \alpha_n$ we have $d(x_\beta,x_\gamma) < 1/n$. Prove that $(x_{\alpha_n})_{n\in\mathbb{Z}_+}$ is a Cauchy sequence, and hence converges to some $x \in X$. Prove that $(x_\alpha)_{\alpha\in I}$ converges to x. (Warning: $(x_{\alpha_n})_{n\in\mathbb{Z}_+}$ is not necessarily a subnet of $(x_\alpha)_{\alpha\in I}$.) □

5.3 Discrete integrals $\sum_{x\in X} f(x)$

In this section, we fix V to be a Banach space over $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$. We fix a (non-necessarily countable) set X. Note that if $f : X \to V$ is a function and X is finite, then $\sum_{x\in X} f(x)$ can be understood in its most obvious way.

Definition 5.39. Let $f : X \to V$ be a map. The expression

$$\sum_{x\in X} f(x)$$

(or simply $\sum_X f$) is called a **discrete integral**. If $v \in V$, we say that $\sum_{x\in X} f(x)$ equals (or converges to) v, if

$$\lim_{A\in\mathrm{fin}(2^X)} \sum_{x\in A} f(x) = v \tag{5.13}$$

In this case, we write

$$\sum_{x\in X} f(x) = v \tag{5.14}$$

Remark 5.40. Recall from Exp. 5.6 that $\mathrm{fin}(2^X)$ is the directed set of finite subsets of 2^X. Its preorder is "$\subset$". So (5.14) means more precisely that:

- For every $\varepsilon > 0$, there exists a finite set $B \subset X$ such that for every finite set A satisfying $B \subset A \subset X$, we have $\|v - \sum_{x\in A} f(x)\| < \varepsilon$.

Remark 5.41. Discrete integrals are one of the most important and representative examples in Moore and Smith's original paper on nets (cf. [MS22]), explaining why nets are called nets: Imagine an infinitely large *fishing net* whose vertices form the set $X = \mathbb{Z}^2$. You grab the net with your hands and pull it up. As you pull it up, the lifted part $A \in \mathrm{fin}(2^X)$ becomes larger and larger.

Remark 5.42. One of the advantages of discrete integrals over series is that discrete integrals are clearly invariant under rearrangement: For every bijection $\varphi : X \to X$, if one side of the following equation converges in V, then the other side converges, and the equation holds true:

$$\sum_{x\in X} f(x) = \sum_{x\in X} f(\varphi(x)) \tag{5.15}$$

or simply $\sum_X f = \sum_X f\circ\varphi$.

Remark 5.43. Let us spell out what Cauchyness means for the net $(\sum_A f)_{A\in \mathrm{fin}(2^X)}$:

(1) For every $\varepsilon > 0$, there exists a finite set $B \subset X$ such that for any finite sets A_1, A_2 satisfying $B \subset A_1 \subset X, B \subset A_2 \subset X$, we have

$$\Big\| \sum_{A_1\backslash A_2} f - \sum_{A_2\backslash A_1} f \Big\| < \varepsilon$$

Note that the term inside the norm is $\sum_{A_1} f - \sum_{A_2} f$. This is also equivalent to:

(2) For every $\varepsilon > 0$, there exists a finite set $B \subset X$ such that for any finite set $E \subset X\backslash B$, we have

$$\Big\| \sum_{E} f \Big\| < \varepsilon$$

We shall mainly use (2) as the Cauchy criterion for the convergence of $\sum_X f$.

Proof of the equivalence. (2) follows from (1) by taking $A_1 = B$ and $A_2 = B \cup E$. (1) follows from (2) by taking $E_1 = A_1\backslash A_2$ and $E_2 = A_2\backslash A_1$ and then concluding $\| \sum_{E_1} f - \sum_{E_2} f\| < 2\varepsilon$. □

Definition 5.44. Let $g : X \to \overline{\mathbb{R}}_{\geqslant 0}$ be a map. Note that the net $(\sum_A g)_{A\in\mathrm{fin}(2^X)}$ is increasing. Hence, its limit exists in $\overline{\mathbb{R}}$ and equals $\sup_{A\in\mathrm{fin}(2^X)}\sum_A g$ (by Exe. 5.16). We write this as $\sum_X g$, or more precisely:

$$\sum_X g \equiv \sum_{x\in X} g(x) \xlongequal{\text{def}} \lim_{A\in\mathrm{fin}(2^X)} \sum_A g = \sup_{A\in\mathrm{fin}(2^X)} \sum_A g \tag{5.16}$$

We say that $\sum_X g$ **converges** or **converges absolutely**, if $\sum_X g < +\infty$.

It is clear that $\sum_X g < +\infty$ iff there exists $C \in \mathbb{R}_{\geqslant 0}$ such that $\sum_A g < C$ for all $A \in \mathrm{fin}(2^X)$.

Remark 5.45. Note that when $g : X \to \mathbb{R}_{\geqslant 0}$, the convergence in Def. 5.44 agrees with that in Def. 5.39. Therefore, Rem. 5.43 still gives a Cauchy criterion for convergence.

Definition 5.46. Let $f : X \to V$. We say that $\sum_X f$ **converges absolutely** if

$$\sum_{x\in X} \|f(x)\| < +\infty$$

Proposition 5.47. *Let $f : X \to V$. If $\sum_X f$ converges absolutely, then it converges, and*

$$\Big\| \sum_{x\in X} f(x) \Big\| \leqslant \sum_{x\in X} \|f(x)\| \tag{5.17}$$

We write this simply as $\| \sum_X f\| \leqslant \sum_X |f|$. (Recall Def. 3.39.)

Proof. (5.17) clearly holds when X is finite. In the general case, assume that $\sum_X f$ converges absolutely. Then by Cauchy criterion (Rem. 5.43-(2)), for every $\varepsilon > 0$ there is $A \in \mathrm{fin}(2^X)$ such that for each finite $E \subset X\backslash A$ we have $\sum_E |f| < \varepsilon$, and hence $\|\sum_E f\| < \varepsilon$. Therefore $\sum_X f$ converges by Cauchy criterion again.

By the continuity of the norm function $v \in V \mapsto \|v\| \in \mathbb{R}_{\geqslant 0}$, and by Thm. 5.15, we have

$$\Big\|\sum_X f\Big\| = \Big\|\lim_A \sum_A f\Big\| = \lim_A \Big\|\sum_A f\Big\|$$

Since $\|\sum_A f\| \leqslant \sum_A |f|$, by Exe. 5.16, the above expression is no less than

$$\lim_A \sum_A |f| = \sum_X |f|$$

□

The following proposition gives another demonstration that discrete integrals are more natural than series. We leave the proof to the readers.

Proposition 5.48. *Let $f : X \to \mathbb{R}^N$ where $N \in \mathbb{Z}_+$. Then*

$$\sum_{x\in X} f(x) \ \text{ converges} \qquad \Longleftrightarrow \qquad \sum_{x\in X} f(x) \ \text{ converges absolutely}$$

Hint. Reduce to the case $N = 1$. Consider $A = \{x \in X : f(x) \geqslant 0\}$ and $B = X\backslash A$. □

When $\mathbb{R}^N$ is replaced by an infinite-dimensional Banach space, the convergence of a discrete integral may not imply absolute convergence. See Pb. 5.6.

5.4 Fubini's theorem for discrete integrals

Fix a Banach space V over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let X, Y be sets.

Theorem 5.49 (Fubini's theorem-A). *Let $f : X \times Y \to V$. Assume that $\sum_{X\times Y} f$ converges. Then $\sum_Y f(x, \cdot)$ converges for each $x \in X$, and $\sum_X f(\cdot, y)$ converges for each $y \in Y$, and*

$$\sum_{x\in X}\sum_{y\in Y} f(x,y) = \sum_{y\in Y}\sum_{x\in X} f(x,y) = \sum_{(x,y)\in X\times Y} f(x,y) \tag{5.18}$$

where all discrete integrals converge in V.

We abbreviate (5.18) to $\sum_X \sum_Y f = \sum_Y \sum_X f = \sum_{X\times Y} f$.

Proof. For each $x \in X$, let $f_x(y) = f(x,y)$. Let us prove that $\sum_Y f_x$ converges. Choose any $\varepsilon > 0$. Since $\sum_{X\times Y} f$ converges, by Cauchy criterion (Rem. 5.43-(2)), there exists a finite $S \subset X \times Y$ such that the sum of f over any finite subset outside S has norm $< \varepsilon$. The projection $X \times Y \to Y$ maps S to a finite set $B \subset Y$. Thus, for each finite $E \subset Y \backslash B$, we have $\|\sum_E f_x\| < \varepsilon$ since $x \times E$ is outside S. Therefore $\sum_Y f_x$ converges. By the same reasoning, $\sum_X f(\cdot, y)$ converges for all y.

Recall that $\sum_{X\times Y} f$ is the limit of the net $(\sum_S f)_{S\in \mathrm{fin}(2^{X\times Y})}$. This net has subnet

$$\Big(\sum_{(x,y)\in A\times B} f(x,y)\Big)_{A\in\mathcal{I},B\in\mathcal{J}} \qquad \text{where } \mathcal{I} = \mathrm{fin}(2^X) \ \ \mathcal{J} = \mathrm{fin}(2^Y)$$

(Its index set is $\mathcal{I} \times \mathcal{J}$.) Thus, by Thm. 5.24,

$$\sum_{(x,y)\in X\times Y} f(x,y) = \lim_{A\in\mathcal{I},B\in\mathcal{J}} \sum_{(x,y)\in A\times B} f(x,y) \tag{5.19}$$

We are now going to use Thm. 5.29 to show that

$$\lim_{A\in\mathcal{I},B\in\mathcal{J}} \sum_{(x,y)\in A\times B} f(x,y) = \lim_{A\in\mathcal{I}} \lim_{B\in\mathcal{J}} \sum_{(x,y)\in A\times B} f(x,y) \tag{5.20}$$

where the RHS limit exists. For that purpose, we need to check for each $A \in \mathcal{I}$ the convergence of $\lim_B \sum_{(x,y)\in A\times B} f(x,y)$. Since $\sum_Y f_x$ converges, we have

$$\begin{aligned}
&\lim_{B\in\mathcal{J}} \sum_{(x,y)\in A\times B} f(x,y) = \lim_{B\in\mathcal{J}} \sum_{x\in A}\sum_{y\in B} f(x,y)\\
=& \sum_{x\in A} \lim_{B\in\mathcal{J}} \sum_{y\in B} f(x,y) = \sum_{x\in A}\sum_{y\in Y} f(x,y)
\end{aligned} \tag{5.21}$$

(Note that $\sum_A$ is a finite sum and hence commutes with $\lim_B$.) Thus, the assumption in Thm. 5.29 ensuring (5.20) has now been proved true. So (5.20) is true. Moreover, combining (5.19), (5.20), (5.21) together, we get

$$\sum_{(x,y)\in X\times Y} f(x,y) = \lim_{A\in\mathcal{I}} \sum_{x\in A}\sum_{y\in Y} f(x,y) = \sum_{x\in X}\sum_{y\in Y} f(x,y)$$

where the second and the third limits exist. This proves a half of (5.18). The other half can be proved in the same way. □

Theorem 5.50 (Fubini's theorem-B). *Let $g : X \times Y \to \overline{\mathbb{R}}_{\geqslant 0}$. Then the five discrete integrals in (5.22) exist in $\overline{\mathbb{R}}_{\geqslant 0}$, and equations (5.22) hold in $\overline{\mathbb{R}}_{\geqslant 0}$:*

$$\sum_{x\in X}\sum_{y\in Y} f(x,y) = \sum_{y\in Y}\sum_{x\in X} f(x,y) = \sum_{(x,y)\in X\times Y} f(x,y) \tag{5.22}$$

Proof. The existence in $\overline{\mathbb{R}}_{\geqslant 0}$ of the five discrete integrals is clear. (Recall Def. 5.44.) Formula (5.22) can be proved in the same way as (5.18). Note that when applying Thm. 5.29 to prove (5.22), the assumption in Thm. 5.29 on the existence of limits is satisfied because all nets involved are increasing in $\overline{\mathbb{R}}$. (Recall Exe. 5.16.) □

Corollary 5.51 (Fubini's theorem-C). *Let $f : X \times Y \to V$. Then the following are equivalent.*

(1) $\sum_{X\times Y} f$ *converges absolutely.*

(2) $\sum_{x\in X}\sum_{y\in Y}\|f(x,y)\| < +\infty$.

(3) $\sum_{y\in Y}\sum_{x\in X}\|f(x,y)\| < +\infty$.

Proof. Immediate from Thm. 5.50. It is also not hard to prove it directly. □

5.5 Parametrization theorem for discrete integrals

We fix a Banach space V over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. In the following sections, we shall apply the results about discrete integrals to the study of series and double series. For the convenience of applications (e.g. the proof of $e^z e^w = e^{z+w}$), we enlarge the concept of series a little:

5.5.1 Series over $\mathbb{Z}$

Definition 5.52. A **series over** $\mathbb{Z}$ is an expression

$$\sum_{n=-\infty}^{+\infty} f(n)$$

where f is a function from $\mathbb{Z}$ to either V or $\overline{\mathbb{R}}_{\geqslant 0}$. We say that this series **converges to** (or equals) $\mu \in V$ resp. equals $\mu \in \overline{\mathbb{R}}_{\geqslant 0}$, if

$$\lim_{m,n\to+\infty}\sum_{i=-m}^{n} f(i) = \mu \tag{5.23}$$

In this case, we write

$$\sum_{n=-\infty}^{+\infty} f(n) = \mu$$

We say that $\sum_{n=-\infty}^{+\infty} f(n)$ **converges absolutely** if $\sum_{n=-\infty}^{+\infty}\|f(n)\| < +\infty$.

Remark 5.53. Note that in the case of $\overline{\mathbb{R}}_{\geqslant 0}$, the limit on the LHS of (5.23) must exist in $\overline{\mathbb{R}}_{\geqslant 0}$. Again, this is due to the fact that the involved net is increasing, and so one can use Exe. 5.16.

Exercise 5.54. Let $\sum_{n=-\infty}^{+\infty} f(n)$ be a series in either V or $\overline{\mathbb{R}}_{\geqslant 0}$.

1. Fix $k \in \mathbb{Z}$. Prove that $\sum_{n=-\infty}^{+\infty} f(n)$ converges iff the following limits converge:

$$\sum_{n=k}^{+\infty} f(n) = \lim_{n\to+\infty} \sum_{i=k}^{n} f(i) \tag{5.24a}$$

$$\sum_{n=-\infty}^{k-1} f(n) = \lim_{m\to+\infty} \sum_{i=-m}^{k-1} f(i) \tag{5.24b}$$

   Moreover, if these limits converge, then

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{n=k}^{+\infty} f(n) + \sum_{n=-\infty}^{k-1} f(n) \tag{5.25}$$

2. In the case that f has codomain V, prove that $\sum_{n=-\infty}^{+\infty} f(n)$ converges if it converges absolutely.

3. Prove that if f is zero outside $\mathbb{Z}_+$, then

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{n=1}^{+\infty} f(n) \tag{5.26}$$

Thus, by (5.26), our following results about series over $\mathbb{Z}$ can be directly applied to series over $\mathbb{Z}_+$ (or over $\mathbb{Z}_{\geqslant k}$ where $k \in \mathbb{Z}$).

5.5.2 Parametrization theorem

The following theorem relates series and discrete integrals. The structure of this theorem is similar to that of Fubini's theorem-A,B,C in Sec. 5.4.

Theorem 5.55 (Parametrization theorem). *Let X be an infinite countable set. Let $\varphi : \mathbb{Z} \to X$ be a bijection (called a* ***parametrization*** *of X). The following are true.*

1. *Let $f : X \to V$. If the RHS of (5.27) converges in V, then the LHS converges, and (5.27) holds:*

$$\sum_{n=-\infty}^{+\infty} f \circ \varphi(n) = \sum_{x\in X} f(x) \tag{5.27}$$

2. *Let $f : X \to \overline{\mathbb{R}}_{\geqslant 0}$. Then (5.27) holds in $\overline{\mathbb{R}}_{\geqslant 0}$.*

3. *Let $f : X \to V$. Then the discrete integral $\sum\limits_{x\in X} f(x)$ converges absolutely iff the series $\sum\limits_{n=-\infty}^{+\infty} f\circ\varphi(n)$ converges absolutely.*

The same conclusions hold if we assume that $\varphi : \mathbb{Z}_+ \to X$ is a bijection.

Proof. We prove the case $\varphi : \mathbb{Z} \to X$; the other case is similar.

Assume that $\sum_X f$ converges, which means that the limit of the net $(\sum_A f)_{A\in\mathrm{fin}(2^X)}$ converges to some $v \in V$. Therefore, by Thm. 5.24, the subnet

$$\Big(\sum_{x\in A_{m,n}} f(x)\Big)_{m,n\in\mathbb{Z}_+} = \Big(\sum_{i=-m}^{n} f\circ\varphi(i)\Big)_{m,n\in\mathbb{Z}_+}$$

converges to v, where $A_{m,n} = \{\varphi(i) : i \in \mathbb{Z}, -m \leqslant i \leqslant n\}$. This proves part 1. The same method proves part 2. Part 3 follows directly from part 2. □

5.6 Application to (double) series and power series; $e^z e^w = e^{z+w}$

Fix a Banach space V over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

5.6.1 General results about series and double series

Corollary 5.56. *Let $f : \mathbb{Z} \to V$, and let $\psi : \mathbb{Z} \to \mathbb{Z}$ be a bijection. Suppose that*

$$\sum_{n=-\infty}^{+\infty} \|f(n)\| < +\infty \tag{5.28}$$

Then (5.29) holds true, where the RHS of (5.29) converges absolutely:

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{n=-\infty}^{+\infty} f\circ\psi(n) \tag{5.29}$$

The same conclusion clearly holds if $\mathbb{Z}$ is replaced by $\mathbb{Z}_+$.

Proof. By (5.28) and Thm. 5.55-3, the discrete integral $\sum_{\mathbb{Z}} f$ converges absolutely, and hence converges. By Thm. 5.55-1, the LHS resp. RHS of (5.29) converges to the value of $\sum_{\mathbb{Z}} f$ if we choose the parametrization to be $\mathrm{id}_{\mathbb{Z}}$ resp. ψ. This proves (5.29) and the convergence of the RHS of (5.29). Applying the same conclusion to $\|f(\cdot)\|$ proves the absolute convergence of the RHS of (5.29). □

Corollary 5.57. *Let $f : \mathbb{Z}^2 \to V$. Let $\Phi : \mathbb{Z}^2 \to \mathbb{Z}^2$ be a bijection. Suppose that*

$$\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} \|f(m,n)\| < +\infty \tag{5.30}$$

Then (5.31) *holds true, where the six series involved in* (5.31) *converge absolutely.*

$$\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} f(m,n)=\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} f(m,n)=\sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} f\circ\Phi(k,l) \tag{5.31}$$

Similar results clearly hold if $\mathbb{Z}^2$ is replaced by $\mathbb{Z}_+^2$: One extends the domain of f and the domain and codomain of Φ from $\mathbb{Z}_+^2$ to $\mathbb{Z}^2$. Then one apply Cor. 5.57.

Also, note that the second term of (5.31) is redundant: it follows from the equality of the first and the third terms of (5.31) if we choose $\Phi(k,l)=(l,k)$.

Proof. By Thm. 5.55-2 and Thm. 5.50, we have

$$\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} \|f(m,n)\|=\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}}\|f(x,y)\|=\sum_{(x,y)\in\mathbb{Z}^2}\|f(x,y)\| \tag{5.32}$$

where all the limits exist in $\overline{\mathbb{R}}_{\geqslant 0}$. Therefore, by (5.30), the discrete integral $\sum_{\mathbb{Z}^2} f$ is absolutely convergent and hence convergent.

Similar to the argument for (5.32), Thm. 5.55-1 and Thm. 5.49 imply that the first two terms of (5.31) exist and are both equal to the discrete integral $\sum_{\mathbb{Z}^2} f$. Since $\sum_{\mathbb{Z}^2} f=\sum_{\mathbb{Z}^2} f\circ\Phi$ (recall Rem. 5.42), by Thm. 5.55-1 and Thm. 5.49 again, the last term of (5.31) converges to $\sum_{\mathbb{Z}^2} f\circ\Phi$.

We have proved that the six series in (5.31) converge, and (5.31) holds. Replacing $f(\cdot,\cdot)$ with $\|f(\cdot,\cdot)\|$ and applying a similar argument, we see that the six series in (5.31) converge absolutely. □

Remark 5.58. Using the same method as in the above proof, one can easily prove a more general version of Cor. 5.57: Let $N\in\mathbb{Z}_+$. Let $f:\mathbb{Z}^2\to V$ such that (5.30) holds true. Let $\Psi:\mathbb{Z}^N\to\mathbb{Z}^2$ be a bijection. Then the N series involved in the expression of (5.33) (from innermost to outermost) converge absolutely:

$$\sum_{n_1=-\infty}^{+\infty}\cdots\sum_{n_N=-\infty}^{+\infty} f\circ\Psi(n_1,\dots,n_N) \tag{5.33}$$

Moverover, the outermost series of (5.33) converges to (5.31). And of course, a similar result holds if $\mathbb{Z}^2$ is replaced by $\mathbb{Z}^M$ for every $M\in\mathbb{Z}_+$. We leave it to the readers to fill in the details.

Corollary 5.59. *Assume that*

$$A=\sum_{n=-\infty}^{+\infty} a_n \qquad B=\sum_{n=-\infty}^{+\infty} b_n$$

are absolutely convergent series in $\mathbb{C}$. Then for each $k \in \mathbb{Z}$, the series

$$c_k = \sum_{l=-\infty}^{+\infty} a_{k-l}b_l$$

converges absolutely. Moreover, the LHS of the (5.34) converges absolutely to the RHS:

$$\sum_{k=-\infty}^{+\infty} c_k = AB \tag{5.34}$$

Proof. Apply Cor. 5.57 to the case that $f(m,n) = a_m b_n$ and $\Phi(k,l) = (k-l,l)$. □

5.6.2 Application to power series

Corollary 5.60. *Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{+\infty} b_n z^n$ be power series in $\mathbb{C}$ with radii of convergence R_1, R_2 respectively. Let $R = \min\{R_1, R_2\}$. For each $k \in \mathbb{Z}_+$, let*

$$c_k = \sum_{l=0}^{k} a_{k-l}b_l$$

Then the power series $h(z) = \sum_{k=0}^{+\infty} c_k z^k$ has radius of convergence $\geqslant R$. Moreover, for each $z \in \mathbb{C}$ satisfying $0 \leqslant |z| < R$, we have

$$h(z) = f(z) \cdot g(z)$$

Proof. For each $0 \leqslant |z| < R$, apply Cor. 5.59 by replacing the a_n, b_n, c_k of Cor. 5.59 with $a_n z^n, b_n z^n, c_k z^k$. This shows that $h(z)$ converges absolutely to $f(z) \cdot g(z)$. Since this is true for all $|z| < R$, $h(z)$ must have radius of convergence at least R by Rem. 4.21. □

The above result also holds more generally for Laurent series. See Exe. 5.66.

Corollary 5.61. *For each $z, w \in \mathbb{C}$ we have*

$$e^z e^w = e^{z+w}$$

Proof. Apply Cor. 5.59 to the case $a_n = z^n/n!$ and $b_n = w^n/n!$. (We set $a_n = b_n = 0$ if $n < 0$.) Then

$$c_k = \sum_{l=0}^{k} \frac{z^{k-l}}{(k-l)!} \cdot \frac{w^l}{l!} = \sum_{l=0}^{k} \binom{k}{l} \frac{z^{k-l}w^l}{k!} = \frac{(z+w)^k}{k!}$$

by (4.5). □

5.7 Summary

The following are some fundamental questions about series and double series:

(a) Are they invariant under rearrangement? (Cf. (5.29).)

(b) Does the value of an interated double series remain unchanged if the order of the two infinite sums is changed? (Cf. the first equality in (5.31).)

(c) A mixture of the above two questions. (Cf. the last term of (5.31).)

We address these questions by relating them to discrete integral, a version of infinite sums which is parametrization-independent. The following are some key features of this theory.

1. (General principle) A discrete integral is to a series (defined by parametrization) as a net to a subnet.

2. (Net ⇒ Subnet) All subnets of a convergent net converge to the same value: the limit of the original net.

3. (Discrete integral ⇒ Series) Therefore, different series converge to the same value if they are different parametrizations of the same convergent discrete integral.

4. (Discrete integrals ⇒ Series) Fubini-type theorems (any theorems about exchanging the orders of iterated sums/integrals) hold for convergent double discrete integrals. Therefore, they hold when passing to subnets, in particular, when passing to double series.

5. (Subnet ⇒ Net) Every increasing net in $\overline{\mathbb{R}}_{\geqslant 0}$ has a limit in $\overline{\mathbb{R}}_{\geqslant 0}$. Therefore, if an increasing net in $\mathbb{R}_{\geqslant 0}$ has a subnet converging to a number $< +\infty$, then the original net converges in $\mathbb{R}_{\geqslant 0}$ (to a finite number).

6. (General principle) The discrete integral $\sum_{x\in X}\|f(x)\|$ is defined by the limit of an increasing net in $\overline{\mathbb{R}}_{\geqslant 0}$.

7. (Series ⇒ Discrete integral) Therefore, if any series or double series corresponds in a reasonable way to a discrete integral, then the absolute convergence of this (double) series (more specifically: (5.28) or (5.30)) implies the absolute convergence (and hence convergence) of the original discrete integral. This implies the absolute convergence of any other (double) series arising from that discrete integral.

8. (Conclusion) Thus, when a (double) series converges absolutely (in the form of (5.28) or (5.30)), the three problems (a), (b), (c) have satisfying answers. The reason absolutely convergent (double) series are so good is because increasing nets in $\overline{\mathbb{R}}_{\geqslant 0}$ are very good!

9. (Counterexamples) Non-absolutely convergent series in $\mathbb{R}$ have rearrangements converging to different values. This is because non-convergent nets may have two subnets converging to different values, cf. Pb. 5.5. (Recall from Prop. 5.48 that for discrete integrals in $\mathbb{R}$, absolute convergence is equivalent to convergence.)

5.8 Problems and supplementary material

Let X be a set, and let V be a Banach space over $\mathbb{R} \in \{\mathbb{R}, \mathbb{C}\}$.

Problem 5.1. Compute $\lim_{p,q\to+\infty} a_{p,q}$ where $(a_{p,q})_{p,q\in\mathbb{Z}_+}$ are given below. Or explain why the limit does not exist.

$$a_{p,q} = \frac{(-1)^p \cdot p}{p+q} \qquad a_{p,q} = \frac{(-1)^p}{p} \qquad a_{p,q} = \frac{\cos(p\pi/4)}{p+q}$$

Problem 5.2. Prove Thm. 5.38. (Every Cauchy net in a complete metric space converges.)

Problem 5.3. Let $f : X \to V$. Define the **support** of f to be

$$\mathrm{Supp}(f) = \{x \in X : f(x) \neq 0\} \tag{5.35}$$

Prove that if $\sum_X f$ converges absolutely, then $\mathrm{Supp}(f)$ is a countable set.

Hint. Consider $\{x \in X : |f(x)| \geqslant \varepsilon\}$ where $\varepsilon > 0$. □

Problem 5.4. Prove Prop. 5.48.

⋆ **Problem 5.5.** Prove **Riemann rearrangement theorem**, which says the following: Let $\sum_{n=1}^{+\infty} x_n$ be a series in $\mathbb{R}$ which converges and which does not converge absolutely. Choose any $A \in \overline{\mathbb{R}}$. Then $\sum_{n=1}^{+\infty} x_n$ has a rearrangement converging to A (i.e., there is bijection $\varphi : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $\sum_{n=1}^{+\infty} f \circ \varphi(n) = A$).

Remark 5.62. By Riemann rearrangement theorem, it is clear that every convergent series in $\mathbb{R}^N$ which is not absolutely convergent must have two rearrangements converging to two different points. However, when $\mathbb{R}^N$ is replaced by an infinite dimensional Banach space, one may find a series $\sum_{n=1}^{+\infty} v_n$ which does not converge absolutely but converge to some v, and every rearrangement of $\sum_{n=1}^{+\infty} v_n$ converges to v. See Pb. 5.6.

Problem 5.6. Consider the case that V is the real Banach space $V = l^\infty(\mathbb{Z}_+, \mathbb{R})$. For each $n \in \mathbb{Z}_+$, let $e_n \in V$ be the characteristic function $\chi_{\{n\}}$. Namey, e_n takes value 1 at n, and takes 0 at the other points. Prove that the discrete integral

$$\sum_{n\in\mathbb{Z}_+} \frac{1}{n} e_n \tag{5.36}$$

converges in V, and find the limit. Prove that (5.36) does not converge absolutely.

Remark 5.63. A more important example that will be considered later is $V = l^2(\mathbb{Z}_+, \mathbb{C})$, the set of all functions $f : \mathbb{Z}_+ \to \mathbb{C}$ satisfying that the ***l*²-norm** $\|f\|_{l^2} = \sqrt{\sum_{n\in\mathbb{Z}_+} |f(n)|^2}$ is finite. Then V is in fact a Banach space. (Actually, it is a so-called **Hilbert space**.) Again, let $e_n = \chi_{\{n\}}$. (These e_n will be called an **orthonormal basis** of V.) Then for each $f \in V$, the discrete integral $\sum_{n\in\mathbb{Z}_+} f(n) \cdot e_n$ converges to f. But it does not converge absolutely if $\sum_{n\in\mathbb{Z}_+} |f(n)| = +\infty$. Take for example $f(n) = n^{-1}$. We will study these objects in the second semester.

Problem 5.7. Define $(x_{j,k})_{(j,k)\in\mathbb{Z}_+\times\mathbb{Z}_+}$ to be

$$x_{j,k} = \begin{cases} \dfrac{1}{j^2 - k^2} & \text{if } j \neq k \\ 0 & \text{if } j = k \end{cases}$$

Prove that the discrete integral $\displaystyle\sum_{(j,k)\in\mathbb{Z}_+^2} x_{j,k}$ does not converge in $\mathbb{R}$.

Hint. Consider $(x_{j,k})$ as a net over $\mathbb{Z}^2$. Find a good bijection $\Phi : \mathbb{Z}^2 \to \mathbb{Z}^2$. □

Definition 5.64. For each $f \in V^X$, define the ***l*¹-norm**

$$\|f\|_{l^1(X,V)} \equiv \|f\|_{l^1} \equiv \|f\|_1 = \sum_{x\in X} \|f(x)\|$$

Define the l^1-space

$$l^1(X, V) = \{f \in V^X : \|f\|_{l^1} < +\infty\}$$

Namely, $l^1(X, V)$ is the set of all $f \in V^X$ where $\sum_X f$ converges absolutely. In particular, $\sum_X f$ converges for such f.

Exercise 5.65. Prove that for each $f, g \in V^X$ and $\lambda \in \mathbb{F}$, we have

$$\|f + g\|_1 \leqslant \|f\|_1 + \|g\|_1 \qquad \|\lambda f\|_1 = |\lambda| \cdot \|f\|_1 \tag{5.37}$$

Show that $l^1(X, V)$ is a linear subspace of $l^\infty(X, V)$, and that $\|\cdot\|_{l^1}$ is a norm on $l^1(X, V)$

Problem 5.8. Prove that $l^1(X, V)$ is a Banach space. Namely, prove that the metric on $l^1(X, V)$ defined by the l^1-norm is complete.

Problem 5.9. Prove the **dominated convergence theorem** for discrete integrals: Let $(f_\alpha)_{\alpha\in I}$ be a net in V^X satisfying the following conditions:

(1) There exists $g \in l^1(X, \mathbb{R})$ satisfying $g \geqslant 0$ (i.e. $g(x) \geqslant 0$ for all $x \in X$) such that for every $\alpha \in I, x \in X$ we have

$$\|f_\alpha(x)\| \leqslant g(x)$$

We simply write the above condition as $|f_\alpha| \leqslant g$.

(2) $(f_\alpha)_{\alpha\in I}$ converges pointwise some $f \in V^X$. Namely, $\lim_\alpha f_\alpha(x) = f(x)$ for every $x \in X$.

Prove that $f \in l^1(X,V)$. Prove that the LHS of (5.38) exists and equals the RHS:

$$\lim_{\alpha\in I} \sum_{x\in X} f_\alpha(x) = \sum_{x\in X} f(x) \tag{5.38}$$

⋆ **Problem 5.10.** Assume $v_n \in V$ for each n. Let z be a complex variable. Then the expression

$$f(z) = \sum_{n=-\infty}^{+\infty} v_n z^n$$

is called a **Laurent series** in V.

Prove that there exist unique $r, R \in \overline{\mathbb{R}}_{\geqslant 0}$ such that $f(z)$ converges absolutely when $|r| < z < |R|$, and that $f(z)$ diverges when $|z| < r$ or $|z| > R$. Prove that

$$r = \limsup_{n\to+\infty} \sqrt[n]{\|v_{-n}\|} \qquad R = \frac{1}{\limsup_{n\to+\infty} \sqrt[n]{\|v_n\|}} \tag{5.39}$$

(Recall that by Exe. 5.54, $f(z)$ diverges iff either $\sum_{n=0}^{\infty} v_n z^n$ or $\sum_{n=-\infty}^{-1} v_n z^n$ diverges.) We call r and R the **radii of convergence** of $f(z)$. □

⋆ **Exercise 5.66.** Consider Laurent series $f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$ (with radii of convergence $r_1 < R_1$) and $g(z) = \sum_{n=-\infty}^{+\infty} b_n z^n$ (with radii of convergence $r_2 < R_2$) in $\mathbb{C}$. Let

$$r = \max\{r_1, r_2\} \qquad R = \min\{R_1, R_2\}$$

Assume that $r < R$. Prove that for each $k \in \mathbb{Z}$, the series

$$c_k = \sum_{l=-\infty}^{+\infty} a_{k-l} b_l$$

converges absolutely. Prove that for each $z \in \mathbb{C}$ satisfying $r < |z| < R$, the LHS of the following equation converges absolutely to the RHS:

$$\sum_{k=-\infty}^{+\infty} c_k z^k = f(z)g(z) \tag{5.40}$$

6 $\star$ Construction of $\mathbb{R}$ from $\mathbb{Q}$

The goal of this chapter is to construct real numbers from rationals. More precisely, our goal is to prove Thm. 1.34. We use the method of Cantor to construct real numbers using equivalence classes of Cauchy sequences in $\mathbb{Q}$. The idea is quite simple: If we admit the existence of $\mathbb{R}$ satisfying Thm. 1.34, then by Prop. 1.29, each $x \in \mathbb{R}$ is the limit of a sequence (x_n) in $\mathbb{Q}$, which must be a Cauchy sequence. Moreover, if (y_n) is a sequence in $\mathbb{Q}$ converging to $y \in \mathbb{R}$, then by Exe. 5.37 we have $x = y$ iff (x_n) and (y_n) are Cauchy equivalent. Motivated by this, we now do not assume the existence of $\mathbb{R}$, and make the following definition:

Definition 6.1. We let $\mathscr{R}$ be the set of Cauchy sequences in $\mathbb{Q}$, [1] namely, the set of $(x_n)_{n\in\mathbb{Z}_+} \in \mathbb{Q}^{\mathbb{Z}_+}$ satisfying

$$\lim_{m,n\to+\infty}(x_m - x_n) = 0$$

We say that two elements $(x_n), (y_n)$ of $\mathscr{R}$ are Cauchy-equivalent and write $(x_n) \sim (y_n)$ if $\lim_{n\to\infty}(x_n - y_n) = 0$.

Note that the above definition does not rely on the existence of $\mathbb{R}$, because the limit of nets in $\mathbb{Q}$ can be defined using only rational numbers: a net $(\xi_\alpha)_{\alpha\in I}$ converges to ξ iff for every $\varepsilon \in \mathbb{Q}_{>0}$, ξ_α is eventually satisfies $|\xi_\alpha - \xi| < \varepsilon$. The readers can check that all the properties about limit used in this chapter does not rely on the existence of $\mathbb{R}$.

Cauchy-equivalence is clearly an equivalence condition on $\mathscr{R}$: For example, if $\lim(x_n - y_n) = \lim(y_n - z_n) = 0$ then $|x_n - z_n| \leqslant |x_n - y_n| + |y_n - z_n| \to 0$. So $|x_n - z_n| \to 0$. This proves the transitivity. The other two conditions are obvious. Therefore, we can define:

Definition 6.2. We let $\mathbb{R} = \mathscr{R}/\sim$ where $\sim$ is the Cauchy-equivalence relation. (Recall Def. 1.18). The equivalence class of $(x_n)_{n\in\mathbb{Z}_+}$ is denoted by $[x_n]_{n\in\mathbb{Z}_+} = [x_1, x_2, \dots]$, simply written as $[x_n]$. The zero element 0 of $\mathbb{R}$ is defined to be $[0, 0, \dots]$.

Exercise 6.3. Choose $[x_n] \in \mathbb{R}$ (i.e. $(x_n) \in \mathscr{R}$) . The following are equivalent:

(1) $[x_n] \neq 0$. Namely, $\lim_n x_n$ does not converge to 0.

(2) There exists $\varepsilon \in \mathbb{Q}_{>0}$ such that either $x_n > \varepsilon$ eventually, or $x_n < -\varepsilon$ eventually. In particular, $x_n \neq 0$ eventually.

Consequently, the map $a \in \mathbb{Q} \mapsto [a, a, \dots] \in \mathbb{R}$ is injective. With the help of this injective map, $\mathbb{Q}$ can be viewed as a subset of $\mathbb{R}$.

[1] Only in this chapter do we use $\mathscr{R}$ for this meaning.

Exercise 6.4. Let $[x_n] \in \mathbb{R}$ and $a \in \mathbb{Q}$. Suppose that $x_n \geqslant a$ (resp. $x_n \leqslant a$) frequently. Then for every $\varepsilon \in \mathbb{Q}_{>0}$, we have that $x_n \geqslant a - \varepsilon$ (resp. $x_n \leqslant a + \varepsilon$) eventually.

Definition 6.5. If $\xi, \eta \in \mathbb{R}$, write $\xi = [x_n]$ and $\eta = [y_n]$. In the case that $\eta \neq 0$, we assume $y_n \neq 0$ for all n, which is possible by Exe. 6.3. Define

$$\begin{gathered}
[x_n] + [y_n] = [x_n + y_n] \\
-[x_n] = [-x_n] \\
[x_n] \cdot [y_n] = [x_n y_n] \\
1/[y_n] = [1/y_n] \qquad (\text{if } [y_n] \neq 0)
\end{gathered}$$

Exercise 6.6. Prove that the above formulas are well-defined: For example, if $(x_n) \sim (x'_n)$ in $\mathscr{R}$, then $(x_n y_n) \sim (x'_n y_n)$. (You may need the easy fact that every Cauchy sequence is bounded.)

Remark 6.7. It is clear that Def. 6.5 makes $\mathbb{R}$ a **field**, which means that for every $\alpha, \beta, \gamma \in \mathbb{R}$, the following are satisfied:

$$\begin{gathered}
\alpha + \beta = \beta + \alpha \qquad (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) \qquad 0 + \alpha = \alpha \qquad \alpha + (-\alpha) = 0 \\
\alpha\beta = \beta\alpha \qquad (\alpha\beta)\gamma = \alpha(\beta\gamma) \qquad 1 \cdot \alpha = \alpha \qquad (\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma \\
\alpha \cdot \frac{1}{\alpha} = 1 \qquad (\text{if } \alpha \neq 0)
\end{gathered}$$

Moreover, $\mathbb{Q}$ is a subfield of $\mathbb{R}$ where the addition, taking negative, multiplication, and inverse of $\mathbb{R}$ restrict to those of $\mathbb{Q}$.

Definition 6.8. Let $[x_n], [y_n] \in \mathbb{R}$. We write $[x_n] < [y_n]$ if one of the following equivalent (due to Exe. 6.4) statements hold:

- There exists $\varepsilon \in \mathbb{Q}_{>0}$ such that $y_n - x_n > \varepsilon$ eventually.
- There exists $\varepsilon \in \mathbb{Q}_{>0}$ such that $y_n - x_n > \varepsilon$ frequently.

It is not hard show that " $<$ " is well-defined, and that (by Exe. 6.3) if $[x_n] < [y_n]$ then $[x_n] \neq [y_n]$. We write $[x_n] \leqslant [y_n]$ if $[x_n] < [y_n]$ or $x_n = y_n$.

Lemma 6.9. *$(\mathbb{R}, \leqslant)$ is a totally ordered set.*

Proof. Choose $[x_n], [y_n], [z_n] \in \mathbb{R}$. If $[x_n] < [y_n]$ and $[y_n] < [z_n]$, then clearly $[x_n] < [z_n]$. This proves that $\leqslant$ is a preorder.

Suppose $[x_n] \leqslant [y_n]$ and $[y_n] \leqslant [x_n]$. Let us prove $[x_n] = [y_n]$. Suppose not. Then $[x_n] < [y_n]$ and $[y_n] < [x_n]$ by the definition of "$\leqslant$". So there is $\varepsilon > 0$ such that $y_n - x_n > \varepsilon$ eventually, and $x_n - y_n > \varepsilon$ eventually. Impossible. So $\leqslant$ is a partial order.

Suppose $[x_n] \neq [y_n]$. Then $(x_n - y_n) \nsim 0$. So Exe. 6.3 implies that either $[x_n] < [y_n]$ or $[y_n] < [x_n]$. So $\leqslant$ is a total order. □

Lemma 6.10. *Let $[x_n], [y_n] \in \mathbb{R}$. Then the following are equivalent.*

- $[x_n] \geqslant [y_n]$.
- *For every $\varepsilon \in \mathbb{Q}_{>0}$, $x_n - y_n \geqslant -\varepsilon$ frequently.*
- *For every $\varepsilon \in \mathbb{Q}_{>0}$, $x_n - y_n \geqslant -\varepsilon$ eventually.*

Proof. Since $\leqslant$ is a total order, the negation of $>$ is $\leqslant$. So the statements follow immediately by negating Def. 6.8. □

Lemma 6.11. *$\mathbb{R}$ is an ordered field extension of $\mathbb{Q}$, and $\mathbb{R}$ is Archimedean.*

Proof. The order of $\mathbb{R}$ clearly restricts to that of $\mathbb{Q}$. We want to prove that $\mathbb{R}$ is an ordered field. (Recall Def. 1.23). Clearly, if $[x_n] < [y_n]$ and $[z_n] \in \mathbb{R}$, then $[x_n] + [z_n] = [x_n + z_n] < [y_n + z_n] = [y_n] + [z_n]$. If $[x_n] > 0$ and $[y_n] > 0$, then there are $\varepsilon > 0$ such that $x_n > \varepsilon$ eventually and $y_n > \varepsilon$ eventually. So $x_n y_n > \varepsilon^2$ eventually. So $[x_n][y_n] > 0$. This proves that $\mathbb{R}$ is an ordered field.

Now let $[x_n] > 0$ and $[y_n] \in \mathbb{R}$. So there exist $\varepsilon \in \mathbb{Q}_{>0}$ such that $x_n > \varepsilon$ eventually. Since (y_n) is Cauchy, one checks easily that (y_n) is bounded. So there is $M \in \mathbb{Q}_{>0}$ such that $|y_n| \leqslant M$ for all n. Since $\mathbb{Q}$ is Archimedean, there exists $k \in \mathbb{Z}_+$ such that $k\varepsilon > M + 1$. So $kx_n > M + 1$ eventually. So $k[x_n] > M$. This proves that $\mathbb{R}$ is Archimedean. □

To finish the proof of Thm. 1.34, it remains to prove that $\mathbb{R}$ satisfies the least-upper-bound property.

Lemma 6.12. *Thm. 1.34 holds if every bounded increasing sequence in $\mathbb{R}$ converges.*

Proof. Suppose that every bounded increasing sequence in $\mathbb{R}$ converges. Choose any nonempty $E \subset \mathbb{R}$ bounded from above. We shall show that E has a least upper bound.

Let F be the set of upper bounds of E. Namely, $F = \{\eta \in \mathbb{R} : \eta \geqslant \xi, \forall \xi \in E\}$. So $F \neq \varnothing$. We construct an increasing sequence (ξ_k) in E and an decreasing sequence (η_k) in F as follows. Since E, F are nonempty, we choose arbitrary $\xi_1 \in E$ and $\eta_1 \in F$. Then $\xi_1 \leqslant \eta_1$. Suppose $\xi_1 \leqslant \cdots \leqslant \xi_k \in E$ and $\eta_1 \geqslant \cdots \geqslant \eta_k \in F$ have been constructed. Let $\psi_k = (\xi_k + \eta_k)/2$. Let

$$\begin{cases} \xi_{k+1} = \psi_k, \eta_{k+1} = \eta_k & \text{if } \psi_k \in E \\ \xi_{k+1} = \xi_k, \eta_{k+1} = \psi_k & \text{if } \psi_k \in F \end{cases}$$

Then the sequences we have constructed satisfy $\lim_{k\to\infty}(\eta_k - \psi_k) = 0$.

By assumption, $\alpha = \lim_{k\to\infty} \xi_k$ exists, and it equals $\lim_k \eta_k$. So α is an upper bound of E. (If $\lambda \in E$, then $\lambda \leqslant \eta_k$ for all k since $\eta_k \in F$. So $\lambda \leqslant \lim_k \eta_k = \alpha$.) We now show that α is the least upper bound. Let $\varepsilon > 0$. Since $\xi_k \to \alpha$, there is k such that $\alpha - \xi_k < \varepsilon$. So $\xi_k > \alpha - \varepsilon$, and hence $\alpha - \varepsilon$ is not an upper bound of E. □

Lemma 6.13. *Thm. 1.34 holds if every bounded increasing sequence in $\mathbb{Q}$ converges to an element of $\mathbb{R}$.*

Proof. Suppose that every increasing sequence in $\mathbb{Q}$ converges in $\mathbb{R}$. By Lem. 6.12, it suffices to prove that every increasing sequence (ξ_k) in $\mathbb{R}$ converges. If $\{\xi_k : k \in \mathbb{Z}_+\}$ is a finite subset of $\mathbb{R}$, then (ξ_k) clearly converges. If $\{\xi_k : k \in \mathbb{Z}_+\}$ is infinite, then (ξ_k) clearly has a strictly increasing subsequence (ξ_{k_l}). If we can prove that (ξ_{k_l}) converges to some $\psi \in \mathbb{R}$, then (ξ_k) converges to ψ. (Choose any $\varepsilon > 0$. Choose $L \in \mathbb{Z}_+$ such that $|\psi - \xi_{k_L}| < \varepsilon$ and hence $0 \leqslant \psi - \xi_{k_L} < \varepsilon$. Then for all $k \geqslant k_L$ we have $0 \leqslant \psi - \xi_k < \varepsilon$.)

Thus, it remains to prove that every strictly increasing sequence (η_k) in $\mathbb{R}$ converges. Since we have proved that $\mathbb{R}$ is an Archimedean ordered field extension of $\mathbb{Q}$, by Prop. 1.29, for each k, there exists $a_k \in \mathbb{Q}$ such that $\xi_k < a_k < \xi_{k+1}$. By assumption, (a_k) converges to some $\alpha \in \mathbb{R}$. Since $a_{k-1} < \xi_k < a_k$, by squeeze theorem, (ξ_k) converges to α. $\square$

Proof of Thm. 1.34. By Lem. 6.13, it suffices to show that every bounded increasing sequence (a_k) in $\mathbb{Q}$ converges in $\mathbb{R}$. Let $M \in \mathbb{Q}$ such that $a_k \leqslant M$ for all k.

We first prove that (a_k) is a Cauchy sequence. If not, then there exists $\varepsilon \in \mathbb{Q}_{>0}$ such that for every $K \in \mathbb{Z}_+$ there is $k > K$ such that $|a_k - a_K| > \varepsilon$, and hence $a_k - a_K > \varepsilon$. Thus, we can find a subsequence (a_{k_l}) such that $a_{k_{l+1}} - a_{k_l} > \varepsilon$. By the Archimedean property for $\mathbb{Q}$, there is $l \in \mathbb{Z}_+$ such that $a_{k_1} + l \cdot \varepsilon > M$. So $a_{k_{l+1}} > M$, impossible.

Note that each a_k is identified with $\xi_k = [a_k, a_k, \dots]$. Let $\psi = [a_1, a_2, a_3, \dots]$, which is an element of $\mathbb{R}$ since we just proved that $(a_n) \in \mathscr{R}$. Then for each k, $\psi - \xi_k = [a_1 - a_k, a_2 - a_k, \dots]$, where the terms are eventually $\geqslant 0$. So $\xi_k \leqslant \psi$ by Lem. 6.10. We have proved that ψ is an upper bound for the sequence (ξ_k).

Let us prove that $\lim_k \xi_k = \psi$. Choose any $\varepsilon \in \mathbb{Q}_{>0}$. Let us prove that there exists k such that $\psi - \varepsilon < \xi_k$. Then for every $k' \geqslant k$ we have $\psi - \varepsilon < \xi_{k'} \leqslant \psi$, finishing the proof of $\lim_k \xi_k = \psi$.

We have proved that $a_1, a_2, \dots$ is a Cauchy sequence in $\mathbb{Q}$. So there exists k such that $a_l - a_k < \varepsilon/2$ for all $l \geqslant k$. Thus, for all $l \geqslant k$ we have $a_k - (a_l - \varepsilon) > \varepsilon/2$. Thus, the l-th term of $\xi_k = [a_k, a_k, \dots]$ minus that of $\psi - \varepsilon = [a_1 - \varepsilon, a_2 - \varepsilon, \dots]$ is $> \varepsilon/2$ for sufficiently large l. By Def. 6.8, we have that $\psi - \varepsilon < \xi_k$. $\square$

7 Topological spaces

7.1 The topologies of metric spaces

In this chapter, we begin our study of topological spaces, which were introduced by Hausdorff in 1914 [Hau14] as a generalization of metric spaces. As we have seen, focusing on metrics in order to study convergence and continuity is often distracting. For example, in $\overline{\mathbb{R}}$, we only care about how the convergence of sequences look like, but not about the particular metrics. The same is true about the countable product of metric spaces $S = \prod_{i\in\mathbb{Z}_+} X_i$: the metrics (2.14) and (2.15) give the same topology, although they look very different. Moreover, the shapes of the open balls defined by these two metrics are not very simple. This makes it more difficult to study the continuity of functions on S by using (2) or (2′) of Def. 2.38.

Topological spaces generalize metric spaces by giving a set of axioms satisfied by the open sets of the spaces.

Definition 7.1. Let X be a metric space, and let $E \subset X$. A point $x \in E$ is called an **interior point** of E if $B_X(x,r) \subset E$ for some $r > 0$. We say that E is an **open (sub)set** of X, if every point of E is an interior point.

Definition 7.2. Let $\mathcal{T}$ be the set of open sets of X. We call $\mathcal{T}$ the **topology of the metric space** X.

Example 7.3. By triangle inequality, every open ball of a metric space X is open. $\varnothing$ and X are open subsets of X. If $p, q \in \mathbb{R}^N$ and $d(p,x) = r$ (where $0 \leqslant r < +\infty$), then p is not an interior point of $\overline{B}_{\mathbb{R}^N}(x,r)$. So the closed balls of $\mathbb{R}^N$ are not open sets. In particular, $[a,b]$ are not open subsets of $\mathbb{R}$ since a, b are not interior points.

Example 7.4. It is not hard to see that a finite intersection of open sets is open.

In topological spaces, open sets play the role of open balls in metric spaces due to the following facts:

Exercise 7.5. Let (x_n) be a sequence in a metric space X. Let $x \in X$. Show that the following are equivalent:

(1) (x_n) converges to x.

(2) For every **neighborhood** U **of** x (i.e. every open set containing x) there is $N \in \mathbb{Z}_+$ such that for every $n \geqslant N$ we have $x_n \in U$.

Exercise 7.6. Let $f : X \to Y$ be a map of metric spaces. Let $x \in X$ and $y = f(x)$. Prove that the following are equivalent.

(1) f is continuous at x.

(2) For every neighborhood V of y there is a neighborhood U of x such that $f(U) \subset V$ (equivalently, $U \subset f^{-1}(V)$.)

But there is an important difference between the intuitions of open sets and open balls: We want the open balls at a point x to be small so that they can be used to describe the approximation to x. However, an arbitrary open set can be very large. For example, when studying convergence and continuity in $\mathbb{R}$, we really want a neighborhood of 1 to be $(1-\varepsilon, 1+\varepsilon)$ but not the more complicated and bigger one $(-\infty, -2) \cup (0, 100-\varepsilon)$. Indeed, open sets can be very big:

Lemma 7.7. *Let X be a metric space. If $(U_\alpha)_{\alpha\in I}$ is a family of open subsets of X, then $W = \bigcup_{\alpha\in I} U_\alpha$ is open in X.*

Proof. Choose $x \in W$. Then $x \in U_\alpha$ for some α. So $B_X(x,r) \subset U_\alpha$ for some $r > 0$. So x is an interior point of W. □

Thus, people very often choose a class $\mathcal{B}$ of smaller open sets (such as the set of open balls) to study the analytic properties of a topological space.

Definition 7.8. Let $\mathcal{B}$ be a set of open sets of a metric space (or more generally, a topological space) X. We say that $\mathcal{B}$ is a **basis for the topology** $\mathcal{T}$ of X if one of the following (clearly) equivalent statements holds:

- For every point $x \in X$ and every neighborhood W of x there exists $U \in \mathcal{B}$ such that $x \in U$ and $U \subset W$.
- Every open subset of X is a union of some members of $\mathcal{B}$.

Thus, according to Def. 7.1, the set of open balls of a metric space X form a basis for the topology of X. Nevertheless, even in the case of metric spaces, we sometimes consider more convenient bases than the set of open balls. We will see this when we study the topologies of infinite product spaces.

7.2 Topological spaces

7.2.1 Definitions and basic examples

Definition 7.9. We say that a pair $(X, \mathcal{T})$ (or simply X) is a **topological space** if X is a set, and if $\mathcal{T}$ (called the **topology** of X) is a set of subsets of X satisfying the following conditions

- $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$.
- (Union property) If $(U_\alpha)_{\alpha\in I}$ is a family of elements of $\mathcal{T}$, then $\bigcup_{\alpha\in I} U_\alpha$ is an element of $\mathcal{T}$.

- (Finite intersection property) If $n \in \mathbb{Z}_+$ and $U_1, \ldots, U_n \in \mathcal{T}$, then $U_1 \cap \cdots \cap U_n$ is an element of $\mathcal{T}$.

Elements of $\mathcal{T}$ are called **open (sub)sets** of X.

Definition 7.10. Let X be a topological space, and $x \in X$. A subset $U \subset X$ is called a **neighborhood** of x, if U is an open subset of X containing x.[1] We define $(\mathrm{Nbh}_X(x), \leqslant)$, the **directed set of neighborhoods of** x, to be

$$\begin{gathered}\mathrm{Nbh}_X(x) = \{\text{neighborhoods of } x \text{ in } X\} \\ U \leqslant U' \qquad \Longleftrightarrow \qquad U \supset U'\end{gathered} \tag{7.1}$$

(Note that one needs the finite intersection property to show that $\mathrm{Nbh}_X(x)$ is a directed set.) We abbreviate this set to $\mathrm{Nbh}_X(x)$ or simply $\mathrm{Nbh}(x)$.

Example 7.11. In Subsec. 7.1, we have proved that the topology of a metric space satisfies the above axioms of a topological space.

In particular, if X is a normed vector space, the topology induced by the metric $d(x, x') = \|x - x'\|$ is called the **norm topology**. If X is a subset of $\mathbb{R}^N$ or $\mathbb{C}^N$, the topology on X induced by the Euclidean metric is called the **Euclidean topology**. □

Definition 7.12. A topological space $(X, \mathcal{T})$ is called **metrizable**, if there is a metric on X inducing the topology $\mathcal{T}$.

We have seen that the open balls of a metric space generate a topology. In general, one may ask what possible subsets of 2^X generate a topology on a set X. Here is a description, whose proof is left to the readers as an exercise.

Proposition 7.13. *Let X be a set, and let $\mathcal{B} \subset 2^X$. Define*

$$\mathcal{T} = \{\textit{Unions of elements of } \mathcal{B}\} \tag{7.2}$$

The following are equivalent.

(1) $(X, \mathcal{T})$ is a topological space.

(2) The following are satisfied:

(2-a) $X = \bigcup_{U \in \mathcal{B}} U$.

(2-b) If $U_1, U_2 \in \mathcal{B}$, then $U_1 \cap U_2 \in \mathcal{T}$ (i.e., for each $x \in U_1 \cap U_2$ there exists $V \in \mathcal{B}$ such that $x \in V$ and $V \subset U_1 \cap U_2$).

[1] We are following the convention in [Mun, Rud-R]. But many people refer to the word "neighborhood" with slightly different meaning: a subset A is called a neighborhood of x if there is an open set U such that $x \in U \subset A$. And our neighborhoods are called "open neighborhoods" by them.

When (1) or (2) holds, we call $\mathcal{T}$ the **topology generated by** $\mathcal{B}$. Clearly, $\mathcal{B}$ is a basis for $\mathcal{T}$ (cf. Def. 7.8).

Exercise 7.14. Let X be a set. Let $\mathcal{B}, \mathcal{B}'$ be subsets of 2^X generating topologies $\mathcal{T}, \mathcal{T}'$ respectively. Prove that the following are equivalent.

(1) $\mathcal{T} = \mathcal{T}'$.

(2) Each $U \in \mathcal{B}$ is a union of elements of $\mathcal{B}'$. Each $U' \in \mathcal{B}'$ is a union of elements of $\mathcal{B}$.

(3) For each $U \in \mathcal{B}$ and $x \in U$, there exists $U' \in \mathcal{B}'$ such that $x \in U' \subset U$. For each $U' \in \mathcal{B}$ and $x \in U'$, there exists $U \in \mathcal{B}$ such that $x \in U \subset U'$.

Example 7.15. If $(X, \mathcal{T})$ is a topological space, then $\mathcal{T}$ is a basis for $\mathcal{T}$.

Example 7.16. Let (X, d) be a metric space with topology $\mathcal{T}$. Let $\mathcal{B} = \{B_X(x, r) : x \in X, 0 < r < +\infty\}$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$. For each $\varepsilon > 0$, the set $\mathcal{B}' = \{B_X(x, r) : x \in X, 0 < r < \varepsilon\}$ is also a basis for $\mathcal{T}$.

Example 7.17. Let $\mathcal{B} \subset 2^{\overline{\mathbb{R}}}$ be defined by

$$\mathcal{B} = \{(a, b), (c, +\infty], [-\infty, d) : a, b, c, d \in \mathbb{R}\} \tag{7.3}$$

Using Prop. 7.13, one easily checks that $\mathcal{B}$ is a basis for a topology $\mathcal{T}$. We call this the **standard topology** of $\overline{\mathbb{R}}$.

Let $\varphi : \overline{\mathbb{R}} \to [u, v]$ be a strictly increasing bijection where $-\infty < u < v < +\infty$. Let $d_{[u,v]}$ be the Euclidean metric, and let $\mathcal{T}'$ be the topology on $\overline{\mathbb{R}}$ defined by $d_{\overline{\mathbb{R}}} = \varphi^* d_{[u,v]}$. Then the set of open balls under $\mathcal{T}'$ is

$$\begin{aligned}\mathcal{B}' = \{&(\varphi^{-1}(y - \varepsilon), \varphi^{-1}(y + \varepsilon)), (\varphi^{-1}(v - \varepsilon'), +\infty], [-\infty, \varphi^{-1}(u + \varepsilon'')) : \\ &y \in (u, v) \text{ and } \varepsilon, \varepsilon', \varepsilon'' \in \mathbb{R}_{>0}\}\end{aligned}$$

(Note that the three types of intervals in the definition of $\mathcal{B}'$ are open balls centered at $\varphi^{-1}(y), +\infty, -\infty$ respectively.) Using Exe. 7.14, one easily checks $\mathcal{T} = \mathcal{T}'$. $\square$

Convention 7.18. Unless otherwise stated, the topology on $\overline{\mathbb{R}}$ is defined to be the standard one, i.e., the one generated by (7.3). We shall forget about the metric on $\overline{\mathbb{R}}$, and view $\overline{\mathbb{R}}$ only as a (metrizable) topological space.

Definition 7.19. Let A be a subset of a topological space $(X, \mathcal{T}_X)$. Then

$$\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}_X\}$$

is clearly a topology on A, called the **subspace topology**. Unless otherwise stated, when viewing a subset as a topological subspace, we always choose the subspace topology for the subset.

Exercise 7.20. Let (X, d_X) be a metric space, inducing a topology $\mathcal{T}_X$. Let A be a metric subspace of X. (So $A \subset X$, and d_X restricts to d_A.) Prove that the topology on A induced by d_A is the subspace topology.

According to the above exercise, if X is a metric space, then viewing a subset A as a topological subspace is compatible with viewing A as a metric subspace.

Exercise 7.21. Let A be a subset of a topological space X. Let $\mathcal{B}$ be a basis for the topology of X. Show that $\{U \cap A : U \in \mathcal{B}\}$ is a basis for the subspace topology of A.

7.2.2 Convergence of nets

Definition 7.22. Let $(x_\alpha)_{\alpha \in I}$ be a net in a topological space X. Let $x \in X$. We say that (x_α) **converges to** x and write

$$\lim_{\alpha \in I} x_\alpha \equiv \lim_\alpha x_\alpha = x$$

or simply write $x_\alpha \to x$, if the following statement holds:

- For every $U \in \mathrm{Nbh}_X(x)$, we have that x_α is eventually in U.

Clearly, if $\mathcal{B}$ is a basis for the topology, then $x_\alpha \to x$ iff:

- For every $U \in \mathcal{B}$ containing x, we have that x_α is eventually in U.

In the case that X is a metric space (and the topology of X is induced by the metric), the definition here agrees with Def. 5.11.

Exercise 7.23. Let (x_α) be a net in X converging to $x \in X$. Prove that every subnet of (x_α) converges to x.

Exercise 7.24. Let A be a subset of a topological space X, equipped with the subspace topology. Let (x_α) be a net in A, and let $x \in A$. Show that $x_\alpha \to x$ in A iff $x_\alpha \to x$ in X.

Example 7.25. Let X be a set. Let $\mathcal{T} = \{\emptyset, X\}$. Then every net in X converges to every point of X. Thus, if X has at least two elements, then the limit of a net in X is not unique. Therefore, a general topological space might be very pathological. To avoid this uniqueness issue, we introduce the following notion:

Definition 7.26. Let X be a topological space with a basis for the topology $\mathcal{B}$. We say that X is a **Hausdorff space** if the following equivalent conditions are satisfied:

(1) (Hausdorff condition) If $x, y \in X$ and $x \neq y$, then there exist neighborhoods U of x and V of y such that $U \cap V = \emptyset$.

(1′) If $x, y \in X$ and $x \neq y$, then there exist $U \in \mathcal{B}$ containing x and $V \in \mathcal{B}$ containing y such that $U \cap V = \emptyset$.

(2) If $(x_\alpha)_{\alpha \in I}$ is a net in X converging to both x and y, then $x = y$.

Proof of the equivalence. (1)⇔(1′): Obvious.

(1)⇒(2): Suppose that (x_α) converges to x and y. Suppose $x \neq y$. By (1), we have disjoint neighborhoods $U \ni x$ and $V \ni y$. Since $x_\alpha \to x$, x_α is eventually in U. Similarly, x_α is eventually in V. Therefore, by the logic (5.6b), x_α is eventually in $U \cap V = \emptyset$, impossible.

¬(1)⇒ ¬(2): Suppose that (1) is not true. Then there exist $x \neq y$ such that every neighborhood of x intersects every neighborhood of y. Let $I = \mathrm{Nbh}_X(x) \times \mathrm{Nbh}_X(y)$. For each $\alpha = (U, V) \in I$, by assumption, there exists $x_\alpha \in U \cap V$. Then $(x_\alpha)_{\alpha \in I}$ is a net in X. We leave it to the readers to check that $x_\alpha \to x$ and $x_\alpha \to y$. □

Remark 7.27. In Hausdorff's 1914 paper introducing topological spaces, the Hausdorff condition is one of the axioms of topological spaces. Non-Hausdorff topological spaces were studied much later. The reason that Hausdorff spaces appeared first may be as follows: The original motivation for topological spaces lies in the study of analysis (especially functional analysis). But in analysis, most spaces are Hausdorff, because we want the limits of sequences or nets to be unique.

In differential geometry and in topology[2], people are also mainly concerned with topological spaces that are Hausdorff. This is related to the fact that in these areas people often use tools from analysis. But in algebraic geometry, the main examples of topological spaces (e.g. varieties and schemes, whose topologies are called **Zariski topology**) are not Hausdorff. As a related fact, sequences and nets are not effective tools in the study of algebraic geometry. □

7.3 Closures, interiors, and closed sets

In this section, we fix a topological space X.

7.3.1 Closure points; dense subsets

Definition 7.28. Let A be a subset of X. We say that $x \in X$ is a **closure point** of A, if the following equivalent conditions hold:

(1) There is a net $(x_\alpha)_{\alpha \in I}$ in A converging to x.

(2) Each $U \in \mathrm{Nbh}_X(x)$ intersects A.

[2]Here, I mean genuine topology, such as algebraic topology, differential topology, geometric topology, etc., but not point-set topology, which is analysis under the guise of topology.

The **closure** of A is defined to be

$$\overline{A} \equiv \mathrm{Cl}(A) \equiv \mathrm{Cl}_X(A) = \{\text{closure points of } A\}$$

Clearly $A \subset \overline{A}$. Clearly, if $A \subset B \subset X$, then $\overline{A} \subset \overline{B}$.

Unless otherwise stated, if several subsets are involved, we always understand $\overline{A}$ as $\mathrm{Cl}_X(A)$ where X is the ambient topological space.

Proof of equivalence. (1)⇒(2): Assume (1). Choose any $U \in \mathrm{Nbh}_X(x)$. Since $x_\alpha \to x$, we have that x_α is eventually in U. So U must contain some x_α. But $x_\alpha \in A$. So $U \cap A \neq \emptyset$.

(2)⇒(1): By (2), for each $U \in \mathrm{Nbh}_X(x)$ we can choose $x_U \in U \cap A$. Then $(x_U)_{U \in \mathrm{Nbh}_X(x)}$ is a net in A converging to x. □

Exercise 7.29. Let $\mathcal{B}$ be a basis for the topology of X. Show that $x \in X$ is a closure point of A iff every $U \in \mathcal{B}$ containing x must intersect A.

Exercise 7.30. Let A be a subset of a metric space. Show that $x \in X$ is a closure point of A iff there is a sequence $(x_n)_{n \in \mathbb{Z}_+}$ in A converging to x.

Exercise 7.31. Recall that if X is a metric space, then $\overline{B}_X(x,r) = \{y \in X : d(x,y) \leqslant r\}$. Show that

$$\overline{B_X(x,r)} \subset \overline{B}_X(x,r) \tag{7.4}$$

and that these two sets are not necessarily equal.

Remark 7.32. Our proof of (2)⇒(1) in Def. 7.28 is an indirect proof, because it uses axiom of choice. (Given $U \in \mathrm{Nbh}_X(x)$, the choice of $x_U \in U \cap A$ is highly arbitrary.) Here is a direct proof: Assume (2). Define a direct set $(I, \leqslant)$ where

$$\begin{gathered} I = \{(p,U) : U \in \mathrm{Nbh}_X(x), p \in U \cap A\} \\ (p,U) \leqslant (p',U') \qquad \Longleftrightarrow \qquad U \supset U' \end{gathered}$$

The fact that I is a directed set is due to (2). Then $(p)_{(p,U)\in I}$ is a net in A converging to x.

We will often prove results about nets in topological spaces using axiom of choice, not only because it is simpler than direct proofs (as above), but also because it is parallel to our use of sequences in metric spaces. (For example, see the proof of (1)⇒(2) in Def. 2.38.) However, it is important to know how to give a direct proof. This is because the studies of topological spaces using nets and using open sets are often equivalent, and direct proofs using nets can be more easily translated into proofs using open sets and vice versa. □

Exercise 7.33. Prove ¬(1)⇒ ¬(2) of Def. 7.26 without using axiom of choice.

Remark 7.34. There is a notion closely related to closure points, called accumulation points. Let A be a subset of X. A point $x \in X$ is called a **accumulation point** (or **limit point** or **cluster point**) of A, if x is a closure point of $A\backslash\{x\}$.

We will not use the notion of accumulation points, although this concept is widely used in many textbooks on analysis or point-set topology. We use closure points instead. (But note that if $x \notin A$, then x is a closure point iff x is an accumulation point.) On the other hand, the following opposite notion of accumulation points is important and has a clear geometric picture: □

Definition 7.35. We say that $x \in X$ is an **isolated point** of X, if the following (clearly) equivalent conditions hold:

(1) $x \notin \overline{X\backslash\{x\}}$.

(2) There is no net in $X\backslash\{x\}$ converging to x.

(3) There is a neighborhood of x disjoint from $X\backslash\{x\}$.

If X is a metric space, then x is an isolated point iff there is no sequence in $X\backslash\{x\}$ converging to x.

We return to the study of closures.

Proposition 7.36. *Let A be a subset of X. Then $\overline{\overline{A}} = \overline{A}$.*

Proof. Choose any $x \in \overline{\overline{A}}$. To prove $x \in \overline{A}$, we choose any $U \in \mathrm{Nbh}_X(x)$, and try to prove $U \cap A \neq \emptyset$. Since x is a closure point of $\overline{A}$, U intersects $\overline{A}$. Pick $y \in U \cap \overline{A}$. Then y is a closure point of A, and $U \in \mathrm{Nbh}_X(y)$. So U intersects A. □

One should think of $\overline{\overline{A}} = \overline{A}$ not only as a "geometric" fact about closures. Instead, one should also understand its analytic content: A closure point of A is a point which can be approximated by elements of A. Thus, $\overline{\overline{A}} = \overline{A}$ says that "approximation is transitive": If x can be approximated by some elements which can be approximated by elements of A, then x can be approximated by elements of A. Alternatively, one can use the language of density:

Definition 7.37. A subset A of X is called **dense** (in X) if $\overline{A} = X$.

Exercise 7.38. Show that A is dense in X iff every nonempty open subset of X intersects A.

Remark 7.39. Let $A \subset B \subset X$. From Def. 7.28-(1), it is clear that

$$\mathrm{Cl}_B(A) = \mathrm{Cl}_X(A) \cap B \tag{7.5}$$

Thus, A is dense in B iff $B \subset \mathrm{Cl}_X(A)$.

Thus, the following property has the same meaning as $\overline{\overline{A}} = A$.

Corollary 7.40. *Let $A \subset B \subset X$. Assume that A is dense in B, and B is dense in X, then A is dense in X.*

Proof. Choose any $x \in X$. Then $x \in \mathrm{Cl}_X(B)$ since B is dense in X. Since A is dense in B, we have $B \subset \mathrm{Cl}_X(A)$. Therefore $x \in \mathrm{Cl}_X(\mathrm{Cl}_X(A))$, and hence $x \in \mathrm{Cl}_X(A)$ by Prop. 7.36. □

Example 7.41. Let $X = C([0,1],\mathbb{R})$, equipped with the l^∞-norm. Let B be the set of polynomials with real coefficients, regarded as continuous functions on $[0,1]$. By Weierstrass approximation theorem (which will be studied in the future), B is a dense subset of X. Then the set A of polynomials with rational coefficients is clearly a dense subset of B under the l^∞-norm. (Proof: Let $f(x) = a_0 + a_1x + \cdots + a_kx^k$. For each $0 \leqslant i \leqslant k$, choose a sequence $(a_{i,n})_{n\in\mathbb{Z}_+}$ in $\mathbb{Q}$ converging to a_i. Let $f_n(x) = a_{0,n} + a_{1,n}x + \cdots + a_{k,n}x^k$. Then $f_n \rightrightarrows f$ on $[0,1]$.) Therefore, A is dense in X. To summarize:

- Since each continuous function on $[0,1]$ can be uniformly approximated by polynomials with $\mathbb{R}$-coefficients, and since each polynomial can be uniformly approximated polynomials with $\mathbb{Q}$-coefficients, therefore each continuous function on $[0,1]$ can be uniformly approximated by polynomials with $\mathbb{Q}$-coefficients.

7.3.2 Interior points

Interior points are dual to closure points:

Definition 7.42. Let A be a subset of X. A point $x \in X$ is called an **interior point** of A if the following equivalent conditions hold:

(1) There exists $U \in \mathrm{Nbh}_X(x)$ such that $U \subset A$.

(2) x is not a closure point of $X\backslash A$.

The set of interior points of A is called the **interior** of A and is denoted by $\mathrm{Int}_X(A)$ or simply $\mathrm{Int}(A)$. So

$$X\backslash\mathrm{Int}(A) = \overline{X\backslash A} \qquad \text{(or simply } \mathrm{Int}(A)^c = \overline{A^c}\text{)} \tag{7.6}$$

according to (2). In particular, $\mathrm{Int}(A) \subset A$.

Proof of equivalence. A contains no neighborhoods of x with respect to X iff A^c intersects every neighborhood of x iff x is a closure point of A^c. □

It is clear that if $\mathcal{B}$ is a basis for the topology, then $x \in \mathrm{Int}(A)$ iff there exists $U \in \mathcal{B}$ such that $x \in U \subset A$.

In analysis, interior points are not as commonly used as closure points. The following property is an important situation where interior points are used:

Proposition 7.43. *Let U be a subset of X. Then U is open iff every point of U is an interior point.*

In other words, U is open iff $U = \mathrm{Int}_X(U)$.

Proof. If U is open and $x \in U$, then $U \in \mathrm{Nbh}_X(x)$. So x is an interior point of U.

Conversely, suppose that each $x \in U$ is interior. Choose $V_x \in \mathrm{Nbh}_X(x)$. Then $U = \bigcup_{x \in U} V_x$. So U is open by the union property in Def. 7.9. □

Note that this is the first time we seriously use the fact that a union of open sets is open.

7.3.3 Closed sets and open sets

Definition 7.44. We say that $A \subset X$ is a **closed (sub)set** of X if $\overline{A} = A$.

Exercise 7.45. Show that the above definition of closed subsets agrees with Def. 3.26 when X is a metric space.

Exercise 7.46. Show that a finite subset of a Hausdorff space is closed. Give an example of non-closed finite subset of a non-Hausdorff topological space.

Remark 7.47. The closure $\overline{A}$ is the smallest closed set containing A. (Proof: By Prop. 7.36, $\overline{A}$ is closed. If B is closed and contains A, then $\overline{A} \subset \overline{B} = B$.)

Theorem 7.48. *Let A be a subset of X. Then A is closed iff $X \backslash A$ is open.*

Proof. Let $B = X \backslash A$. Then A is closed iff every closure point of A is in A, iff every non-interior point of B is not in B, iff every point in B is an interior point of B. By Prop. 7.43, this is equivalent to that B is open. □

Corollary 7.49. *$\varnothing$ and X are closed subsets of X. An intersection of closed subsets is closed. A finite union of closed subsets is closed.*

Proof. Take the complement of Def. 7.9, and apply Thm. 7.48. (Of course, they can also be proved directly using the condition $A = \overline{A}$ for closedness.) □

Corollary 7.50. *X is Hausdorff iff for every distinct $x, y \in X$ there exists $U \in \mathrm{Nbh}_X(x)$ such that $y \notin \overline{U}$.*

Proof. "$\Leftarrow$": Let $x \neq y$. Choose $U \in \mathrm{Nbh}(x)$ such that $y \notin \overline{U}$. Then $X \backslash \overline{U} \in \mathrm{Nbh}(y)$ by Thm. 7.48. So x and y are separated by neighborhoods $U, X \backslash \overline{U}$.

"$\Rightarrow$": Let $x \neq y$. Choose disjoint $U \in \mathrm{Nbh}(x)$ and $V \in \mathrm{Nbh}(y)$. Then $X \backslash V$ is closed by Thm. 7.48. So $\overline{U} \subset X \backslash V$ by Rem. 7.47. So $y \notin \overline{U}$. □

Corollary 7.51. *Let Y be a subset of X, and let $A \subset Y$. Then the following are equivalent.*

(1) A is a closed subset of Y.

(2) $A = B \cap Y$ for some closed subset B of X.

Note that the "open subset" version of this corollary is true due to the definition of the subspace topology of Y (cf. Def. 7.19).

First proof. A is closed in Y iff $Y \backslash A = Y \cap A^c$ is open in Y, iff $Y \cap A^c$ equals $Y \cap U$ for some open subset $U \subset X$, iff $Y \cap A$ (which is A) equals $Y \cap U^c$ for some open subset $U \subset X$. This finishes the proof, thanks to Thm. 7.48. □

Second proof. Recall by (7.5) that $\overline{A} \cap Y$ is the closure of A in Y. Then A is closed in Y iff $A = \overline{A} \cap Y$. This proves (1)⇒(2) since $\overline{A}$ is closed by Prop. 7.36. Assume (2). Then $A = B \cap Y$ where $B = \overline{B}$. So $\overline{A} \cap Y = \overline{B \cap Y} \cap Y \subset \overline{B} \cap Y = B \cap Y = A$. This proves (1). □

As an immediate consequence of Def. 7.9 and Cor. 7.51, we have:

Exercise 7.52. Let $A \subset B \subset X$.

1. Prove that if B is open in X, then A is open in B iff A is open in X.

2. Prove that if B is closed in X, then A is closed in B iff A is closed in X.

Remark 7.53. Many people define a closed set to be the complement of an open set, and then proves that a set A is closed iff $A = \overline{A}$. I went the other way because I believe that $A = \overline{A}$ is more essential for understanding of closedness from the viewpoint of analysis. In Thm. 3.48, we have already seen a classical example of closed set in analysis: $C([0,1],\mathbb{R})$ is a closed subset of $l^\infty([0,1],\mathbb{R})$, which has the clear analytic meaning that the uniform limit of a sequence/net of continuous functions $[0,1] \to \mathbb{R}$ is continuous. And we will see many more examples of this type in the future.

Remark 7.54. I defined closedness using $A = \overline{A}$, and hence using the limits of nets. This is because the intuition of closed sets is very closely related to the intuition of limits of nets/sequences. On the other hand, the intuition of open sets is very different. Let me say a few words about this.

Without a doubt, the keyword I give for the intuition of limits of nets is "approximation": Limit is not only a dynamic process, but also gives an impression of "getting smaller and smaller". When dealing with closed sets, we often do the same thing! We take an intersection of possibly infinitely many closed subsets, and the result we get is still a closed set (cf. Cor. 2.4).

The keyword I give for open sets is "local", or more precisely, "local-to-global" (as opposed to "getting smaller and smaller"!). This is not only because a union of open sets is open, but also because open sets are very often used to prove a global result by reducing to local problems. One easy example is Exe. 7.119, which says that in order to prove that a function is continuous on the whole space X, it suffices to prove this locally. (We have already used this strategy in Sec. 2.4.) Here is a more advanced example: to define the integral for a function on a large set, one can first define it locally (i.e. on small enough open subsets), and then patch these local values together. We will see many examples in the future, for example, in the following chapter about compactness.

Remark 7.55. Very often, a theorem is an important result establishing two seemingly different (systems of) intuitions, and hence two different ways of mathematical thinking. This is why I call "closed sets are the complements of open sets" a theorem. The term "complement" implies that this theorem often manifests itself in the following way: If solving a problem using open sets is a direct proof, then solving the problem using limits of sequences/nets is a proof by contradiction/contrapositive. And vise versa.

7.4 Continuous maps and homeomorphisms

Unless otherwise stated, X and Y are topological spaces.

7.4.1 Continuous maps

Definition 7.56. Let $f: X \to Y$ be a map. Let $x \in X$. We say that f is **continuous at** x if the following equivalent conditions hold:

(1) For every net $(x_\alpha)_{\alpha\in I}$ in X converging to x, we have $\lim_{\alpha\in I} f(x_\alpha) = f(x)$.

(2) For every $V \in \mathrm{Nbh}_Y(f(x))$, there exists $U \in \mathrm{Nbh}_X(x)$ such that for every $p \in U$ we have $f(p) \in V$.

(2') For every $V \in \mathrm{Nbh}_Y(f(x))$, the point x is an interior point of $f^{-1}(V)$.

We say that f is a **continuous** function/map, if f is continuous at every point of X.

It is clear that "for every $V \in \mathrm{Nbh}_Y(f(x))$" in (2) and (2') can be replaced by "for every $V \in \mathcal{B}$ containing $f(x)$" if $\mathcal{B}$ is a basis for the topology of Y.

Note that in the case that (x_α) or $(f(x_\alpha))$ has more than one limits (which could happen when X or Y is not Hausdorff), condition (1) means that $f(x)$ is one of the limits of $(f(x_\alpha))_{\alpha\in I}$ if x is one of the limits of (x_α).

Proof of equivalence. Clearly (2) is equivalent to (2'). The proof of (2)⇒(1) is similar to the case of sequences in metric spaces. (See Def. 2.38.) We leave the details to the reader.

$\neg$(2)$\Rightarrow$ $\neg$(1): Assume that (2) is not true. Then there is a neighborhood V of $f(x)$ such that for every neighborhood U of x there exists $x_U \in U$ such that $f(x_U) \notin V$. Then $(x_U)_{U \in \mathrm{Nbh}_X(x)}$ is a net in X, and $\lim_U x_U = x$ since $x_U \in U$. However, for each U we have $f(x_U) \in Y \backslash V$. So $\lim_U f(x_U)$ cannot converge to $f(x)$. □

Exercise 7.57. Show that when X, Y are metric spaces, Def. 7.56 agrees with Def. 2.38.

Exercise 7.58. Let $f : X \to Y$ and $g : Y \to Z$ be maps of topological spaces. Assume that f is continuous at $x \in X$, and g is continuous at $f(x)$. Prove that $g \circ f : X \to Z$ is continuous at x.

The proof of (1)$\Rightarrow$(2) in Def. 7.56 is indirect, since it uses the axiom of choice. (The merit of this proof is that it is parallel to the proof for metric spaces in Sec. 2.4.) One can also give a direct proof. Indeed, there is a particular net (x_α) converging to x such that $\lim f(x_\alpha) = f(x)$ iff (2) is true:

Exercise 7.59. Define $(\mathrm{PNbh}_X(x), \leqslant)$, the **directed set of pointed neighborhoods** of x, to be

$$\begin{gathered}\mathrm{PNbh}_X(x) = \big\{(p, U) : U \in \mathrm{Nbh}_X(x), p \in U\big\} \\ (p, U) \leqslant (p', U') \qquad \Longleftrightarrow \qquad U \supset U'\end{gathered} \tag{7.7}$$

For each $\alpha = (p, U) \in \mathrm{PNbh}_X(x)$, let $x_\alpha = p$. Then $(x_\alpha)_{\alpha \in \mathrm{PNbh}_X(x)}$ is a net in X converging to x. Prove that f is continuous at x iff $\lim_\alpha f(x_\alpha) = f(x)$.

Proposition 7.60. *Let $f : X \to Y$ be a map. The following are equivalent:*

(1) f is continuous.

(2) If $V \subset Y$ is open in Y, then $f^{-1}(V)$ is open in X.

(3) If $F \subset Y$ is closed in Y, then $f^{-1}(F)$ is closed in Y.

Proof. (1)$\Leftrightarrow$(2): By Def. 7.56-(2′) and Prop. 7.43. (2)$\Leftrightarrow$(3): By Thm. 7.48 and the fact that $f^{-1}(B^c) = f^{-1}(B)^c$ for every $B \subset Y$. □

Remark 7.61. We first defined the continuity of f at a point, and then used this to define a continuous function f to be one continuous at every point. However, it seems that the notion of continuity at a point is used only in analysis. In geometry and in topology, only continuous maps (but not a map continuous at a point) are used, and they are defined by Prop. 7.60-(2).

One might think that continuous functions are special cases of functions which are continuous at given points. But in fact, the latter notion can also be derived from the former: □

Exercise 7.62. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a map of topological spaces. Let $x \in X$. Define a new topological space $(X_x, \mathcal{T}_x)$ as follows. X_x equals X as a set. The topology $\mathcal{T}_x$ of X_x is generated by the basis

$$\mathcal{B}_x = \mathrm{Nbh}_X(x) \cup \{\{p\} : p \neq x\} \tag{7.8}$$

Prove that if X is Hausdorff, then X_x is Hausdorff. Prove that the following are equivalent:

(1) $f : X \to Y$ is continuous at x.

(2) $f : X_x \to Y$ is continuous.

Continuous functions are determined by their values on a dense subset:

Proposition 7.63. *Let A be a subset of X, and let $f, g : \overline{A} \to Y$ be continuous. Then*

$$f(\overline{A}) \subset \overline{f(A)} \tag{7.9a}$$

If Y is moreover Hausdorff, then

$$f = g \qquad \Longleftrightarrow \qquad f|_A = g|_A \tag{7.9b}$$

Proof. If $y \in f(\overline{A})$, then $y = f(x)$ for some $x \in \overline{A}$. Choose a net (x_α) in A converging to x. Then $\lim_\alpha f(x_\alpha) = f(x)$, and hence $f(x) \in \overline{f(A)}$.

Assume that Y is Hausdorff. If $f = g$ then clearly $f|_A = g|_A$. Assume that $f|_A = g|_A$. For each $x \in \overline{A}$, choose a net (x_α) in A converging to x. Then $f(x) = \lim f(x_\alpha) = \lim g(x_\alpha) = g(x)$. So $f = g$. $\square$

You are encouraged to prove Prop. 7.63 using open sets instead of using nets.

7.4.2 Homeomorphisms

Definition 7.64. A map $f : X \to Y$ is called **open** (resp. **closed**) if for every open (resp. closed) subset $A \subset X$, the image $f(A)$ is open (resp. closed) in Y.

Definition 7.65. A bijection $f : X \to Y$ is called a **homeomorphism** if the following clearly equivalent conditions hold:

(1) f and f^{-1} are continuous.

(2) For every net (x_α) in X and each $x \in X$, we have that $\lim_\alpha x_\alpha = x$ iff $\lim_\alpha f(x_\alpha) = f(x)$.

(3) f is continuous and open.

(4) f is continuous and closed.

If a homeomorphism $f : X \to Y$ exists, we say that X, Y are **homeomorphic**.

Recall from Def. 2.60 that when X, Y are metric spaces, the sequential version of (2) holds.

Remark 7.66. Let $\mathcal{T}_1, \mathcal{T}_2$ be two topologies on a set X. Clearly, we have $\mathcal{T}_1 = \mathcal{T}_2$ iff

$$\varphi : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2) \qquad x \mapsto x \tag{7.10}$$

is a homeomorphism. Thus, the following are equivalent:

(1) $\mathcal{T}_1 = \mathcal{T}_2$.

(2) For every net (x_α) in X and $x \in X$, we have that $\lim_\alpha x_\alpha = x$ under $\mathcal{T}_1$ iff $\lim_\alpha x_\alpha = x$ under $\mathcal{T}_2$.

This equivalence implies that

$$\boxed{\text{topologies are determined by net convergence}}$$

Therefore, instead of using open sets or bases of topologies to describe a topology, one can also describe a topology $\mathcal{T}$ on a set X in the following way:

$$\begin{gathered}\mathcal{T} \text{ is the unique topology on } X \text{ such that} \\ \text{a net } (x_\alpha) \text{ in } X \text{ converges to } x \in X \text{ iff ...}\end{gathered} \tag{7.11}$$

Similarly, by Def. 2.60, metrizable topologies are determined by sequential convergence. Therefore, metrizable topologies can be described in the following way:

$$\begin{gathered}\mathcal{T} \text{ is the unique metrizable topology on } X \text{ such that} \\ \text{a sequence } (x_n) \text{ in } X \text{ converges to } x \in X \text{ iff ...}\end{gathered} \tag{7.12}$$

7.5 Examples of topological spaces described by net convergence

Example 7.67. Let A be a subset of a topological space $(X, \mathcal{T}_X)$. Then the subspace topology $\mathcal{T}_A$ of A is the unique topology such that a net (x_α) in A converges to $x \in A$ under $\mathcal{T}_A$ iff it converges to x under $\mathcal{T}_X$.

$\star$ **Example 7.68.** Let $X = \bigsqcup_{\alpha \in \mathscr{A}} X_\alpha$ be a disjoint union where each $(X_\alpha, \mathcal{T}_\alpha)$ is a topology space. Then

$$\mathcal{B} = \bigcup_{\alpha \in \mathscr{A}} \mathcal{T}_\alpha$$

is clearly a basis generating a topology $\mathcal{T}$ on X, called **disjoint union topology**. $\mathcal{T}$ is the unique topology on X such that for every net $(x_\mu)_{\mu \in I}$ in X and any $x \in X$, the following are equivalent:

(1) $(x_\mu)_{\mu\in I}$ converges to x under $\mathcal{T}$.

(2) There exists $\nu \in I$ such that for every $\mu \geqslant \nu$, the element x_μ belongs to the unique X_α containing x. Moreover, $\lim_{\mu\in I_{\geqslant\nu}} x_\mu = x$ in X_α.

We call $(X, \mathcal{T})$ the **disjoint union topological space** of $(X_\alpha)_{\alpha\in\mathscr{A}}$.

The following exercise says that "disjoint union of topological spaces" is synonymous with "disjoint union of open subsets".

⋆ **Exercise 7.69.** Assume that $X = \bigsqcup_{\alpha\in\mathscr{A}} X_\alpha$. Assume that X has a topology $\mathcal{T}$, and equip each X_α with the subspace topology. Show that $(X, \mathcal{T})$ is the disjoint union topological space of $(X_\alpha)_{\alpha\in\mathscr{A}}$ iff each X_α is an open subset of $(X, \mathcal{T})$.

Thus, for example, $\bigcup_{n\in\mathbb{N}}[2n, 2n+1)$ (under the Euclidean topology) is the disjoint union topological space of the family $\big([2n, 2n+1)\big)_{n\in\mathbb{N}}$.

⋆ **Exercise 7.70.** In Exp. 7.68, assume that each $(X_\alpha, \mathcal{T}_\alpha)$ is metrizable. Prove that $(X, \mathcal{T})$ is metrizable. More precisely: Choose a metric d_α inducing $\mathcal{T}_\alpha$, and assume that $d_\alpha \leqslant 1$ (cf. Prop. 2.74). Define a metric d on X as in Pb. 2.4. Solve *the net version* of part 2 of Pb. 2.4. Conclude from this that d induces the topology $\mathcal{T}$. (Warning: we cannot conclude this from the original sequential version of Pb. 2.4-2.)

Definition 7.71. Let $(X_\alpha)_{\alpha\in\mathcal{A}}$ be a family of topological spaces. Elements of the product space

$$S = \prod_{\alpha\in\mathscr{A}} X_\alpha$$

are denoted by $x = (x(\alpha))_{\alpha\in\mathscr{A}}$. One checks easily that

$$\begin{aligned}\mathcal{B} = \Big\{ \prod_{\alpha\in\mathscr{A}} U_\alpha : \text{each } U_\alpha \text{ is open in } X_\alpha,& \\ U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \Big\}&\end{aligned} \tag{7.13}$$

is a basis for a topology $\mathcal{T}$, called the **product topology** or **pointwise convergence topology** of S. We call $(S, \mathcal{T})$ the **product topological space**.

Equivalently, let

$$\pi_\alpha : S \to X_\alpha \qquad x \mapsto x(\alpha) \tag{7.14}$$

be the **projection map onto the X_α component**. Then

$$\mathcal{B} = \Big\{ \bigcap_{\alpha\in E} \pi_\alpha^{-1}(U_\alpha) : E \in \mathrm{fin}(2^{\mathscr{A}}),\ U_\alpha \text{ is open in } X_\alpha \text{ for each } \alpha \in E \Big\} \tag{7.15}$$

Unless otherwise stated, a product of topological spaces is equipped with the product topology. □

Example 7.72. Let $S = X_1 \times \cdots \times X_N$ be a finite product of topological spaces. Then the product topology has a basis

$$\mathcal{B} = \{U_1 \times \cdots \times U_N : \text{ each } U_i \text{ is open in } X_i\} \tag{7.16}$$

Theorem 7.73. *Let $S = \prod_{\alpha \in \mathscr{A}} X_\alpha$ be a product of topological spaces, equipped with the product topology. Then each projection map π_α is continuous. Moreover, for every net $(x_\mu)_{\mu \in I}$ in S and every $x \in S$, the following are equivalent.*

(1) $\lim_{\mu \in I} x_\mu = x$ *in* S.

(2) For every $\alpha \in \mathscr{A}$, *we have* $\lim_{\mu \in I} x_\mu(\alpha) = x(\alpha)$ *in* X_α.

If (x_μ) satisfies (1) or (2), we say that x_μ **converges pointwise** to x if we view (x_μ) as a net of functions with domain $\mathscr{A}$.

Proof. We leave the proof to the readers. Note that the continuity of π_α follows easily from the basis-for-topology version of Prop. 7.60-(2). And from the continuity of π_α one easily deduce (1)⇒(2). □

Remark 7.74. In the spirit of (7.11), one says that:

- The product topology $\mathcal{T}$ on $S = \prod_{\alpha \in \mathscr{A}} X_\alpha$ is the unique topology such that a net (x_μ) converges to x under $\mathcal{T}$ iff x_μ converges pointwise to x as a net of functions on $\mathscr{A}$.

Corollary 7.75. *If each X_α is a Hausdorff space, then $S = \prod_{\alpha \in \mathscr{A}} X_\alpha$ is Hausdorff.*

Proof. Either prove this directly using the basis for the topology, or prove that any net cannot converge to two different values using Thm. 7.73. □

Corollary 7.76. *Let $X_1, X_2, \dots$ be a possibly finite sequence of metric spaces. Then $S = \prod_i X_i$ is metrizable. More precisely, for each i, choose a metric d_i on X_i topologically equivalent to the original one such that $d_i \leqslant 1$ (cf. Prop. 2.4). Then the metric d on S defined by*

$$d(f,g) = \sup_i \frac{d_i(f(i),g(i))}{i} \tag{7.17}$$

induce the product topology.

We note that the product topology is also induced by

$$\delta(f,g) = \sum_i 2^{-i} d_i(f(i),g(i)) \tag{7.18}$$

Proof. The same method for solving Pb. 2.3 also applies to its net version: One shows that a net (f_α) in S converges to f under d (or under δ) iff (f_α) converges pointwise to f. Thus, by Thm. 7.73, d and δ induce the product topology. □

The next example discusses the topologies induced by uniform convergence metrics. (Recall Def. 3.58.) In this example, Y is usually a normed vector space.

Example 7.77. Let X be a set, and let (Y, d_Y) be a metric space. Then there is a unique topology $\mathcal{T}$ on Y^X such that for every net $(f_\alpha)_{\alpha\in I}$ in Y^X and every $f \in Y^X$, the following are equivalent:

(1) The net (f_α) converges to f under $\mathcal{T}$.

(2) We have $\lim\limits_{\alpha\in I} \sup\limits_{x\in X} d_Y(f_\alpha(x), f(x)) = 0$.

(If (f_α) satisfies (2), we say that f_α **converges uniformly** to f.) For example, one checks easily that $\mathcal{T}$ is induced by any **uniform convergence metric**, i.e., any metric on Y^X equivalent to d where

$$d(f,g) = \min\Big\{1, \sup_{x\in X} d_Y(f(x), g(x))\Big\} \tag{7.19}$$

So $\mathcal{T}$ is metrizable. We call $\mathcal{T}$ the **uniform convergence topology** on Y^X.

Theorem 7.78. *Let Y be a complete metric space. Let X be a set. Then Y^X, equipped with the metric (7.19), is complete.*

Proof. It can be proved in a similar way as Thm. 3.45. We leave the details to the readers. □

Theorem 7.79. *Let V be a normed vector space over $\mathbb{R}$ or $\mathbb{C}$. Let X be a topological space. Equip V^X with the uniform convergence topology. Then $C(X,V)$ is a closed subset of V^X.*

Proof. This is similar to the proof of Thm. 3.48. Let (f_α) be a net $C(X,V)$ converging uniformly to $f : X \to V$. Choose any $x \in X$ and $\varepsilon > 0$. Then there is $\alpha \in I$ such that $\sup_{p\in X} \|f(x) - f_\alpha(x)\| < \varepsilon$. Since f_α is continuous, there is $U \in \mathrm{Nbh}_X(x)$ such that for each $p \in U$ we have $\|f_\alpha(x) - f_\alpha(p)\| < \varepsilon$. Thus, for each $p \in U$ we have

$$\|f(x) - f(p)\| \leqslant \|f(x) - f_\alpha(x)\| + \|f_\alpha(x) - f_\alpha(p)\| + \|f_\alpha(p) - f(p)\| < 3\varepsilon$$

So f is continuous. □

Remark 7.80. Note that the uniform convergence topology depends on the equivalence class (not just the topological equivalence class) of d_Y. Thus, one needs metrics when talking about uniform convergence. On the other hand, the study of pointwise convergence does not require metrics.

7.6 Limits of functions

By Prop. 7.63, if $A \subset X$, and if $f : \overline{A} \to Y$ is continuous, then the value of f is uniquely determined by $f|_A$ provided that Y is Hausdorff (cf. Prop. 7.63). We now consider the opposite question of extension of continuous functions: Suppose that $f : A \to Y$ is continuous. Can we extend f to a continuous function $f : \overline{A} \to Y$? (We know that such extension must be unique if it exists.) The classical concept of the limits of functions can be understood in this light.

Definition 7.81. Let A be a subset of X. Let $f : A \to Y$ be a map. Let $x \in \overline{A} \backslash A$. Let $y \in Y$. We say that the **limit of the function** f at x is y and write

$$\lim_{\substack{p \in A \\ p \to x}} f(p) \equiv \lim_{p \to x} f(p) = y$$

if the following equivalent conditions hold:

(1) If we extend f to a function $A \cup \{x\} \to Y$ satisfying $f(x) = y$, then $f : A \cup \{x\} \to Y$ is continuous at x.

(2) For every $V \in \mathrm{Nbh}_Y(y)$, there exists $U \in \mathrm{Nbh}_X(x)$ such that for every $p \in U \cap A$, we have $f(p) \in V$.

(3) For every net $(x_\alpha)_{\alpha \in I}$ in A converging to x, we have $\lim_{\alpha \in I} f(x_\alpha) = y$.

When X, Y are metric spaces, the above three conditions and the following two are equivalent:

(2m) For every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $p \in A$, if $d(p, x) < \delta$ then $d(f(p), y) < \varepsilon$.

(3m) For every sequence $(x_n)_{n \in \mathbb{Z}_+}$ in A converging to x, we have $\lim_{n \to \infty} f(x_n) = y$.

Recall that by the definition of subspace topology, we have

$$\mathrm{Nbh}_A(x) = \{U \cap A : A \in \mathrm{Nbh}_X(x)\} \tag{7.20}$$

Proof of equivalence. Extend f to $\widetilde{f} : A \cup \{x\} \to Y$ by setting $\widetilde{f}(x) = y$. Then by Def. 7.56-(2), condition (1) of Def. 7.81 means that for every $V \in \mathrm{Nbh}_Y(y)$ there is a neighborhood of x in $A \cup \{x\}$ (which, by (7.20), must be of the form $U \cap (A \cup \{x\})$ where $U \in \mathrm{Nbh}_X(x)$) such that for every $p \in U \cap (A \cup \{x\})$ we have $\widetilde{f}(p) \in V$. This is clearly equivalent to (2), since $\widetilde{f}(x) = y \in V$. The equivalence (2)⇔(3) can be proved in a similar way as the equivalence of (1) and (2) in Def. 7.56. We leave the details to the readers. When X, Y are metric spaces, (2) is clearly equivalent to (2m). The equivalence (2m)⇔(3m) can be proved in a similar way as the equivalence of (1) and (2) in Def. 2.38. □

The following remarks show that the limit of a function at a point is the limit of a single net, rather than the limit of many nets (as in Def. 7.81-(3)).

Remark 7.82. Assume the setting of Def. 7.81. In the same spirit of Exe. 7.59, define a directed set $(\mathrm{PNbh}_A(x), \leqslant)$ where

$$\begin{aligned} \mathrm{PNbh}_A(x) &= \big\{(p,U) : U \in \mathrm{Nbh}_X(x), p \in U \cap A\big\} \\ (p,U) \leqslant (p',U') \qquad &\Longleftrightarrow \qquad U \supset U' \end{aligned} \tag{7.21}$$

(That it is a directed set is due to $x \in \overline{A}$.) We have seen this directed set in Rem. 7.32. Then $(p)_{(p,U)\in\mathrm{PNbh}_A(x)}$ is a net converging to x, and

$$\lim_{p\to x} f(p) = \lim_{(p,U)\in\mathrm{PNbh}_A(x)} f(p) \tag{7.22}$$

where the convergence of the LHS is equivalent to that of the RHS.

Remark 7.83. In the setting of Def. 7.81, assume moreover that X is a metric space, then $\lim_{p\to x} f(p)$ can be described by the limit of a simpler net. Define a directed set $(A_x, \leqslant)$ where

$$\begin{aligned} A_x &= A \text{ as sets} \\ p \leqslant p' \qquad &\Longleftrightarrow \qquad d(p',x) \geqslant d(p,x) \end{aligned} \tag{7.23}$$

Then $(p)_{p\in A_x}$ is a net in A converging to x, and

$$\lim_{p\to x} f(p) = \lim_{p\in A_x} f(p) \tag{7.24}$$

where the convergence of the LHS is equivalent to that of the RHS.

Remark 7.84. Thanks to the above two remarks, limits of functions enjoy all the properties that limits of nets enjoy. For example, they satisfy Squeeze theorem; if $f : A \to \mathbb{F}$ and $g : A \to V$ (where V is a normed vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$), if $x \in \overline{A}\backslash A$, and if $\lambda = \lim_{p\to x} f(p)$ and $v = \lim_{p\to x} g(p)$ exist, then $\lim_{p\to x} f(p)g(p)$ converges to λv.

Of course, you can also conclude them by using Def. 7.81-(3) instead of Rem. 7.82. In practice, it makes no difference whether you view $\lim_{p\to x} f(p)$ as the limit of $f(x_\alpha)$ for an arbitrary net x_α in A converging to x, or whether you view $\lim_{p\to x} f(p)$ as the limit of the particular net in Rem. 7.82 or Rem. 7.83. The explicit constructions of nets in these two remarks are not important for proving results about limits of functions. □

Remark 7.85. Let $f : A \to Y$, and let $x \in \overline{A}\backslash A$. Suppose that Y is Hausdorff. Assume that there exist two nets $(x_\alpha)_{\alpha\in I}$ and $(y_\beta)_{\beta\in J}$ in A converging to x such that $(f(x_\alpha))$ and $(f(y_\beta))$ converge to two different values. Then by Def. 7.81-(2), the limit $\lim_{p\to x} f(p)$ does not exist.

The above remark gives a useful criterion for the non-convergence of limits of functions. The following proposition, on the other hand, gives a method of computing limits of functions by decomposing the domain into (non-necessarily mutually disjoint) subsets.

Proposition 7.86. *Assume the setting of Def. 7.81. Assume that* $A = A_1 \cup \cdots \cup A_N$. *Then the following are equivalent.*

(1) We have $\lim\limits_{p\to x} f(p) = y$.

(2) For every $1 \leqslant i \leqslant N$ *such that* $x \in \overline{A_i}$, *we have* $\lim\limits_{p\to x} f|_{A_i}(p) = y$

Proof. (1)⇒(2): Assume (1). Extend $f : A \to Y$ to a function $\widetilde{f} : A \cup \{x\} \to Y$ by setting $\widetilde{f}(x) = y$. Then $\widetilde{f}$ is continuous by (1). Thus, if $x \in \overline{A_i}$, then $\widetilde{f}|_{A_i\cup\{x\}}$ is continuous. This proves (2).

(2)⇒(1): Assume (2). Choose any $V \in \mathrm{Nbh}_Y(y)$. By (2), for each i, either $x \in \overline{A_i}$ so that there is $U_i \in \mathrm{Nbh}_X(x)$ satisfying $U_i \cap A_i \subset f^{-1}(V)$ (recall (7.20)), or that $x \notin \overline{A_i}$ so that there is $U_i \in \mathrm{Nbh}_X(x)$ disjoint from A_i. In either case, we have $U_i \cap A_i \subset f^{-1}(V)$. Let $U = U_1 \cap \cdots \cap U_N$. Then

$$U \cap A = U \cap (A_1 \cup \cdots \cup A_N) = \bigcup_i U \cap A_i \subset \bigcup_i U_i \cap A_i$$

which is therefore a subset of $f^{-1}(V)$.

Another proof of $\neg$(1)⇒ $\neg$(2): Assume (1) is not true. Then there is a net $(x_\alpha)_{\alpha\in I}$ in A converging to x such that $f(x_\alpha)$ does not converge to y. So there exists $V \in \mathrm{Nbh}_Y(y)$ such that $f(x_\alpha)$ is not eventually in V, i.e., $f(x_\alpha)$ is frequently in V^c. Then $(f(x_\alpha))$ has a subnet $(f(x_\beta))_{\beta\in J}$ which is always in V^c. For example, take

$$J = \{\beta \in I : f(x_\beta) \in V^c\}$$

Since $(x_\beta)_{\beta\in J}$ is always in A, by the logic (5.6c), there is $1 \leqslant i \leqslant N$ such that (x_β) is frequently in A_i. Thus, by the same argument as above, (x_β) has a subnet $(x_\gamma)_{\gamma\in K}$ which is always in A_i. Since $x_\alpha \to x$, we have $x_\gamma \to x$, and hence $x \in \overline{A_i}$. But $f(x_\gamma) \in V^c$. So we have found a net (x_γ) in A_i converging to x such that $(f(x_\gamma))$ does not converge to y. This disproves (2). □

Remark 7.87. In many textbooks, $\lim_{p\to x} f(x)$ is also defined more generally when x is an accumulation point of A, i.e., when $x \in \overline{A\backslash\{x\}}$. In this case, the limit of f at x simply means

$$\lim_{p\to x} f(p) \xlongequal{\text{def}} \lim_{p\to x} f|_{A\backslash\{x\}}(p) \tag{7.25}$$

This more general case is important in classical analysis, but is less useful in abstract analysis. (As a matter of fact, accumulation points are less convenient than closure points.) In order not to deviate too far from the traditional analysis textbooks, let's take a look at some examples.

Definition 7.88. Let $A \subset \mathbb{R}$ and $x \in \mathbb{R}$. Let $f : A \to Y$ be a function. If x is a closure point of $A \cap \mathbb{R}_{<x}$ resp. $A \cap \mathbb{R}_{>x}$, we define the **left limit** resp. **right limit** to be

$$\lim_{t \to x^-} f(t) = \lim_{t \to x} f|_{A \cap \mathbb{R}_{<x}}(t) \tag{7.26a}$$

$$\lim_{t \to x^+} f(t) = \lim_{t \to x} f|_{A \cap \mathbb{R}_{>x}}(t) \tag{7.26b}$$

If $x \in \overline{\mathbb{R}}$ and x is a closure point of $A \backslash \{x\}$, then $\lim_{t \to x} f(t)$ is understood by (7.25).

Remark 7.89. In Def. 7.88, if $x \in \mathbb{R}$ is a closure point of $A \backslash \{x\}$, then by Prop. 7.86,

$$\lim_{p \to x} f(p) = y \qquad \Longleftrightarrow \qquad \lim_{p \to x^-} f(p) = \lim_{p \to x^+} f(p) = y \tag{7.27}$$

In particular, the existence of the limit on the LHS is equivalent to the existence and the equality of the two limits on the RHS.

Example 7.90. Let $g, h : \mathbb{R} \to \mathbb{R}$ be continuous functions. Let $c \in \mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ to be

$$f(x) = \begin{cases} g(x) & \text{if } x < 0 \\ c & \text{if } x = 0 \\ h(x) & \text{if } x > 0 \end{cases}$$

Since $g|_{(-\infty,0]}$ is continuous, by Def. 7.81-(1) we have that $\lim_{x \to 0^-} f(x) = \lim_{x \to 0, t<0} g(t) = g(0)$. Similarly, $\lim_{x \to 0^+} f(x) = h(0)$. Therefore, by Rem. 7.89, $\lim_{x \to 0} f(x)$ exists iff $g(0) = h(0)$, and it converges to $g(0)$ if $g(0) = h(0)$. The value c is irrelevant to the limits.

Example 7.91. Let $f : X = \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ be $f(x,y) = \frac{x}{x+y}$. Then $(1/n, 0)$ and $(0, 1/n)$ are sequences in X converging to 0. But $f(1/n, 0) = 1$ and $f(0, 1/n) = 0$. So $\lim_{(x,y) \to (0,0)} f(x,y)$ does not exist by Rem. 7.85.

7.7 Connected spaces

Let X be a topological space. In this section, we shall define a notion of connected space. Based on our usual geometric intuition, one might attempt to define a connected space as one satisfying that any two points can be linked by a path. Such spaces are actually called **path-connected spaces** and is stronger than connected spaces. In fact, connected spaces arise from the study of intermediate value problem.

7.7.1 Connected $\Leftrightarrow$ IVP

Definition 7.92. We say that the topological space X is **connected** if X can not be written as the disjoint of two nonempty open sets. Namely, if $X = U \sqcup V$ where U, V are open subsets of X, then either $U = \emptyset$ (and hence $V = X$) or $V = \emptyset$ (and hence $U = X$).

Equivalently (by Thm. 7.48), X is connected iff every $U \subset X$ which is both closed and open must be $\emptyset$ or X.

Definition 7.93. We say that X satisfies the **intermediate value property** (abbreviated to **IVP**) if for every continuous function $f : X \to \mathbb{R}$ and every $x, y \in X$ we have

$$f(x) < f(y) \qquad \Longrightarrow \qquad [f(x), f(y)] \subset f(X) \tag{7.28}$$

Theorem 7.94. *X is connected iff X satisfies IVP. Moreover, if X is not connected, then there is a continuous $f : X \to \mathbb{R}$ such that $f(X) = \{0, 1\}$.*

Proof. First, assume that X does not satisfy IVP. Choose a continuous $f : X \to \mathbb{R}$ with real numbers $a < b < c$ such that $a, c \in f(X)$ but $b \notin f(X)$. So $U = f^{-1}(-\infty, b)$ and $V = f^{-1}(b, +\infty)$ are disjoint non-empty open subsets of X, and $X = U \sqcup V$. They are open, because f is continuous (cf. Prop. 7.60). So X is not connected.

Next, assume that X is not connected. Then $X = U \sqcup V$ where U, V are open subsets of X. Define $f : X \to \mathbb{R}$ to be constantly 0 on U and constantly 1 on V. It is easy to check that f is continuous. (See also Rem. 7.120.) That $f(X) = \{0, 1\}$ means that X does not satisfy IVP. $\square$

We now give a couple of elementary examples.

7.7.2 Connected subsets of $\overline{\mathbb{R}}$ are precisely intervals

Proposition 7.95. *Let A be a dense subset of X. Assume that A is connected. Then X is connected.*

Proof. If $X = \overline{A}$ is not connected, then by Thm. 7.94, there exists a continuous surjection $f : \overline{A} \to \{0, 1\}$. By Prop. 7.63, $\overline{f(A)}$ contains $f(\overline{A})$. So $f(A)$ has closure $\{0, 1\}$. So $f(A) = \{0, 1\}$. A does not satisfy IVP, and hence is not connected. $\square$

Theorem 7.96. *Let A be a nonempty subset of $\overline{\mathbb{R}}$. Then A is connected iff A is an interval.*

Proof. Step 1. Suppose that A is connected. Let $a = \inf A$ and $b = \sup B$. To show that A is one of (a, b), $(a, b]$, $[a, b)$, $[a, b]$, it suffices to show that every $c \in (a, b)$ belongs to A. Suppose that some $c \in (a, b)$ does not belong to A. Then A is the disjoint union of two nonempty open subsets $A \cap [-\infty, c)$ and $A \cap (c, +\infty]$, impossible.

Step 2. Every single point is clearly connected. Since every interval containing at least two points is homeomorphic to one of $[0,1]$, $(0,1]$, $[0,1)$, $(0,1)$, it suffices to prove that these four intervals are connected. Since $(0,1)$ is dense in the other three intervals, by Prop. 7.95, it suffices to prove that $(0,1)$ is connected.

Suppose that $(0,1)$ is not connected. Then $(0,1) = U \sqcup V$ where U, V are disjoint open nonempty subsets. Choose $x_1 \in U$ and $y_1 \in V$, and assume WLOG that $x_1 < y_1$. In the following, we construct an increasing sequence (x_n) in U and a decreasing one (y_n) in V satisfying $x_n < y_n$ for all n by induction. Suppose x_n, y_n has been constructed. Let $z_n = (x_n + y_n)/2$.

- If $z_n \in U$, then let $x_{n+1} = z_n$ and $y_{n+1} = y_n$.
- If $z_n \in V$, then let $x_{n+1} = x_n$ and $y_{n+1} = z_n$.

Then $y_n - x_n$ converges to 0. So x_n and y_n converge to the same point $\xi \in \mathbb{R}$. We have $\xi \in (0,1)$ since $x_1 < \xi < y_1$. Since V is open, $U = (0,1)\backslash V$ is closed in $(0,1)$ by Thm. 7.48. So $\xi \in \mathrm{Cl}_{(0,1)}(U) = U$. Similarly, $\xi \in \mathrm{Cl}_{(0,1)}(V) = V$. This is impossible. $\square$

7.7.3 More examples of connected spaces

Definition 7.97. Let $x, y \in X$. A **path** in X from x to y is defined to be a continuous map $\gamma : [a,b] \to X$ where $-\infty < a < b < +\infty$, such that $\gamma(a) = x$ and $\gamma(b) = y$. Unless otherwise stated, we take $[a,b]$ to be $[0,1]$. We call x and y respectively the **initial point** and the **terminal point** of γ.

Definition 7.98. We say that X is **path-connected** if for every $x, y \in X$ there is a path in X from x to y.

Example 7.99. $\mathbb{R}^N$ is path-connected. $B_{\mathbb{R}^N}(0,R)$ and $\overline{B}_{\mathbb{R}^N}(0,R)$ (where $R < +\infty$) are path connected. $\{x \in \mathbb{R}^N : r < x < R\}$ (where $0 \leqslant r < R < +\infty$) are path connected. The region enclosed by a triangle is a connected subset of $\mathbb{R}^2$. $[0,1]^N$ is connected.

Theorem 7.100. *Assume that X is path-connected. Then X is connected.*

Proof. If X is not connected, then $X = U \sqcup V$ where U, V are nonempty open subsets of X. Since X is path-connected, there is a path γ from a point of U to a point of V. So $[0,1] = \gamma^{-1}(U) \sqcup \gamma^{-1}(V)$ where $\gamma^{-1}(U), \gamma^{-1}(V)$ are open (by Prop. 7.60) and nonempty. This contradicts the fact that $[0,1]$ is connected (cf. Thm. 7.96). $\square$

Proposition 7.101. *Let $f : X \to Y$ be a continuous map of topological spaces. Suppose that X is connected. Then $f(X)$ is connected.*

Proof. By replacing $f : X \to Y$ by the restricted continuous map $f : X \to f(X)$, it suffices to assume $Y = f(X)$. If Y is not connected, then $Y = U \sqcup V$ where U, V are open and nonempty. Then $X = f^{-1}(U) \sqcup f^{-1}(V)$ are open (by Prop. 7.60) and nonempty subsets of X. So X is not connected, impossible. (One can also use Thm. 7.94 to prove that $f(X)$ is connected.) $\square$

Remark 7.102. When $Y = \mathbb{R}$, Prop. 7.101 and Thm. 7.96 imply that $f(X)$ is an interval. So f satisfies (7.28). Therefore, Prop. 7.101 can be viewed as a generalization of IVP for connected spaces.

Corollary 7.103. *Let I be an interval, and let $f : I \to \overline{\mathbb{R}}$ be a strictly increasing continuous map. Then $J = f(I)$ is an interval, and the restriction $f : I \to J$ is a homeomorphism.*

Proof. By Thm. 7.96 and Prop. 7.101, J is connected and hence is an interval. Therefore, by Thm. 2.72, f is a homeomorphism. $\square$

$\star$ **Example 7.104.** Not all connected spaces are path-connected. Let $f : (0, 1] \to \mathbb{R}^2$ be defined by $f(x) = (x, \sin(x^{-1}))$. Then the range $f((0, 1])$ is connected by Prop. 7.101. Since $f((0, 1])$ is a dense subset of $X = f((0, 1]) \cup \{(0, 0)\}$, by Prop. 7.95, X is connected. However, it can be checked that X is not path-connected. (Prove it yourself, or see [Mun, Sec. 24].) X is called the **topologist's sine curve**.

The following proposition can be used to decompose (for example) an open subset of $\mathbb{R}^N$ into open connected subsets. (See Pb. 7.13.)

Proposition 7.105. *Assume that $X = \bigcup_{\alpha\in\mathscr{A}} X_\alpha$ where each X_α is connected. Assume that $\bigcap_{\alpha\in\mathscr{A}} X_\alpha \neq \varnothing$. Then X is connected.*

Proof. Suppose that X is not connected. By Thm. 7.94, there is a continuous surjection $f : X \to \{0, 1\}$. Let $p \in \bigcap_\alpha X_\alpha$. Then $f(p)$ is 0 or 1. Assume WLOG that $f(p) = 0$. Choose $x \in X$ such that $f(x) = 1$. Choose α such that $x \in X_\alpha$. Then $f|_{X_\alpha} : X_\alpha \to \{0, 1\}$ is a continuous surjection. So X_α does not satisfy IVP, and hence is not connected. $\square$

Exercise 7.106. Prove a path-connected version of Prop. 7.105.

Exercise 7.107. Prove Prop. 7.95 and 7.105 directly using Def. 7.92 (but not using IVP).

7.8 Rigorous constructions of $\sqrt[n]{x}$, $\log x$, and a^x

With the help of Cor. 7.103, one can construct a lot of well-known functions rigorously.

Example 7.108. Let $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be $f(x) = x^n$ where $n \in \mathbb{Z}_+$. Then by Cor. 7.103, $J = f(\mathbb{R}_{\geqslant 0})$ is an interval. Clearly $J \subset [0, +\infty)$. Since $0 \in J$ and $\sup J = +\infty$, we have $J = [0, +\infty)$. Therefore f is a homeomorphism. Its inverse function is a homeomorphism: the **n-th root** function

$$\sqrt[n]{\ } : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0} \qquad x \mapsto \sqrt[n]{x}$$

This gives the rigorous construction of $\sqrt[n]{x}$.

A similar method gives the rigorous construction of $\log$.

Example 7.109. By Exp. 4.28, the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is continuous. We claim that $\exp$ is a strictly increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}_{>0}$. Its inverse function is called the **logarithmic function**

$$\log : \mathbb{R}_{>0} \to \mathbb{R} \qquad x \mapsto \log x$$

Proof. From $e^x = \sum_{n=0}^{\infty} x^n/n!$ we clearly have $e^0 = 1$ and $e^x > 1$ if $x > 0$. From $e^{x+y} = e^x e^y$ proved in Cor. 5.61, we have $e^x e^{-x} = e^0 = 1$, which shows that $e^x \in \mathbb{R}_{>0}$ for all $x \in \mathbb{R}$. If $x < y$, then $e^y > e^x > 0$ since $e^y = e^{y-x}e^x$ and $e^{y-x} > 1$. So $\exp$ is strictly increasing. Thus, by Cor. 7.103, $\exp$ is a homeomorphism from $\mathbb{R}$ to $J = \exp(\mathbb{R})$, and J is an interval.

When $x \geqslant 0$, we have $e^x \geqslant x$ from the definition of e^x. So $\sup_{x \geqslant 0} e^x = +\infty$. When $x \leqslant 0$, since $e^x e^{-x} = 1$, we have $\inf_{x \leqslant 0} e^x = 1/\sup_{x \geqslant 0} e^x = 0$. So $\sup J = +\infty$ and $\inf J = 0$. Since $0 \notin \exp(\mathbb{R})$ (if $e^x = 0$, then $1 = e^x e^{-x} = 0$, impossible), we have $J = \mathbb{R}_{>0}$. $\square$

Example 7.110. Let $a \in \mathbb{R}_{>0}$. For each $z \in \mathbb{C}$, define

$$a^z = e^{z \log a} \tag{7.29}$$

By Exp. 7.109, if $a > 1$ (resp. $0 < a < 1$), the map

$$\mathbb{R} \to \mathbb{R}_{>0} \qquad x \mapsto a^x \tag{7.30}$$

is an increasing (resp. decreasing) homeomorphism, since it is the composition of the increasing (resp. decreasing) homeomorphism $x \in \mathbb{R} \mapsto x \log a \in \mathbb{R}$ and the increasing one $\exp : \mathbb{R} \to \mathbb{R}_{>0}$. By the proof of Exp. 7.109, we have

$$a^0 = 1 \qquad a^x a^{-x} = 1 \qquad a^x a^y = a^{x+y} \tag{7.31}$$

And clearly

$$a^1 = a. \tag{7.32}$$

It follows that for every $n \in \mathbb{Z}_+$, $a^n = a^{1+\cdots+1} = a \cdots a$. Namely, $a^n = e^{n \log a}$ agrees with the usual understanding of a^n. Thus, since $(a^{1/n})^n$ equals $a^{1/n} \cdots a^{1/n} = a^{1/n+\cdots+1/n} = a^1 = a$, we conclude

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Example 7.111. By Exp. 7.109, if $p > 0$ (resp. $p < 0$), then

$$\mathbb{R}_{>0} \to \mathbb{R}_{>0} \qquad x \mapsto x^p = e^{p\log x} \tag{7.33}$$

is an increasing (resp. decreasing) homeomorphism, since it is the composition of the increasing (resp. decreasing) homeomorphism $x \in \mathbb{R}_{>0} \to p\log x \in \mathbb{R}$ and the increasing homeomorphism $\exp : \mathbb{R} \to \mathbb{R}_{>0}$.

7.9 Problems and supplementary material

Let X and Y be topological spaces.

7.9.1 Open sets, closed sets, closures

Problem 7.1. Let $A, B \in X$. Let $(A_\alpha)_{\alpha\in\mathscr{A}}$ be a family of subsets of X. Prove that

$$\overline{A \cup B} = \overline{A} \cup \overline{B} \tag{7.34a}$$

$$\overline{\bigcap_{\alpha\in\mathscr{A}} A_\alpha} \subset \bigcap_{\alpha\in\mathscr{A}} \overline{A_\alpha} \tag{7.34b}$$

The following problem is crucial to the study of compactness. (See Sec. 8.3 for instance.)

Problem 7.2. Let $(x_\alpha)_{\alpha\in I}$ be a net in X. Let $x \in X$. Prove that the following statements are equivalent:

(1) $(x_\alpha)_{\alpha\in I}$ has a subnet converging to x.

(2) For every neighborhood U of x, we have that x_α is frequently in U.

(3) x belongs to $\bigcap_{\alpha\in I} \overline{\{x_\beta : \beta \geqslant \alpha\}}$.

Any $x \in X$ satisfying one of these three conditions is called a **cluster point** of $(x_\alpha)_{\alpha\in I}$. (Compare Pb. 3.1.)

Hint. (2)⇔(3) is a direct translation. Assume (2). To prove (1), show that $(J, \leqslant)$ is a directed set, where

$$\begin{gathered} J = \{(\alpha, U) \in I \times \mathrm{Nbh}_X(x) : x_\alpha \in U\} \\ (\alpha, U) \leqslant (\alpha', U') \qquad \Longleftrightarrow \qquad \alpha \leqslant \alpha' \text{ and } U \supset U' \end{gathered} \tag{7.35a}$$

Prove that $(x_\mu)_{\mu\in J}$ is a subnet of $(x_\alpha)_{\alpha\in I}$ if for each $(\alpha, U) \in J$ we set

$$x_{(\alpha,U)} = x_\alpha \tag{7.35b}$$

(Namely, the increasing map $J \to I$ is defined to be $(\alpha, U) \mapsto \alpha$.) Prove that $(x_\mu)_{\mu\in J}$ converges to x. You should point out where (2) is used in your proofs. □

Remark 7.112. By Pb. 7.2-(3), the set of cluster points of (x_α) is a closed subset, since intersections of closed subsets are closed (cf. Cor. 7.49, or by (7.34b)).

For the reader's convenience, we present below the sequential version of Pb. 7.2. "(1)⇔(2)" is due to Pb. 3.1. "(2)⇔(3)" is due to Pb. 7.2.

Proposition 7.113. *Assume that X is a metric space. Let $(x_n)_{n\in\mathbb{Z}_+}$ be a sequence in X. Let $x \in X$. The following statements are equivalent:*

(1) $(x_n)_{n\in\mathbb{Z}_+}$ has a subsequence converging to x.

(2) For every neighborhood U of x, we have that x_n is frequently in U.

(3) x belongs to $\displaystyle\bigcap_{n\in\mathbb{Z}_+} \overline{\{x_k : k \geqslant n\}}$.

*Any $x \in X$ satisfying one of these three conditions is called a **cluster point** of $(x_n)_{n\in\mathbb{Z}_+}$.*

Prop. 7.113-(3) should remind you of the definitions of $\limsup$ and $\liminf$.

⋆ **Remark 7.114.** It is not hard to show that *the (1,2,3) of Prop. 7.113 are equivalent in the more general case that X is a first countable topological space* (see below for the definition). The proof is similar to that for metric spaces, and is left to the readers as an exercise.

Definition 7.115. Let X be a topological space. A subset $\mathcal{B}_x$ of $\mathrm{Nbh}_X(x)$ is called a **neighborhood basis** of x, if for every $U \in \mathrm{Nbh}_X(x)$ there exists $V \in \mathcal{B}_x$ such that $V \subset U$. We say that X **first countable** if every point x has a neighborhood basis $\mathcal{B}_x$ which is a countable set.

Example 7.116. If X is a metric space, then X is first countable, since for every $x \in X$, $\{B_X(x, 1/n) : n \in \mathbb{Z}_+\}$ is a neighborhood basis of x.

Remark 7.117. By Pb. 7.2-(2) and Prop. 7.113-(2), if (x_n) is a sequence in a metric space X, then (x_n) has a subsequence converging to $x \in X$ iff (x_n) has a subnet converging to x. This is not necessarily true when X is a general topological space. Note that in the general case, cluster points of a sequence (x_n) mean cluster points of (x_n) as a net. Thus, they are not the limits of convergent *subsequences* of (x_n).

Problem 7.3. Assume that X is a metric space. Let $E \subset X$. Recall that $d(x, E) = d(E, x) = \inf_{e\in E} d(e, x)$. Prove that

$$\{x \in X : d(x, E) = 0\} = \overline{E} \tag{7.36}$$

Remark 7.118. If E, F are disjoint subsets of a metric space X, a continuous function $f : X \to [0, 1]$ is called an **Urysohn function** with respect to E, F, if

$$f^{-1}(1) = E \qquad f^{-1}(0) = F$$

For example, it is easy to check that

$$f : X \to [0,1] \qquad f(x) = \frac{d(x,F)}{d(x,E) + d(x,F)} \tag{7.37}$$

is an Urysohn function.

⋆ **Problem 7.4.** A topological space X is called **normal** if for every closed disjoint $E, F \subset X$, there exist disjoint open subsets $U, V \subset X$ such that $E \subset U$ and $F \subset V$.

1. Prove that X is normal iff for each $E \subset W \subset X$ where E is closed and W is open, there exists an open subset $U \subset X$ such that $E \subset U \subset \overline{U} \subset W$.

2. Prove that if X is metrizable, then X is normal.

7.9.2 Continuous maps

Exercise 7.119. Let $f : X \to Y$ be a map.

1. Suppose that F is a subset of Y containing $f(X)$. Show that $f : X \to Y$ is continuous iff $f : X \to F$ is continuous.

2. (Local to global principle) Suppose that $X = \bigcup_{\alpha \in I} U_\alpha$ where each U_α is an open subset of X, Prove that f is continuous iff $f|_{U_\alpha} : U_\alpha \to Y$ is continuous for every α.

Remark 7.120. The above local-to-global principle for continuous functions can be rephrased in the following way. Suppose that $X = \bigcup_{\alpha \in I} U_\alpha$ where each U_α is an open subset of X. Suppose that for each α we have a continuous map $f_\alpha : U_\alpha \to Y$. Assume that for each $\alpha, \beta \in I$ we have

$$f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$$

Then there is a (necessarily unique) continuous function $f : X \to Y$ such that $f|_{U_\alpha} = f_\alpha$ for every α.

Problem 7.5. Let $f : X \to Y$ be a map. Suppose that $X = A_1 \cup \cdots \cup A_N$ where $N \in \mathbb{Z}_+$ and $A_1, \dots, A_N$ are closed subsets of X. Suppose that $f|_{A_i} : A_i \to Y$ is continuous for each $1 \leqslant i \leqslant N$. Prove that f is continuous.

Does the conclusion remain true if $A_1, \dots, A_N$ are not assumed closed? If no, find a counterexample. □

Note. Do not use Prop. 7.86 in your proof. But you can think about how this problem is related to Prop. 7.86. □

Definition 7.121. Let $(I, \leqslant)$ be a directed set. Let ∞_I (often abbreviated to ∞) be a new symbol not in I. Then

$$I^* = I \cup \{\infty_I\}$$

is also a directed set if we extend the preorder $\leqslant$ of I to I^* by setting

$$\alpha \leqslant \infty_I \qquad (\forall \alpha \in I^*)$$

For each $\alpha \in I$, let

$$I^*_{\geqslant \alpha} = \{\beta \in I^* : \beta \geqslant \alpha\}$$

The **standard topology** on I^* is defined to be the one induced by the basis

$$\mathcal{B} = \big\{\{\alpha\} : \alpha \in I\big\} \cup \big\{I^*_{\geqslant \alpha} : \alpha \in I\big\} \tag{7.38}$$

Problem 7.6. Let $(I, \leqslant)$ be a directed set. Let I^* be as in Def. 7.121.

1. Check that $\mathcal{B}$ (defined by (7.38)) is a basis for a topology. (Therefore, $\mathcal{B}$ generates a topology $\mathcal{T}$ on I^*.)

2. Let $(x_\alpha)_{\alpha \in I}$ be a net in a topological space X. Let $x_\infty \in X$. (So we have a function $x : I^* \to X$.) Prove that

$$x \text{ is a continuous function} \qquad \Longleftrightarrow \qquad \lim_{\alpha \in I} x_\alpha = x_\infty \tag{7.39}$$

3. Is I^* Hausdorff? Prove it, or find a counterexample.

7.9.3 Product spaces

Problem 7.7. Prove Thm. 7.73.

Problem 7.8. Let $(X_\alpha)_{\alpha \in \mathscr{A}}$ and $(Y_\alpha)_{\alpha \in \mathscr{A}}$ be families of nonempty topological spaces. Let Z be a nonempty topological space. For each $\alpha \in \mathscr{A}$, choose maps $f_\alpha : X_\alpha \to Y_\alpha$ and $g_\alpha : Z \to X_\alpha$.

1. Use Thm. 7.73 to prove that

$$\prod_{\alpha \in \mathscr{A}} f_\alpha : \prod_\alpha X_\alpha \to \prod_\alpha Y_\alpha \qquad (x(\alpha))_{\alpha \in \mathscr{A}} \mapsto (f_\alpha(x(\alpha)))_{\alpha \in \mathscr{A}} \tag{7.40}$$

is continuous iff each f_α is continuous.

2. Use Thm. 7.73 to prove that

$$\bigvee_{\alpha \in \mathscr{A}} g_\alpha : Z \to \prod_\alpha X_\alpha \qquad z \mapsto (g_\alpha(z))_{\alpha \in \mathscr{A}} \tag{7.41}$$

is continuous iff each g_α is continuous.

Problem 7.9. Let $(X_\alpha)_{\alpha \in \mathscr{A}}$ be an uncountable family of metric spaces, where each X_α has at least two elements. Let $S = \prod_{\alpha \in \mathscr{A}} X_\alpha$ be the product space, equipped with the product topology. Prove that S is not first countable (recall Def. 7.115), and hence not metrizable.

7.9.4 Limits of functions

Problem 7.10. Prove the equivalence of (2) and (3) in Def. 7.81.

Problem 7.11. Find $\lim_{(x,y)\to(0,0)} f(x,y)$, or explain why it does not exist:

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

$$f(x,y) = \frac{(xy)^2}{(xy)^2 + (x-y)^2}$$

$$f(x,y) = \frac{x^6y^2}{(x^4 + y^2)^2}$$

7.9.5 Connectedness

Problem 7.12. Assume that X, Y are not empty. Prove that X and Y are connected iff $X \times Y$ is connected.

Hint. Write $X \times Y$ as a union of sets of the form $(X \times \{y\}) \cup (\{x\} \times Y)$. □

Definition 7.122. A topological space X is called **locally connected** if every $x \in X$ has a neighborhood basis $\mathcal{B}_x$ (recall Def. 7.115) whose members are all connected.

Example 7.123. Every open subset of a locally connected space is clearly locally connected.

Example 7.124. $\mathbb{R}^N$ is locally connected, since open balls are path-connected and hence connected (by Thm. 7.100). Therefore, every open subset of $\mathbb{R}^N$ is locally connected.

Problem 7.13. Suppose that X is locally connected. Prove that X has a unique (disjoint) decomposition $X = \bigsqcup_{\alpha\in\mathscr{A}} X_\alpha$ where each X_α is a nonempty connected open subset of X. (Each X_α is called a **connected component** of X.)

Hint. For each $x \in X$, consider the union of all connected neighborhoods containing x. □

8 Compactness

8.1 Overture: two flavors of compactness

Let X be a topological space.

Definition 8.1. An **open cover** of X means a family $\mathfrak{U} = (U_\alpha)_{\alpha\in\mathscr{A}}$ of open subsets of X such that $X = \bigcup_{\alpha\in\mathscr{A}} U_\alpha$. $\mathfrak{U}$ is called **finite** resp. **countable** if $\mathscr{A}$ is finite resp. countable. A **subcover** of $\mathfrak{U}$ means an open cover $\mathfrak{V} = (V_\beta)_{\beta\in\mathscr{B}}$ of X such that each V_β equals U_α for some $\alpha \in \mathscr{A}$.

Definition 8.2. We say that X is a **compact space** if every open cover of X has a finite subcover. We say that X is a **Lindelöf space** if every open cover of X has a countable subcover.

Remark 8.3. For the purpose of this chapter, it suffices to consider open covers $\mathfrak{U} = (U_\alpha)_{\alpha\in\mathscr{A}}$ such that $\alpha \in \mathscr{A} \mapsto U_\alpha \in 2^X$ is injective. (Namely, one can throw away repeated open sets.) In this case, we write

$$\mathfrak{U} \subset 2^X$$

and view $\mathfrak{U}$ as a subset of 2^X. A **subcover** of such $\mathfrak{U}$ is an open cover $\mathfrak{V}$ such that $\mathfrak{V} \subset \mathfrak{U}$. The readers can check that this assumption does not affect the definition of compact spaces and Lindelöf spaces.

Remark 8.4. Compactness can also be formulated in a relevant version: If A is a compact subset of X, and if $\mathfrak{U}$ is a set of open subsets of X such that $A \subset \bigcup_{U\in\mathfrak{U}} U$, then since $\{U \cap A : U \in \mathfrak{U}\}$ gives an open cover of A, we know that

$$A \subset U_1 \cup \cdots \cup U_n$$

some $U_1, \ldots, U_n \in \mathfrak{U}$. Conversely, any A satisfying such property is a compact subspace of X.

Exercise 8.5. Show that a finite union of compact spaces is compact. Show that a finite set is compact.

Exercise 8.6. Show that X is compact iff X satisfies the **finite intersection property**: The intersection of a family of non-empty closed subsets is nonempty.

A main goal of this chapter is to prove the following theorem. Its proof will be finished at the end of Sec. 8.5. In fact, the formal proof will only be in Sec. 8.3 and Sec. 8.5.

Theorem 8.7. *Let X be a metric space. Then X is sequentially compact iff X is compact.*

This is the first fundamental theorem we prove in this course. Its significance lies in the fact that it connects two seemingly very different notions of compactness, and hence two strikingly different intuitions. We hope that the reader can not only follow the logical chains of the proofs, but also understand the pictures behind the proof. More precisely, we hope that the readers can have an intuitive understanding of the following questions:

- Why are these two compactness both powerful in solving certain problems? What roles do they play in the proof? How are the roles played by these two compactness related? (This is more important than just knowing why these notions are *logically* equivalent.)
- Why is one version of compactness more powerful than the other one in solving certain problems?

Thus, I feel that it is better to look at some applications of compactness before we prove Thm. 8.7 rigorously. For pedagogical purposes, we also introduce two related notions of compactness:

Definition 8.8. We say that X is **net-compact**, if every net $(x_\alpha)_{\alpha\in I}$ in X has at least one cluster point (recall Pb. 7.2), equivalently, at least one convergent subnet. We say that X is **countably compact**, if every countable open cover of X has a finite subcover.

Sequential compactness and net-compactness clearly share the same intuition. Countable compactness is intuitively similar to compactness. Also, compactness clearly implies countable compactness. But net-compactness does not imply sequential compactness in general: In a net-compact space, every sequence has a convergent subnet, but not necessarily a convergent subsequence.

The relationship between these four versions of compactness is as follows. (We will prove this in the course of proving Thm. 8.7.)

$$\text{Metric spaces:} \qquad \text{all the four versions of compactness are equivalent} \tag{8.1a}$$

$$\text{Topological spaces:} \qquad \text{net-compact} \Longleftrightarrow \text{compact} \tag{8.1b}$$

After proving (8.1), we will not use the notions of countable compactness and net-compactness. This is because net-compactness is equivalent to compactness, and countable compactness is more difficult to use than compactness. (Nevertheless, proving/using compactness by proving/using the existence of cluster points is often helpful.)

8.2 Act 1: case studies

8.2.1 Extreme value theorem (EVT)

Lemma 8.9 (Extreme value theorem). *Let X be a compact topological space. Let $f : X \to \mathbb{R}$ be a continuous function. Then f attains its maximum and minimum at points* *of X. In particular, $f(X)$ is a bounded subset of $\mathbb{R}$.*

We have seen the sequential compactness version of **EVT** (extreme value theorem) in Lem. 3.2. There, we find the point $x \in X$ at which f attains its maximum by first finding a sequence (x_n) such that $f(x_n)$ converges to $\sup f(X)$. Then we choose any convergent subsequence (x_{n_k}), which converges to the desired point x. The same method can be used to prove EVT for net-compact spaces if we replace sequences by nets.

For compact spaces, EVT is proved in a completely different way. In fact, without the tools of sequences and nets, one can not easily find x at which f attains it maximum. The argument is rather indirect:

Proof of Lem. 8.9. It suffices to prove that $f(X)$ is bounded for all continuous maps $f : X \to \mathbb{R}$. Then $a = \sup f(X)$ is in $\mathbb{R}$. If $a \notin f(X)$, we choose a homeomorphism $\varphi : (-\infty, a) \to \mathbb{R}$. So $\varphi \circ f : X \to \mathbb{R}$ is continuous but has no upper bound. This is impossible. □

Thus, to prove EVT, it remains to prove:

Example 8.10. Assume that X is compact and $f : X \to \mathbb{R}$ is continuous. Then $f(X)$ is a bounded subset of $\mathbb{R}$.

Proof. For each $x \in X$, since f is continuous at x, there exists $U_x \in \mathrm{Nbh}(x)$ such that $|f(p) - f(x)| < 1$ for all $p \in U_x$. In particular, $f(U_x)$ is bounded. Since $X = \bigcup_{x \in X} U_x$ is an open cover of X, by compactness, $X = U_{x_1} \cup \cdots \cup U_{x_n}$ for some $x_1, \dots, x_n \in X$. Thus $X = f(U_{x_1}) \cup \cdots \cup f(U_{x_n})$ is bounded. □

The above proof is typical. It suggests that compactness is powerful for proving finiteness properties rather than finding solutions of functions satisfying certain requirements. Thus, if you want to prove a finiteness property using sequential or net-compactness, you have to prove it indirectly. For example, you need to prove by contradiction:

Example 8.11. Assume that X is net-compact or sequentially compact. Assume that $f : X \to \mathbb{R}$ is continuous. Then $f(X)$ is a bounded subset of $\mathbb{R}$.

Proof. Assume that X is net-compact. If $f(X)$ is not bounded above, then there is a sequence (x_α) in X (viewed as a net) such that $\lim_\alpha f(x_\alpha) = +\infty$. By net-compactness, (x_α) has a subnet (x'_β) converging to $x \in X$. So $f(x) = \lim_\beta f(x'_\beta) = +\infty$, impossible. So f is bounded above, and hence bounded below by a similar argument. The case where X is sequentially compact can be proved by a similar method. □

Let us look at a more complicated example.

8.2.2 Uniform convergence in multivariable functions

Example 8.12. Let X, Y be topological spaces. Assume that Y is compact. Let V be a normed vector space. Choose $f \in C(X \times Y, V)$. For each $x \in X$, let

$$f_x : Y \to V \qquad y \mapsto f(x, y)$$

Equip $C(Y, V)$ with the l^∞-norm. Then the following map is continuous:

$$\Phi(f) : X \to C(Y, V) \qquad x \mapsto f_x \tag{8.2}$$

Remark 8.13. Note that since Y is compact, each $g \in C(Y, V)$ is bounded by EVT (applied to $|g|$). So $C(Y, V) \subset l^\infty(Y, V)$.

The continuity of $\Phi(f)$ means that for each $x \in X$, the following statement holds:

❦ For every $\varepsilon > 0$ there exists $U \in \mathrm{Nbh}_X(x)$ such that for all $p \in U$ and all $y \in Y$ we have $\|f(p, y) - f(x, y)\| < \varepsilon$.

This is clearly a finiteness property. Thus, its sequentially compact version or net-compact version should be proved indirectly. Indeed, when X, Y are metric spaces and Y is sequentially compact, we have proved this in Pb. 3.7 by contradiction. The same method (with sequences replaced by nets) also works for the net-compact case:

Example 8.14. Example 8.12 is true, assuming that Y is net-compact rather than compact.

Proof. Suppose "❦" is not true. Then there is $\varepsilon > 0$ such that for every $U \in \mathrm{Nbh}_X(x)$ there is $x_U \in U$ and $y_U \in Y$ such that $\|f(x_U, y_U) - f(x, y_U)\| \geqslant \varepsilon$. Then $(x_\alpha)_{\alpha \in \mathrm{Nbh}_X(x)}$ is a net converging to x, where $x_\alpha = x_U$ if $\alpha = U$. Since Y is net-compact, (y_α) has a subnet (y_β) converging to some $y \in Y$. Since the subnet (x_β) also converges to x, we have $\lim_\beta f(x_\beta, y_\beta) = f(x, y)$ and $\lim_\beta f(x, y_\beta) = f(x, y)$ by the continuity of f. This contradicts the fact that $\|f(x_\beta, y_\beta) - f(x, y_\beta)\| \geqslant \varepsilon$ for all β. □

On the other hand, the solution of Exp. 8.12 using open covers is a direct proof:

Proof of Exp. 8.12. "❦" is a finiteness property global over Y. We prove "❦" by first proving it locally, and then passing to the global space Y using the compactness of Y.

Fix $x \in X$. Choose any $\varepsilon > 0$. For each $y \in Y$, since f is continuous at (x, y), there is a neighborhood W of (x, y) such that for every $(p, q) \in W$ we have $\|f(p, q) - f(x, y)\| < \varepsilon/2$. By Exp. 7.72, we can shrink W to a smaller neighborhood of the form $U_y \times V_y$ where $U_y \in \mathrm{Nbh}_X(x)$ and $V_y \in \mathrm{Nbh}_Y(y)$. Then for each $p \in U_y$

and $q \in U_y$ we have $\|f(p,q) - f(x,y)\| < \varepsilon/2$, and hence $\|f(p,q) - f(x,q)\| < \varepsilon$. This proves the special case of "$\varnothing$" where Y is replaced by U_y.

Now we pass from local to global in the same way as in the proof of Exp. 8.10. Since $Y = \bigcup_{y \in Y} V_y$ is an open cover of Y, by the compactness of Y, there is a finite subset $F \subset Y$ such that $Y = \bigcup_{y \in F} V_y$. Then "$\varnothing$" is true if we let $U = \bigcap_{y \in F} U_y$. $\square$

8.2.3 Conclusions

1. Sequential compactness and net-compactness are useful for finding solutions of a function satisfying some given conditions.

2. Compactness is useful for proving finiteness properties. The proof is usually a local-to-global argument. It is usually a direct argument (rather than proof by contradiction).

3. If one uses sequential/net-compactness to prove a finiteness property, one usually proves it by contradiction: Assume that this finiteness is not true. Find a sequence/net (x_α) that violates this finiteness property, and pass to a convergent subsequence/subnet to find a contradiction.

4. Therefore, for sequential/net-compact spaces, the argument is in the direction of "getting smaller and smaller", opposite to the argument for compact spaces.

Let me emphasize that the proof for sequentially/net-compact spaces is opposite to the one for compact spaces in two aspects: (1) If one argument is direct, the other is a proof by contradiction for the same problem. (2) The former has the intuition of "getting smaller", while the latter local-to-global argument has the intuition of "getting larger".

I have already touched on this phenomenon in Rem. 7.54: *The reason that sequences and nets run in the opposite direction to that of open sets is because closed sets are opposite to open sets*, as proved in Thm. 7.48. Thus, you can expect that the transition between closed and open sets plays a crucial role in the following proof of Thm. 8.7.

8.3 Act 2: "sequentially compact $\Leftrightarrow$ countably compact" for metric spaces, just as "net-compact $\Leftrightarrow$ compact"

> The road up and the road down is one and the same.
>
> Heraclitus

As mentioned before, our goal of this chapter is to prove "sequentially compact ⇔ compact" for metric spaces. Our strategy is as follows: We reformulate the sequential compactness condition in terms of decreasing chains of closed sets, and reformulate the compactness condition in terms of increasing chains of open sets. Then we relate these two pictures easily using Thm. 7.48.

The following difficulty arises when carrying out this strategy. Sequences are countable by nature, whereas open covers can have arbitrarily large cardinality. Thus, sequences are related to countable decreasing chains, and hence countable open covers. Therefore, the above idea only implies the equivalence

$$\text{sequentially compact} \quad \Longleftrightarrow \quad \text{countably compact} \qquad \text{(for metric spaces)} \tag{8.3a}$$

Accordingly, it will only imply

$$\text{net-compact} \quad \Longleftrightarrow \quad \text{compact} \tag{8.3b}$$

since there are no constraints on the cardinalities of indexed sets of nets. We will prove (8.3) in this section, and leave the proof of "countably compact ⇔ compact" for metric spaces to Sec. 8.5.

Since the proofs of (8.3a) and (8.3b) are similar, we first discuss (8.3b).

Proposition 8.15. *Let X be a topological space. Then the following are equivalent.*

(1) X is compact.

*(2) (**Increasing chain property**) If $(U_\mu)_{\mu\in I}$ is an increasing net of open subsets of X satisfying $\bigcup_{\mu\in I} U_\mu = X$, then $U_\mu = X$ for some μ.*

*(3) (**Decreasing chain property**) If $(E_\mu)_{\mu\in I}$ is a decreasing net of nonempty closed subsets of X, then $\bigcap_{\mu\in I} E_\mu \neq \emptyset$.*

Here, "increasing net" means $U_\mu \subset U_\nu$ if $\mu \leqslant \nu$, and "decreasing net" means the opposite.

Proof. (1)⇒(2): Assume (1). Then $X = \bigcup_\mu U_\mu$ is an open cover of X. So, by the compactness of X, we have $X = U_{\mu_1} \cup \cdots \cup U_{\mu_n}$ for some $\mu_1, \dots, \mu_n \in I$. Choose $\mu \in I$ which is $\geqslant \mu_1, \dots, \mu_n$. Then $X = U_\mu$.

(2)⇒(1): Assume (2). Let $X = \bigcup_{\alpha\in\mathscr{A}} W_\alpha$ be an open cover of X. Let $I = \mathrm{fin}(2^{\mathscr{A}})$. For each $\mu = \{\alpha_1, \dots, \alpha_n\} \in I$, let $U_\mu = W_{\alpha_1} \cup \cdots \cup W_{\alpha_n}$. Then $(U_\mu)_{\mu\in I}$ is an increasing net of open sets covering X. Thus, by (2), we have $U_\mu = X$ for some μ. This proves (1).

(2)⇔(3): If we let $E_\mu = X\backslash U_\mu$, then (2) says that if (E_μ) is a decreasing net of closed sets whose intersection is $\emptyset$, then $E_\mu = \emptyset$ for some μ. This is the contraposition of (3). □

We now relate decreasing chain property and cluster points of nets using (8.4).

Theorem 8.16. *Let X be a topological space. Then X is net-compact iff X is compact.*

Proof. Assume that X is net-compact. By Prop. 8.15, it suffices to prove that X satisfies the decreasing chain property. Let $(E_\mu)_{\mu\in I}$ be a decreasing net of nonempty closed subsets of X. For each μ we choose $x_\mu \in E_\mu$, which gives a net $(x_\mu)_{\mu\in I}$ in X. The fact that (E_μ) is decreasing implies that $F_\mu \subset E_\mu$ if we set

$$F_\mu = \{x_\nu : \nu \in I, \nu \geqslant \mu\} \tag{8.4a}$$

Thus, the closure $\overline{F}_\mu$ is a subset of E_μ since E_μ is closed. It suffices to prove that $\bigcap_{\mu\in I} \overline{F}_\mu \neq \varnothing$. By Pb. 7.2,

$$\bigcap_\mu \overline{F}_\mu = \{\text{the cluster points of the net } (x_\mu)_{\mu\in I}\} \tag{8.4b}$$

which is nonempty because X is net-compact. This finishes the proof that X is compact.

Now we assume that X is compact. Let $(x_\mu)_{\mu\in I}$ be a net in X. Define F_μ by (8.4a). Then $(\overline{F}_\mu)_{\mu\in I}$ is a decreasing net of nonempty closed subsets. So $\bigcap_\mu \overline{F}_\mu$, the set of cluster points of $(x_\mu)_{\mu\in I}$, is nonempty by the decreasing chain property (cf. Prop. 8.15). So X is net-compact. □

Proposition 8.17. *Let X be a topological space. Then the following are equivalent.*

(1) X is countably compact.

*(2) (**Increasing chain property**) If $(U_n)_{n\in\mathbb{Z}_+}$ is an increasing sequence of open subsets of X satisfying $\bigcup_{n\in\mathbb{Z}_+} U_n = X$, then $U_n = X$ for some n.*

*(3) (**Decreasing chain property**) If $(E_n)_{n\in\mathbb{Z}_+}$ is a decreasing sequence of nonempty closed subsets of X, then $\bigcap_{n\in\mathbb{Z}_+} E_n \neq \varnothing$.*

Proof. Similar to the proof of Prop. 8.15. □

Exercise 8.18. Fill in the details of the proof of Prop. 8.17.

Lemma 8.19. *Let X be a metric space. Then X is sequentially compact iff X is countably compact.*

Proof. This lemma can be proved in a similar way as Thm. 8.17. The only difference is that one should use the sequential version of Pb. 7.2, namely, Prop. 7.113. Note that the "(1)⇔(2)" of Prop. 7.113 does not hold for general topological spaces. Thus, sequential compactness is not equivalent to countable compactness for general topological spaces. □

⋆ **Remark 8.20.** As pointed out in Rem. 7.114, Prop. 7.113 holds more generally for first countable topological spaces (Def. 7.115). Therefore, Lem. 8.19 also holds for such spaces:

$$\begin{gathered}\text{sequentially compact} \quad\Longleftrightarrow\quad \text{countably compact}\\ (\text{for first countable spaces})\end{gathered} \tag{8.5}$$

8.4 Intermezzo: elementary properties about compactness

Proposition 8.21. *Suppose that $X \subset Y$ where Y is a topological space. The following are true.*

1. *Assume that Y is compact and X is closed in Y. Then X is compact.*
2. *Assume that Y is Hausdorff and X is compact, then X is a closed subset of Y.*

Proof. Part 1. Let (x_α) be a net in X. Since Y is compact, (x_α) has a subnet (x'_β) converging to some $p \in Y$. Since (x'_β) is in X, we have $p \in \overline{X} = X$. So X is compact.

Part 2. To prove $\overline{X} = X$, we choose any net (x_α) in X converging to $p \in Y$ and show that $p \in X$. Indeed, since X is compact, (x_α) has a subnet (x'_β) converging to some $x \in X$. Since (x'_β) also converges to p, we have $p = x$ because Y is Hausdorff. So $p \in X$. □

I have mentioned that non-Hausdorff spaces are not often used in analysis. Thus, we mainly use the following special case of Prop. 8.21:

Corollary 8.22. *Let Y be a Hausdorff space and $X \subset Y$. If X is compact, then X is closed in Y. If X is closed in Y and if Y is compact, then X is compact.*

Recall that a similar property holds for complete metric spaces, cf. Prop. 3.27.

Theorem 8.23. *Suppose that $f : X \to Y$ is a continuous map of topological spaces where X is compact. Then $f(X)$ is compact. Moreover, if f is injective and X, Y are Hausdorff, then f restricts to a homeomorphism $f : X \to f(X)$.*

Proof. Choose any net $(f(x_\alpha))$ in $f(X)$ where $x_\alpha \in X$. Since X is compact, (x_α) has a subnet (x'_β) converging to some $x \in X$. Then $f(x'_\beta)$ converges to $f(x)$. So $f(X)$ is compact.

Now assume that f is injective and Y is Hausdorff. Then the subspace $f(X)$ is also Hausdorff. By replacing Y by $f(X)$, we assume that f is bijective. To show that f^{-1} is continuous, by Prop. 7.60, it suffices to prove that f is a closed map, i.e., f sends every closed $E \subset X$ to a closed subset $f(E)$. Indeed, since X is compact, E is also compact by Cor. 8.22. So $f(E)$ is compact by the first paragraph. So $f(E)$ is closed in X by Cor. 8.22. □

The second part of Thm. 8.23 can also be proved in a similar way as Pb. 3.3 by replacing sequences with nets. But that argument relies on the fact that every net in a compact Hausdorff space with only one cluster point is convergent. We leave the proof of this fact to the readers (cf. Pb. 8.1).

Exercise 8.24. The first part of Thm. 8.23 can be viewed as a generalization of extreme value theorem. Why?

Proposition 8.25. *Suppose that X, Y are compact topological spaces. Then $X \times Y$ is compact.*

Proof. Take a net (x_α, y_α) in $X \times Y$. Since X is compact, (x_α) has a convergent subnet (x_{α_β}). Since Y is compact, (y_{α_β}) has a convergent subnet $(y_{\alpha_{\beta_\gamma}})$. So $(x_{\alpha_{\beta_\gamma}}, y_{\alpha_{\beta_\gamma}})$ is a convergent subnet of (x_α, y_α). $\square$

Remark 8.26. Note that if $X \times Y$ is compact, then X, as the image of $X \times Y$ under the projection map, is compact by Thm. 8.23. Therefore, we conclude that $X \times Y$ is compact iff X and Y are compact.

8.5 Act 3: "countably compact ⇔ compact" for Lindelöf spaces

Assume in this section that X is a topological space. Recall that X is Lindelöf iff every open cover has a countable subcover. Thus, it is obvious that

$$\text{countably compact} \quad \Longleftrightarrow \quad \text{compact} \qquad \text{(for Lindelöf spaces)} \tag{8.6}$$

Thus, by Lem. 8.19, to prove that sequentially/countably compact metric spaces are compact, it suffices to prove that they are Lindelöf.

We introduce two related concepts that are more useful than Lindelöf spaces:

Definition 8.27. We say that a topological space X is **separable** if X has a countable dense subset. We say that X is **second countable** if the topology of X has a countable basis (i.e., a basis $\mathcal{B}$ with countably many elements).

Example 8.28. $\mathbb{R}^N$ is separable, since $\mathbb{Q}^N$ is a dense subset.

As we shall immediately see, these two notions are equivalent for metric spaces. It is often easier to visualize and prove separability for concrete examples (such as Exp. 8.28). Indeed, Exp. 8.28 is the typical example that helps us imagine more general separable spaces. However, for general topological spaces, second countability behaves better than separability. The following property gives one reason.

Proposition 8.29. *If Y is a subset of a second countable space X, then Y is second countable.*

Proof. Let $\mathcal{B}$ be a countable basis for the topology of X. Then $\{Y \cap U : U \in \mathcal{B}\}$ is a countable basis for the topology of Y. $\square$

Another reason that second countability is better is because it implies Lindelöf property. This is mainly due to the following fact:

Proposition 8.30. *Let $\mathcal{B}$ be a basis for the topology of X. Then the following are equivalent.*

(1) X is Lindelöf (resp. compact).

(2) If X has open cover $\mathfrak{U}$ where each member of $\mathfrak{U}$ is an element of $\mathcal{B}$, then $\mathfrak{U}$ has a countable (resp. finite) subcover.

Proof. "(1)⇒(2)" is obvious. Assume (2). Let $\mathfrak{W}$ be an open cover of X. Let

$$\mathfrak{U} = \{U \in \mathcal{B} : U \subset W \text{ for some } W \in \mathfrak{W}\}$$

For each $x \in X$, since there is $W \in \mathfrak{W}$ containing x, and since $\mathcal{B}$ is a basis, there is $U \in \mathcal{B}$ such that $x \in U \subset W$. This proves that $\mathfrak{U}$ is an open cover of X. So $\mathfrak{U}$ has a countable (resp. finite) subcover $\mathfrak{U}_0$. For each $U \in \mathfrak{U}_0$, choose $W_U \in \mathfrak{W}$ containing U. Then $\{W_U : U \in \mathfrak{U}_0\}$ is a countable (resp. finite) subcover of $\mathfrak{W}$. □

Corollary 8.31. *Every second countable topological space X is Lindelöf.*

Proof. Let $\mathcal{B}$ be a countable basis for the topology of X. Let $\mathfrak{U}$ be an open cover of X such that each member of $\mathfrak{U}$ is in $\mathcal{B}$. Then by discarding duplicated terms, $\mathfrak{U}$ becomes countable. This verifies (2) of Prop. 8.30. □

Theorem 8.32. *Consider the following statements:*

(1) X is second countable.

(2) X is separable.

Then (1)⇒(2). If X is metrizable, then (1)⇔(2).

Proof. Assume that X is second countable with countable basis $\mathcal{B}$. For each $U \in \mathcal{B}$, choose $x_U \in U$. Then one checks easily that $\{x_U : U \in \mathcal{B}\}$ is dense by checking that it intersects every nonempty open subset of X. This proves (1)⇒(2).

Assume that X is a metric space with countable dense subset E. Let us prove that the countable set

$$\mathcal{B} = \{B_X(e, 1/n) : e \in E, n \in \mathbb{Z}_+\}$$

is a basis for the topology of X. Choose any open $W \subset X$ with $x \in W$. We want to show that some member of $\mathcal{B}$ contains x and is in W. By shrinking W, we may assume that $W = B_X(x, 1/n)$ for some $n \in \mathbb{Z}_+$. Since E is dense, $B(x, 1/2n)$ contains some $e \in E$. So $d(x, e) < 1/2n$. Therefore, $B(e, 1/2n)$ contains x and is inside W by triangle inequality. □

It can be proved that Lindelöf metric spaces are separable. (Therefore, the three notions agree for metric spaces.) We will not use this fact. So we leave its proof to the readers as an exercise (cf. Pb. 8.12). The following chart is a summary of the relationships between the various topological properties about countability.

$$
\begin{array}{lll}
\text{Topological spaces:} & \text{second countable} \Longrightarrow & \begin{cases} \text{subset is second countable} \\ \text{separable} \\ \text{Lindelöf} \end{cases} \\
\text{Metric spaces:} & \text{second countable} \Longleftrightarrow & \text{separable} \Longleftrightarrow \text{Lindelöf}
\end{array}
\tag{8.7}
$$

Example 8.33. Since $\mathbb{R}^N$ is separable and hence second countable, every subset of $\mathbb{R}^N$ is second countable, and hence is separable.

Theorem 8.34. *Let X be a sequentially compact metric space. Then X is separable, and hence second countable and Lindelöf.*

Proof. We claim that for every $\delta > 0$, there is a finite set $E \subset X$ such that for every $x \in E$, the distance $d(x, E) < \delta$. Suppose that there is no such a finite set for some given number $\delta > 0$. Pick any $x_1 \in X$. Suppose $x_1, \ldots, x_k \in X$ have been constructed. Then there is a point $x_{k+1} \in X$ whose distance to $\{x_1, \ldots, x_k\}$ is $\geqslant \delta$. This defines inductively a sequence $(x_k)_{k \in \mathbb{Z}_+}$ in X such that any two elements have distance $\geqslant \delta$. So (x_k) has no convergent subsequence, contradicting the sequential compactness of X.

Thus, for each $n \in \mathbb{Z}_+$, we can choose a finite $E_n \subset X$ satisfying $d(x, E_n) < 1/n$ for all $x \in X$. Let $E = \bigcup_{n \in \mathbb{Z}_+} E_n$, which is countable. Then for each $x \in X$, $d(x, E) \leqslant d(x, E_n) < 1/n$ for every n, which implies $d(x, E) = 0$ and hence $x \in \overline{E}$ by Pb. 7.3. So E is dense in X. □

Remark 8.35. The above proof is indirect because it proves the existence of E_n by contradiction but not by explicit construction. However, if X is a bounded closed subset of $\mathbb{R}^N$, one can find an explicit countable basis for the topology of X:

$$
\mathcal{B} = \{X \cap B(x, 1/n) : n \in \mathbb{Z}_+, x \in \mathbb{Q}^N\}
$$

and hence has an explicit countable dense subset $\{x_U : U \in \mathcal{B}, U \neq \emptyset\}$ where for each U we choose some $x_U \in U$. More generally, if X is a closed subset of the sequentially compact space $[0, 1]^{\mathbb{Z}_+}$, one can find a countable basis for the topology of X and hence a countable dense subset of X in a similar way. (You will be asked to construct them in Pb. 8.13.)

We shall see in Thm. 8.45 that every sequentially compact metric space is homeomorphic to a closed subset of $[0, 1]^{\mathbb{Z}_+}$. Therefore, for any sequentially compact metric space X you will see in the real (mathematical) life, you don't need the indirect construction in the proof of Thm. 8.34 to prove the separability of X. So what is the point of giving an indirect proof of Thm. 8.34? Well, you need Thm. 8.34 to prove Thm. 8.45. □

Proof of Thm. 8.7. Let X be a metric space. Assume that X is compact. Then X is clearly countably compact, and hence sequentially compact by Lem. 8.19. Conversely, assume that X is sequentially compact. Then by Lem. 8.19 and Thm. 8.34, X is countably compact and Lindelöf, and hence compact. $\square$

⋆ **Remark 8.36.** Since second countable spaces are first countable, by Rem. 8.20 and Cor. 8.31, we have

$$\begin{gathered}\text{sequentially compact} \quad \Longleftrightarrow \quad \text{countably compact} \quad \Longleftrightarrow \quad \text{compact} \\ \text{(for second countable topological spaces)}\end{gathered} \tag{8.8}$$

Relation (8.8) not only generalizes Thm. 8.7, but also tells us what are the crucial properties that ensure the equivalence of compactness and sequential compactness for metric spaces.

8.6 Problems and supplementary material

Let X, Y be topological spaces.

8.6.1 Compactness

Problem 8.1. Let $(x_\alpha)_{\alpha\in I}$ be a net in a compact Hausdorff space X. Prove that (x_α) is convergent iff (x_α) has exactly one cluster point.

Problem 8.2. Let $(x_\alpha)_{\alpha\in I}$ be a net in the compact space $\overline{\mathbb{R}}$. Let S be the (automatically nonempty) set of cluster points of (x_α) in $\overline{\mathbb{R}}$. Recall that S is a closed subset by Rem. 7.112. For each $\alpha \in I$, define

$$A_\alpha = \inf\{x_\beta : \beta \geqslant \alpha\} \qquad B_\alpha = \sup\{x_\beta : \beta \geqslant \alpha\} \tag{8.9}$$

Then (A_α) is increasing and (B_α) is decreasing. So they converge in $\overline{\mathbb{R}}$. Define

$$\liminf_{\alpha\in I} x_\alpha = \sup\{A_\alpha : \alpha \in I\} = \lim_{\alpha\in I} A_\alpha \tag{8.10a}$$

$$\limsup_{\alpha\in I} x_\alpha = \inf\{B_\alpha : \alpha \in I\} = \lim_{\alpha\in I} B_\alpha \tag{8.10b}$$

Prove that

$$\liminf_{\alpha\in I} x_\alpha = \inf S \qquad \limsup_{\alpha\in I} x_\alpha = \sup S \tag{8.11}$$

Note. You will get a quick proof by choosing the right one of the three equivalent definitions of cluster points in Pb. 7.2. A wrong choice will take you much more effort. $\square$

Corollary 8.37. *Let (x_α) be a net in $\overline{\mathbb{R}}$. Then (x_α) converges in $\overline{\mathbb{R}}$ iff* $\limsup_\alpha x_\alpha = \liminf_\alpha x_\alpha$.

Proof. $\overline{\mathbb{R}}$ is compact. Therefore, $\limsup_\alpha x_\alpha = \liminf_\alpha x_\alpha$ iff (x_α) has only one cluster point (by Pb. 8.2), iff (x_α) converges (by Pb. 8.1). □

Problem 8.3. Given a general topological space X, which one implies the other between the conditions of "sequential compactness" and "countable compactness"? Prove your conclusion with details.

Hint. Check the proof of Thm. 8.16, and think about the question: If (x_n) is a sequence in X, what is the inclusion relation between $\bigcap_n \overline{\{x_k : k \geqslant n\}}$ and the set of limits of the convergent subsequences (rather than subnets) of (x_n)? □

Problem 8.4. Prove Prop. 8.21 using the original definition of compact spaces (i.e. every open cover has a finite subcover) instead of using nets.

⋆ **Problem 8.5.** Prove Prop. 8.25 using the original definition of compact spaces instead of using nets.

Problem 8.6. Assume that Y is compact. Let $(x_\alpha, y_\alpha)_{\alpha\in I}$ be a net in $X \times Y$. Assume that $x \in X$ is a cluster point of (x_α). Prove that there exists $y \in Y$ such that (x, y) is a cluster point of (x_α, y_α).

Problem 8.7. (**Tychonoff theorem, countable version**) Let $(X_n)_{n\in\mathbb{Z}_+}$ be a sequence of compact topological spaces. Prove that the product space $S = \prod_{n\in\mathbb{Z}_+} X_n$ (equipped with the product topology) is compact using the following hint.

Hint. Let $(f_\alpha)_{\alpha\in I}$ be a net in S where $f_\alpha = (f_\alpha(1), f_\alpha(2), \dots)$. Use Pb. 8.6 to construct inductively an element $x = (x(1), x(2), \dots) \in S$ such that for every $n \in \mathbb{Z}_+$, the element $(x(1), \dots, x(n))$ is a cluster point of $(f_\alpha(1), \dots, f_\alpha(n))_{\alpha\in I}$ in $X_1 \times \cdots \times X_n$. Prove that x is a cluster point of $(f_\alpha)_{\alpha\in I}$ in S. □

Remark 8.38. The same idea as above can be used to prove the general Tychonoff theorem (the version where the index set $\mathbb{Z}_+$ in Pb. 8.7 is replaced by an arbitrary set) by replacing mathematical induction by Zorn's lemma.

⋆ **Problem 8.8.** Let X be the set of two elements: $X = \{0, 1\}$, viewed as a metric subspace of $\mathbb{R}$. Let $S = X^{[0,1]}$, the product space of uncountably many X, where the index set is the interval $[0, 1]$. S is equipped with the product topology. According to Tychonoff theorem (to be proved in the future), S is compact. Prove that S is not sequentially compact.

Hint. Use binary representations in $[0, 1]$. □

8.6.2 LCH spaces

Definition 8.39. A subset A of a Hausdorff space X is called **precompact** if its closure $\overline{A}$ is compact. This is equivalent to saying that A is contained in a compact subset of X.

Proof of equivalence. If $\overline{A}$ is compact, then $\overline{A}$ is a compact subset of X (cf. Cor. 8.22) containing A. Conversely, if $A \subset K$ where K is a compact subset of X, then K is closed by Cor. 8.22. Since $\overline{A}$ is the smallest closed subset of X containing A, we have $\overline{A} \subset K$. By Exe. 7.52, $\overline{A}$ is closed in K. By Cor. 8.22, $\overline{A}$ is compact. □

It is clear that a subset of a precompact set is precompact.

Problem 8.9. Suppose that X is metric space. Let $A \subset X$. Prove that the following are equivalent.

(1) A is precompact, i.e., $\overline{A}$ is compact.

(2) Every sequence in A has a subsequence converging to some point of X.

Definition 8.40. A Hausdorff space X is called a **locally compact Hausdorff (LCH) space** if every point has a precompact neighborhood.

Proposition 8.41. *Let X be an LCH space. Then the closed subsets and the open subsets of X are LCH.*

Proof. Let $E \subset X$ be closed. Let $x \in E$. Then there is $U \in \mathrm{Nbh}_X(x)$ with compact closure $\mathrm{Cl}_X(U)$. So $\mathrm{Cl}_X(U) \cap E$ is compact by Cor. 8.22. Note that $U \cap E \in \mathrm{Nbh}_E(x)$, and $U \cap E$ is a subset of $\mathrm{Cl}_X(U) \cap E$. So $U \cap E$ is precompact in E. This proves that E is LCH.

Next, we let W be an open subset of X. Let $x \in W$. We want to show that there exists $U \in \mathrm{Nbh}_X(x)$ such that $\overline{U} = \mathrm{Cl}_X(U)$ is compact and $\overline{U} \subset W$. Then U is clearly precompact in W.

Since X is LCH, there is $\Omega \in \mathrm{Nbh}_X(x)$ with compact closure $\overline{\Omega} = \mathrm{Cl}_X(\Omega)$. Then every open subset U of Ω containing x has compact closure in X. Thus, it suffices to prove that there exists $U \in \mathcal{I} = \mathrm{Nbh}_\Omega(x)$ such that $\overline{U} \subset W$. Suppose that this is not true. Then for each $U \in \mathcal{I}$ there is $x_U \in \overline{U} \backslash W$. So $(x_U)_{U \in \mathcal{I}}$ is a net in $\overline{\Omega} \backslash W$, where $\overline{\Omega} \backslash W$ is compact by Cor. 8.22. So $(x_U)_{U \in \mathcal{I}}$ has a cluster point $y \in \overline{\Omega} \backslash W$. In particular, $y \neq x$.

Since X is Hausdorff, by Cor. 7.50, there is $V_0 \in \mathrm{Nbh}_X(x)$ such that $y \notin \overline{V}_0$. Let $V = V_0 \cap \Omega$. Then $V \in \mathcal{I}$ and $y \notin \overline{V}$. For every $U \in \mathcal{I}$ satisfying $U \subset V$ we have $x_U \in \overline{V}$. So $(x_U)_{U \in \mathcal{I}}$ is eventually not in the neighborhood $X \backslash \overline{V}$ of y. This is impossible. □

Definition 8.42. Let X be LCH. Let Y be a metric space. Let $(f_\alpha)_{\alpha \in I}$ be a net in Y^X. Let $f \in Y^X$. We say that (f_α) **converges locally uniformly** to f if the following equivalent conditions are satisfied:

(1) For each $x \in X$, there exists $U \in \mathrm{Nbh}_X(x)$ such that $(f_\alpha|_U)$ converges uniformly to $f|_U$.

(2) For each precompact open subset $W \subset X$, the net $(f_\alpha|_W)$ converges uniformly to $f|_W$.

Problem 8.10. Prove that in Def. 8.42, conditions (1) and (2) are equivalent.

Exercise 8.43. Let X be LCH. Let $\mathcal{V}$ be a normed vector space. Let $(f_\alpha)_{\alpha\in I}$ be a net in $C(X,\mathcal{V})$ converging pointwise to $f : X \to \mathcal{V}$. Assume that the net (f_α) converges locally uniformly on X (clearly to f). Prove that f is continuous.

An example of locally uniform convergence was given in Thm. 4.27.

8.6.3 Countability in topological spaces

Problem 8.11. Let $(X,\mathcal{T})$ be a second countable LCH space. Prove that the topology $\mathcal{T}$ has a countable basis $\mathcal{B}$ whose members are all precompact.

Hint. Use Lindelöf property for open subsets of X. □

⋆ **Problem 8.12.** Let X be a Lindelöf metric space. Prove that X is separable.

Problem 8.13. Let $(X_n)_{n\in\mathbb{Z}_+}$ be a sequence of topological spaces. Equip $S = \prod_{n\in\mathbb{Z}_+} X_n$ with the product topology.

1. Prove that S is second countable if each X_n is second countable.

2. Prove that S is separable if each X_n is separable

Recall from Pb. 7.13 the basic facts about connected components.

Problem 8.14. Let X be a locally connected topological space. Prove that if X is second countable, then X has countably many connected components. Use this result to show that every open subset of $\mathbb{R}$ is a countable disjoint union of open intervals.

8.6.4 The problem of embedding

Definition 8.44. Let $\mathscr{F}$ be a set of functions $X \to Y$. We say that $\mathscr{F}$ **separates points** of X, if for every distinct $x_1, x_2 \in X$ there exists $f \in \mathscr{F}$ such that $f(x_1) \neq f(x_2)$.

Problem 8.15. Let X be a nonempty compact metric space.

1. Prove that there is a sequence of continuous functions $(f_n)_{n\in\mathbb{Z}_+}$ from X to $[0,1]$ separating points of X.

2. Prove that

$$\Phi : X \to [0,1]^{\mathbb{Z}_+} \qquad x \mapsto (f_1(x), f_2(x), \dots)$$

gives a homeomorphism $\Phi : X \to \Phi(X)$ where $\Phi(X)$ is a closed subspace of $[0,1]^{\mathbb{Z}_+}$ (equipped with the subspace topology).

The topological space $[0,1]^{\mathbb{Z}_+}$, equipped with the product topology, is called the **Hilbert cube**.

Hint. Part 1: Choose an infinite countable basis $\mathcal{B} = (U_1, U_2, \dots)$ for the topology of X where each U_n is nonempty. (Why can you do so?) Use Urysohn functions (Rem. 7.118) to construct $f_n : X \to [0,1]$ such that $f^{-1}(0) = X \backslash U_n$.

Part 2: Notice Pb. 7.8. □

Theorem 8.45. *Let X be a topological space. The following are equivalent.*

(1) X is a compact metrizable space.

(2) X is homeomorphic to a closed subset of the Hilbert cube $[0,1]^{\mathbb{Z}_+}$.

Proof. (1)⇒(2): By Pb. 8.15. (2)⇒(1): By Cor. 7.76, $[0,1]^{\mathbb{Z}_+}$ is metrizable. By countable Tychonoff theorem (Thm. 3.54 or Pb. 8.7), $[0,1]^{\mathbb{Z}_+}$ is compact. So its closed subsets are compact by Cor. 8.22. □

⋆ **Exercise 8.46.** Let $N \in \mathbb{Z}_+$. Prove that the following are equivalent.

(1) X is a compact Hausdorff space. Moreover, there exist $f_1, \dots, f_N \in C(X, \mathbb{R})$ separating points of X.

(2) X is homeomorphic to a bounded closed subset of $\mathbb{R}^N$.

9 The injection $\Psi : C(X, C(Y, \mathcal{V})) \to C(X \times Y, \mathcal{V})$

In this chapter, unless otherwise stated, X, Y are topological spaces, all normed vector spaces are over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and $\mathcal{V}$ is a normed vector space.

Remark 9.1. Let $\Phi : \mathcal{V} \to \mathcal{W}$ be a linear map of normed vector spaces. Recall that Φ is an isometry (in the category of metric spaces) iff $\|\Phi(u) - \Phi(v)\| = \|u - v\|$ for all $u, v \in \mathcal{V}$. By linearity, we have

$$\Phi \text{ is an isometry} \qquad \Longleftrightarrow \qquad \|\Phi(v)\| = \|v\| \text{ for all } v \in \mathcal{V} \tag{9.1}$$

Definition 9.2. A linear map $\Phi : \mathcal{V} \to \mathcal{W}$ of normed vector spaces over $\mathbb{F}$ is a called an **isomorphism of normed vector spaces** if Φ is an isometric isomorphism. If Φ is an isomorphism, and if one of $\mathcal{V}, \mathcal{W}$ is complete, then the other one is also complete. In this case, we call Φ an **isomorphism of Banach spaces**.

Recall that if X is compact, then the norm on $C(X, \mathcal{V})$ is assumed to be the l^∞-norm (cf. Conv. 3.49).

9.1 Ψ is bijective when Y is compact

Theorem 9.3. *Equip $C(Y, \mathcal{V})$ with the uniform convergence topology (cf. Exp. 7.77). Then there is a well-defined injective linear map*

$$\begin{gathered} \Psi : C(X, C(Y, \mathcal{V})) \to C(X \times Y, \mathcal{V}) \\ \Psi(F)(x, y) = F(x)(y) \end{gathered} \tag{9.2}$$

(for each $F \in C(X, C(Y, \mathcal{V}))$). Moreover, the following are true:

(a) If Y is compact, then Ψ is a linear isomorphism of vector spaces.

(b) If X, Y are compact, then Ψ is an isomorphism of normed vector spaces.

Proof. Step 1. Write $\mathcal{W} = C(Y, \mathcal{V})$ for simplicity. To prove that (9.2) is a well-defined map, we need to prove that for each $F \in C(X, \mathcal{W})$, the map $f = \Psi(F) : X \times Y \to \mathcal{V}$ sending (x, y) to $f(x, y) = F(x)(y)$ is continuous.

The continuity of $F : X \to \mathcal{W}$ means that if $(x_\alpha)_{\alpha \in I}$ is a net in X converging to x, then $F(x_\alpha)$ converges to $F(x)$, i.e.,

$$\lim_{\alpha \in I} A_\alpha = 0 \tag{9.3a}$$

where each $A_\alpha \in \overline{\mathbb{R}}_{\geqslant 0}$ is

$$A_\alpha = \sup_{y \in Y} \|f(x_\alpha, y) - f(x, y)\| \tag{9.3b}$$

Now, suppose that $(x_\alpha, y_\alpha)_{\alpha\in I}$ is a net in $X \times Y$ converging to (x, y). Then

$$\begin{aligned}
&\|f(x_\alpha, y_\alpha) - f(x, y)\| \leqslant \|f(x_\alpha, y_\alpha) - f(x, y_\alpha)\| + \|f(x, y_\alpha) - f(x, y)\| \\
\leqslant &A_\alpha + \|f(x, y_\alpha) - f(x, y)\|
\end{aligned}$$

Since $F(x) : y \in Y \mapsto f(x, y) \in \mathcal{V}$ is continuous, we have $\lim_\alpha \|f(x, y_\alpha) - f(x, y)\| = 0$. Therefore, by Squeeze theorem, we have

$$\lim_{\alpha\in I} \|f(x_\alpha, y_\alpha) - f(x, y)\| = 0 \tag{9.4}$$

Thus, f is continuous at every (x, y).

Step 2. Clearly Ψ is linear and injective. Assume that Y is compact. Then the surjectivity of Ψ follows from Exp. 8.12. This proves (a). Assume that X is also compact. Choose any $F \in C(X, \mathcal{W})$ and write $f = \Psi(F)$. Then

$$\sup_{x\in X} \sup_{y\in Y} \|f(x, y)\| = \sup_{x\in X, y\in Y} \|f(x, y)\|$$

by the easy Lem. 9.4. This proves that Φ is an isometry, and hence proves (b). □

Lemma 9.4. *Let $g : A \times B \to \overline{\mathbb{R}}$ be a function where A, B are sets. Then*

$$\sup_{a\in A} \sup_{b\in B} g(a, b) = \sup_{(a,b)\in A\times B} g(a, b)$$

Proof. Write $\lambda_a = \sup_{b\in B} g(a, b)$ and $\rho = \sup_{(a,b)\in A\times B} g(a, b)$. Then, clearly $\lambda_a \leqslant \rho$ for each a. So $\sup_a \lambda_a \leqslant \rho$. For each a, b we have $g(a, b) \leqslant \lambda_a$, and hence $g(a, b) \leqslant \sup_a \lambda_a$. Taking sup over a, b yields $\rho \leqslant \sup_a \lambda_a$. □

Remark 9.5. Thanks to Thm. 9.3, we can reduce many problems about multivariable functions to problems about single-variable functions. Here is an example we will study in the future: If $I = [a, b]$, $J = [c, d]$ are compact intervals and $F \in C(I \times J, \mathbb{R})$, then with the help of Thm. 9.3, the Fubini's theorem

$$\int_a^b \int_c^d F(x, y) dx dy = \int_c^d \int_a^b F(x, y) dx dy$$

for Riemann integrals follows directly from the easy general fact

$$\int_a^b \Lambda \circ f(x) dx = \Lambda\Big(\int_a^b f(x) dx\Big)$$

where $f : [a, b] \to \mathcal{W}$ is a continuous map to a real Banach space $\mathcal{W}$, and $\Lambda : \mathcal{W} \to \mathbb{R}$ is a continuous linear map.

Thm. 9.3 can be used the other way round: We will prove in the future that every Banach space $\mathcal{V}$ is isomorphic to a closed linear subspace of $C(Y, \mathbb{F})$ for some compact Hausdorff space Y. Thus, a problem about continuous maps $X \to \mathcal{V}$ (where $\mathcal{V}$ is an abstract Banach space) can be reduced to a problem about continuous scalar-valued functions $X \times Y \to \mathbb{F}$. □

In the following sections, we give a surprising application of Thm. 9.3: We show that uniform-convergence and equicontinuity, two closely related but different notions, can be understood in the same context.

9.2 Equicontinuity

Let I be a set not necessarily preordered or directed.

Definition 9.6. Assume that Y is a metric space. A family of functions $(f_\alpha)_{\alpha\in I}$ from X to Y is called **equicontinuous at** $x \in X$ if the following equivalent conditions hold:

(1) The function

$$X \to Y^I \qquad x \mapsto (f_\alpha(x))_{\alpha\in I} \tag{9.5}$$

is continuous at x, where Y^I is equipped with the uniform convergence topology (Exp. 7.77).

(2) For every $\varepsilon > 0$, there exists $U \in \mathrm{Nbh}_X(x)$ such that for every $p \in U$ we have

$$\sup_{\alpha\in I} d_Y(f_\alpha(p), f_\alpha(x)) < \varepsilon$$

Clearly, if $(f_\alpha)_{\alpha\in I}$ is equicontinuous at x, then $f_\alpha : X \to Y$ is continuous at x for every $\alpha \in I$. We say that $(f_\alpha)_{\alpha\in I}$ is **(pointwise) equicontinuous** if it is equicontinuous at every point of X.

Proof of equivalence. This is immediate if we choose the uniform convergence metric on Y^I to be

$$d((y_\alpha), (y'_\alpha)) = \min\Big\{\sup_{\alpha\in I} d_Y(y_\alpha, y'_\alpha), 1\Big\}$$

and use (the base version of) Def. 7.56-(2). □

Remark 9.7. Warning: The above definition of equicontinuity is weaker than the one in Rudin's book [Rud-P, Ch. 7], which will be called **uniform equicontinuity** in this course (cf. Def. 10.11).

Example 9.8. Assume that X and Y are metric spaces. Fix $C \geqslant 0$. Then

$$\{f \in Y^X : f \text{ has Lipschitz constant } C\}$$

is an equicontinuous family of functions $X \to Y$.

In the future, we will see that if $f : [a,b] \to \mathbb{R}$ (where $[a,b] \subset \mathbb{R}$) is differentiable and satisfies $|f'(x)| \leqslant C$ for all $x \in [a,b]$, then f has Lipschitz constant C. (For example, if we assume moreover that f' is continuous, then for each $a \leqslant x < y \leqslant b$ we have $|f(y) - f(x)| = |\int_x^y f'| \leqslant \int_x^y |f'| \leqslant C(y-x)$.) Therefore, all such functions form an equicontinuous family of functions. □

Equicontinuity is important for several reasons. First, equicontinuity is closely related to compactness, both under the uniform convergence topology and under the pointwise convergence topology. This is hinted at in Rem. 9.22, and will be explored in more detail in a future chapter. Second, equicontinuity and uniform convergence are symmetric notions:

Remark 9.9. Note that if $x \in \overline{X\backslash x}$, a map $\varphi : X \to Y$ is continuous at x iff $\lim_{p\to x} f(p) = f(x)$, where $\lim_{p\to x} f(p)$ is the limit of a net (cf. Rem. 7.82). Now, assume that Y is a metric space. Then we see that a family $(f_\alpha)_{\alpha\in I}$ in Y^X satisfies that

$$(f_\alpha)_{\alpha\in I} \text{ is equicontinuous at } x \qquad \Longleftrightarrow \qquad \lim_{p\to x}\sup_{\alpha\in I} d(f_\alpha(p), f_\alpha(x)) = 0 \tag{9.6}$$

If we compare this with

$$f_\alpha \rightrightarrows f \qquad \Longleftrightarrow \qquad \lim_{\alpha\in I}\sup_{x\in X} d(f_\alpha(x), f(x)) = 0 \tag{9.7}$$

(if I is a directed set and $f \in Y^X$), we see that equicontinuity and uniform convergence are "symmetric about the diagonal line of the Cartesian product $I \times X$": Equicontinuity is a uniform convergence over the index set I, and the uniform convergence $f_\alpha \rightrightarrows f$ is uniform over X.

The symmetry of equicontinuity and uniform convergence will be further studied in Sec. 9.3. (Indeed, we will see that it is better to view "uniform convergence + continuity" and "pointwise convergence + equicontinuity" as symmetric conditions.) As an application, we will see that equicontinuity is equivalent to uniform convergence for any sequence (f_n) of pointwise convergent continuous functions on a compact topological space. (See Cor. 9.26.) Thus, in this case, one can prove the uniform convergence of (f_n) by proving for instance that it converges pointwise and has a uniform Lipschitz constant.

9.3 Uniform convergence and equicontinuity: two faces of Ψ

9.3.1 Main results

In this subsection, we fix a directed set I, and let

$$I^* = I \cup \{\infty_I\}$$

where ∞_I is a new symbol not in I. Write ∞_I as ∞ for simplicity. Equip I^* with the standard topology as in Def. 7.121. Recall that this topology has basis

$$\mathcal{B} = \big\{\{\alpha\} : \alpha \in I\big\} \cup \big\{I^*_{\geqslant\alpha} : \alpha \in I\big\}$$

Clearly $I \times X$ is dense in $I^* \times X$. Equip $C(X, \mathcal{V})$ and $C(I^*, \mathcal{V})$ with the uniform convergence topologies.

Throughout this subsection, we fix a net $(f_\alpha)_{\alpha \in I}$ in $\mathcal{V}^X$ and an element $f_\infty \in \mathcal{V}^X$. Define

$$F : I^* \times X \to \mathcal{V} \qquad F(\mu, x) = f_\mu(x) \tag{9.8}$$

The meaning of the title of this section is illustrated by the following theorem.

Theorem 9.10. *We have (a)⇔(b) and (1)⇔(2) where:*

(a) $(f_\alpha)_{\alpha \in I}$ *converges uniformly to* f_∞*, and* $f_\alpha : X \to \mathcal{V}$ *is continuous for each* $\alpha \in I$*.*

(b) F *gives rise to a continuous map* $I^* \to C(X, \mathcal{V})$

(1) $(f_\alpha)_{\alpha \in I}$ *is equicontinuous and converges pointwise to* f_∞*.*

(2) F *gives rise to a continuous map* $X \to C(I^*, \mathcal{V})$*.*

Proof. Assume (a). Then we have $f_\infty \in C(X, \mathcal{V})$ due to Thm. 7.79. So F gives rise to a map $I^* \to C(X, \mathcal{V})$. Since $f_\alpha \rightrightarrows f_\infty$, we see that F is continuous by Pb. 7.6-2. This proves (b).

Assume (b), then the fact that the map $I^* \to C(X, \mathcal{V})$ has range in $C(X, \mathcal{V})$ means precisely that each f_α and f_∞ are continuous. The continuity of the map $I^* \to C(X, \mathcal{V})$ at ∞ means $f_\alpha \rightrightarrows f_\infty$. This proves (a).

(1)⇒(2): Assume (1). The equicontinuity of (f_α) is equivalent to that

$$x \in X \mapsto (f_\alpha(x))_{\alpha \in I} \in \mathcal{V}^I \tag{9.9}$$

is continuous where $\mathcal{V}^I$ is equipped with the uniform convergence topology. Equivalently, for each $x \in X$ and $\varepsilon > 0$ there is $U \in \mathrm{Nbh}_X(x)$ such that for all $p \in U$ and all $\alpha \in I$ we have

$$\|f_\alpha(p) - f_\alpha(x)\| \leqslant \varepsilon \tag{9.10}$$

Since $(f_\alpha)_{\alpha \in I}$ converges pointwise to f_∞, by applying $\lim_{\alpha \in I}$ to (9.10), we see that $\|f_\mu(p) - f_\mu(x)\| \leqslant \varepsilon$ for all $p \in U$ and $\mu \in I^*$. So

$$x \in X \mapsto (f_\mu(x))_{\mu \in I^*} \in \mathcal{V}^{I^*} \tag{9.11}$$

is continuous, where $\mathcal{V}^{I^*}$ is given the uniform convergence metric. By Pb. 7.6-2, the pointwise convergence of $(f_\alpha)_{\alpha \in I}$ to f_∞ is equivalent to the continuity of

$$\mu \in I^* \to f_\mu(x) \in \mathcal{V} \tag{9.12}$$

for each $x \in X$. So the map (9.11) has range inside $C(I^*, \mathcal{V})$.

(2)⇒(1): Assume (2). The continuity of $X \to C(I^*, \mathcal{V})$ implies that of (9.9). So $(f_\alpha)_{\alpha \in I}$ is equicontinuous. Its pointwise convergence to f is due to the continuity of (9.12) for each x, which is clearly true by (2). □

Remark 9.11. Conditions (a) and (1) in Thm. 9.10 are symmetric. Condition (a) says roughly that F converges uniformly under the limit over I, and converges pointwise under the limit over X. Condition (1) says roughly that F converges uniformly under the limit over X, and pointwise under the limit over I. The next theorem clarifies the relationship between these two symmetric conditions. We will see a similar condition in Thm. 9.28.

Theorem 9.12. *Consider the following statements:*

(1) The function $F : I^* \times X \to \mathcal{V}$ is continuous.

(2) $(f_\alpha)_{\alpha\in I}$ converges uniformly to f_∞, and $f_\alpha : X \to \mathcal{V}$ is continuous for each $\alpha \in I$.

(3) $(f_\alpha)_{\alpha\in I}$ is equicontinuous and converges pointwise to f_∞.

Then we have

$$\begin{gathered}(2) \Longrightarrow (1) \Longleftarrow (3)\\ (2) \Longleftrightarrow (1) \qquad \text{if } X \text{ is compact}\\ (1) \Longleftrightarrow (3) \qquad \text{if } I = \mathbb{N}\end{gathered}$$

where $\mathbb{N}$ is equipped with the usual order.

A more explicit description of condition (1) will be given in Prop. 9.16.

Proof. This follows immediately from Thm. 9.10, Thm. 9.3, and the fact that I^* is compact if $I = \mathbb{N}$. □

Remark 9.13. From the above proof, we see that the equivalence (1)⇔(3) holds in the more general case that I^* is compact. See Pb. 9.1 for an equivalent description of the compactness of I^*.

Example 9.14. Define $f_n : (0,1) \to \mathbb{R}$ by $f_n(x) = x^n$. Then $(f_n)_{n\in\mathbb{Z}_+}$ is equicontinuous (by Exp. 9.8) and pointwise convergent, but not uniformly convergent. Accordingly, $(0,1)$ is not compact.

Example 9.15. Let $I = \mathbb{Z}_+ \times \mathbb{Z}_+$, equipped with the product preorder: $(k_1,n_1) \leqslant (k_2,n_2)$ means $k_1 \leqslant k_2$ and $n_1 \leqslant n_2$. Consider $(f_{k,n})_{(k,n)\in I}$, where $f_{k,n} : [0,1] \to \mathbb{R}$ is defined by $f_{k,n}(x) = \dfrac{x^n}{k}$. Then $\lim_{(k,n)\in I} f_{k,n}$ converges uniformly to 0 by squeeze theorem and $0 \leqslant f_{k,n} \leqslant k^{-1}$. Thus, condition (1) of Thm. 9.12 is satisfied. However, this net of functions is not equicontinuous at $x = 1$. Moreover, for every $(k_0,n_0) \in I$, the net of functions $(f_{k,n})_{(k,n)\in I_{\geqslant(k_0,n_0)}}$ is not equicontinuous at 1.

Proof. Choose any $x \in [0,1)$. Then (recalling Lem. 9.4)

$$\sup_{k\geqslant k_0}\sup_{n\geqslant n_0}|f_{k,n}(1) - f_{k,n}(x)| = \sup_{k\geqslant k_0}\sup_{n\geqslant n_0}\frac{1-x^n}{k} = \sup_{k\geqslant k_0}\frac{1}{k} = \frac{1}{k_0}$$

□

In the following, let me give a more explicit description of condition (1) of Thm. 9.12 in terms of (f_α) and f_∞. More precisely, we show that (1) can be interpreted as the convergence of a double limit.

Proposition 9.16. *Assume that $f_\alpha : X \to \mathcal{V}$ is continuous for each $\alpha \in I$. Then part (1) of Thm. 9.12 is equivalent to both (1′) and (1″), where*

(1′) For each $x \in X$, we have

$$\lim_{\substack{(\alpha,p)\in I\times X\\(\alpha,p)\to(\infty,x)}} f_\alpha(p) = f_\infty(x) \tag{9.13}$$

(1″) For each $x \in X$ and $\varepsilon > 0$, there exist $\beta \in I$ and $U \in \mathrm{Nbh}_X(x)$ such that

$$\|f_\alpha(p) - f_\infty(x)\| < \varepsilon \qquad (\text{for all } \alpha \in I_{\geqslant\beta} \text{ and } p \in U) \tag{9.14}$$

Note that one can make sense of the LHS of (9.13) because $I \times X$ is clearly dense in $I^* \times X$.

Proof. The equivalence of (1′) and (1″) is clear by Def. 7.81. The continuity of f_α for all $\alpha \in I$ means precisely that $F|_{I\times X}$ is continuous. Therefore, (1′) is equivalent to part (1) of Thm. 9.12, thanks to the following Thm. 9.17. □

See Exp. 9.32 for an elementary example satisfying (1′) of Prop. 9.16 (or equivalently, (1) of Thm. 9.12), but not satisfying (2) or (3) of Thm. 9.12. (See also Pb. 9.5 and Thm. 9.31 for related facts.)

9.3.2 Proving continuity using limits

Theorem 9.17. *Let $\varphi : X \to Y$ be a map of topological spaces where Y is metrizable. Let A be a dense subset of X. Then the following are equivalent.*

(1) $\varphi : X \to Y$ is continuous.

(2) The restriction $\varphi|_A : A \to Y$ is continuous. Moreover, for each $x \in X\backslash A$, the restriction $\varphi|_{A\cup\{x\}}$ is continuous at x, namely (cf. Def. 7.81) ,

$$\lim_{\substack{p\in A\\p\to x}} \varphi(p) = \varphi(x) \tag{9.15}$$

Proof. Clearly (2) is equivalent to

$$\varphi|_{A\cup\{x\}} \text{ is continuous at } x \text{ for all } x \in X \tag{9.16}$$

So (1)⇒(2). Assume (2) and that Y is a metric space. Choose any $x \in X$. We want to show that $\varphi : X \to Y$ is continuous at x. Choose any $\varepsilon > 0$. Recall that an open subset of A is precisely the intersection of A and an open subset of X. Thus, by (9.16), there is $U \in \mathrm{Nbh}_X(x)$ such that for all $p \in A \cap U$ we have

$$d(\varphi(p), \varphi(x)) \leqslant \varepsilon/2 \tag{9.17}$$

It remains to prove (9.17) for all $p \in U$. Choose any $p \in U$. By Lem. 9.18, $A \cap U$ is dense in U. Thus, there is a net (p_α) in $A \cap U$ converging to p. In particular, for each α we have

$$d(\varphi(p_\alpha), \varphi(x)) \leqslant \varepsilon/2$$

Since $\varphi|_{A \cup \{p\}}$ is continuous (by (9.16)), we have $\lim_\alpha \varphi(p_\alpha) = \varphi(p)$. Thus, applying $\lim_\alpha$ to the above inequality proves (9.17) for $p \in U$. □

Lemma 9.18. *Suppose that A is a dense subset of X. Let U be an open subset of A. Then $A \cap U$ is dense in U.*

This lemma is clearly false if U is not assumed to be open: simply take $U = X \backslash A$.

First proof. Choose any $x \in U$. We want to find a net in $A \cap U$ converging to x. Since A is dense in X, there is a net $(x_\alpha)_{\alpha \in J}$ in X converging to x. Since U is a neighborhood of x, (x_α) is eventually in U, say $x_\alpha \in U$ whenever $\alpha \geqslant \beta$. Then $(x_\alpha)_{\alpha \in J_{\geqslant \beta}}$ is a net in $A \cap U$ converging to x. □

Second proof. We want to show that every nonempty open subset of U intersects $A \cap U$. But an open subset of U is precisely an open subset of X contained inside U (Exe. 7.52). So this set (when nonempty) must intersect A because A is dense in X. □

⋆ **Exercise 9.19.** In Thm. 9.17, weaken the metrizability of Y to the condition that Y is regular. (See the definition below.) Prove the conclusion of Thm. 9.17.

⋆ **Definition 9.20.** A topological space Y is called **regular** if for every $y \in Y$ and every $U \in \mathrm{Nbh}_Y(y)$ there exists $V \in \mathrm{Nbh}_Y(y)$ such that $\mathrm{Cl}_Y(V) \subset U$.

9.3.3 Immediate consequences of Thm. 9.12

The following result is parallel to Thm. 7.79 for uniformly convergent nets of continuous functions.

Corollary 9.21. *Let $(f_\alpha)_{\alpha \in I}$ be an equicontinuous net of functions $X \to \mathcal{V}$. Assume that (f_α) converges pointwise to $f : X \to \mathcal{V}$. Then f is continuous.*

Proof. Write $f_\infty = f$ and define F by (9.8). Then F is continuous by (3)⇒(1) of Thm. 9.12. So $f = f_\infty$ is continuous, since f is the composition of F with the inclusion map $x \in X \mapsto (\infty, x) \in I^* \times X$. □

Remark 9.22. In the future, we will study the general Tychonoff theorem, which says for example that if $(f_\alpha)_{\alpha\in I}$ is a net of functions $X \to \mathbb{R}^N$ which is pointwise bounded, i.e. $\sup_{\alpha\in I} \|f(x)\| < +\infty$ for all $x \in X$, then (f_α) has a subnet converging poinwise. However, if we assume moreover that each f_α is continuous, we cannot conclude in general that (f_α) has a subnet converging pointwise to a *continuous* function. But we can make such a conclusion when (f_α) is equicontinuous, thanks to Cor. 9.21. Therefore, Cor. 9.21 tells us that equicontinuity is useful for studying the problems of compactness of families of continuous functions (under the pointwise convergence topology). Cf. Thm. 17.7.

In fact, we have a slightly stronger version of Cor. 9.21:

Corollary 9.23. *Let $(f_\alpha)_{\alpha\in I}$ be a net of functions $X \to \mathcal{V}$ equicontinuous at x. Assume that (f_α) converges pointwise to $f : X \to \mathcal{V}$. Then f is continuous at x.*

Proof. Define X_x to be the same as X as a set, but has a different topology: the one generated by the basis

$$\mathcal{B}_x = \mathrm{Nbh}_X(x) \cup \big\{\{p\} : p \neq x\big\}$$

(cf. Exe. 7.62). Define $g_\alpha : X_x \to \mathcal{V}$ and $g : X_x \to \mathcal{V}$ to be the same as f_α and f. Then (g_α) is a net of equicontinuous functions converging pointwise to g. Therefore, by Cor. 9.21, g is continuous. So f is continuous at x. □

Remark 9.24. By a similar argument, we can generalize Thm. 7.79 to the following form: Let (f_α) be a net of functions $X \to \mathcal{V}$ converging uniformly to $f : X \to \mathcal{V}$. Suppose that each f_α is continuous at x. Then f is continuous at x.

Example 9.25. In this example, we pretend to know derivatives. Let (f_n) be a sequence of functions $\mathbb{R}_{\geqslant 0} \to \mathbb{R}$ defined by $f_n(x) = x^{1/n}$. (We understand $0^{\frac{1}{n}} = 0$.) Find all $x \in \mathbb{R}_{\geqslant 0}$ at which (f_n) is equicontinuous.

Proof. We prove that $\mathbb{R}_{>0}$ is the set of all points at which (f_n) is equicontinuous. First, assume $x > 0$. Choose $0 < a < 1 < b$ such that $a < x < b$. Then, on $[a,b]$, $f_n'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$ is bounded by $C = \max\{a^{-1}, b\}$. So $(f_n|_{[a,b]})$ has Lipschitz constant C by Exp. 9.8. So (f_n) is equicontinuous at x.

Note that (f_n) converges pointwise to f where $f(x) = 1$ if $x > 0$ and $f(0) = 0$. But f is not continuous at 0. So (f_n) is not equicontinuous at 0 due to Cor. 9.23. □

One can also prove that (f_n) is equicontinuous on $(0,1)\cup(1,+\infty)$ without using derivatives: See Thm. 9.31-1.

Corollary 9.26. *Let $(f_\alpha)_{\alpha\in I}$ be a net in $C(X,\mathcal{V})$ converging pointwise to $f \in \mathcal{V}^X$. Consider the following statements:*

(1) $(f_\alpha)_{\alpha\in I}$ converges uniformly to f.

(2) $(f_\alpha)_{\alpha\in I}$ is equicontinuous.

Then the following are true.

1. *If $(f_\alpha)_{\alpha\in I}$ is a sequence $(f_n)_{n\in\mathbb{Z}_+}$, then (1)⇒(2).*

2. *If X is compact, then (2)⇒(1).*

Proof. Immediate from Thm. 9.12. □

9.4 Uniform convergence and double limits

In view of Prop. 9.16, Thm. 9.12 implies that the convergence of a double limit is the consequence of uniform convergence. In this section, instead of using the language of topological spaces, we formulate this result in terms of double nets so that it can be applied to a broader context. We first give a useful criterion for uniform convergence.

Proposition 9.27. *Assume that $\mathcal{V}$ is a Banach space. Let $(f_\alpha)_{\alpha\in I}$ be a net in $C(X,\mathcal{V})$. Assume that (f_α) converges uniformly on a dense subset E of X. Then (f_α) converges uniformly on X to some $f\in C(X,\mathcal{V})$.*

The completeness of $\mathcal{V}$ is important here.

Proof. Since E is dense, by (7.9a) (applied to the function $|f_\alpha - f_\beta|$), we have

$$\sup_{x\in X}\|f_\alpha(x)-f_\beta(x)\| = \sup_{x\in E}\|f_\alpha(x)-f_\beta(x)\|$$

where the RHS converges to 0 under $\lim_{\alpha,\beta}$. Thus, (f_α) is a Cauchy net in $\mathcal{V}^X$ where $\mathcal{V}^X$ is equipped with the uniform convergence metric as in Exp. 7.77. So $(f_\alpha)_{\alpha\in I}$ converges uniformly on X to some $f:X\to\mathcal{V}$ by Thm. 7.78. By Thm. 7.79, f is continuous. □

Theorem 9.28 (Moore-Osgood theorem). *Let $(f_{\alpha,\beta})_{(\alpha,\beta)\in I\times J}$ be a net in a Banach space $\mathcal{V}$ with index set $I\times J$ where I,J are directed sets. Assume the following conditions:*

(1) For each $\alpha\in I$, there exists $f_{\alpha,\infty}\in\mathcal{V}$ such that

$$\lim_{\beta\in J} f_{\alpha,\beta} = f_{\alpha,\infty} \tag{9.18}$$

(2) For each $\beta\in J$, there exists $f_{\infty,\beta}\in\mathcal{V}$ such that

$$\lim_{\alpha\in I}\sup_{\beta\in J}\|f_{\alpha,\beta}-f_{\infty,\beta}\| = 0 \tag{9.19}$$

Then the following are true:

1. *The following limits exist and are equal:*

$$\lim_{(\alpha,\beta)\in I\times J} f_{\alpha,\beta} = \lim_{\alpha\in I} f_{\alpha,\infty} = \lim_{\beta\in J} f_{\infty,\beta} \tag{9.20}$$

2. *If $I = \mathbb{N}$ where $\mathbb{N}$ is equipped with the usual order, then*

$$\lim_{\beta\in J} \sup_{\alpha\in I} \|f_{\alpha,\beta} - f_{\alpha,\infty}\| = 0 \tag{9.21}$$

Conditions (1) and (2) in Thm. 9.28, which say that the limit of $f_{\alpha,\beta}$ is pointwise over one index and uniform in another index, should remind you of conditions (2) and (3) in Thm. 9.12 (cf. Rem. 9.11). In fact, we shall use Thm. 9.12 to understand and prove the Moore-Osgood theorem.

Proof. Part 1: By Thm. 5.29, it suffices to prove that $\lim_{\alpha,\beta} f_{\alpha,\beta}$ converges. Define topological spaces $I^* = I \cup \{\infty_I\}$ and $J^* = J \cup \{\infty_J\}$ as in Subsec. 9.3.1. Define

$$g_\alpha : J^* \to \mathcal{V} \qquad g_\alpha(\nu) = f_{\alpha,\nu}$$

where $g_\alpha(\infty_J) = f_{\alpha,\infty}$. By (1), for each $\alpha \in I$, the function g_α is continuous at ∞_J, and hence $g_\alpha \in C(J^*, \mathcal{V})$. By (2), $(g_\alpha)_{\alpha\in I}$ converges uniformly on J. Since J is a dense subset of J^*, by Prop. 9.27, $(g_\alpha)_{\alpha\in I}$ converges uniformly on J^* to some $g_{\infty_I} : J^* \to \mathcal{V}$. Thus, by Thm. 9.12, the function

$$F : I^* \times J^* \to \mathcal{V} \qquad F(\mu, \nu) = g_\mu(\nu)$$

is continuous. Its continuity at (∞_I, ∞_J) implies that $\lim_{\alpha\in I,\beta\in J} f_{\alpha,\beta} = \lim_{\alpha\in I,\beta\in J} F(\alpha, \beta)$ converges to $F(\infty_I, \infty_J)$.

Part 2: By Thm. 9.12, $(g_\alpha)_{\alpha\in I}$ is an equicontinuous family of functions $J^* \to \mathcal{V}$. Its equicontinuity at ∞_J means precisely (9.21). □

Remark 9.29. Whenever you see a theorem stated in very plain language but proved using a huge machinery, you should always ask yourself if a direct proof is possible. A huge machinery or fancy language is not always necessary for the proof, but often helps to understand the nature of the problem.

Now, since Thm. 9.28 is stated without using the language of topological spaces and continuous maps, it is desirable to have a direct and elementary proof. It could be done by directly translating the above proof (and the proof of the results cited in that proof) into the pure language of nets. However, we prefer to give a simpler proof which is related to, but is not a direct translation of, the above topological proof. □

A direct proof of Thm. 9.28. Part 1: By Thm. 5.29, it suffices to prove that $\lim_{\alpha,\beta} f_{\alpha,\beta}$ converges. Since V is complete, it suffices to prove the Cauchy condition:

(i) For each $\varepsilon > 0$, there exists $(\alpha, \beta) \in I \times J$ such that for all $(\mu, \nu) \in I_{\geqslant\alpha} \times J_{\geqslant\beta}$, we have $\|f_{\alpha,\beta} - f_{\mu,\nu}\| < \varepsilon$.

Choose $\varepsilon > 0$. By condition (2) of Thm. 9.28, $(f_{\alpha,\beta})$, as a net of functions $J \to \mathcal{V}$ with index set I, converges uniformly. Thus, by the Cauchy condition for the uniform convergence metric as in Exp. 7.77 (which is available due to Thm. 7.78), we have:

(ii) There exists $\alpha \in I$ such that for all $\mu \geqslant \alpha$, we have $\sup_{\nu\in J} \|f_{\alpha,\nu} - f_{\mu,\nu}\| < \varepsilon/2$.

Fix α as above. By condition (1), the limit $\lim_{\beta\in J} f_{\alpha,\beta}$ exists. Thus:

(iii) There exists $\beta \in J$ such that for all $\nu \geqslant \beta$ we have $\|f_{\alpha,\beta} - f_{\alpha,\nu}\| < \varepsilon/2$.

Combining (ii) and (iii) and using triangle inequality, we see that for each $\mu \geqslant \alpha$ and $\nu \geqslant \beta$,

$$\|f_{\alpha,\beta} - f_{\mu,\nu}\| \leqslant \|f_{\alpha,\beta} - f_{\alpha,\nu}\| + \|f_{\alpha,\nu} - f_{\mu,\nu}\| < \varepsilon$$

This proves (i).

Part 2: Assume $I = \mathbb{N}$. Since $(f_{\alpha,\beta})$ is a Cauchy net, for each $\varepsilon > 0$ there exist $\alpha_0 \in I, \beta_0 \in J$ such that for all $\alpha \geqslant \alpha_0$ and $\beta, \nu \geqslant \beta_0$ we have $\|f_{\alpha,\beta} - f_{\alpha,\nu}\| < \varepsilon$. Applying $\lim_\nu$ gives

$$\|f_{\alpha,\beta} - f_{\alpha,\infty}\| \leqslant \varepsilon \qquad (\forall \alpha \geqslant \alpha_0, \beta \geqslant \beta_0)$$

Since $I = \mathbb{N}$, there are finitely many α not $\geqslant \alpha_0$. For any such α, by condition (1) of Thm. 9.28, there exists $\beta_\alpha \in J$ such that for all $\beta \geqslant \beta_\alpha$, we have $\|f_{\alpha,\beta} - f_{\alpha,\infty}\| \leqslant \varepsilon$. Choose $\widetilde{\beta}$ greater than or equal to β_0 and all these (finitely many) β_α. Thus, we have $\|f_{\alpha,\beta} - f_{\alpha,\infty}\| \leqslant \varepsilon$ for all $\alpha \in I$ and all $\beta \geqslant \widetilde{\beta}$. This proves (9.21). □

Remark 9.30. The main theme of this chapter is the study of the relationship between the convergence of $\lim_{\alpha,\beta} f_{\alpha,\beta}$ and the uniform convergence of $\lim_\alpha f_{\alpha,\beta}$ and $\lim_\beta f_{\alpha,\beta}$. The main results of this chapter (i.e. Thm. 9.3, Thm. 9.12 (together with Prop. 9.16), and Moore-Osgood Thm. 9.28) can be summarized as follows:

(1) If one of $\lim_\alpha f_{\alpha,\beta}$ and $\lim_\beta f_{\alpha,\beta}$ converges uniformly and the other one converges pointwise, then $\lim_{\alpha,\beta} f_{\alpha,\beta}$ converges. (Consequently, by Thm. 5.29, we have $\lim_\alpha \lim_\beta f_{\alpha,\beta} = \lim_\beta \lim_\alpha f_{\alpha,\beta} = \lim_{\alpha,\beta} f_{\alpha,\beta}$.)

(2) If $\lim_\alpha f_{\alpha,\beta}$, $\lim_\beta f_{\alpha,\beta}$, and $\lim_{\alpha,\beta} f_{\alpha,\beta}$ all converge pointwise, and if "there is a compactness on β", then $\lim_\alpha f_{\alpha,\beta}$ converges uniformly over all β.

The detailed statements of (1) and (2) are given in Thm. 9.12 ((together with Prop. 9.16)), or equivalently, in Thm. 9.3. (See also Rem. 9.11.) Although Thm. 9.3 and Thm. 9.12 look very different, they are actually telling the same story. (We have proved Thm. 9.12 from Thm. 9.3. But it is not hard to see that Thm. 9.3 also implies Thm. 9.12.) The Moore-Osgood theorem is only about part (1), but not about part (2). (Or, more accurately, the second part of Moore-Osgood corresponds to a very weak version of (2).)

9.5 Problems and supplementary materials

Recall from the beginning of this chapter that X is a topological space and $\mathcal{V}$ is a normed vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Problem 9.1. Let I be a directed set. Let $I^* = I \cup \{\infty\}$, equipped with the standard topology as in Subsec. 9.3.1. Prove that the following are equivalent:

(1) I^* is compact.

(2) For each $\alpha \in I$, the complement of $I^*_{\geqslant \alpha} = \{\beta \in I^* : \beta \geqslant \alpha\}$ is a finite set.

Note. Prop. 8.30 can make your proof shorter. □

Problem 9.2. Let (f_n) be a sequence of functions $\mathbb{R}_{\geqslant 0} \to \mathbb{R}$ defined by $f(x) = x^{\frac{1}{n}}$ (as in Exp. 9.25). Give a direct proof that (f_n) is not equicontinuous at 0 using the definition of equicontinuity. Do not use Cor. 9.23.

Problem 9.3. Give a direct proof of Cor. 9.21 without using Thm. 9.3 (and its consequences) or using Thm. 9.28.

Problem 9.4. Give a direct proof of Cor. 9.26 without using Thm. 9.3 (and its consequences) or using Thm. 9.28.

⋆ **Problem 9.5.** Let $(f_\alpha)_{\alpha\in I}$ be a net in $C(X, \mathbb{R})$. Assume that $(f_\alpha)_{\alpha\in I}$ is increasing, i.e., $f_\alpha \leqslant f_\beta$ whenever $\alpha \leqslant \beta$. Assume that $(f_\alpha)_{\alpha\in I}$ converges pointwise to $f \in C(X, \mathbb{R})$. Prove that for every $x \in X$,

$$\lim_{\substack{\alpha\in I\\ p\to x}} f_\alpha(p) = f(x) \tag{9.22}$$

in the sense of Prop. 9.16. Namely, prove that for every $x \in X$ and $\varepsilon > 0$ there exist $\beta \in I$ and $U \in \mathrm{Nbh}_X(x)$ such that $|f_\alpha(p) - f(x)| < \varepsilon$ for all $\alpha \geqslant \beta$ and all $p \in U$.

Note. Let $g_\alpha = f - f_\alpha$. Then $(g_\alpha)_{\alpha\in I}$ is a decreasing net of continuous functions converging pointwise to 0. It suffices to prove the easier statement that $\lim\limits_{\alpha\in I, p\to x} g_\alpha(p) = 0$ for every $x \in X$. (Why is this sufficient?) □

⋆ **Theorem 9.31.** *Let $(f_\alpha)_{\alpha\in I}$ be a net in $C(X, \mathbb{R})$. Assume that $(f_\alpha)_{\alpha\in I}$ is increasing and converges pointwise to $f \in C(X, \mathbb{R})$. The following statements are true.*

1. *If $(f_\alpha)_{\alpha\in I}$ is a sequence $(f_n)_{n\in\mathbb{Z}_+}$, then $(f_n)_{n\in\mathbb{Z}_+}$ is equicontinuous.*

2. *(**Dini's theorem**) If X is compact, then $(f_\alpha)_{\alpha\in I}$ converges uniformly to f.*

Proof. By Pb. 9.5, $(f_\alpha)_{\alpha\in I}$ and $f_\infty = f$ satisfy (1′) of Prop. 9.16. Therefore, the two statements follow directly from Thm. 9.12. □

Example 9.32. Let $f_n : (0,1) \to \mathbb{R}$ be $f_n(x) = x^n$, where $n \in \mathbb{Z}$. Then $(f_n)_{n\in\mathbb{Z}}$ is a decreasing net of continuous functions converging pointwise to 0. Here, $\mathbb{Z}$ is given the usual order "$\leqslant$". By Pb. 9.5, (f_n) satisfies

$$\lim_{(n,p)\to(+\infty,x)} f_n(p) = f(x)$$

for all $x \in (0,1)$. (So it satisfies condition (1) of Thm. 9.12.) However, $(f_n)_{n\in\mathbb{Z}}$ is neither equicontinuous (since $\sup_{n\in\mathbb{Z}} |f_n(p) - f_n(x)| = +\infty$ whenever $0 < p < x < 1$) nor converging uniformly to 0 (since $\sup_{x\in(0,1)} |f_n(x)| = 1$ if $n > 0$). Accordingly, $(0,1)$ is not compact, and $I^* = I \cup \{\infty_I\}$ is not compact if $I = \mathbb{Z}$.

However, if we replace $\mathbb{Z}$ by $\mathbb{Z}_+$, then $(f_n)_{n\in\mathbb{Z}_+}$ is equicontinuous by Exp. 9.8 (applied to any compact subinterval of $(0,1)$), or by Thm. 9.31. But $(f_n)_{n\in\mathbb{Z}_+}$ is still not uniformly convergent. □

Problem 9.6. Let $X_1, X_2, \dots$ be a sequence of nonempty topological spaces. Let $S = \prod_{n\in\mathbb{N}_+} X_n$, equipped with the product topology. Let $f : S \to \mathbb{R}$ be continuous. Fix $(p_n)_{n\in\mathbb{Z}_+} \in S$. For each $n \in \mathbb{Z}_+$, define

$$\begin{gathered}\varphi_n : S \to S \\ (x_1, x_2, \dots, x_{n-1}, x_n, x_{n+1}, \dots) \mapsto (x_1, x_2, \dots, x_{n-1}, p_n, p_{n+1}, \dots)\end{gathered}$$

Prove for every $x_\bullet = (x_n)_{n\in\mathbb{Z}_+}$ that

$$\lim_{\substack{(n,y_\bullet)\in\mathbb{Z}_+\times S \\ (n,y_\bullet)\to(\infty,x_\bullet)}} f\circ\varphi_n(y_\bullet) = f(x_\bullet)$$

in the sense of Prop. 9.16. Conclude that $(f\circ\varphi_n)_{n\in\mathbb{Z}_+}$ is equicontinuous, and that if each X_n is compact then $(f\circ\varphi_n)$ converges uniformly to f. (Recall that in this case S is compact by the countable Tychonoff theorem, cf. Pb. 8.7.)

10 Extending continuous functions to the closures

10.1 Introduction

Let $\widetilde{X}, Y$ be topological spaces, let $X \subset \widetilde{X}$, and let $f : X \to Y$ be a continuous map. The **extension problem** asks whether f can be extended to a continuous map $\widetilde{f} : \widetilde{X} \to Y$. "Extended" means that $\widetilde{f}|_X = f$. Since we can try to first extend f from X to its closure $\mathrm{Cl}_{\widetilde{X}}(X)$, and then from $\mathrm{Cl}_{\widetilde{X}}(X)$ to $\widetilde{X}$, the extension problem can naturally be divided into two cases: (1) X is dense in $\widetilde{X}$. (2) X is closed in $\widetilde{X}$.

In this chapter, we study the first case. (The second case will be discussed in Sec. 15.4.) Assume that X is dense in $\widetilde{X}$. Then by Prop. 7.63, we know that f can have at most one extension if Y is Hausdorff. So there is essentially no uniqueness issue.

The study of extension problem in this case has a long history. As we have seen in Subsec. 7.6, the limits of functions can be understood in this light: If $x \in \widetilde{X}\backslash X$, then f can be extended to a continuous function on $X \cup \{x\}$ iff $\lim_{p\to x} f(p)$ exists. Of course, this is simply a rephrasing of the definition of $\lim_{p\to x} f(p)$. But the idea of "extending f to $\widetilde{X}$ by first extending f to a slightly larger set with one extra point $\{x\}$" is helpful and can sometimes simplify proofs.

Indeed, recall that in the proof of Prop. 9.16 we used Thm. 9.17, which tells us that if $\lim_{p\to x} f(p)$ converges for all $x \in \widetilde{X}\backslash X$, then f can be extended (necessarily uniquely) to a continuous $\widetilde{f} : \widetilde{X} \to Y$ where Y is assumed metrizable. Thus, the extensibility of f to $\widetilde{X}$ can be checked pointwise. Thm. 9.17 is our first important general result on the extension problem. Let me state Thm. 9.17 in the following equivalent way, which is more convenient for the study of extension problems.

Corollary 10.1. *Let $f : X \to Y$ be a continuous map of topological spaces where Y is metrizable and X is a dense subspace of a topological space $\widetilde{X}$. The following are equivalent:*

(1) There exists a continuous map $\widetilde{f} : \widetilde{X} \to Y$ such that $\widetilde{f}|_X = f$.

(2) For each $x \in \widetilde{X}\backslash X$, the limit $\lim_{p\to x} f(p)$ exists.

Proof. If (1) is true, then $\widetilde{f}|_{X\cup\{x\}}$ is continuous whenever $x \in \widetilde{X}\backslash X$. This proves (2). Conversely, assume (2). Extend f to a map $\widetilde{f} : \widetilde{X} \to Y$ by setting $\widetilde{f}(x) = \lim_{p\to x} f(p)$ if $x \in \widetilde{X}\backslash X$. Then $\widetilde{f}$ is continuous by Thm. 9.17. □

This chapter will focus on another useful method for extending continuous functions in the setting of metric spaces. A main result (cf. Cor. 10.9) is that if $\widetilde{X}, Y$ are metric spaces and Y is complete, then a sufficient condition for the extensibility of f onto $\widetilde{X}$ is that f is uniformly continuous. Moreover, uniform continuity is also a necessary condition if $\widetilde{X}$ is compact. (Recall that compact metric spaces are complete, cf. Thm. 3.23.) It is worth mentioning that the extensibility of f is a purely topological question, whereas the uniform continuity of f depends on the metric on $\widetilde{X}$.

10.2 Uniform continuity

Fix metric spaces $\widetilde{X}, Y$ and a dense subset $X \subset \widetilde{X}$.

10.2.1 Basics

We first give some examples of $f \in C(X, Y)$ that cannot be extended to a continuous function on $\widetilde{X}$. Since uniform continuity is a sufficient condition for the extensibility (when Y is complete), these examples are not uniformly continuous. Recall our convention that subsets of $\mathbb{R}$ or $\mathbb{R}^N$ are equipped with the Euclidean metrics.

Example 10.2. Assume $X = (0, +\infty)$, $\widetilde{X} = [0, +\infty)$, $Y = \mathbb{R}$, and $f : X \to Y$ is defined by $f(x) = 1/x$. Then $\lim_{x\to 0} f(x)$ is $+\infty$ in $\overline{\mathbb{R}}$, and hence does not converge in Y. So f cannot be extended onto $\widetilde{X}$.

Example 10.3. $X = (0, 1]$, $\widetilde{X} = [0, 1]$, $Y = \mathbb{R}$, $f : X \to Y$ is defined by $f(x) = \sin(1/x)$. Then $\lim_{n\to\infty} f(x_n)$ equals 0 if the sequence (x_n) in X converging to 0 is defined by $x_n = 1/2n\pi$, and equals 1 if $x_n = 1/(2n + \frac{1}{2})\pi$. Thus, by Rem. 7.85, $\lim_{x\to 0} f(x)$ does not exist in Y. So f cannot be extended onto $\widetilde{X}$.

Definition 10.4. A map $f : X \to Y$ is called **uniformly continuous** if the following equivalent conditions are satisfied:

(1) For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, x' \in X$ we have

$$d(x, x') < \delta \qquad \Longrightarrow \qquad d(f(x), f(x')) < \varepsilon \tag{10.1a}$$

(2) For every nets $(x_\alpha)_{\alpha\in I}, (x'_\alpha)_{\alpha\in I}$ in X (with the same index set I) we have

$$\lim_{\alpha\in I} d(x_\alpha, x'_\alpha) = 0 \qquad \Longrightarrow \qquad \lim_{\alpha\in I} d(f(x_\alpha), f(x'_\alpha)) = 0 \tag{10.1b}$$

(3) For every sequences $(x_n)_{n\in\mathbb{Z}_+}, (x'_n)_{n\in\mathbb{Z}_+}$ in X we have

$$\lim_{n\to\infty} d(x_n, x'_n) = 0 \qquad \Longrightarrow \qquad \lim_{n\to\infty} d(f(x_n), f(x'_n)) = 0 \tag{10.1c}$$

Uniformly continuous maps are clearly continuous. Def. 10.4-(2) says that uniformly continuous functions are those sending Cauchy-equivalent nets (cf. Def. 5.36) to Cauchy-equivalent nets.

Proof of equivalence. (1)⇒(2): Assume (1). Choose nets $(x_\alpha)_{\alpha\in I}, (x'_\alpha)_{\alpha\in I}$ in X. Choose any $\varepsilon > 0$. Choose $\delta > 0$ such that (10.1a) holds. If $d(x_\alpha, x'_\alpha) \to 0$, then $d(x_\alpha, x'_\alpha) < \delta$ for sufficiently large α. So $d(f(x_\alpha), f(x'_\alpha)) < \varepsilon$ for sufficiently large α. This proves (2).

(2)⇒(3): Obvious.

¬(1) ⇒ ¬(3): Assume that (1) is false. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exist $x, x' \in X$ with $d(x, x') < \delta$ such that $d(f(x), f(x')) \geqslant \varepsilon$. Thus, for each $n \in \mathbb{Z}_+$, there exist $x_n, x'_n \in X$ such that $d(x_n, x'_n) < 1/n$ and $d(f(x_n), f(x'_n)) \geqslant \varepsilon$. The sequences $(x_n), (x'_n)$ imply that (10.1c) is false. □

Corollary 10.5. *Assume that $f : X \to Y$ is uniformly continuous. Let $(x_\alpha)_{\alpha\in I}$ be a Cauchy net in X. Then $(f(x_\alpha))_{\alpha\in I}$ is a Cauchy net in Y.*

Proof. Apply Def. 10.4-(2) to the nets $(x_\alpha)_{(\alpha,\beta)\in I^2}$ and $(x_\beta)_{(\alpha,\beta)\in I^2}$ of X. □

10.2.2 Extensibility of uniformly continuous functions

The following theorem can be viewed as the uniform continuity version of Prop. 9.27. In particular, both results assume the completeness of the codomain.

Theorem 10.6. *Let $f : X \to Y$ be uniformly continuous, and assume that Y is complete. Then there exists a (necessarily unique) uniformly continuous $\widetilde{f} : \widetilde{X} \to Y$ satisfying $\widetilde{f}|_X = f$.*

Proof. Choose any $x \in \widetilde{X}$. Since X is dense in $\widetilde{X}$, we can choose a sequence (x_n) in X converging to x in $\widetilde{X}$. In particular, (x_n) is a Cauchy sequence. Therefore, by Cor. 10.5, $(f(x_n))$ is a Cauchy sequence in Y, which converges to some point $\widetilde{f}(x) \in Y$ by the completeness of Y. If $x \in X$, we assume that (x_n) is the constant sequence x. This shows that $\widetilde{f}(x) = f(x)$ if $x \in X$.

We have constructed a function $\widetilde{f} : \widetilde{X} \to Y$ satisfying $\widetilde{f}|_X = f$. Let us prove that $\widetilde{f}$ is uniformly continuous. Choose any $\varepsilon > 0$. Since f is uniformly continuous, there is $\delta > 0$ such that for all $p, q \in X$ we have

$$d(p,q) < 2\delta \qquad \Longrightarrow \qquad d(f(p), f(q)) < \varepsilon/2$$

Choose any $x, x' \in \widetilde{X}$ satisfying $d(x, x') < \delta$. By our construction of $\widetilde{f}$, there are sequences (x_n) in X converging to x and (x'_n) in X converging to x' such that $f(x_n) \to \widetilde{f}(x)$ and $f(x'_n) \to \widetilde{f}(x')$. Thus, there exist $N \in \mathbb{Z}_+$ such that for all $n \geqslant N$ we have

$$d(x, x_n) < \frac{\delta}{2} \qquad d(x', x'_n) < \frac{\delta}{2} \qquad d(\widetilde{f}(x), f(x_n)) < \frac{\varepsilon}{4} \qquad d(\widetilde{f}(x'), f(x'_n)) < \frac{\varepsilon}{4}$$

Choose $n = N$. Then by triangle inequality, we have $d(x_n, x'_n) < 2\delta$, and hence $d(f(x_n), f(x'_n)) < \varepsilon/2$. So $d(\widetilde{f}(x), \widetilde{f}(x')) < \varepsilon$ by triangle inequality again. □

We now study the other direction. The following theorem implies that if a continuous $f : X \to Y$ can be extended to a continuous $\widetilde{f} : \widetilde{X} \to Y$, and if $\widetilde{X}$ is compact, then f is uniformly continuous.

Theorem 10.7. *Suppose that $f : X \to Y$ is continuous and X is compact. Then f is uniformly continuous.*

In the same spirit as in Sec. 8.2, we give two proofs for this theorem, one using sequences and the other using open covers.

First proof. Assume that f is not uniformly continuous. By Def. 10.4-(3), there exist sequences (x_n) and (x'_n) in X such that $\lim_{n\to\infty} d(x_n, x'_n) = 0$, and that $d(f(x_n), f(x'_n)) \nrightarrow 0$. The latter means that there is $\varepsilon > 0$ such that $d(f(x_n), f(x'_n))$ is frequently $\geqslant \varepsilon$. Thus, by passing to a subsequence, we may assume that $d(f(x_n), f(x'_n)) \geqslant \varepsilon$ for all n. Since $X \times X$ is sequentially compact, by replacing (x_n, x'_n) with a convergent subsequence, we assume that $x_n \to x$ and $x'_n \to x'$ where $x, x' \in X$. Since $d(x_n, x'_n) \to 0$, we must have $x = x'$. By the continuity of f, $f(x_n)$ and $f(x'_n)$ converge to $f(x) = f(x')$, contradicting the fact that $d(f(x_n), f(x'_n)) \geqslant \varepsilon$ for all n. □

To prove Thm. 10.7 using open covers, we prove a more general result instead. The following theorem is useful for proving properties of the form "there exists $\delta > 0$ such that for all $x, x' \in X$ satisfying $d(x, x') < \delta$, we have ...".

Theorem 10.8 (Lebesgue number lemma). *Assume that the metric space X is compact. Let $\mathfrak{U} \subset 2^X$ be an open cover of X. Then there exists $\delta > 0$ satisfying the following conditions:*

- *For every $x \in X$ there exists $U \in \mathfrak{U}$ such that $B_X(x, \delta) \subset U$.*

The number δ in Thm. 10.8 is called a **Lebesgue number of** $\mathfrak{U}$. In the following proof, we follow the local-to-global strategy as in Sec. 8.2.

Proof. Choose any $p \in X$. Then there is $U \in \mathfrak{U}$ containing p. So there is $\delta_p > 0$ such that $B(p, 2\delta_p) \subset U$. Therefore, there exists $V_p \in \mathrm{Nbh}_X(p)$ such that for every x in V_p we have $B(x, \delta_p) \subset U$. (Simply take $V_p = B(p, \delta_p)$.) This solves the problem locally: for each $x \in V_p$, the ball $B(x, \delta_p)$ is a subset of some member of $\mathfrak{U}$.

Since $X = \bigcup_{p\in X} V_p$ and since X is compact, there is a finite subset $E \subset X$ such that $X = \bigcup_{p\in E} V_p$. Take $\delta = \min\{\delta_p : p \in E\}$. For each $x \in X$, choose $p \in E$ such that $x \in V_p$. Then $B(x, \delta_p)$ is a subset of some member of $\mathfrak{U}$ by the last paragraph. So the same is true for $B(x, \delta)$. □

Of course, similar to the examples studied in Sec. 8.2, Thm. 10.8 can also be proved by contradiction and by using sequential compactness. See Pb. 10.2.

Second proof of Thm. 10.7. Let us verify Def. 10.4-(1). Choose any $\varepsilon > 0$. For each $x \in X$, the set $U_x = f^{-1}(B_Y(f(x), \varepsilon/2))$ is a neighborhood of x by Prop. 7.60. So $\{U_x : x \in X\}$ is an open cover of X. Let δ be a Lebesgue number of $\mathfrak{U}$. Choose any $x, y \in X$ satisfying $d(x, y) < \delta$. Then $B_X(x, \delta) \subset U_z$ for some $z \in X$. So $x, y \in B_X(x, \delta)$ and hence $x, y \in U_z$. Therefore,

$$d(f(x), f(y)) \leqslant d(f(x), f(z)) + d(f(z), f(y)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

□

Corollary 10.9. *Choose $f \in C(X, Y)$. Consider the following statements:*

(1) f is uniformly continuous.

(2) There exists $\widetilde{f} \in C(\widetilde{X}, Y)$ such that $\widetilde{f}|_X = f$.

Then (1)⇒(2) if Y is complete, and (2)⇒(1) if $\widetilde{X}$ is compact.

Proof. Immediate from Thm. 10.6 and Thm. 10.7. □

Example 10.10. Let $D = B_{\mathbb{C}}(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let $f : D \to Y$ be a continuous function where Y is a complete metric space. Then f is uniformly continuous iff $\lim_{w \to z} f(w)$ exists for every $z \in \mathbb{S}^1$.

Proof. By Cor. 10.1, the limit $\lim_{w \to z} f(w)$ exists for every $z \in \mathbb{S}^1$ iff f can be extended to a continuous function $\widetilde{f} : \overline{D} = D \cup \mathbb{S}^1 \to Y$. By Cor. 10.9, this is equivalent to that f is uniformly continuous (because $\overline{D}$ is compact and Y is compete). □

10.2.3 Uniform equicontinuity

Although the notion of uniform equicontinuity will rarely be used in our notes, it is used in many textbooks. So let me give a brief account of uniform equicontinuity.

Definition 10.11. Let $(f_\alpha)_{\alpha \in I}$ be a family of functions $X \to Y$. (Here, the index set I is not necessarily directed.) Define a metric d on Y^I in a similar way as (7.19), namely, if $\mathbf{y}, \mathbf{y}' \in Y^I$ then

$$d(\mathbf{y}, \mathbf{y}') = \min \Big\{1, \sup_{\alpha \in I} d_Y(\mathbf{y}(\alpha), \mathbf{y}'(\alpha))\Big\}$$

We say that $(f_\alpha)_{\alpha \in I}$ is **uniformly equicontinuous** if the map

$$X \to Y^I \qquad x \mapsto (f_\alpha(x))_{\alpha \in I} \tag{10.2}$$

is uniformly continuous with respect to the metric d. Clearly, this is equivalent to saying that:

- For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, x' \in X$ satisfying $d_(x, x') < \delta$, we have

$$\sup_{\alpha \in I} d_Y(f_\alpha(x), f_\alpha(x')) < \varepsilon$$

Uniformly equicontinuous families of functions are equicontinuous, because uniformly continuous functions are continuous. Conversely, we have:

Proposition 10.12. *Assume that X is compact and $(f_\alpha)_{\alpha \in I}$ is a family of functions $X \to Y$. Then $(f_\alpha)_{\alpha \in I}$ is equicontinuous iff it is uniformly equicontinuous.*

Proof. "$\Leftarrow$" is obvious, as mentioned above. "$\Rightarrow$" follows immediately by applying Thm. 10.7 to the continuous map (10.2). □

10.3 Completion of metric spaces

Fix a metric space X in this section. We are going to apply uniform continuity to the study of completions of metric spaces. Roughly speaking, a completion of X is a complete metric space $\widehat{X}$ containing X as a dense subspace. However, completions are not unique, but are unique "up to equivalence". So we want to show that two completions $\widehat{X}, \widetilde{X}$ of the same metric space X are equivalent. However, it is confusing to view X as a subset of $\widetilde{X}$ and $\widetilde{X}$ simultaneously. A better approach is to consider (automatically injective) isometries $\varphi : X \to \widehat{X}, \psi : X \to \widetilde{X}$, and to show that φ and ψ are equivalent using the language of commutative diagrams (cf. Sec. 1.2).

Definition 10.13. A **completion** of the metric space X is an isometry $\varphi : X \to \widehat{X}$ where $\widehat{X}$ is a complete metric space, and $\varphi(X)$ is dense in $\widehat{X}$. We sometimes just say that $\widehat{X}$ is a completion of X.

Thus, if A is a dense subset of a complete metric space B, then the inclusion map $A \hookrightarrow B$ is a completion. Therefore, $\mathbb{R}$ is a completion of $\mathbb{Q}$, and $[0, 1]$ is a completion of $(0, 1)$, $[0, 1)$, $[0, 1] \cap \mathbb{Q}$.

Example 10.14. Let A be a dense subset of a metric space X. Suppose that $\varphi : X \to \widehat{X}$ is a completion of X. Then $\varphi|_A : A \to \widehat{X}$ is clearly a completion of A.

Example 10.15. Let X be a subset of a complete metric space Y. Then $X \hookrightarrow \mathrm{Cl}_Y(X)$ is a completion of X because every closed subset of Y is complete (cf. Prop. 3.27), and hence $\mathrm{Cl}_Y(X)$ is complete. For example, $\{(x, y) \in \mathbb{R}^2 : x \geqslant 0\}$ is a completion of both $A = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ and $B = A \cap \mathbb{Q}^2$.

We want to prove that every metric space X has a completion $\widehat{X}$. First, we need a lemma, which can be viewed as analogous to Pb. 8.9.

Lemma 10.16. *Suppose that X is a dense subspace of a metric space $\widehat{X}$. Suppose that every Cauchy sequence in X converges to an element of $\widehat{X}$. Then $\widehat{X}$ is complete.*

Proof. Let (x_n) be a Cauchy sequence in $\widehat{X}$. Since X is dense, there exists $x'_n \in X$ such that $d(x_n, x'_n) < 1/n$. So (x'_n) is Cauchy-equivalent to (x_n). Thus (x'_n) is a Cauchy sequence by Exe. 5.37. By assumption, (x'_n) converges to some $x \in X$. So (x'_n) also converges to x by Exe. 5.37. □

Theorem 10.17. *Every metric space X has a completion $\varphi : X \to \widehat{X}$. Moreover, any completion $\psi : X \to \widetilde{X}$ is* ***equivalent*** *(also called* ***isomorphic****) to φ in the sense that there is an isometric isomorphism of metric spaces $\Phi : \widehat{X} \to \widetilde{X}$ such that the following diagram commutes:*

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ \widehat{X} \arrow[rr, "\Phi", "\simeq"'] & & \widetilde{X} \end{tikzcd} \tag{10.3}$$

Recall that the commutativity of (10.3) means that $\psi = \Phi \circ \varphi$.

Proof of existence. The construction of $\varphi : X \to \widehat{X}$ is similar to the construction of $\mathbb{R}$ from $\mathbb{Q}$ in Ch. 6. Let $\mathscr{C}$ be the set of Cauchy sequences in X. Let $\widehat{X} = \mathscr{C}/\sim$ be the quotient set (cf. Def. 1.18) where $\sim$ is the Cauchy-equivalence relation: $(x_n) \sim (y_n)$ iff $\lim_{n\to\infty} d(x_n, y_n) = 0$. We let $[x_n]_{n\in\mathbb{Z}_+}$ or simply let $[x_n]$ denote the equivalence class of (x_n) in $\widehat{X}$. The map φ is defined by

$$\varphi : X \to \widehat{X} \qquad x \mapsto [x]_{n\in\mathbb{Z}_+}$$

where $[x]_{n\in\mathbb{Z}_+}$ is the equivalence class of the constant sequence $(x, x, \dots)$.

Step 1: Let us define a metric $d_{\widehat{X}}$ on $\widehat{X}$. Note that if $(x_n), (y_n) \in \mathscr{C}$, then by triangle inequality,

$$|d(x_m, y_m) - d(x_n, y_n)| \leqslant d(x_m, x_n) + d(y_m, y_n)$$

where the RHS converges to 0 as $m, n \to +\infty$. Therefore, the LHS also converges to 0. This shows that $(d(x_n, y_n))_{n\in\mathbb{Z}_+}$ is a Cauchy sequence in $\mathbb{R}_{\geqslant 0}$, and hence converges. Therefore, we define

$$d_{\widehat{X}} : \widehat{X} \times \widehat{X} \to \mathbb{R}_{\geqslant 0}$$
$$d_{\widehat{X}}([x_n], [y_n]) = \lim_{n\to\infty} d(x_n, y_n)$$

This is well-defined: If $[x_n] = [x'_n]$, and $[y_n] = [y'_n]$, then

$$|d(x_n, y_n) - d(x'_n, y'_n)| \leqslant d(x_n, x'_n) + d(y_n, y'_n)$$

which converge to 0 as $n \to \infty$. So $(d(x'_n, y'_n))_{n\in\mathbb{Z}_+}$ and $(d(x_n, y_n))_{n\in\mathbb{Z}_+}$ are Cauchy-equivalent, and hence converge to the same number.

Clearly $d_{\widehat{X}}([x_n],[y_n]) = 0$ iff $(x_n) \sim (y_n)$ iff $[x_n] = [y_n]$. And clearly $d_{\widehat{X}}([x_n],[y_n]) = d_{\widehat{X}}([y_n],[x_n])$. If $[x_n],[y_n],[z_n]$ are in $\mathscr{C}$, applying $\lim_{n\to\infty}$ to

$$d(x_n, z_n) \leqslant d(x_n, y_n) + d(y_n, z_n)$$

yields $d_{\widehat{X}}([x_n],[z_n]) \leqslant d_{\widehat{X}}([x_n],[y_n]) + d_{\widehat{X}}([y_n],[z_n])$. So $d_{\widehat{X}}$ is a metric.

Step 2. The map $\varphi : X \to \widehat{X}$ is clearly an isometry. Let us show that it has dense range. Choose any $[x_n]_{n\in\mathbb{Z}_+} \in \widehat{X}$. We shall show that $\varphi(x_k) = [x_k, x_k, \dots]$ approaches $[x_n]_{n\in\mathbb{Z}_+}$ as $k \to \infty$.

For each k, we have

$$d_{\widehat{X}}(\varphi(x_k), [x_n]_{n\in\mathbb{Z}_+}) = \lim_{n\to\infty} d(x_k, x_n) \tag{10.4}$$

where the RHS converges because $d_{\widehat{X}}$ is defined. Since $(x_n)_{n\in\mathbb{Z}_+}$ is a Cauchy sequence in X, we have

$$\lim_{k,n\to\infty} d(x_k, x_n) = 0 \tag{10.5}$$

Therefore, by (10.5) and the convergence of the RHS of (10.4), we can use Thm. 5.29 to conclude that

$$\lim_{k\to\infty} d_{\widehat{X}}(\varphi(x_k), [x_n]_{n\in\mathbb{Z}_+}) = \lim_{k\to\infty}\lim_{n\to\infty} d(x_k, x_n) = \lim_{k,n\to\infty} d(x_k, x_n) = 0$$

Step 3. It remains to prove that $\widehat{X}$ is complete. By Lem. 10.16 (applied to $\varphi(X) \subset \widehat{X}$) and the fact that φ is an isometry, it suffices to prove that for every Cauchy sequence $(x_k)_{k\in\mathbb{Z}_+}$ in X, the sequence $(\varphi(x_k))_{k\in\mathbb{Z}_+}$ converges in $\widehat{X}$. But this is true: we have shown in Step 2 that $(\varphi(x_k))_{k\in\mathbb{Z}_+}$ converges to $[x_n]_{n\in\mathbb{Z}_+}$. $\square$

Proof of equivalence. Suppose that $\psi : X \to \widetilde{X}$ is another completion. The map

$$\Phi : \varphi(X) \to \psi(X) \qquad \varphi(x) \mapsto \psi(x)$$

is well-defined since φ is injective. Moreover, Φ is an isometry since φ and ψ are isometries. In particular, Φ is uniformly continuous. Therefore, by Cor. 10.9, Φ can be extended to a uniformly continuous map $\Phi : \widehat{X} \to \widetilde{X}$. Clearly $\psi = \Phi \circ \varphi$. The continuous map

$$\widehat{X} \times \widehat{X} \to \mathbb{R}$$
$$(p, q) \mapsto d_{\widetilde{X}}(\Phi(p), \Phi(q)) - d_{\widehat{X}}(p, q)$$

is zero on the dense subset $\varphi(X) \times \varphi(X)$ of its domain. Therefore it is constantly zero by Prop. 7.63. This proves that Φ is an isometry.

It remains to prove that Φ is surjective. Since $\widehat{X}$ is complete and Φ restricts to an isometric isomorphism of metric spaces $\widehat{X} \to \Phi(\widehat{X})$, $\Phi(\widehat{X})$ is a complete metric subspace of $\widehat{X}$. Thus, by Prop. 3.27, $\Phi(\widehat{X})$ is a closed subset of $\widetilde{X}$. But $\Phi(\widehat{X})$ is dense in $\widetilde{X}$ since it contains $\psi(X)$. Therefore $\Phi(\widehat{X}) = \widetilde{X}$. □

The proof of Thm. 10.17 is complete.

10.4 Why did Hausdorff believe in completion?

Thm. 10.17, the existence and uniqueness of completion of an arbitrary metric space, was proved by Hausdorff in his 1914 work introducing Hausdorff topological spaces [Hau14, Sec. 8.8, p.315]. The construction of completion using equivalence classes of Cauchy sequences is quite abstract: Although it mimics Cantor's construction of real numbers (cf. Ch. 6), its main application is not in the realm of finite-dimensional geometric objects, but in the world of function spaces. But what is the practical significance of the equivalence classes of Cauchy sequences of functions? In concrete analysis problems about functions, these objects are much more difficult to deal with than functions themselves.

Strangely enough, the only nontrivial examples we have now is $\mathbb{Q} \hookrightarrow \mathbb{R}$. Besides this, we do not yet have any exciting new metric spaces arising from completion. For example: we know that $[0, 1]$ is the completion of $(0, 1)$ under the Euclidean metric, and that $C([0, 1], \mathbb{R})$ is the completion of the set of real polynomials $\mathbb{R}[x]$ under the norm $\sup_{x\in[0,1]} |f(x)|$. (The density of the set of polynomials in $C([0, 1], \mathbb{R})$ is due to Weierstrass.) But $[0, 1]$ and $C([0, 1], \mathbb{R})$ are examples we are already familiar with.

Mathematicians do not generalize just for the sake of generalization. They want to solve problems by generalizing old concepts to a broader context. Moreover, mathematicians do not randomly choose a way of generalization and then build a huge theory. Instead, they develop a theory only in the direction that has already proved useful in solving explicit problems. Thus, Hausdorff proved Thm. 10.17 because he was already convinced of the importance of abstract completion by certain powerful examples. For the moment, we are not ready to study these examples rigorously. (We will do this in the next semester.) But I want to give an informal introduction to one of these examples which historically has paved the way for many important ideas in analysis.

10.4.1 The Fourier series method in integral equations

In the years of 1900-1907, many important progress has been made in integral equations, which originated from the study of Dirichlet problems (finding solutions of harmonic equation $\Delta\varphi(x, y) = 0$ with given boundary condition, where $\Delta = (\partial_x)^2 + (\partial_y)^2$). For example, one asks if there is $\lambda \in \mathbb{R}$ and a single-variable complex valued function $f : [0, 2\pi] \to \mathbb{C}$ satisfying the eigenvalue problem $Tf = \lambda f$ where

$$(Tf)(x) = \int_0^{2\pi} K(x,y) f(y) dx$$

and $K : [0, 2\pi]^2 \to \mathbb{R}$ is given. (Cf. also Sec. 2.1.)

Hilbert studied this problem using Fourier series. In the modern language, the theory of Fourier series claims that $C([0, 2\pi], \mathbb{C})$, under the L^2-norm

$$\|f\|_{L^2} = \sqrt{\int_0^{2\pi} |f(x)|^2 \cdot \frac{dx}{2\pi}} \tag{10.6}$$

has a completion

$$\boxed{\begin{gathered}\Phi : C([0, 2\pi], \mathbb{C}) \to l^2(\mathbb{Z}, \mathbb{C}) \\ \Phi(f) = \widehat{f}\end{gathered}} \tag{10.7}$$

where $l^2(\mathbb{Z}, \mathbb{C})$ is the space of all $g : \mathbb{Z} \to \mathbb{C}$ satisfying $\sqrt{\sum_{n \in \mathbb{Z}} |g(n)|^2} < +\infty$, and

$$\widehat{f} : \mathbb{Z} \to \mathbb{C} \qquad \widehat{f}(n) = \int_0^{2\pi} f(x) e^{-\mathbf{i}nx} \cdot \frac{dx}{2\pi}$$

gives the **Fourier coefficients** of f. See Cor. 20.42.

Hilbert studied the eigenvalue problem $Tf = \lambda f$ by transforming it into a linear algebra problem on $l^2(\mathbb{Z}, \mathbb{C})$ (so that T becomes an $\infty \times \infty$ discrete matrix $\widehat{T}$), finding the possible eigenvectors $\widehat{f}$ and eigenvalues λ satisfying

$$\widehat{T}\widehat{f} = \lambda \widehat{f}$$

and returning to the original problems by finding the function f whose Fourier coefficents are $\widehat{f}$. It is easy to return: if $f \in C([0, 2\pi], \mathbb{C})$, then one gets f from $\widehat{f}$ by the formula

$$f(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{\mathbf{i}nx} \tag{10.8}$$

where the RHS is called the **Fourier series** of f.

Here comes the crucial point: the range of Φ, namely $\{\widehat{f} : f \in C([0,1], \mathbb{C})\}$, is not the whole space $l^2(\mathbb{Z}, \mathbb{C})$ but only its dense subspace. This remains true if we enlarge $C([0,1], \mathbb{C})$ to the space of Riemann integrable functions $\mathscr{R}([0,1], \mathbb{C})$. However, the eigenvectors of $\widehat{T}$ found by Hilbert were only known to be elements of $l^2(\mathbb{Z},\mathbb{C})$. If we use (10.8) to find the eigenvectors of the original T, namely, if for an arbitrary $g \in l^2(\mathbb{Z},\mathbb{C})$ we write

$$\Psi(g) = \sum_{n\in\mathbb{Z}} g(n)e^{\mathbf{i}nx} \tag{10.9}$$

then $\Psi(g)$ is not necessarily continuous or even Riemann integrable. It seems that, in the light of the proof of Thm. 10.17, the only way to make sense of (10.9) is as follows:

- One views $\Psi(g)$ as the Cauchy-equivalence class of the Cauchy sequence $(s_n)_{n\in\mathbb{Z}_+}$ in $C([0,2\pi],\mathbb{C})$ under the L^2-norm, where $s_n(x) = \sum_{k=-n}^{n} g(k)e^{\mathbf{i}kx}$.

But how can we understand $\Psi(g)$ as an actual function on $[0,2\pi]$ solving the eigenvalue problem $T\Psi(g) = \lambda\Psi(g)$?

Therefore, (10.7) and (10.9) are the very first example of abstract completion, and also one of the most important examples motivating Hausdorff's study of completion in general.

10.4.2 Riesz-Fischer theorem

Hilbert established these results by 1906, the same year Fréchet defined metric spaces. The story was finished by Riesz and Fischer, who proved in 1907 that $\Psi(g)$ can actually be represented by a **Lebesgue measurable** $f : [0,2\pi] \to \mathbb{C}$ satisfying $\|f\|_{L^2} < +\infty$, where the norm $\|f\|_{L^2}$ is defined by (10.6) using Lebesgue integral instead of Riemann integral. Let $L^2([0,2\pi],\mathbb{C})$ denote the space of all such functions, and view $C([0,2\pi],\mathbb{C})$ as its subspace. Then using the language of Thm. 10.17, we have a commutative diagram

$$\begin{tikzcd} & C([0,2\pi],\mathbb{C}) \arrow[dl, "\Phi"'] \arrow[dr, hook, "\iota"] & \\ l^2(\mathbb{Z},\mathbb{C}) \arrow[rr, "\Psi", "\simeq"'] & & L^2([0,2\pi],\mathbb{C}) \end{tikzcd} \tag{10.10}$$

where ι is the inclusion map. Thus:

- The abstract completion $\Phi : C([0,2\pi],\mathbb{C}) \to l^2(\mathbb{Z},\mathbb{C})$ is equivalent to the concrete completion $C([0,2\pi],\mathbb{C}) \subset L^2([0,2\pi],\mathbb{C})$, where "concrete" means that it is function-theoretic.

This equivalence of abstract and concrete completions, connecting Hilbert's algebraic approach to integral equations and Lebesgue's function-theoretic approach to measure theory, is the original form of **Riesz-Fischer theorem** (proved in 1907),[1] one of the most significant theorems in early 20th century.

Thanks to Riesz-Fischer theorem, people were convinced that abstract completions of function spaces could deepen one's understanding of analysis in much the same way that Lebesgue's measure theory broadened one's understanding of functions by harmonizing with Hilbert's l^2 theory. So what is the hesitation in trying to prove an abstract theorem like Thm. 10.17?

Hausdorff proved the existence and uniqueness of completion in general, but it was Riesz-Fischer theorem that proved the value of abstract completion.

10.5 Completion of normed vector spaces

Fix a normed vector space V over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 10.18. A **completion of the normed vector space** V (or a **Banach space completion** of V) is a *linear* isomertry $\varphi : V \to \widehat{V}$ such that $\widehat{V}$ is a Banach space, and $\varphi(V)$ is dense in $\widehat{V}$.

Thus, if $\varphi : V \to \widehat{V}$ is a completion, then V is isomorphic to $\varphi(V)$ as normed vector spaces, and hence can be viewed as equivalently a dense normed subspace of $\widehat{V}$.

Theorem 10.19. *Every normed vector space V has a completion $\varphi : V \to \widehat{V}$. Morevoer, every completion $\psi : V \to \widetilde{V}$ is **isomorphic** to φ in the sense that there is an isomorphism of Banach spaces (cf. Def. 9.2) $\Phi : \widehat{V} \to \widetilde{V}$ such that the following diagram commutes:*

$$\begin{tikzcd}
 & V \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\
\widehat{V} \arrow[rr, "\Phi", "\simeq"'] & & \widetilde{V}
\end{tikzcd} \tag{10.11}$$

Proof of existence. By Thm. 10.17, we have a completion $\varphi : V \to \widehat{V}$ in the context of metric spaces. So $\widehat{V}$ is a completion of metric space with metric $d_{\widehat{V}}$, and φ is an isometry of metric spaces with dense range. We need to show that $\widehat{V}$ is a complete normed vector space, and that φ is linear with dense range. Indeed, we shall identify V with $\varphi(\widehat{V})$ via φ so that V is the metric subspace of $\widehat{V}$. We shall extend the structure of normed vector space from V to $\widehat{V}$. In the following, the

[1]More precisely, the original form of Riesz-Fischer theorem says that the Ψ in (10.10) (whose explicit description is as in (10.9)) is an isometric isomorphism. As a consequence, the space L^2 is complete, because it is fairly easy to show that l^2 is complete. However, most modern analysis textbooks present Riesz-Fischer theorem in the following simple form: "L^2 is a complete metric space".

convergence in $\widehat{V}$ and the continuity of the maps about $\widehat{V}$ are understood using the metric $d_{\widehat{V}}$.

The map

$$+ : V \times V \to V \qquad (u, v) \mapsto u + v$$

is Lipschitz continuous, and hence uniformly continuous. Therefore, by Cor. 10.9, it can be extended (uniquely) to a continuous map $+ : \widehat{V} \times \widehat{V} \to \widehat{V}$. Similarly, if $\lambda \in \mathbb{F}$, the Lipschitz continuous map

$$V \to V \qquad v \mapsto \lambda \cdot v$$

can be extended to a continuous map $\widehat{V} \to \widehat{V}$. Thus, we have defined the addition $+$ and the scalar multiplication $\cdot$ for $\widehat{V}$. Similarly, the map

$$\|\cdot\| : V \to \mathbb{R}_{\geqslant 0} \qquad v \mapsto \|v\|$$

is uniformly continuous and hence can be extended to a map $\widehat{V} \to \mathbb{R}_{\geqslant 0}$.

We want to show that the above addition, scalar multiplication, and norm function make $\widehat{V}$ a normed vector space. For example, suppose that $\lambda \in \mathbb{F}$. We want to prove that $\lambda(u + v) = \lambda u + \lambda v$ for all $u, v \in \widehat{V}$, and we know that this is true when $u, v \in V$. Indeed, since $V \times V$ is dense in $\widehat{V} \times \widehat{V}$, and since the following two continuous maps

$$(u, v) \in \widehat{V} \times \widehat{V} \mapsto \lambda(u + v) \in \widehat{V}$$
$$(u, v) \in \widehat{V} \times \widehat{V} \mapsto \lambda u + \lambda v \in \widehat{V}$$

are equal on $V \times V$, these two maps are the same by Prop. 7.63. The same argument proves that $\widehat{V}$ is a vector space.

Since the following continuous map

$$\varphi : (u, v) \in \widehat{V} \times \widehat{V} \mapsto \|u\| + \|v\| - \|u + v\| \in \mathbb{R}$$

satisfies $\varphi(V \times V) \subset \mathbb{R}_{\geqslant 0}$, by $\varphi(\widehat{V} \times \widehat{V}) \subset \overline{\varphi(V \times V)}$ (due to Prop. 7.63 again), we conclude that $\varphi(\widehat{V} \times \widehat{V}) \subset \mathbb{R}_{\geqslant 0}$. So $\|u + v\| \leqslant \|u\| + \|v\|$ for all $u, v \in \widehat{V}$. A similar argument shows $\|\lambda v\| = |\lambda| \cdot \|v\|$.

Since the metric on V is induced by the norm of V, the map

$$\widehat{V} \times \widehat{V} \to \mathbb{R} \qquad (u, v) \mapsto d_{\widehat{V}}(u, v) - \|u - v\|$$

is zero on the dense subset $V \times V$ of $\widehat{V} \times \widehat{V}$. Since this map is continuous, it is constantly zero. In particular, if $v \in \widehat{V}$ satisfies $\|v\| = 0$, then $d_{\widehat{V}}(v, 0) = 0$, and hence $v = 0$. So $\|\cdot\|$ is a norm on $\widehat{V}$, and the complete metric $d_{\widehat{V}}$ on $\widehat{V}$ (arising from the metric-space-completion of V) is defined by this norm. So this norm is complete. Therefore, $\widehat{V}$ is a Banach space. Since V is dense in $\widehat{V}$ under $d_{\widehat{V}}$, V is dense in $\widehat{V}$ under the norm of $\widehat{V}$. $\square$

Proof of uniqueness. By Thm. 10.17, there is a unique isometric isomorphism of metric spaces $\Phi : \widehat{V} \to \widetilde{V}$ such that $\psi = \Phi \circ \varphi$. Since φ and ψ are linear injections, Φ is a linear isomorphism when restricted to $\varphi(V) \to \psi(V)$. Since Φ is continuous and $\varphi(V)$ is dense in $\widehat{V}$, we conclude that Φ is linear thanks to the following property. $\square$

Proposition 10.20. *Let $T : V \to W$ be a continuous map of normed vector spaces. Assume that V_0 is a dense linear subspace of V. Assume that $T|_{V_0} : V_0 \to W$ is linear. Then T is linear.*

Proof. This is same as the proof of the existence part of Thm. 10.19. Choose any $\alpha, \beta \in \mathbb{F}$. Then the following continuous map

$$(u, v) \in V \times V \mapsto T(\alpha u + \beta v) - (\alpha T(u) + \beta T(v))$$

is zero on the dense subset $V_0 \times V_0$. So it is zero on $V \times V$. $\square$

Exercise 10.21. Let V and W be normed vector spaces over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ (resp. $\mathbb{F} = \mathbb{C}$). Let $T : V \to W$ be a continuous map. Assume that V_0 is a dense $\mathbb{K}$-linear subspace of V, where $\mathbb{K} = \mathbb{Q}$ (resp. $\mathbb{K} = \mathbb{Q} + \mathbf{i}\mathbb{Q}$). Assume that the restriction $T|_{V_0} : V_0 \to W$ is $\mathbb{K}$-linear. Prove that $T : V \to W$ is $\mathbb{F}$-linear.

Remark 10.22. So far in this course, we have proved a lot of results about functions whose codomains are normed vector spaces. Some results assume that these spaces are Banach (i.e. complete), some do not. Thanks to Thm. 10.19, we can assume that all these results are stated only for Banach spaces, and then check whether they also hold for normed vector spaces in general (which is not difficult). This will make us easier to remember theorems.

For example, suppose that we know that Thm. 7.79 holds only for Banach spaces: Namely, suppose we know that for any Banach space V and topological space X, if (f_α) is a net in $C(X, V)$ converging uniformly to $f : X \to V$, then f is continuous. Then we know that this result also holds when V is a normed vector space. To see this, consider the completion $V \subset \widehat{V}$. Then (f_α) is a net of continuous functions $X \to \widehat{V}$ converging uniformly to some $f : X \to \widehat{V}$ (satisfying $f(X) \subset V$). Then $f : X \to \widehat{V}$ is continuous. Hence $f : X \to V$ is continuous.

Consider Prop. 9.27 as another example. It tells us that if V is a Banach space and (f_α) is a net in $C(X, V)$ converging uniformly on a dense subset $E \subset X$, then (f_α) converges uniformly on X. Now assume that V is only a normed vector space, and take completion $V \subset \widehat{V}$. Then (f_α) is a net in $C(X, \widehat{V})$ converges uniformly on E to a function $E \to V$. Thus, by Prop. 9.27, it converges uniformly to a function $f : X \to \widehat{V}$. Although we know $f(E) \subset V$ by assumption, we do not know whether $f(X) \subset V$. So we cannot prove the normed vector space version of Prop. 9.27. $\square$

10.6 Bounded linear maps

In this section, we consider normed vector spaces over a given field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

We have seen that uniform continuity is crucial to the construction of addition and scalar multiplication in Thm. 10.19. Indeed, uniform continuity is ubiquitous in the world of normed vector spaces: We shall see that every continuous linear map of normed vector spaces is uniformly continuous.

Definition 10.23. Let $T : V \to W$ be a linear map of normed vector spaces. The **operator norm** $\|T\|$ is defined to be

$$\|T\| \xlongequal{\text{def}} \sup_{v \in \overline{B}_V(0,1)} \|Tv\| \tag{10.12}$$

Remark 10.24. $\|T\|$ is the smallest number in $\overline{\mathbb{R}}_{\geqslant 0}$ satisfying

$$\|Tv\| \leqslant \|T\| \cdot \|v\| \qquad (\forall v \in V) \tag{10.13}$$

Proof. (10.13) is clearly true when $v = 0$. Assume $v \neq 0$. Since $v/\|v\| \in \overline{B}_V(0,1)$, we have $\|T(v/\|v\|)\| \leqslant \|T\|$. This proves (10.13).

Now suppose that $C \in \overline{\mathbb{R}}_{\geqslant 0}$ satisfies that $\|Tv\| \leqslant C\|v\|$ for all v. Then for each $v \in \overline{B}_V(0,1)$ we have $\|Tv\| \leqslant C$. So $\|T\| \leqslant C$. □

Proposition 10.25. *Let $T : V \to W$ be a linear map of normed vector spaces. Then the following are equivalent:*

(1) T is Lipschitz continuous.

(2) T is continuous.

(3) T is continuous at 0.

(4) $\|T\| < +\infty$.

Moreover, if one of these conditions holds, then T has Lipschitz constant $\|T\|$.

Proof. Clearly (1)⇒(2) and (2)⇒(3). Suppose (3) is true. Note that $T0 = 0$. So there is $\delta > 0$ such that $Tv \in \overline{B}_W(0,1)$ for all $v \in \overline{B}_V(0,\delta)$. Namely, for all $v \in V$ satisfying $\|v\| \leqslant \delta$ we have $\|Tv\| = \|Tv - T0\| \leqslant 1$. Thus, if $\|v\| \leqslant 1$, then $\|\delta v\| \leqslant \delta$. So

$$\|Tv\| = \delta^{-1}\|T(\delta v)\| \leqslant \delta^{-1}$$

This proves $\|T\| \leqslant \delta^{-1}$. So (4) is proved.

Assume (4). By Rem. 10.24, for each $u, v \in V$ we have

$$\|Tu - Tv\| = \|T(u - v)\| \leqslant \|T\| \cdot \|u - v\|$$

This proves that T has Lipschitz constant $\|T\|$. □

Due to Prop. 10.25-(4), we make the following definition:

Definition 10.26. Let V, W be normed vector spaces over $\mathbb{F}$. We call $T : V \to W$ to be a **bounded linear map** if T is a continuous linear map. We write

$$\mathfrak{L}(V, W) = \{\text{bounded linear maps } V \to W\} \qquad \mathfrak{L}(V) = \mathfrak{L}(V, V) \tag{10.14}$$

Thus, the word "bounded" means that the linear map T is bounded on $\overline{B}_V(0, 1)$, but not that T is bounded on V.

Proposition 10.27. *$\mathfrak{L}(V, W)$ is a linear subspace of W^V, and the operator norm $\|\cdot\|$ is a norm on $\mathfrak{L}(V, W)$.*

Proof. By Rem. 10.24, for each linear $S, T : V \to W$ and $\lambda \in \mathbb{F}$, and for each $v \in V$ we have

$$\|(S + T)v\| \leqslant \|Sv\| + \|Tv\| \leqslant (\|S\| + \|T\|)\|v\|$$
$$\|\lambda Tv\| = |\lambda| \cdot \|Tv\| \leqslant |\lambda| \cdot \|T\| \cdot \|v\|$$

Thus, by Rem. 10.24 again, we have

$$\|S + T\| \leqslant \|S\| + \|T\| \qquad \|\lambda T\| \leqslant |\lambda| \cdot \|T\| \tag{10.15}$$

The proposition now follows easily from the above inequalities. (Notice Rem. 3.34) □

Since Lipschitz continuous functions are uniformly continuous, we have:

Proposition 10.28. *Let V_0 be a dense linear subspace of a normed vector space V. Let W be a Banach space. Let $T_0 : V_0 \to W$ be a bounded linear map. Then there is a unique bounded linear map $T : V \to W$ such that $T|_{V_0} = T_0$.*

Proof. Uniqueness is clear from the density of V_0. By Prop. 10.25, T_0 is uniformly continuous. Therefore, by Cor. 10.9, T_0 can be extended to a continuous map $T : V \to W$, which is linear by Prop. 10.20. □

10.7 Problems and supplementary material

Problem 10.1. Give a direct proof of Thm. 10.7 using open covers instead of using subsequences. Do not use Lebesgue numbers.

The following Pb. 10.2 gives another proof that sequentially compact metric spaces are compact. Therefore, do not use this fact in your solution of Pb. 10.2.

⋆ **Problem 10.2.** Let X be a sequentially compact metric space. Let $\mathfrak{U} \subset 2^X$ be an open cover of X.

1. Prove that $\mathfrak{U}$ has a Lebesgue number. Namely, prove that there exists $\delta > 0$ such that for every $x \in X$ there is $U \in \mathfrak{U}$ satisfying $B_X(x, \delta) \subset U$.

2. In our proof that X is separable (cf. Thm. 8.34), we showed that for every $\delta > 0$ there exists a finite set $E \subset X$ such that $d(x, E) < \delta$ for all $x \in X$. Use this fact and Part 1 to prove that $\mathfrak{U}$ has a finite subcover.

Definition 10.29. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space $\mathbb{V}$ over $\mathbb{R}$ or $\mathbb{C}$ are called **equivalent** if there exist $\alpha, \beta > 0$ such that for all $v \in V$ we have

$$\|v\|_1 \leqslant \alpha\|v\|_2 \qquad \|v\|_2 \leqslant \beta\|v\|_1$$

Clearly, two equivalent norms induce equivalent metrics, and hence induce the same topology.

Problem 10.3. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{F}^n$. Let $\nu : \mathbb{F}^n \to \mathbb{R}_{\geqslant 0}$ be a norm on $\mathbb{F}^n$.

1. Prove that there exists $\alpha > 0$ such that $\nu(\mathbf{x}) \leqslant \alpha\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{F}^n$. In particular, show that ν is continuous (under the Euclidean topology).

2. Let $\beta = \inf\{\nu(\mathbf{x}) : \mathbf{x} \in \mathbb{F}^n, \|\mathbf{x}\| \leqslant 1\}$. Prove that $\beta > 0$. Prove that $\|\mathbf{x}\| \leqslant \beta^{-1} \cdot \nu(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{F}^n$.

The above problem proves

Theorem 10.30. *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Then any norm on $\mathbb{F}^n$ is equivalent to the Euclidean norm. In particular, the operator norm on $\mathbb{F}^{m \times n}$ (if we view an $m \times n$ matrix as an element of $\mathfrak{L}(\mathbb{F}^n, \mathbb{F}^m)$) is equivalent to the Euclidean norm.*

11 Derivatives

11.1 Basic properties of derivatives

Fix a Banach space V over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let Ω be a nonempty open subset of $\mathbb{C}$. The following assumption will be often considered:

Definition 11.1. Let $f : [a,b] \to V$ (where $-\infty < a < b < +\infty$) be a map with variable t, and let $x \in [a,b]$. The **derivative** of f at x is

$$f'(x) \equiv \frac{df}{dt}(x) \xlongequal{\text{def}} \lim_{\substack{t\in[a,b]\setminus\{x\}\\ t\to x}} \frac{f(t)-f(x)}{t-x} = \lim_{\substack{h\in[a-x,b-x]\setminus\{0\}\\ h\to 0}} \frac{f(x+h)-f(x)}{h}$$

provided that the limits converge. In other words (cf. Def. 7.81),

- $f'(x)$ converges to $v \in V$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $t \in [a,b]$ satisfying $0 < |t-x| < \delta$ we have
$$\Big\|\frac{f(t)-f(x)}{t-x} - v\Big\| < \varepsilon$$

If $f'(x)$ exists for some x, we say that f is **differentiable** at x. If $f'(x)$ exists for every $x \in [a,b]$, we say that f is a **differentiable function** and view f' as a function $[a,b] \to V$.

Derivatives on intervals $[a,b), (a,b], (a,b)$ are understood in a similar way. $\square$

Definition 11.2. Let E be a subset of $\mathbb{R}^n$. Let $f : E \to V$ be a function with variables $t_1, \dots, t_n$, and $\mathbf{x} = (x_1, \dots, x_n) \in E$. Let $1 \leqslant i \leqslant n$. Suppose that there are a, b satisfying $-\infty < a < x_i < b < +\infty$ such that $(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$ belongs to E for all $t \in [a,b]$. The derivative of the function $t_i \mapsto f((x_1, \dots, x_{i-1}, t_i, x_{i+1}, \dots, x_n)$ at $t = x_i$ is denoted by

$$\frac{\partial f}{\partial t_i}(\mathbf{x}) \equiv \partial_i f(\mathbf{x})$$

and called the **partial derivative** of f at $\mathbf{x}$ with respect to the variable t_i.

Definition 11.3. If V is over $\mathbb{C}$, and if $f : \Omega \to V$ and $z \in \Omega$, we define the **derivative** of f at z to be

$$f'(z) = \lim_{\substack{w\in\Omega\setminus\{z\}\\ w\to z}} \frac{f(w)-f(z)}{w-z}$$

provided that the RHS exists, and simply write it as

$$\lim_{w\to z} \frac{f(w)-f(z)}{w-z} = \lim_{\zeta\to 0} \frac{f(z+\zeta)-f(z)}{\zeta}$$

Convention 11.4. Unless otherwise stated, when talking about derivatives of a function defined on an interval I, we always assume that I is inside $\mathbb{R}$ and has at least two points. When talking about derivatives of a function $f : \Omega \to V$, we always assume that V is over $\mathbb{C}$.

Definition 11.5. Given a function $f : E \to V$ where E is an interval in $\mathbb{R}$ with at least two points or $E = \Omega$, if $n \in \mathbb{N}$ and $x \in E$, we define the ***n*-th derivative** $f^{(n)}(x)$ inductively by $f^{(0)} = f$ and $f^{(n)}(x) = (f^{(n-1)})'(x)$ if $f^{(n-1)}$ exists on some neighborhood of x with respect to E. $f'', f''', f'''', \dots$ mean $f^{(2)}, f^{(3)}, f^{(4)}, \dots$.

The n-th partial derivative on the i-th variable is written as $\partial_i^n f$. □

It is desirable to use sequences or nets to study derivatives. For that purpose, the following lemma is useful:

Lemma 11.6. *Let E be an interval in $\mathbb{R}$ with at least two elements, or let $E = \Omega$. Let $z \in E$. Let $f : E \to V$. Let $v \in V$. The following are equivalent.*

(1) *We have $f'(z) = v$.*

(2) *For any sequence $(z_n)_{n\in\mathbb{Z}_+}$ in $E\backslash\{z\}$ converging to z, we have $\lim\limits_{n\to\infty} \dfrac{f(z_n) - f(z)}{z_n - z} = v$.*

(3) *For any net $(z_\alpha)_{\alpha\in\mathcal{I}}$ in E converging to z, we have $\lim\limits_{\alpha\in\mathcal{I}} \varphi(z_\alpha) = v$, where $\varphi : E \to V$ is defined by*

$$\varphi(w) = \begin{cases} \dfrac{f(w) - f(z)}{w - z} & \text{if } w \neq z \\ v & \text{if } w = z \end{cases} \tag{11.1}$$

Proof. The equivalence of (1)⇔(2) is due to Def. 7.81-(3m). Also, by Def. 7.81-(1), that $f'(z) = v$ is equivalent to that the function φ in (11.1) is continuous at z. So it is equivalent (3) by Def. 7.56-(1). □

Proposition 11.7. *Let E be an interval in $\mathbb{R}$ or $E = \Omega$. If $f : E \to V$ is differentiable at $z \in E$, then f is continuous at z.*

Proof. We consider the case $f : \Omega \to V$; the other case is similar. Choose any sequence (z_n) in $\Omega\backslash\{z\}$ converging to z. By Lem. 11.6, we have $\lim_n \frac{f(z_n)-f(z)}{z_n-z} = v$. Since $\lim_n(z_n - z) = 0$, by the continuity of scalar multiplication (Prop. 3.38), we have

$$\lim_{n\to\infty} f(z_n) - f(z) = 0 \cdot v = 0$$

Thus, by Def. 7.81-(3m), we obtain $\lim_{w\neq z, w\to z} f(w) = f(z)$, which means by Def. 7.81-(1) that f is continuous at z. □

Proposition 11.8. *Let E be an interval in $\mathbb{R}$ or $E = \Omega$, and $x \in E$. Suppose that f, g are functions $E \to V$ such that $f'(x)$ and $g'(x)$ exist. Then $(f+g)'(x)$ exists, and*

$$(f+g)'(x) = f'(x) + g'(x) \tag{11.2a}$$

If λ is a function $E \to \mathbb{F}$ such that $\lambda'(x)$ exists, then $(\lambda f)'(x)$ exists and satisfies the ***Leibniz rule***

$$(\lambda f)'(x) = \lambda'(x)f(x) + \lambda(x)f'(x) \tag{11.2b}$$

Assume moreover that λ does not have value 0. Then

$$\Big(\frac{1}{\lambda}\cdot f\Big)'(x) = \frac{-\lambda'(x)f(x) + \lambda(x)f'(x)}{\lambda(x)^2} \tag{11.2c}$$

Proof. The first formula is easy. To compute the second one, we choose any sequence (x_n) in $[a,b]\backslash\{x\}$ or $\Omega\backslash\{x\}$ converging to x. Then

$$\frac{\lambda(x_n)f(x_n) - \lambda(x)f(x)}{x_n - x} = \frac{(\lambda(x_n) - \lambda(x))}{x_n - x}\cdot f(x_n) + \lambda(x)\frac{(f(x_n) - f(x))}{x_n - x}$$

which, by Lem. 11.6 and Prop. 3.38 and the continuity of f at x (Prop. 11.7), converges to the RHS of (11.2b). This proves (11.2b), thanks to Lem. 11.6.

The third formula will follow from the second one if we can prove that $1/\lambda$ has derivative $-\frac{\lambda'(x)}{\lambda(x)^2}$ at x. This is not hard: Choose any sequence $x_n \to x$ but $x_n \neq x$. Then

$$\Big(\frac{1}{\lambda(x_n)} - \frac{1}{\lambda(x)}\Big)\Big/(x_n - x) = -\frac{\lambda(x_n) - \lambda(x)}{x_n - x}\cdot\frac{1}{\lambda(x_n)\lambda(x)}$$

converges to $-\lambda'(x)\cdot\frac{1}{\lambda(x)^2}$ as $n \to \infty$. Here, we have used the continuity of λ at x and Prop. 3.38 again. □

Example 11.9. The derivative of a constant function is 0. Thus, by Leibniz rule, if λ is a scalar, and if $f'(z)$ exists, then $(\lambda f)'(z) = \lambda \cdot f'(z)$.

Example 11.10. The identity map $f : z \in \mathbb{C} \mapsto z \in \mathbb{C}$ has derivative $\lim_{w\to z}\frac{w-z}{w-z} = 1$. Thus, by induction and Prop. 11.8, we have $(z^n)' = nz^{n-1}$ if $n \in \mathbb{Z}_+$. If $-n \in \mathbb{Z}_+$, then when $z \neq 0$ we have

$$(z^n)' = (1/z^{-n})' = -\frac{(z^{-n})'}{z^{-2n}} = -\frac{-nz^{-n-1}}{z^{-2n}} = nz^{n-1}$$

We conclude that $(z^n)' = nz^{n-1}$ whenever $n \in \mathbb{N}$, or whenever $n = -1, -2, \dots$ and $z \neq 0$. The same conclusion holds for the real variable function x^n.

Example 11.11. Let $f : z \in \mathbb{C} \mapsto \overline{z} \in \mathbb{C}$. We claim that for every $z \in \mathbb{C}$, the limit

$$f'(z) = \lim_{w \to z} \frac{\overline{w} - \overline{z}}{w - z} = \lim_{h \to 0} \overline{h}/h \tag{11.3}$$

does not exist with the help of Rem. 7.85: Take $h_n = 1/n$. Then $h_n \to 0$ and $\overline{h_n}/h_n = 1 \to 1$ as $n \to \infty$. Take $h_n = \mathbf{i}/n$. Then $h_n \to 0$ and $\overline{h_n}/h_n = -\mathbf{i}/\mathbf{i} = -1 \to -1$ as $n \to \infty$. So $f'(z)$ does not exist.

Theorem 11.12 (Chain rule). *Let Ω, Γ be nonempty open subsets of $\mathbb{C}$. Assume that $f : \Omega \to \Gamma$ is differentiable at $z \in \Omega$, and that $g : \Gamma \to V$ is differentiable at $f(z)$. Then $g \circ f$ is differentiable at z, and*

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z) \tag{11.4}$$

The same conclusion holds if Ω is replaced by an interval in $\mathbb{R}$, or if both Ω and Γ are replaced by intervals in $\mathbb{R}$.

Recall Conv. 11.4 for the assumption on the field $\mathbb{F}$.

Proof. Define a function $A : \Gamma \to \mathbb{C}$ by

$$A(\zeta) = \begin{cases} \dfrac{g(\zeta) - g \circ f(z)}{\zeta - f(z)} & \text{if } \zeta \neq f(z) \\ g'(f(z)) & \text{if } \zeta = f(z) \end{cases} \tag{11.5}$$

Choose any sequence (z_n) in $\Omega \backslash \{z\}$ converging to z. Then

$$\frac{g \circ f(z_n) - g \circ f(z)}{z_n - z} = A(f(z_n)) \cdot \frac{f(z_n) - f(z)}{z_n - z}$$

By Prop. 11.7, f is continuous at z. So $\lim_n f(z_n) = f(z)$. Thus, by Lem. 11.6-(3), the above expression converges to $g'(f(z))f'(z)$ as $n \to \infty$. □

Proposition 11.13. *Let Ω, Γ be nonempty open subsets of $\mathbb{C}$. Let $f : \Omega \to \Gamma$ be a bijection. Let $z \in \Omega$. Suppose that $f'(z)$ exists and $f'(z) \neq 0$. Suppose also that $f^{-1} : \Gamma \to \Omega$ is continuous at $f(z)$. Then f^{-1} is differentiable at $f(z)$, and*

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)} \tag{11.6}$$

The same conclusion holds if Ω and Γ are replaced by intervals of $\mathbb{R}$.

Proof. Choose any sequence (w_n) in $\Gamma \backslash \{f(z)\}$ converging to $f(z)$. Then, as $n \to \infty$, we have $f^{-1}(w_n) \to f^{-1}(f(z)) = z$ since f^{-1} is continuous at $f(z)$, and hence

$$\frac{f^{-1}(w_n) - f^{-1}(f(z))}{w_n - f(z)} = \frac{f^{-1}(w_n) - z}{f(f^{-1}(w_n)) - f(z)} \tag{11.7}$$

converges to $1/f'(z)$ by Lem. 11.6 and the continuity of the map $\zeta \in \mathbb{C}^\times \mapsto 1/\zeta \in \mathbb{C}$. This proves $(f^{-1})'(f(z)) = 1/f'(z)$, thanks to Lem. 11.6. □

11.2 Rolle's and Lagrange's mean value theorems (MVT)

Fix a Banach space V over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Assume $-\infty < a < b < +\infty$.

11.2.1 MVTs

Definition 11.14. Let $f : X \to \mathbb{R}$ where X is a topological space. We say that f has a **local maximum** (resp. **local minimum**) at $x \in X$, if there exists $U \in \mathrm{Nbh}_X(x)$ such that $f|_U$ attains its maximum (resp. minimum) at x. The word "**local extremum**" refers to either local maximum or local minimum.

You must know that derivatives can be used to find the monotonicity of real-valued real variable functions. Here is the precise statement:

Proposition 11.15. *Assume that $f : [a, b] \to \mathbb{R}$ is differentiable at $x \in [a, b]$, and $f'(x) > 0$. Then there exists $\delta > 0$ such that for any $y \in (x - \delta, x + \delta) \cap [a, b]$, we have*

$$y > x \Rightarrow f(y) > f(x) \qquad y < x \Rightarrow f(y) < f(x) \tag{11.8}$$

We leave it to the readers the find the analogous statement for the case $f'(x) < 0$.

Proof. Let $A = f'(x) > 0$. Then there exists $\delta > 0$ such that for all $y \in (x-\delta, x+\delta) \cap [a, b]$ not equal to x, we have $\left|\frac{f(y) - f(x)}{y - x} - A\right| < \frac{A}{2}$, and hence $\frac{f(y) - f(x)}{y - x} > \frac{A}{2}$. This proves (11.8). □

Corollary 11.16. *Assume that $f : [a, b] \to \mathbb{R}$ has a local extremum at $x \in (a, b)$. Assume that $f'(x)$ exists. Then $f'(x) = 0$.*

Proof. If $f'(x)$ is a non-zero number, then either $f'(x) > 0$ or $(-f)'(x) > 0$. In either case, Prop. 11.15 indicates that f cannot have a local extremum at x. □

From Prop. 11.15, it is clear that if $f' > 0$ on (a, b), then f is strictly increasing. However, to prove that if $f' \geqslant 0$ then f is increasing, we need more preparation.

Lemma 11.17 (Rolle's MVT). *Suppose that $f \in C([a, b], \mathbb{R})$ is differentiable on (a, b). Suppose moreover that $f(a) = f(b)$. Then there exists $x \in (a, b)$ such that $f'(x) = 0$.*

Proof. If f is constant than $f' = 0$. Suppose that f is not constant. Then there is $x \in (a, b)$ at which f attains its maximum (if $f(t) > f(a)$ for some $t \in (a, b)$) or minimum (if $f(t) < f(b)$ for some $t \in (a, b)$). So $f'(x) = 0$ by Cor. 11.16. □

Example 11.18. Rolle's MVT does not hold for vector-valued functions. Especially, it does hot hold for functions to $\mathbb{C} \simeq \mathbb{R}^2$. Consider $f : [0, 2\pi] \to \mathbb{C}$ defined by $f(t) = e^{\mathbf{i}t} = \cos t + \mathbf{i} \sin t$. We assume that $\sin' = \cos$ and $\cos' = -\sin$ are proved. Then $f(0) = f(2\pi) = 1$, whereas $f'(t) = -\sin t + \mathbf{i} \cos t$ is never zero.

Theorem 11.19 (Lagrange's MVT). *Suppose that $f \in C([a,b],\mathbb{R})$ is differentiable on (a,b). Then there is $x \in (a,b)$ such that*

$$f'(x) = \frac{f(b) - f(a)}{b-a} \tag{11.9}$$

Proof. The case $f(a) = f(b)$ is just Rolle's MVT. When $f(a) \neq f(b)$, we can "shift the function f vertically" so that its two end points have the same height. Technically, we consider $g(x) = f(x) - kx$ where $k = \frac{f(b)-f(a)}{b-a}$ is the slope of the interval from $(a, f(a))$ to $(b, f(b))$. Then $g(a) = g(b)$. By Rolle's MVT, there is $x \in (a,b)$ such that $g'(x) = 0$, i.e. $f'(x) - k = 0$. □

11.2.2 Applications of MVTs

Corollary 11.20. *Assume that $f \in C([a,b],\mathbb{R})$ is differentiable on (a,b). Then*

$$f' \geqslant 0 \text{ on } (a,b) \qquad \Longleftrightarrow \qquad f \text{ is increasing on } [a,b] \tag{11.10}$$

Moreover, if $f' > 0$ on (a,b), then f is strictly increasing on $[a,b]$.

Proof. If $f'(x) < 0$ for some $x \in (a,b)$, then Prop. 11.15 implies that x has a neighborhood on which f is not increasing. Conversely, suppose that $f'(x) \geqslant 0$ for all $x \in (a,b)$. Choose $x, y \in [a,b]$ satisfying $a \leqslant x < y \leqslant b$. Then by Lagrange's MVT, there is $z \in (x,y)$ such that $f(y) - f(x) = f'(z)(y-x) \geqslant 0$. So f is increasing on $[a,b]$. We have finished proving (11.10).

Suppose that $f' > 0$ on (a,b). Then by Prop. 11.15, f is strictly increasing on (a,b). If $a < x < b$, then there is $\varepsilon > 0$ such that $f(a+1/n) \leqslant f(x) - \varepsilon$ for sufficiently large n. Since f is continuous at a, by taking $\lim_{n\to\infty}$ we get $f(a) \leqslant f(x) - \varepsilon$ and hence $f(a) < f(x)$. A similar argument proves $f(b) > f(x)$. So f is strictly increasing on $[a,b]$. □

Corollary 11.21. *Suppose that $f \in C([a,b],\mathbb{R})$ is differentiable on (a,b), and $f'(x) \neq 0$ for all $x \in (a,b)$. Then f is strictly monotonic (i.e. strictly increasing or strictly decreasing). In particular, by Cor. 11.20, either $f' \geqslant 0$ on (a,b) or $f' \leqslant 0$ on (a,b).*

Proof. Rolle's MVT implies $f(b) - f(a) \neq 0$, and similarly, $f(y) - f(x) \neq 0$ whenever $a \leqslant x < y \leqslant b$. Therefore f is injective. The strict monotonicity of f follows from the following general fact: □

Proposition 11.22. *Let $-\infty \leqslant a < b \leqslant +\infty$, and let $f : [a,b] \to \overline{\mathbb{R}}$ be a continuous injective function. Then f is strictly monotonic.*

Proof. Choose a strictly increasing homeophism $\varphi : [0,1] \to [a,b]$. It suffices to prove that $g = f \circ \varphi : [0,1] \to \overline{\mathbb{R}}$ is strictly monotonic. Assume for simplicity that $g(0) \leqslant g(1)$. Let us show that g is increasing. Then the injectivity implies that g is strictly increasing.

We claim that for every $x \in (0,1)$ we have $g(0) \leqslant g(x) \leqslant g(1)$. Suppose the claim is true. Then for every $0 < x < y < 1$ we have $g(0) \leqslant g(y) \leqslant g(1)$. Applying the claim to the interval $[0,y]$ shows $g(0) \leqslant g(x) \leqslant g(y)$. So f is increasing.

Let us prove the claim. Suppose the claim is not true. Then there is $x \in (0,1)$ such that either $g(0) \leqslant g(1) < g(x)$ or $g(x) < g(0) \leqslant g(1)$. In the first case, by intermediate value theorem (applied to $f|_{[0,x]}$), there is $p \in [0,x]$ such that $g(p) = g(1)$. So g is not injective since $p < 1$. This is impossible. Similarly, the second case is also impossible. $\square$

Besides monotonicity, the uniqueness of antiderivatives is another classical application of MVT.

Definition 11.23. An **antiderivative** of a function $f : E \to V$ is a differentiable function $g : E \to V$ satisfying $g' = f$ on E.

The renowned fact that any two antiderivatives g_1, g_2 of $f : [a,b] \to V$ differ by a constant is immediate from the following fact (applied to $g_1 - g_2$):

Corollary 11.24. *Suppose that $f \in C([a,b],V)$ is differentiable on (a,b) and satisfies $f' = 0$ on (a,b). Then f is a constant function.*

Proof for $V = \mathbb{F}^N$. Since $\mathbb{C}^N \simeq \mathbb{R}^{2N}$, it suffices to prove the case $V = \mathbb{R}^N$. Since a sequence in $\mathbb{R}^N$ converges to a vector iff each component of the sequence converges to the corresponding component of the vector, it suffices to prove the case $f : [a,b] \to \mathbb{R}$.

Choose any $x \in (a,b]$. By Lagrange's MVT, there exists $y \in (a,x)$ such that $0 = f'(y) = \dfrac{f(x) - f(a)}{x - a}$. This proves $f(x) = f(a)$ for all $x \in (a,b]$. $\square$

Now we discuss the general case.

Remark 11.25. When V is not necessarily finite-dimensional, the method of reducing Cor. 11.24 to the case of scalar-valued functions is quite subtle: How should we understand the "components" of an element v in the Banach space V? In fact, in the general case, we should view

$$\langle \varphi, v \rangle \equiv \langle v, \varphi \rangle \xlongequal{\text{def}} \varphi(v) \tag{11.11}$$

as a component of v, where φ is an element inside the **dual (Banach) space** of V, defined by

$$V^* = \mathfrak{L}(V, \mathbb{F}) \tag{11.12}$$

An element φ in V^* is called a **bounded linear functional** on V. (In general, a **linear functional** on a vector space W over a field $\mathbb{K}$ is simply a linear map $W \to \mathbb{K}$.)

The vector space V^*, equipped with the operator norm, is indeed a Banach space. (This is why we call V^* the dual *Banach* space.) For the moment we do not need this fact. And we will discuss this topic in a later chapter.

Remark 11.26. In the future, we will prove that V^* separates points of V. (Recall Def. 8.44.) By linearity, this is equivalent to saying that for any $v \in V$ we have

$$v = 0 \qquad \Longleftrightarrow \qquad \langle \varphi, v \rangle = 0 \text{ for all } \varphi \in V^* \tag{11.13}$$

In fact, in the future we will prove the famous **Hahn-Banach extension theorem**, which implies (cf. Cor. 16.6) that if $v \in V$ then

$$\exists \varphi \in V^* \backslash \{0\} \qquad \text{such that} \qquad \langle \varphi, v \rangle = \|\varphi\| \cdot \|v\| \tag{11.14}$$

where $\|\varphi\|$ is the operator norm of φ. (Note that, by Rem. 10.24, we have $|\langle \varphi, v \rangle| \leqslant \|\varphi\| \cdot \|v\|$ for all $v \in V, \varphi \in V^*$. It is nontrivial that "$\leqslant$" can be "$=$" for some $\varphi \neq 0$.)

Proof of Cor. 11.24 assuming Hahn-Banach. Let us prove that $f(x) = f(a)$ for any $x \in [a, b]$. Since V^* separates points of V, it suffices to choose an arbitrary $\varphi \in V^*$ and prove that $\varphi \circ f(x) = \varphi \circ f(a)$. Indeed, since φ is continuous, for each sequence (x_n) in $[a, b] \backslash \{x\}$ converging to x, in view of Lem. 11.6 we have

$$\lim_{n \to \infty} \frac{\varphi \circ f(x_n) - \varphi \circ f(x)}{x_n - x} = \varphi\Big(\lim_{n \to \infty} \frac{f(x_n) - f(x)}{x_n - x}\Big) = \varphi(f'(x)) = \varphi(0) = 0$$

if $a < x < b$. Thus $(\varphi \circ f)' = 0$ on (a, b). Therefore, by the finite-dimensional version of Cor. 11.24, we have that $\varphi \circ f$ is constant on $[a, b]$. $\square$

Hahn-Banach theorem is extremely useful for reducing a problem about vector-valued functions to one about scalar-valued functions. In this course, we will also use Hahn-Banach to prove another fun fact: every Banach space over $\mathbb{F}$ is isormorphic to a closed linear subspace of $C(X, \mathbb{F})$ where X is a compact Hausdorff space. However, in the following section we would like to give an elementary proof of Cor. 11.24 without using Hahn-Banach.

11.3 Finite-increment theorem

Fix a Banach space V over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Assume $-\infty < a < b < +\infty$.

Cor. 11.24 follows immediately from the following theorem by taking $g = 0$.

Theorem 11.27. *Suppose that $f \in C([a, b], V)$ and $g \in C([a, b], \mathbb{R})$ are differentiable on (a, b). Assume that for every $x \in (a, b)$ we have $\|f'(x)\| \leqslant g'(x)$. Then*

$$\|f(b) - f(a)\| \leqslant g(b) - g(a) \tag{11.15}$$

Proof. By continuity, it suffices to prove $\|f(\beta) - f(\alpha)\| \leqslant g(\beta) - g(\alpha)$ for all α, β satisfying $a < \alpha < \beta < b$. Therefore, by replacing $[a,b]$ by $[\alpha,\beta]$, it suffices to assume that f, g are differentiable on $[a,b]$.

Step 1. For each $\varepsilon > 0$, consider the condition on $x \in [a,b]$:

$$\|f(x) - f(a)\| \leqslant g(x) - g(a) + \varepsilon(x-a) \tag{11.16}$$

The set

$$E_\varepsilon = \{x \in [a,b] : x \text{ satisfies } (11.16)\}$$

is nonempty because $a \in E_\varepsilon$. One checks easily that E_ε is a closed subset of $[a,b]$: This is because, for example, E_ε is the inverse image of the closed subset $(-\infty, 0]$ of $\mathbb{R}$ under the map

$$[a,b] \to \mathbb{R} \qquad x \mapsto \|f(x) - f(a)\| - g(x) + g(a) - \varepsilon(x-a) \tag{11.17}$$

We now fix $x = \sup E_\varepsilon$. Then $x \in E_\varepsilon$ because E_ε is closed. We shall prove that $x = b$. Then the fact that $b \in E_\varepsilon$ for all $\varepsilon > 0$ proves (11.15).

Step 2. Suppose that $x \neq b$. Then $a \leqslant x < b$. We shall prove that there exists $y \in (x,b)$ such that

$$\|f(y) - f(x)\| \leqslant g(y) - g(x) + \varepsilon(y-x) \tag{11.18}$$

Add this inequality to (11.16) (which holds because $x \in E_\varepsilon$). Then by triangle inequality, we obtain $y \in E_\varepsilon$, contradicting $x = \sup E_\varepsilon$.

Let us prove the existence of such y. For each $y \in (x,b)$, define $v(y) \in V, \lambda(y) \in \mathbb{R}$ such that

$$\begin{aligned} f(y) - f(x) &= f'(x)(y-x) + v(y)(y-x) \\ g(y) - g(x) &= g'(x)(y-x) + \lambda(y)(y-x) \end{aligned}$$

The definition of $f'(x)$ and $g'(x)$ implies that $v(y) \to 0$ and $\lambda(y) \to 0$ as $y \to x$. Therefore, there exists $y \in (x,b)$ such that $\|v(u)\| < \varepsilon/2$ and $|\lambda(y)| < \varepsilon/2$. Thus

$$\begin{aligned} &\|f(y) - f(x)\| - (g(y) - g(x)) \\ \leqslant&(\|f'(x)\| - g'(x))(y-x) + (\|v(y)\| - \lambda(y)) \cdot (y-x) \\ \leqslant&0 \cdot (y-x) + \Big(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\Big) \cdot (y-x) = \varepsilon(y-x) \end{aligned}$$

This proves (11.18). □

Remark 11.28. If we apply Thm. 11.27 to the special case that $f = 0$, we see that if $g \in C([a,b],\mathbb{R})$ satisfies $g' \geqslant 0$ on (a,b), then g is increasing. This gives another proof of (11.10) besides the one via Lagrange's MVT.

An important special case of Thm. 11.27 is:

Corollary 11.29 (Finite-increment theorem). *Suppose that $f \in C([a,b],V)$ is differentiable on (a,b), and that there exists $M \in \mathbb{R}_{\geqslant 0}$ such that $\|f'(x)\| \leqslant M$ for all $x \in (a,b)$. Then*

$$\|f(b) - f(a)\| \leqslant M|b-a| \tag{11.19}$$

Proof. Choose $g(x) = Mx$ in Thm. 11.27. □

It is fairly easy to prove finite-increment theorem for complex-variable functions.

Definition 11.30. A subset E of a real vector space is called **convex**, if for every $x, y \in E$, the interval

$$[x,y] = \{tx + (1-t)y : t \in [0,1]\} \tag{11.20}$$

is a subset of E.

When talking about convex subsets of $\mathbb{C}$, we view $\mathbb{C}$ as $\mathbb{R}^2$. Then, for example, all open disks in $\mathbb{C}$ are convex.

Corollary 11.31 (Finite-increment theorem). *Assume $\mathbb{F} = \mathbb{C}$. Let Ω be a nonempty open convex subset of $\mathbb{C}$. Let $f : \Omega \to V$ be differentiable on Ω. Choose any $z_1, z_2 \in \Omega$. Suppose that there exists $M \in \mathbb{R}_{\geqslant 0}$ such that $\|f'(z)\| \leqslant M$ for all z in the interval $[z_1, z_2]$. Then*

$$\|f(z_2) - f(z_1)\| \leqslant M|z_2 - z_1| \tag{11.21}$$

Proof. Define $\gamma : [0,1] \to \mathbb{C}$ by $\gamma(t) = (1-t)z_1 + tz_2$. Then $\gamma([0,1]) \subset \Omega$ because Ω is convex. By chain rule (Thm. 11.12), we have

$$(f \circ \gamma)'(t) = f'(\gamma(t)) \cdot \gamma'(t) = (z_2 - z_1)f'(\gamma(t))$$

whose norm is bounded by $M|z_2 - z_1|$. Applying Cor. 11.29 to $f \circ \gamma$ finishes the proof. □

Example 11.32. Let X be an interval in $\mathbb{R}$, or let $X = \Omega$ where Ω is convex. Let $\mathscr{A}$ be the set of differentiable functions $f : \Omega \to V$ satisfying $\|f'\|_{l^\infty} \leqslant M$. Then elements of $\mathscr{A}$ have Lipschitz constant M by Finite-increment theorems. So $\mathscr{A}$ is equicontinuous.

11.4 Commutativity of derivatives and limits

Fix a Banach space V over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let Ω be a nonempty open subset of $\mathbb{C}$. Recall Conv. 11.4. Recall Def. 8.42 for the meaning of locally uniform convergence.

11.4.1 Main theorem

Theorem 11.33. *Let X be Ω or an interval in $\mathbb{R}$. Let $(f_\alpha)_{\alpha\in I}$ be a net of differentiable functions $X \to V$. Suppose that the following are true:*

(a) The net $(f_\alpha)_{\alpha\in I}$ converges pointwise to some $f : X \to V$.

(b) The net $(f'_\alpha)_{\alpha\in I}$ converges uniformly to some $g : X \to V$.

Then f is differentiable on X, and $f' = g$.

Proof. We prove the case $X = \Omega$. The other case is similar. Choose any $z \in \Omega$. By shrinking Ω, we assume that Ω is an open disk centered at z. We know $\lim_{w\to z} \frac{f_\alpha(w)-f_\alpha(z)}{w-z}$ converges to $f'_\alpha(z)$ for each α. Therefore, if we can show that $\frac{f_\alpha(w)-f_\alpha(z)}{w-z}$ converges uniformly (over all $w \in \Omega\backslash\{z\}$) under $\lim_\alpha$, then it must converge uniformly to $\frac{f(w)-f(z)}{w-z}$ since it converges pointwise to $\frac{f(w)-f(z)}{w-z}$ by (a). Then, by Moore-Osgood Thm. 9.28, we have

$$\lim_{w\to z}\lim_{\alpha} \frac{f_\alpha(w)-f_\alpha(z)}{w-z} = \lim_{\alpha}\lim_{w\to z} \frac{f_\alpha(w)-f_\alpha(z)}{w-z} = \lim_\alpha f'_\alpha(z) = g(z)$$

finishing the proof.

To prove the uniform convergence, by the Cauchy condition on $V^{\Omega\backslash\{z\}}$ (equipped with a complete uniform convergence metric as in Exp. 7.77), it suffices to prove that

$$\sup_{w\in\Omega\backslash\{z\}} \Big\| \frac{f_\alpha(w)-f_\alpha(z)}{w-z} - \frac{f_\beta(w)-f_\beta(z)}{w-z} \Big\| \tag{11.22}$$

converges to 0 under $\lim_{\alpha,\beta\in I}$. Applying Cor. 11.31 to the function $f_\alpha - f_\beta$, we see

$$\begin{aligned}&\|f_\alpha(w) - f_\beta(w) - f_\alpha(z) + f_\beta(z)\| \leqslant |w-z| \cdot \sup_{\zeta\in[z,w]} \|f'_\alpha(\zeta) - f'_\beta(\zeta)\|\\ \leqslant&|w-z|\cdot\|f'_\alpha - f'_\beta\|_{l^\infty(\Omega,V)}\end{aligned} \tag{11.23}$$

Thus, we have

$$(11.22) \leqslant \|f'_\alpha - f'_\beta\|_{l^\infty(\Omega,V)}$$

where the RHS converges to 0 under $\lim_{\alpha,\beta}$ due to (b). This finishes the proof. $\square$

We didn't assume the uniform convergence of (f_α) in Thm. 11.33 because it is often redundant:

Lemma 11.34. *Assume that either X is a bounded interval in $\mathbb{R}$, or $X = \Omega$ where Ω is assumed to be bounded and convex. Then under the assumptions in Thm. 11.33, the net $(f_\alpha)_{\alpha\in I}$ converges uniformly to f.*

Proof. We already know that (f_α) converges pointwise to f. In fact, we shall only use the fact that $\lim_\alpha f_\alpha(z) = f(z)$ for some $z \in X$. Let z be such a point. Motivated by the proof of Thm. 11.33, let us prove the Cauchy condition that

$$\lim_{\alpha,\beta\in I} \sup_{w\in\Omega} \|f_\alpha(w) - f_\beta(w) - f_\alpha(z) + f_\beta(z)\| = 0$$

Then $(f_\alpha - f_\alpha(z))$ converges uniformly, and hence (f_α) converges uniformly. Indeed, the Cauchy condition follows easily from (11.23), in which $\sup_{w\in\Omega}|w-z|$ is a finite number because X is bounded. □

Thus, whether or not Ω satisfies boundedness and convexity, the net (f_α) must converge locally uniformly to f. Knowing the locally uniform convergence is often enough for applications. And here is another proof of this fact:

***Another proof** that (f_α) converges locally uniformly to f.* We consider the case $X = \Omega$; the other case is similar. For each $z \in X$, choose a convex precompact $U \in \mathrm{Nbh}_X(x)$. (Namely, $\mathrm{Cl}_X(U)$ is compact.) By (b) of Thm. 11.33, there is $\mu \in I$ such that $\sup_{x\in\overline{U}} \|f'_\alpha(x) - f'_\mu(x)\| \leqslant 1$ for all $\alpha \geqslant \mu$. Let $h_\alpha = f_\alpha - f_\mu$, and replace I by $I_{\geqslant\mu}$. By finite-increment Thm. 11.31, for each $x, y \in U$ we have

$$\|h_\alpha(x) - h_\alpha(y)\| \leqslant \|x - y\|$$

i.e., $(h_\alpha|_U)_{\alpha\in I}$ has uniform Lipschitz constant 1, and hence is an equicontinuous set of functions. Choose a closed ball B centered at z such that $B \subset U$. Then, since $(h_\alpha|_B)_{\alpha\in I}$ is equicontinuous and converges pointwise to $f - f_\mu$ (on B), by Cor. 9.26, $(f_\alpha - f_\mu)_{\alpha\in I}$ converges uniformly on B to $f - f_\mu$. Thus f_α converges uniformly on B to f. □

11.4.2 An interpretation of Thm. 11.33 in terms of Banach spaces

We give a more concise way of understanding the two conditions in Thm. 11.33.

Corollary 11.35. *Let X be Ω or an interval in $\mathbb{R}$. Define*

$$l^{1,\infty}(X, V) = \{f \in V^X : f \text{ is differentiable on } X \text{ and } \|f\|_{l^{1,\infty}} < +\infty\}$$

where $\|\cdot\|_{l^{1,\infty}} = \|\cdot\|_{1,\infty}$ is defined by

$$\|f\|_{1,\infty} = \|f\|_{l^\infty} + \|f'\|_{l^\infty} = \sup_{x\in X}\|f(x)\| + \sup_{x\in X}\|f'(x)\|$$

Then $l^{1,\infty}(X, V)$ is a Banach space with norm $\|\cdot\|_{1,\infty}$.

Proof. It is a routine check that $\|f\|_{l^{1,\infty}}$ defines a norm on $l^{1,\infty}(X,V)$. We now prove that $l^{1,\infty}(X,V)$ is complete. Let (f_n) be a Cauchy sequence in $l^{1,\infty}(X,V)$. So (f_n) and (f_n') are Cauchy sequences in l^∞, converging uniformly to $f,g\in V^X$ respectively. By Lem. 11.33, f is differentiable, and $f'=g$. (In particular, $\|f'\|_\infty<+\infty$.) So $f_n'\rightrightarrows f'$. Thus

$$\|f_n-f\|_{1,\infty}=\|f_n-f\|_\infty+\|f_n'-f'\|_\infty\to 0$$

□

Remark 11.36. Thm. 11.33 and Cor. 11.35 are almost equivalent. We have proved Cor. 11.35 using Thm. 11.33. But we can also prove a slightly weaker version of Thm. 11.33 using Cor. 11.35 as follows: In the setting of Thm. 11.33, assume that each $f_\alpha:X\to V$ is differentiable, and that

$$f_\alpha\rightrightarrows f\qquad f_\alpha'\rightrightarrows g \tag{11.24}$$

where $f,g\in l^\infty(X,V)$. Then Cor. 11.35 implies that f is differentiable and $f'=g$.

⋆ *Proof.* By (11.24), there is $\beta\in I$ such that $\|f_\alpha-f\|_\infty<+\infty$ and $\|f_\alpha'-g\|<+\infty$ for all $\alpha\geqslant\beta$. Thus, by replacing I with I_β, we assume that f_α and f_α' are bounded on X. So $f_\alpha\in l^{1,\infty}(X,V)$.

By (11.24), both (f_α) and (f_α') converge in $l^\infty(X,V)$. So they are Cauchy nets under the l^∞-norm. So (f_α) is a Cauchy net in $l^{1,\infty}(X,V)$. By Cor. 11.35, (f_α) converges to $\widetilde{f}\in l^{1,\infty}(X,V)$ under the $l^{1,\infty}$ norm. In particular, $f_\alpha\rightrightarrows\widetilde{f}$ and $f_\alpha'\rightrightarrows\widetilde{f}'$. Since, by assumption, we have $f_\alpha\rightrightarrows f$ and $f_\alpha'\rightrightarrows g$, we therefore get $f=\widetilde{f}$ and $g=\widetilde{f}'$. □

It is not surprising that Thm. 11.33 can be rephrased in terms of the completeness of a normed vector space. After all, our proof of Thm. 11.33 uses Cauchy nets in an essential way. In the future, we will use the completeness of $l^{1,\infty}$ to understand the commutativity of derivatives and other limit processes.

11.5 Derivatives of power series

This section is a continuation of the previous one. We shall use Thm. 11.33 to compute derivatives of power series.

11.5.1 General result

Corollary 11.37. *Assume that V is over $\mathbb{C}$. Let $f(z)=\sum_{n=0}^\infty v_nz^n$ be a power series in V. Assume that its radius of convergence is $R>0$. Then f is differentiable on $B_{\mathbb{C}}(0,R)$ and*

$$f'(z)=\sum_{n=0}^\infty nv_nz^{n-1}$$

Note that since $\lim_n \sqrt[n]{n} = 1$, $(\sqrt[n]{\|v_n\|})_{n\in\mathbb{Z}_+}$ and $(\sqrt[n]{n\|v_n\|})_{n\in\mathbb{Z}_+}$ have the same cluster points. Therefore

$$\limsup_{n\to\infty} \sqrt[n]{\|v_n\|} = \limsup_{n\to\infty} \sqrt[n]{n\|v_n\|} \tag{11.25}$$

So $\sum v_n z^n$ and $\sum_{n=0}^\infty n v_n z^{n-1}$ have the same radius of convergence.

First proof. For each $n \in \mathbb{N}_+$, let $g_n(z) = v_0 + v_1 z + \cdots + v_n z^n$. Then by Thm. 4.27, for each $0 < r < R$, the sequences (g_n) (resp. (g'_n)) converges uniformly to f (resp. converges uniformly to $h(z) = \sum_0^\infty n v_n z^{n-1}$) on $B_{\mathbb{C}}(0, r)$. Therefore, by Thm. 11.33, we have $f'(z) = h(z)$ for all $z \in B_{\mathbb{C}}(0, r)$, and hence all $z \in B_{\mathbb{C}}(0, R)$ since r is arbitrary. □

Second proof. Choose any $0 < r < R$, and let $X_r = B_{\mathbb{C}}(0, r)$. Consider $\sum v_n z^n$ as a series in the Banach space $l^{1,\infty}(X_r, \mathbb{C})$ (cf. Cor. 11.35). The norm of each term is $\|v_n\| + (n+1)\|v_{n+1}\|$. So the radius of convergence is R. Thus $\sum v_n z^n$ converges in $l^{1,\infty}(X_r, \mathbb{C})$ to some $g \in l^{1,\infty}(X_r, \mathbb{C})$. This means that, on X_r, $\sum v_n z^n$ converges uniformly to g (and hence $f = g$ on X_r) and $\sum n v_n z^{n-1}$ converges uniformly to g'. So $f'(z) = g'(z) = \sum n v_n z^{n-1}$ for all $z \in X_r$. □

11.5.2 Examples

Example 11.38. By Cor. 11.37, the function $\exp : \mathbb{C} \to \mathbb{C}$ is differentiable, and

$$\frac{d}{dz} e^z = e^z$$

Thus, if $\gamma : [a, b] \to \mathbb{C}$ is differentiable, then by chain rule, $\exp \circ \gamma$ is differentiable on $[a, b]$, and

$$\frac{d}{dt} e^{\gamma(t)} = e^{\gamma(t)} \cdot \frac{d}{dt}\gamma(t)$$

For example:

$$(e^{\alpha t})' = \alpha e^{\alpha t} \qquad (e^{t^2 + \mathbf{i}t})' = (2t + \mathbf{i}) e^{t^2 + \mathbf{i}t}$$

Example 11.39. Define functions $\sin : \mathbb{C} \to \mathbb{C}$ and $\cos : \mathbb{C} \to \mathbb{C}$ by

$$\cos z = \frac{e^{\mathbf{i}z} + e^{-\mathbf{i}z}}{2} \qquad \sin z = \frac{e^{\mathbf{i}z} - e^{-\mathbf{i}z}}{2\mathbf{i}}$$

It follows from $e^z e^w = e^{z+w}$ that

$$\cos(z + w) = \cos z \cos w - \sin z \sin w \qquad \sin(z + w) = \sin z \cos w + \cos z \sin w \tag{11.26}$$

By chain rule, we have $(e^{\alpha z})' = \alpha e^{\alpha z}$. So

$$\sin' z = \cos z \qquad \cos' z = -\sin z$$

Remark 11.40. From $e^z = \sum_{n\in\mathbb{N}} z^n/n!$, it is clear that the complex conjugate of e^z is

$$\overline{e^z} = \overline{\sum_{n=0}^{\infty} \frac{z^n}{n!}} = \sum_{n=0}^{\infty} \frac{\overline{z}^n}{n!} = e^{\overline{z}}$$

Here, we have exchanged the order of conjugate and infinite sum, because the function $z \in \mathbb{C} \mapsto \overline{z}$ is continuous. Thus, if $t \in \mathbb{R}$, then

$$\overline{e^{\mathbf{i}t}} = e^{-\mathbf{i}t} \qquad |e^{\mathbf{i}t}|^2 = \overline{e^{\mathbf{i}t}} e^{\mathbf{i}t} = e^{-\mathbf{i}t} e^{\mathbf{i}t} = 1$$

It follows that

$$\cos t = \mathrm{Re}(e^{\mathbf{i}t}) \qquad \sin t = \mathrm{Im}(e^{\mathbf{i}t})$$
$$(\cos t)^2 + (\sin t)^2 = (\mathrm{Re}(e^{\mathbf{i}t}))^2 + (\mathrm{Im}(e^{\mathbf{i}t}))^2 = |e^{\mathbf{i}t}|^2 = 1$$

It also follows from $|e^{\mathbf{i}t}| = 1$ that if $a, b \in \mathbb{R}$ then, by $e^{a+b\mathbf{i}} = e^a e^{b\mathbf{i}}$, we have

$$|e^{a+b\mathbf{i}}| = e^a \qquad \in \mathbb{R}_{>0}$$

Example 11.41. By Prop. 11.13, the function $\log : \mathbb{R}_{>0} \to \mathbb{R}$ is differentiable, and

$$(\log x)' = \frac{1}{x}$$

Example 11.42. We have

$$\lim_{t\to 0} \frac{\log(1+t)}{t} = (\log x)'|_{x=1} = 1$$

and hence $\lim_{x\to+\infty} x\log(1+1/x) = 1$ by Def. 7.81-(3) (since $\lim_{x\to+\infty} 1/x = 0$ as a net limit). Taking exponential, and using the continuity of $\exp$ at 1, we get

$$\lim_{x\to+\infty} \Big(1 + \frac{1}{x}\Big)^x = e$$

Example 11.43. If $a > 0$ and $z \in \mathbb{C}$, recalling $a^z = e^{z\log a}$, we use chain rule to find

$$\frac{d}{dz} a^z = a^z \log a$$

Similarly, if $\alpha \in \mathbb{C}$ and $x > 0$, then the chain rule gives the derivative of x^α:

$$\frac{d}{dx} x^\alpha = \alpha \cdot x^{\alpha-1}$$

Example 11.44. Compute $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{3^n}$

Proof. The series $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{3^n}$ has radius of convergence 3. When $|z| < 3$, it converges absolutely to $(1 - \frac{z}{3})^{-1} = 3(3-z)^{-1}$. So, by Cor. 11.37, when $|z| < 3$ we have

$$\sum_{n=0}^{\infty} \frac{nz^{n-1}}{3^n} = f'(z) = 3(3-z)^{-2}$$

$$\sum_{n=0}^{\infty} \frac{n(n-1)z^{n-2}}{3^n} = f''(z) = 6(3-z)^{-3}$$

So the value of the original series equals

$$f''(1) + f'(1) = \frac{3}{2}$$

□

11.5.3 A proof of (generalized) Leibniz rule

We end this section by giving a fun proof of Leibniz rule for higher derivatives. For simplicity, we consider only scalar-valued functions.

Proposition 11.45 (Leibniz rule). *Let X be either a nonempty interval in $\mathbb{R}$ (resp. a nonempty open subset of $\mathbb{C}$). Let f, g be functions from X to $\mathbb{R}$ (resp. to $\mathbb{C}$). Let $z \in X$. Suppose that $f^{(n)}(z)$ and $g^{(n)}(z)$ exist. Then*

$$(fg)^{(n)}(z) = \sum_{j=0}^{n} \binom{n}{j} f^{(n-j)}(z) g^{(j)}(z) \tag{11.27}$$

The above Leibniz rule is usually proved using the formula

$$\binom{n+1}{j} = \binom{n}{j-1} + \binom{n}{j} \tag{11.28}$$

where $n \in \mathbb{N}$ and $1 \leqslant j \leqslant n$. In the following, we give a proof without using this formula. We need the fact that the function

$$\mathbb{C}^n \to C(\mathbb{R}, \mathbb{C}) \qquad (a_0, \dots, a_{n-1}) \mapsto p(x) = \sum_{j=0}^{n-1} a_j x^j \tag{11.29}$$

is injective: this is because $j! \cdot a_j = f^{(j)}(0)$ by Exp. 11.10.

Proof of Prop. 11.45. By induction on n and by the classical Leibniz rule (Prop. 11.8), we have

$$(fg)^{(n)}(z) = \sum_{j=0}^{n} C_{n,j} \cdot f^{(n-j)}(z)g^{(j)}(z) \tag{11.30}$$

where each $C_{n,j}$ is an integer independent of f and g. (In particular, $C_{n,j}$ is independent of whether the variables are real of complex.) Thus, to determine the value of $C_{n,j}$, we can use some special functions.

We consider $f(x) = e^{sx}$ and $g(x) = e^{tx}$ where $s, t \in \mathbb{R}$. Recall that we have proved that $(e^{\alpha t})' = \alpha e^{\alpha t}$ using chain rule and the derivative formula for exponentials. So (11.30) reads

$$(s+t)^n \cdot e^{(s+t)x} = \sum_{j=0}^{n} C_{n,j} \cdot s^{n-j}t^j \cdot e^{(s+t)x}$$

Taking $x = 0$ gives

$$(s+t)^n = \sum_{j=0}^{n} C_{n,j} \cdot s^{n-j}t^j \tag{11.31}$$

Comparing this with the binomial formula (4.5), and noticing the injectivity of (11.29) (first applied to (11.31) for each fixed t, where (11.31) is viewed as a polynomial of s; then applied to each coefficient before s^{n-j}, which is a polynomial of t), we immediately get $C_{n,j} = \binom{n}{j}$. $\square$

11.6 Problems and supplementary material

Let Ω be nonempty open subset of $\mathbb{C}$. Let V be a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Assume that $\mathbb{F} = \mathbb{C}$ if we take derivatives with respect to complex variables.

Definition 11.46. Let X be either Ω or an interval in $\mathbb{R}$. Define

$$C^n(X, V) = \{f \in V^X : f, f', \dots, f^{(n)} \text{ exist and are continuous}\} \qquad (\text{if } n \in \mathbb{N})$$

$$C^\infty(X, V) = \bigcap_{n \in \mathbb{N}} C^n(X, V)$$

If $X \subset \mathbb{R}$, elements in $C^\infty(X, V)$ are called **smooth functions**.

Problem 11.1. Let X be either Ω or an interval in $\mathbb{R}$. For each $n \in \mathbb{N}$, define

$$l^{n,\infty}(X, V) = \{f \in V^X : f, f', \dots, f^{(n)} \text{ exist, and } \|f\|_{n,\infty} < +\infty\}$$

where $\|f\|_{n,\infty} = \|f\|_{l^{n,\infty}}$ is defined by

$$\|f\|_{n,\infty} = \|f\|_\infty + \|f'\|_\infty + \cdots + \|f^{(n)}\|_\infty$$

(In particular, we understand $l^{0,\infty}$ as l^∞.) Clearly $\|\cdot\|_{n,\infty}$ is a norm. We have proved that $l^{n,\infty}(X, V)$ is complete when $n = 0, 1$.

1. Prove by induction on n that $l^{n,\infty}(X, V)$ is complete for every n.

2. Prove that for each $n \in \mathbb{N}$, $C^n(X, V) \cap l^{n,\infty}(X, V)$ is a closed subset of $l^{n,\infty}(X, V)$ (and hence, is a Banach space by Prop. 3.27). Prove that if X is compact then $C^n(X, V) \subset l^{n,\infty}(X, V)$.

Convention 11.47. Unless otherwise stated, when X is compact, we always choose $l^{n,\infty}$ to be the norm on $C^n(X, V)$.

12 More on derivatives

12.1 Cauchy's MVT and L'Hôpital's rule

The goal of this section is to prove L'Hôpital's rule. For that purpose, we first need to prove Cauchy's MVT.

12.1.1 Main theorems

Theorem 12.1 (Cauchy's MVT). *Let $-\infty < a < b < +\infty$. Let $f, g \in C([a, b], \mathbb{R})$ be differentiable on (a, b). Then there exists $x \in (a, b)$ such that*

$$f'(x)(g(b) - g(a)) = g'(x)(f(b) - f(a))$$

In particular, if $g' \neq 0$ on (a, b), then g is injective (Cor. 11.21), and we can write the above formula as

$$\frac{f'(x)}{g'(x)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof. Cauchy's MVT specializes to Lagrange's MVT if we set $g(x) = x$. Moreover, in the proof of Lagrange's MVT (Thm. 11.19), we applied Rolle's MVT to the function $f(x) - kx$ where k is the slope of a line segment. Motivated by this observation, we consider the function $\psi(x) = f(x) - kg(x)$. If one wants $\psi(a) = \psi(b)$, one then solves that $k = \frac{f(b)-f(a)}{g(b)-g(a)}$. But we would rather consider $(g(b)-g(a))\psi$ in order to avoid the issue that the denominator is possibly zero. So we set

$$h(x) = (g(b) - g(a))f(x) - (f(b) - f(a))g(x)$$

Clearly $h(a) = h(b)$. By Rolle's MVT, there exists $x \in (a, b)$ such that

$$0 = h'(x) = (g(b) - g(a))f'(x) - (f(b) - f(a))g'(x)$$

□

Theorem 12.2 (L'Hôpital's rule). *Let $-\infty \leqslant a < b \leqslant +\infty$. Let $f, g \in C((a, b), \mathbb{R})$ be differentiable on (a, b). Assume that g' is nowhere zero on (a, b). (So g is strictly monotonic, cf. Prop. 11.22.) Assume*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = A \qquad \textit{exists in } \overline{\mathbb{R}} \tag{12.1}$$

Assume that one of the following two cases are satisfied:

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \tag{Case $\frac{0}{0}$}$$

$$\lim_{x\to a}|g(x)|=+\infty \qquad \text{(Case } \tfrac{*}{\infty}\text{)}$$

Then we have $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)}=A$. *The same conclusion holds if "$x\to a$" is replaced by "$x\to b$".*

Remark 12.3. Since g is strictly monotonic, there is at most one $x\in(a,b)$ such that $g(x)=0$. So $\lim_{x\to a}f(x)/g(x)$ means the limit over $x\in(a,b)\backslash g^{-1}(0)$. Alternatively, one can assign an arbitrary value to $f(x)/g(x)$ when $g(x)=0$, and understand $\lim_{x\to a}$ as a limit over $x\in(a,b)$.

Example 12.4. Compute $\displaystyle\lim_{x\to+\infty}\frac{x^n}{e^x}$

Proof. By L'Hôpital's rule in the case $\frac{*}{\infty}$, we have

$$\lim_{x\to+\infty}\frac{x^n}{e^x}=\lim_{x\to+\infty}\frac{nx^{n-1}}{e^x}=\lim_{x\to+\infty}\frac{n(n-1)x^{n-2}}{e^x}=\cdots=\lim_{x\to+\infty}\frac{n!}{e^x}=0$$

where the convergence of the limit is derived from right to left. □

12.1.2 Proof of L'Hôpital's rule

We divide the proof of L'Hôpital's rule into several steps. Also, we only treat the case $x\to a$, since the other case is similar. In the following, (a,b) means an interval, but not an element in the Cartesian product (which will be written as $a\times b$).

Step 1. We let $\dfrac{f(x)-f(y)}{g(x)-g(y)}$ take value $\dfrac{f'(x)}{g'(x)}$ if $x=y$. In this step, we prove

$$\lim_{x\times y\to a\times a}\frac{f(x)-f(y)}{g(x)-g(y)}=A \tag{12.2}$$

where $x\times y$ is defined on $(a,b)^2=(a,b)\times(a,b)$. In view of Def. 7.81-(3m), we pick any sequence $x_n\times y_n$ in $(a,b)^2$. By Cauchy's MVT, there is $\xi_n\in[x_n,y_n]$ (if $x_n\leqslant y_n$) or $\xi_n\in[y_n,x_n]$ (if $x_n\geqslant y_n$) such that

$$f'(\xi_n)(g(x_n)-g(y_n))=g'(\xi_n)(f(x_n)-f(y_n)).$$

So we have

$$\frac{f'(\xi_n)}{g'(\xi_n)}=\frac{f(x_n)-f(y_n)}{g(x_n)-g(y_n)}$$

Since $\lim_n x_n=\lim_n y_n=a$, we clearly have $\lim_n\xi_n=a$. Therefore, by (12.1) (this is the only place where we use (12.1)), we have

$$\lim_{n\to\infty}\frac{f(x_n)-f(y_n)}{g(x_n)-g(y_n)}=\lim_{n\to\infty}\frac{f'(\xi_n)}{g'(\xi_n)}=A$$

This proves (12.2). □

Step 2. It follows from Def. 7.81-(3) that if (x_n) and (y_k) are sequences in (a,b) converging to a, then

$$\lim_{n,k\to\infty} \frac{f(x_n)-f(y_k)}{g(x_n)-g(y_k)} = A \tag{12.3}$$

In other words, we apply Def. 7.81-(3) to the net $(x_n \times y_k)_{n\times k\in\mathbb{Z}_+^2}$ which replaces the sequence $(x_n \times y_n)_{n\in\mathbb{Z}_+}$ in Step 1. We shall use (12.3) to prove the two cases of L'Hôpital's rule.

Let us prove the case $\frac{0}{0}$. This is the easier case. Choose any sequence (x_n) in (a,b) converging to a. We want to prove that $\lim_n f(x_n)/g(x_n) = A$. So we choose any sequence (y_k) in (a,b) converging to a, and we know that the double limit (12.3) exists. Moreover, if n is fixed, then $\lim_k f(y_k) = \lim_k g(y_k) = 0$ by assumption. Thus

$$\lim_{k\to\infty} \frac{f(x_n)-f(y_k)}{g(x_n)-g(y_k)} = \frac{f(x_n)}{g(x_n)} \tag{12.4}$$

Thus, by Thm. 5.29, when $n\to\infty$, the RHS of the above equation converges to (12.3). This finishes the proof for the case $\frac{0}{0}$. □

Step 3. Finally, we address the (more difficult) case $\frac{*}{\infty}$. Assume $\lim_{x\to a}|g(x)| = +\infty$. Again, we choose a sequence (x_n) in (a,b) converging to a. To prove $f(x_n)/g(x_n)\to A$, one may want to pick any sequence (y_k) in (a,b), and compute the limit on the LHS of (12.4). Unfortunately, in this case, we do not know whether this limit converges or not: As one can compute, it is equal to $\lim_{k\to\infty} f(y_k)/g(y_k)$, whose convergence is part of the result we need to prove!

It is not hard to address this issue: Since $\overline{\mathbb{R}}$ is (sequentially) compact, by Thm. 3.15, it suffices to prove that any cluster point $B\in\overline{\mathbb{R}}$ of $(f(x_n)/g(x_n))_{n\in\mathbb{Z}_+}$ is equal to A. Thus, we let (y_k) be any subsequence of (x_n) such that $(f(y_k)/g(y_k))_{k\in\mathbb{Z}_+}$ converges to B. Let us prove $A=B$ using the same method as in Step 2. We compute that

$$\frac{f(x_n)-f(y_k)}{g(x_n)-g(y_k)} = \frac{g(y_k)}{g(x_n)-g(y_k)} \cdot \frac{f(x_n)-f(y_k)}{g(y_k)} \tag{12.5}$$

Since $\lim_k |g(y_k)| = +\infty$, we have $\lim_{k\to\infty} C/g(y_k) \to 0$ for any $C\in\mathbb{R}$ independent of k. Therefore, as $k\to\infty$, the first factor on the RHS of (12.5) converges to -1, and the second factor converges to $-B$. It follows that

$$\lim_{n\to\infty}\lim_{k\to\infty} \frac{f(x_n)-f(y_k)}{g(x_n)-g(y_k)} = \lim_{n\to\infty} B = B$$

Therefore $A=B$ by Thm. 5.29. □

The proof of L'Hôpital's rule is now complete.

Remark 12.5. The above proof can be easily translated into a language without double limits. We consider the case of $\frac{*}{\infty}$ and assume for example that $-\infty < A < +\infty$ and $a \in \mathbb{R}$, and sketch the proof as follows.

By (12.1) and Cauchy' MVT, for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $a < x, y < a + \delta$ (where $x \neq y$) we have

$$A - \varepsilon \leqslant \frac{f(x) - f(y)}{g(x) - g(y)} \leqslant A + \varepsilon$$

Choose any sequence (x_n) in (a, b) converging to a. Let B be any cluster point of $(f(x_n)/g(x_n))_{n\in\mathbb{Z}_+}$ in $\overline{\mathbb{R}}$. We need to prove that $A = B$. Let $(y_k)_{k\in\mathbb{Z}_+}$ be any subsequence of (x_n) such that $\lim_k f(y_k)/g(y_k) = B$. Then there is $N \in \mathbb{N}$ such that for all $n \geqslant N, k \geqslant N$ satisfying $x_n \neq y_k$, we have

$$A - \varepsilon \leqslant \frac{f(x_n) - f(y_k)}{g(x_n) - g(y_k)} \leqslant A + \varepsilon$$

For each $n \geqslant N$, apply $\lim_{k\to\infty}$ to the above inequality, and notice $\lim_k |g(y_k)| = +\infty$. Then we get $A - \varepsilon \leqslant B \leqslant A + \varepsilon$, finishing the proof. □

12.2 Trigonometric functions and π

In this section, we prove that sin, cos, and π satisfy the properties we learned in high schools. Some of them have already been proved in Subsec. 11.5.2. We leave the proof of the basic properties of the other trigonometric functions to the reader.

Let x be a real variable. Recall that $\sin, \cos : \mathbb{R} \to \mathbb{R}$ are determined by the fact that $e^{\mathbf{i}x} = \cos x + \mathbf{i} \sin x$. In particular, that $|e^{\mathbf{i}x}| = 1$ implies $(\cos x)^2 + (\sin x)^2 = 1$. We have proved that

$$\sin' = \cos \qquad \cos' = -\sin$$

Since $e^{\mathbf{i}\cdot 0} = 1$, we have

$$(\sin x)'|_{x=0} = \cos 0 = 1 \qquad (\cos x)'|_{x=0} = -\sin 0 = 0$$

In particular, since $\sin' = \cos$ is strictly positive on a neighborhood of 0, by Cor. 11.20, sin is strictly increasing on that neighborhood.

We shall define $\frac{\pi}{2}$ to be the smallest positive zero cos. However, we must first prove the existence of this number:

Lemma 12.6. *There exists $x \geqslant 0$ such that $\cos x = 0$.*

Proof. Suppose this is not true. Then by $\cos 0 = 1$ and intermediate value theorem, we have $\cos x > 0$ for all $x \geqslant 0$. In other words, $\sin' > 0$ on $\mathbb{R}_{\geqslant 0}$. Thus, by Cor.

11.20, sin is strictly increasing on $\mathbb{R}_{\geqslant 0}$. Therefore $A = \lim_{x\to+\infty} \sin x$ exists in $\overline{\mathbb{R}}$. Since $\sin 0 = 0$, we must have $A \in \overline{\mathbb{R}}_{>0}$. By L'Hôpital's rule in case $\frac{*}{\infty}$, we have

$$\lim_{x\to+\infty} \frac{\cos x}{x} = \lim_{x\to+\infty} -\sin x = -A < 0$$

contradicting the fact that $\cos x > 0$ if $x \geqslant 0$. □

Definition 12.7. We define the number π to be

$$\pi = 2 \cdot \inf\left(\mathbb{R}_{\geqslant 0} \cap \cos^{-1}(0)\right) = 2 \cdot \inf\{x \in \mathbb{R}_{\geqslant 0} : \cos x = 0\}$$

Note that $\left(\mathbb{R}_{\geqslant 0} \cap \cos^{-1}(0)\right)$ is a closed subset of $\mathbb{R}$. So its infinimum belongs to this set. Therefore, $\frac{\pi}{2}$ is the smallest $x \geqslant 0$ satisfying $\cos(x/2) = 0$.

Proposition 12.8. *We have*

$$\sin 0 = 0 \qquad \sin\frac{\pi}{2} = 1 \qquad \cos 0 = 1 \qquad \cos\frac{\pi}{2} = 0 \tag{12.6}$$

On $(0, \pi/2)$, sin *and* cos *are strictly positive. On* $[0, \pi/2]$, sin *is strictly increasing, and* cos *is strictly decreasing.*

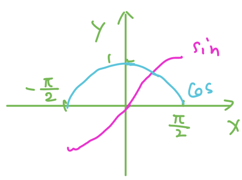
Proof. All the formulas in (12.6), except $\sin(\pi/2) = 1$, has been proved. Since $(\sin\frac{\pi}{2})^2 = 1 - (\cos\frac{\pi}{2})^2$, we have $\sin\frac{\pi}{2} = \pm 1$.

By the definition of π, we know that $\sin' = \cos$ is > 0 on $[0, \frac{\pi}{2})$. So sin is strictly increasing on $[0, \frac{\pi}{2}]$. Thus, $\sin 0 = 0$ implies that $\sin > 0$ on $(0, \frac{\pi}{2}]$. In particular, $\sin\frac{\pi}{2} = 1$. Since $\cos' = -\sin$ is < 0 on $(0, \frac{\pi}{2})$, cos is strictly decreasing on $[0, \frac{\pi}{2}]$. □

Proposition 12.9. *We have* $\sin(-x) = -\sin x$ *and* $\cos(-x) = \cos x$.

Proof. Immediate from $e^{-\mathrm{i}x} = 1/e^{\mathrm{i}x}$ and the definitions of sin and cos. □

From what we have proved, we know that the graph of sin and cos on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ looks as follows.



Proposition 12.10. *We have*

$$\sin x = \cos(x - \frac{\pi}{2}) = \cos(\frac{\pi}{2} - x) \qquad \cos(x) = \sin(\frac{\pi}{2} - x)$$

Proof. Immediate from (11.26) and (12.6). □

Thus, the graph of $\sin$ is the rightward translation of that of $\cos$ by $\frac{\pi}{2}$. Therefore, the graph of $\sin$ on $[-\frac{\pi}{2}, 0]$ is translated to the graph of $\cos$ on $[-\pi, -\frac{\pi}{2}]$. That $\cos(x) = \cos(-x)$ gives us the graph of $\cos$ on $[\frac{\pi}{2}, \pi]$. Thus, we know the graph of $\cos$ on $[-\pi, \pi]$.

Theorem 12.11 (Euler's formula). *We have* $e^{\mathbf{i}\pi} = -1$, *and hence* $e^{2\mathbf{i}\pi} = e^{\mathbf{i}\pi}e^{\mathbf{i}\pi} = 1$.

Proof. We have $e^{\mathbf{i}\pi/2} = \cos(\frac{\pi}{2}) + \mathbf{i}\sin(\frac{\pi}{2}) = \mathbf{i}$. Hence $e^{\mathbf{i}\pi} = \mathbf{i}^2 = -1$. □

Proposition 12.12. *We have* $\sin x = \sin(x + 2\pi)$ *and* $\cos x = \cos(x + 2\pi)$

Proof. Immediate from (11.26) and that $1 = e^{2\mathbf{i}\pi} = \cos(2\pi) + \mathbf{i}\sin(2\pi)$. □

Thus, $\cos$ and $\sin$ have period 2π. This completes the graphs of $\cos x$ and $\sin x = \cos(x - \frac{\pi}{2})$ on $\mathbb{R}$.

The fact that 2π is the length of the unit circle will be proved in Exp. 13.39.

12.3 Taylor's theorems

Assume throughout this section that $-\infty < a < b < +\infty$ and V is a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

In this section, we generalize MVTs and finite-increment theorem to higher derivatives. These generalizations are all under the name "Taylor theorem". Recall Def. 11.46 and Pb. 11.1 for the meaning of $l^{n,\infty}, C^n, C^\infty$. We first discuss the generalization of finite-increment theorem, which can be applied to vector-valued functions.

12.3.1 Taylor's theorems for vector-valued functions

Definition 12.13. Let X be a normed vector space. Let $A \subset X$. Let $a \in \mathrm{Cl}_X(A)\backslash A$ (or more generally, assume $a \in \mathrm{Cl}_X(A\backslash\{a\})$). Let $f : A \to V$. Let $r \in \mathbb{R}_{\geqslant 0}$.

- We write $f(x) = o(\|x - a\|^r)$ if $\displaystyle\lim_{x\to a} \frac{f(x)}{\|x - a\|^r} = 0$.
- We write $f(x) = O(\|x - a\|^r)$ if $\displaystyle\limsup_{x\to a} \frac{\|f(x)\|}{\|x - a\|^r} < +\infty$ where $\limsup$ is the limit superior of a net (cf. Rem. 7.83 and Pb. 8.2). In other words, there exists $U \in \mathrm{Nbh}_X(a)$ such that $\sup_{x\in A\cap U} \|f(x)\|/\|x - a\|^r < +\infty$.

When $r = 0$, we simply write $o(1)$ and $O(1)$. The two symbols o, O are called **Landau symbols**.

In this section, we will frequently use the formula

$$S_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k \tag{12.7}$$

whenever the RHS makes sense.

Theorem 12.14 (Taylor's theorem, Peano form). *Let $f : [a,b] \to V$ and $n \in \mathbb{Z}_+$. Assume that $f', f'', \dots, f^{(n)}$ exist at a (resp. at b). Then for each $x \in (a,b)$ we have*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + o((x-a)^n) \tag{12.8a}$$

resp.

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k + o((b-x)^n) \tag{12.8b}$$

Expressions of the form (12.8a) are called **Taylor expansions** of f with **center** a.

Remark 12.15. In order for $f^{(n)}(a)$ to exist, it is assumed that $f^{(n-1)}$ exists on a neighborhood of a.

Proof of Peano-form. Since the two cases are similar, we only treat the case at a. Define **remainder** $g(x) = f(x) - S_n(x)$. Then

$$g(a) = g'(a) = \cdots = g^{(n)}(a) = 0 \tag{12.9}$$

It suffices to prove Taylor's theorem for g, i.e.

$$g(x) = o((x-a)^n) \tag{12.10}$$

We prove this by induction. The case $n = 1$ is obvious from the definition of derivatives, which says

$$\frac{g(x) - g(a)}{x-a} - g'(a) = o(1) \tag{12.11}$$

and hence $g(x) = (x-a)o(1) = o(x-a)$.

Now assume $n \geqslant 2$. Assume that Peano form has been proved for case $n-1$. Applying this result to g'. We then get $g'(x) = o((x-a)^{n-1})$. Since $g''(a)$ exists, g' exists on $[a,c]$ where $a < c < b$. In particular, g is continuous on $[a,c]$. Therefore, by finite-increment Thm. 11.29, if $x \in (a,c)$, then

$$\|g(x)\| \leqslant (x-a) \cdot \sup_{a<t<x} \|g'(t)\|$$

If we can prove $\sup_{a\leqslant t\leqslant x} \|g'(t)\| = o((x-a)^{n-1})$, then we immediately have $g(x) = o((x-a)^n)$. Thus, the proof of Peano form is finished by the next lemma. □

Lemma 12.16. *Assume that $f : [a,b] \to V$ satisfies $f(x) = o((x-a)^r)$. Define*

$$\widetilde{f}(x) = \sup_{a<t<x} \|f(t)\|$$

Then $\widetilde{f}(x) = o((x-a)^r)$.

Proof. Choose any $\varepsilon > 0$. Since $f(x) = o((x-a)^r)$, we know that there is $c \in (a,b)$ such that for all $a < x < c$ we have $\|f(x)\| \leqslant \varepsilon|x-a|^r$. Thus

$$|\widetilde{f}(x)| \leqslant \sup_{a<t<x} \varepsilon|t-a|^r = \varepsilon|x-a|^r$$

□

Among all the versions of Taylor's theorem discussed in this section, the following one is most useful. (Another useful version, the integral form of Taylor's theorem, will be proved in Thm. 13.32.) Note that if we assume $f \in C^{(n+1)}([a,b],V)$ in Thm. 12.14, then Thm. 12.17 immediately implies Thm. 12.14.

Theorem 12.17 (Higher order finite-increment theorem). *Let $n \in \mathbb{N}$ and $f \in C^n([a,b],V)$. Assume that $f^{(n+1)}$ exists everywhere on (a,b). Then, for every $x \in (a,b]$ resp. $x \in [a,b)$, we have respectively*

$$\Big\|f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\Big\| \leqslant \frac{(x-a)^{n+1}}{(n+1)!} \cdot \sup_{a<t<x} \|f^{(n+1)}(t)\| \tag{12.12a}$$

$$\Big\|f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k\Big\| \leqslant \frac{(b-x)^{n+1}}{(n+1)!} \cdot \sup_{x<t<b} \|f^{(n+1)}(t)\| \tag{12.12b}$$

Proof. We only prove the first formula: applying the first formula to $\widetilde{f}(x) = f(-x)$ implies the second one. We prove (12.12a) by induction on n. Moreover, we shall prove (12.12a) for the case $x = b$. The general case follows by restricting f to $[a,x]$. When $n = 0$, (12.12a) is the content of (classical) finite-increment Thm. 11.29. Assume case $n-1$ has been proved ($n \in \mathbb{Z}_+$). In case n, take $g(x) = f(x) - S_n(x)$. Then (12.9) is true. Let

$$M = \sup_{a<t<b} \|f^{(n+1)}(t)\| = \sup_{a<t<b} \|g^{(n+1)}(t)\|$$

By case $n-1$, for each $a \leqslant x \leqslant b$ we have

$$\|g'(x)\| \leqslant \frac{(x-a)^n}{n!} \cdot M = h'(x) \qquad \text{where } h(x) = \frac{M(x-a)^{n+1}}{(n+1)!}$$

Thus, by Thm. 11.27, we have

$$\|g(b) - g(a)\| \leqslant h(b) - h(a) = \frac{M(b-a)^{n+1}}{(n+1)!}$$

This proves (12.12a) for g, and hence for f. □

12.3.2 Taylor's theorem for real-valued functions

Theorem 12.18 (Taylor's theorem, Lagrange form). *Let $n \in \mathbb{N}$ and $f \in C^n([a,b],\mathbb{R})$. Assume that $f^{(n+1)}$ exists everywhere on (a,b). Then for every $x \in (a,b]$ resp. $x \in [a,b)$, there exists $\xi \in (a,x)$ resp. $\eta \in (x,b)$ such that, respectively,*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \tag{12.13a}$$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k + \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-b)^{n+1} \tag{12.13b}$$

Proof. We only prove (12.13a). Applying (12.13a) to $\widetilde{f}(x) = f(-x)$ (defined on $[-b,-a]$) implies the second formula. Again, it suffices to prove (12.13a) for $g(x) = f(x) - S_n(x)$, which satisfies (12.9). Thus, we want to find $\xi \in (a,x)$ satisfying

$$g(x) = \frac{g^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

This can be proved by applying Cauchy's MVT repeatedly:

$$\begin{aligned}&\frac{g(x)}{(x-a)^{n+1}} \xlongequal{\exists x_1 \in (a,x)} \frac{g'(x_1)}{(n+1)(x_1-a)^n} \xlongequal{\exists x_2 \in (a,x_1)} \frac{g''(x_2)}{(n+1)n(x_2-a)^{n-1}}\\ &= \cdots \xlongequal{\exists x_n \in (a,x_{n-1})} \frac{g^{(n)}(x_n)}{(n+1)!(x_n-a)} \xlongequal{\exists \xi \in (a,x_n)} \frac{g^{(n+1)}(\xi)}{(n+1)!}\end{aligned}$$

□

We will mainly use Thm. 12.17 instead of the Lagrange form, since the latter does not apply directly to vector valued functions. However, Thm. 12.17 can be derived from the Lagrange form and Hahn-Banach theorem. We have used the fact that V^* separates points of V to prove that a function $[a,b] \to V$ is constant if its derivative exists and is constantly zero. (See Cor. 11.24.) However, to prove Thm. 12.17, we need the stronger fact that for every $v \in V$ there exists a nonzero $\varphi \in V^*$ such that $\langle \varphi, v\rangle = \|\varphi\| \cdot \|v\|$. (See Rem. 11.26.) By scaling φ, we can assume that $\|\varphi\| = 1$ and $\langle \varphi, v\rangle = \|v\|$.

We now give the second proof of Thm. 12.17:

Proof of Thm. 12.17 using Lagrange form and Hahn-Banach. The Lagrange form clearly implies Thm. 12.17 in the special case that $V = \mathbb{R}$. Now consider the general case, and view V as a real Banach space if it was originally over $\mathbb{C}$. Take $g(x) = f(x) - S_n(x)$. We need to prove $\|g(x)\| \leqslant \frac{(x-a)^{n+1}}{(n+1)!}M$ where $M = \sup_{a<t<x} \|g^{(n+1)}(t)\|$.

By Hahn-Banach (Rem. 11.26), there exists $\varphi \in V^*$ with $\|\varphi\| = 1$ such that $\varphi \circ g(x) = \|g(x)\|$. Applying the one dimensional special case to $\varphi \circ g$, we have

$$\|g(x)\| = \varphi \circ g(x) \leqslant \frac{(x-a)^{n+1}}{(n+1)!} \cdot \sup_{a<t<x} \left|(\varphi \circ g)^{(n+1)}(t)\right| \tag{12.14}$$

provided that $\varphi \circ g \in C^n([a,b], V)$ and that $(\varphi \circ g)^{(n+1)}$ exists on (a,b).

By the continuity of φ, we have

$$(\varphi \circ g)'(t) = \lim_{s \to t} \frac{\varphi \circ g(s) - \varphi \circ g(t)}{s-t} = \varphi\Big(\lim_{s \to t} \frac{g(s)-g(t)}{s-t}\Big) = \varphi(g'(t)) \tag{12.15}$$

Applying this formula repeatedly, we see that $\varphi \circ g \in C^n([a,b], V)$, that $(\varphi \circ g)^{(n+1)}$ exists on (a,b), and that $(\varphi \circ g)^{(n+1)} = \varphi \circ g^{(n+1)}$. By Rem. 10.24, we have

$$\sup_{a<t<x} \left|(\varphi \circ g)^{(n+1)}(t)\right| = \sup_{a<t<x} \left|\langle \varphi, g^{(n+1)}(t)\rangle\right| \leqslant \sup_{a<t<x} \|g^{(n+1)}(t)\| = M$$

Combining this result with (12.14) finishes the proof. □

When V is $\mathbb{R}^N$ equipped with the Euclidean norm, the Hahn-Banach theorem (in the form of Rem. 11.26) is very easy: Define the linear map $\varphi : \mathbb{R}^N \to \mathbb{R}$ sending each $v \in \mathbb{R}^N$ to its dot product with $g(x)/\|g(x)\|$. Then φ satisfies $\|\varphi\| = 1$ and $\varphi \circ g(x) = \|g(x)\|$. So there is nothing mysterious in the above proof. (You are encouraged to compare this proof with the one of [Rud-P, Thm. 5.19].)

12.4 Functions approximated by their Taylor series

Taylor's theorems do not imply that a smooth function f on an interval of $\mathbb{R}$ can be approximated by its **Taylor series** (with **center** a):

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \tag{12.16}$$

The following is a typical example:

Example 12.19. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \exp\big(-\dfrac{1}{x^2}\big) & (\text{if } x > 0) \\ 0 & (\text{if } x \leqslant 0) \end{cases}$$

Then $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. So the Taylor series of f at 0 is 0, which cannot approximate $f(x)$ when $x > 0$.

On the contrary, if f is defined on an open subset of $\mathbb{C}$, and if f' exists everywhere on its domain, then $f^{(n)}$ exists for all n, and f can be approximated locally uniformly by its Taylor series. This is a deep result in complex analysis. In fact, a thorough understanding of power series is impossible without the help of complex analysis.

In the following, we show that some important real variable functions can be approximated by their Taylor series. Actually, the proof can be simplified by complex analysis, since these examples are the restriction of some differentiable complex variable functions (aka **holomorphic functions**) to the real line.

Example 12.20. Consider the Taylor expansion of $\log : \mathbb{R}_{>0} \to \mathbb{R}$ at $x = 1$. By induction on $n \in \mathbb{Z}_+$, one computes that

$$\log^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{x^n}$$

Therefore, its Taylor expansion in order n is

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(x-1)^k + R_{n+1}(x)$$

where $R_{n+1}(x)$ is the remainder. To show that $\log x$ is approximated uniformly (on certain domain) by its Taylor series, one need to show that R_{n+1} converges uniformly to 0 on that domain.

We would like to prove that for every $0 < r < 1$ we have $R_{n+1} \rightrightarrows 0$ on $[1-r, 1+r]$. This would imply that series on the RHS of the following formula (whose radius of convergence is 1) converges uniformly to f on $[1-r, 1+r]$:

$$\log(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(x-1)^k \tag{12.17}$$

However, using the Taylor's theorems proved in the previous section, one can prove the uniform convergence only when $0 < r \leqslant 1/2$. The general case of $0 < r < 1$ should be proved by another method. □

Proof for the case $0 < r \leqslant \frac{1}{2}$. In fact, we shall prove the uniform convergence on $[1-r, 2]$. By Thm. 12.17, for all $x \in [1-r, 1+r]$ we have

$$|R_{n+1}(x)| \leqslant \frac{|x-1|^{n+1}}{(n+1)!} \cdot \sup_{\substack{1\leqslant t\leqslant x \\ \text{or } x\leqslant t\leqslant 1}} \big|\log^{(n+1)}(t)\big| = \frac{|x-1|^{n+1}}{n+1} \cdot \sup_{\substack{1\leqslant t\leqslant x \\ \text{or } x\leqslant t\leqslant 1}} \frac{1}{t^{n+1}}$$

where the $\sup$ is over $1 \leqslant t \leqslant x$ or $x \leqslant t \leqslant 1$, depending on whether $1 \leqslant x \leqslant 1+r$ or $1-r \leqslant x \leqslant 1$.

If $1 \leqslant x \leqslant 1+r$, then $1 \leqslant t \leqslant x$ implies $1/t^{n+1} \leqslant 1$. So

$$|R_{n+1}(x)| \leqslant \frac{(x-1)^{n+1}}{n+1} \leqslant \frac{r^{n+1}}{n+1}$$

where the RHS converges to 0 as $n \to \infty$ whenever $0 < r \leqslant 1$. In particular, we get the renowned formula

$$\log 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \tag{12.18}$$

If $1-r \leqslant x \leqslant 1$, then $x \leqslant t \leqslant 1$ implies $1/t^{n+1} \leqslant 1/x^{n+1}$. Thus

$$|R_{n+1}(x)| \leqslant \frac{1}{n+1}\Big(\frac{1}{x}-1\Big)^{n+1} \leqslant \frac{1}{n+1}\Big(\frac{1}{1-r}-1\Big)^{n+1} = \frac{1}{n+1}\Big(\frac{r}{1-r}\Big)^{n+1}$$

If $0 < r \leqslant 1/2$, then $0 < r/(1-r) \leqslant 1$, and hence the RHS above converges to 0 as $n \to \infty$. This proves that $R_{n+1} \rightrightarrows 0$ on $[1-r, 1+r]$ when $0 < r \leqslant 1/2$. $\square$

Proof for the case $0 < r < 1$. Assume $0 < r < 1$. For the convenience of discussion, we prove instead that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \tag{12.19}$$

the Taylor series of $\log(1+x)$ at 0, converges uniformly to $\log(1+x)$ on $[-r, r]$

The radius of convergence of the series (12.19) is 1. Thus, by Thm. 4.27, (12.19) converges locally uniformly on $(-1,1)$ to some function $f : (-1,1) \to \mathbb{R}$. By Cor. 11.37, f is differentiable on $(-1,1)$, and the series

$$\sum_{k=0}^{\infty} (-1)^k x^k \tag{12.20}$$

converges locally uniformly on $(-1,1)$ to f'. But we clearly have

$$(12.20) = \frac{1}{1-(-x)} = \frac{1}{1+x} \tag{12.21}$$

Thus $f'(x) = \frac{d}{dz}(\log(1+x))$. Now, (12.19) clearly implies that $f(0) = 0 = \log(1+0)$. Therefore, by Cor. 11.24, we obtain $f(x) = \log(1+x)$ on $(-1,1)$. In other words, (12.19) converges locally uniformly on $(-1,1)$ (and hence uniformly on $[-r,r]$) to $\log(1+x)$. $\square$

Example 12.21. Let $\alpha \in \mathbb{C}$. Then

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \tag{12.22}$$

the Taylor series of $(1+x)^\alpha = \exp(\alpha \log(1+x))$ at $x = 0$, converges locally uniformly on $(-1,1)$ to $(1+x)^\alpha$.

Proof. Using Prop. 4.23, one easily checks that the radius of convergence of (12.22) is 1. So Thm. 4.27 implies that (12.22) converges locally uniformly on $(-1,1)$ to some $f:(-1,1)\to\mathbb{C}$. Let

$$g(x)=(1+x)^\alpha$$

Our goal is to prove $f=g$.

It is clear that $(1+x)g'(x)=\alpha g(x)$. By Cor. 11.37, the series

$$\sum_{k=0}^\infty k\binom{\alpha}{k}x^{k-1}=\sum_{k=0}^\infty(k+1)\binom{\alpha}{k+1}x^k \tag{12.23}$$

converges locally uniformly on $(-1,1)$ to f'. Thus, by Cor. 5.60, we have

$$(1+x)f'(x)=\sum_{k=0}^\infty\left(k\binom{\alpha}{k}+(k+1)\binom{\alpha}{k+1}\right)x^k$$
$$=\sum_{k=0}^\infty\alpha\binom{\alpha}{k}x^k=\alpha f(x)$$

Therefore, f satisfies the same differential equation as g, namely $(1+x)f'=\alpha f$. Moreover, we clearly have $f(0)=1=g(0)$. So we have $f=g$ on $(-1,1)$ if we apply the next lemma to $f-g$. $\square$

Lemma 12.22. *Let I be an interval of $\mathbb{R}$ (with at least two points) such that $0\in I$. Let $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$. Let $\varphi\in C(I,\mathbb{F}^{n\times n})$. Assume that $f\in C^1(I,\mathbb{F}^n)$ satisfies*

$$f'(x)=\varphi(x)\cdot f(x)\qquad f(0)=0 \tag{12.24}$$

Then for all $x\in I$ we have $f(x)=0$.

Proof. We prove this lemma for the case that I is an open interval. The proof for the other types of intervals is similar. Also, we equip $\mathbb{F}^{n\times n}$ with the operator norm by viewing $n\times n$ matrices as elements of $\mathfrak{L}(\mathbb{F}^n)$. (Recall that the operator norm is equivalent to the Eucliden norm, see Thm. 10.30.) Let

$$\Omega=\{x\in I:f(x)=0\}=f^{-1}(0)$$

which is a closed subset of I (since the inverse image under a continuous map of a closed set is closed). Clearly Ω is nonempty since $0\in\Omega$. If we can prove that Ω is open, then we have $\Omega=I$ because I is connected. This will finish the proof.

So let us prove that any $p\in\Omega$ is an interior point of Ω with respect to I. Choose $r>0$ such that $[p-r,p+r]\subset I$. Let

$$C=\sup_{-r\leqslant x\leqslant r}\|\varphi(x)\|$$

which is a finite number by extreme value theorem. Let $\delta = \min\{\frac{1}{2C}, r\}$ which is > 0. It suffices to prove that

$$M = \sup_{x \in [p-\delta, p+\delta]} \|f(x)\|$$

is zero. (Note that $M < +\infty$ again by EVT.) Then we will have $[p-\delta, p+\delta] \subset \Omega$, finishing the proof.

For each $x \in [p-\delta, p+\delta]$, by Rem. 10.24 for operator norms, we have

$$\|f'(x)\| = \|\varphi(x)f(x)\| \leqslant \|\varphi(x)\| \cdot \|f(x)\| \leqslant CM$$

Thus, by finite-increment theorem, we have for all $x \in [p-\delta, p+\delta]$ that

$$\|f(x)\| = \|f(x) - f(p)\| \leqslant CM|x-p| \leqslant CM\delta \leqslant \frac{M}{2}$$

Applying $\sup_{x \in [p-\delta, p+\delta]}$ to $\|f(x)\|$, we get $M \leqslant M/2$. So $M = 0$. □

Remark 12.23. The above Lemma can also be proved in a similar way as Thm. 11.27: Let $x \in I$. Assume WLOG that $x > 0$. Let $E = f^{-1}(0) \cap [0, x]$, which is a closed subset of $[0, x]$. So $t = \sup E$ is in $[0, x]$. To show $f(x) = 0$, one just need to prove $t = x$. If not, then $t < x$. Then, as in the above proof, one can find $\delta > 0$ such that $\varphi(s) = 0$ whenever $s \in [t, t+\delta]$, impossible.

12.5 Convex functions

In this section, we fix an interval $I \subset \mathbb{R}$ containing at least two points.

Definition 12.24. Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, $\mathbf{z} = (z_1, z_2)$ be three points of $\mathbb{R}^2$ satisfying $x_1 < y_1 < z_1$. We say that the ordered triple $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a **convex triple** if the following equivalent conditions are satisfied:

(1) $\mathbf{y}$ is below or on the interval $[\mathbf{x}, \mathbf{z}]$. In other words, we have $y_2 \leqslant tx_2 + (1-t)z_2$ if $t \in [0, 1]$ is such that $y_1 = tx_1 + (1-t)z_1$.

(2) We have $\dfrac{y_2 - x_2}{y_1 - x_1} \leqslant \dfrac{z_2 - y_2}{z_1 - y_1}$.

(3) We have $\dfrac{y_2 - x_2}{y_1 - x_1} \leqslant \dfrac{z_2 - x_2}{z_1 - x_1}$.

(4) We have $\dfrac{z_2 - x_2}{z_1 - x_1} \leqslant \dfrac{z_2 - y_2}{z_1 - y_1}$.

Proof of equivalence. The equivalence of these statements is clear from the picture, and is not hard to check rigorously using inequalities. We leave the details to the reader. □

Recall the definition of convex subsets in real vector spaces (Def. 11.30).

Definition 12.25. A function $f : I \to \mathbb{R}$ is called **convex** if the following equivalent conditions are true:

(1) The set
$$D_f = \{(x,y) \in \mathbb{R}^2 : x \in I, y \geqslant f(x)\} \tag{12.25}$$
is a convex subset of $\mathbb{R}^2$.

(1) For any three points $x < y < z$ of I, the points $(x, f(x)), (y, f(y)), (z, f(z))$ form a convex triple.

Proof of equivalence. Again, the equivalence is clear from the picture. The details are left to the readers. □

The convexity of differentiable functions is easy to characterize:

Theorem 12.26. *Let $f : I \to \mathbb{R}$ be differentiable. Then f is convex iff f' is increasing. In particular, if f'' exists on I, then f is convex iff $f'' \geqslant 0$ on I.*

Proof. Assume that f' is increasing. Choose $x < y < z$ in I. By Lagrange's MVT, there exist $\xi \in (x,y)$ and $\eta \in (y,z)$ such that the slope of the interval $[(x, f(x)), (y, f(y))]$ equals $f'(\xi)$, and that the slope of $[(y, f(y)), (z, f(z))]$ equals $f'(\eta)$. Since $f'(\xi) \leqslant f'(\eta)$, the points $(x, f(x)), (y, f(y)), (z, f(z))$ form a convex triple. This proves that f is convex.

Now assume that f is convex. Choose any $x < y$ in I. We need to prove that $f'(x) \leqslant f'(y)$. Choose any $\varepsilon > 0$. Then there exist x_1 and y_1 such that $x < x_1 < y_1 < y$, that $f'(x) - \varepsilon$ is $\leqslant$ the slope k_1 of $[(x, f(x)), (x_1, f(x_1))]$, and that $f'(y) + \varepsilon$ is $\geqslant$ the slope k_2 of $[(y_1, f(y_1)), (y, f(y))]$. Since f is convex, k_1 is $\leqslant$ the slope k' of $[(x_1, f(x_1)), (y_1, f(y_1))]$, and $k' \leqslant k_2$. Therefore $k_1 \leqslant k_2$. Thus $f'(x) - \varepsilon \leqslant f'(y) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $f'(x) \leqslant f'(y)$.

When f'' exists, the equivalence between $f'' \geqslant 0$ and the increasing monotonicity of f' is due to Cor. 11.20. □

The most important general property about convex functions is:

Theorem 12.27 (Jensen's inequality). *Let $f : I \to \mathbb{R}$ be a convex function. Let $n \in \mathbb{Z}_+$ and $x_1, \dots, x_n \in I$. Choose $t_1, \dots, t_n \in [0,1]$ such that $t_1 + \cdots + t_n = 1$. Then*
$$f(t_1x_1 + \cdots + t_nx_n) \leqslant t_1 f(x_1) + \cdots + t_n f(x_n) \tag{12.26}$$

Proof. The points $(x_1, f(x_1)), \dots, (x_n, f(x_n))$ are in $D_f = $ (12.25). Therefore, by Lem. 12.28, the point
$$t_1(x_1, f(x_1)) + \cdots + t_n(x_n, f(x_n)) = (t_1x_1 + \cdots + t_nx_n, t_1f(x_1) + \cdots + t_nf(x_n))$$
is in the convex set D_f. □

Lemma 12.28. *Let V be a real vector space. Let E be a convex subset of V. Let $n \in \mathbb{Z}_+$ and $p_1, \dots, p_n \in E$. Any* ***convex combination*** *of $p_1, \dots, p_n$ (i.e., any point of the form $t_1p_1 + \cdots + t_np_n$ where $t_1, \dots, t_n \in [0,1]$ and $t_1 + \cdots + t_n = 1$) is inside E.*

The geometric meaning of this lemma is clear: If $V = \mathbb{R}^2$ (which is the main case we are interested in), then $t_1p_1 + \cdots + t_np_n$ stands for an arbitrary point inside the polygon with vertices $p_1, \dots, p_n$. So this point is clearly inside E.

Proof. We prove by induction on n. The case $n = 1$ is obvious. Suppose case $n-1$ has been proved $n \geqslant 2$. Consider case n. It is trivial when $t_1 + \cdots + t_{n-1} = 0$. So let us assume $t_1 + \cdots + t_{n-1} > 0$. By case $n-1$, the point $q = \lambda_1 p_1 + \cdots + \lambda_{n-1}p_{n-1}$ is in E where

$$\lambda_i = \frac{t_i}{t_1 + \cdots + t_{n-1}}$$

Since E is convex, the point $(t_1 + \cdots + t_{n-1})q + t_np_n$ (which equals $t_1p_1 + \cdots + t_np_n$) is in E. $\square$

Example 12.29. Since $(-\log x)'' = x^{-2}$ is positive on $\mathbb{R}_{>0}$, by Thm. 12.26, $-\log$ is a convex function on $\mathbb{R}_{>0}$. Therefore, if $0 < \lambda_1, \dots, \lambda_n \leqslant 1$ and $\lambda_1 + \cdots + \lambda_n = 1$, then by Jensen's inequality, for each $x_1, \dots, x_n > 0$ we have $-\log(\sum_i \lambda_i x_i) \leqslant -\lambda_i \log x_i$. Taking exponentials, we get

$$x_1^{\lambda_1} \cdots x_n^{\lambda_n} \leqslant \lambda_1 x_1 + \cdots + \lambda_n x_n$$

for all $x_1, \dots, x_n > 0$, and hence for all $x_1, \dots, x_n \geqslant 0$. In particular, we get the **inequality of arithmetic and geometric means**

$$(x_1 \cdots x_n)^{\frac{1}{n}} \leqslant \frac{x_1 + \cdots + x_n}{n} \tag{12.27}$$

and the **Young's inequality**

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leqslant \frac{x}{p} + \frac{y}{q} \tag{12.28}$$

if $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 12.30. Let $p \in [1, +\infty]$. We say that $q \in [1, +\infty]$ is the **Hölder conjugate** of p if $\frac{1}{p} + \frac{1}{q} = 1$.

12.6 Hölder's and Minkowski's inequalities; l^p spaces

In this section, we use Young's inequality to prove two inequalities that are of vital importance to the development of modern analysis:

Theorem 12.31. *Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}_{\geqslant 0}$. Let $p, q \in (1, +\infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then the following inequalities (called respectively **Hölder's inequality** and **Minkowski's inequality**) are true:*

$$x_1 y_1 + \cdots + x_n y_n \leqslant \sqrt[p]{x_1^p + \cdots + x_n^p} \cdot \sqrt[q]{y_1^q + \cdots + y_n^q} \tag{12.29a}$$

$$\sqrt[p]{(x_1+y_1)^p + \cdots + (x_n+y_n)^p} \leqslant \sqrt[p]{x_1^p + \cdots + x_n^p} + \sqrt[p]{x_1^p + \cdots + x_n^p} \tag{12.29b}$$

In the special case that $p = q = 2$, Hölder's inequality is called the **Cauchy-Schwarz inequality**.

The following proof is quite tricky. In Sec. 29.6, we will give a more straightforward proof of the two inequalities using Lagrange multipliers.

$\star\star$ *Proof.* Assume WLOG that $x_i > 0$ and $y_j > 0$ for some i, j. Young's inequality implies that for each i,

$$\frac{x_i}{\sqrt[p]{\sum_k x_k^p}} \cdot \frac{y_i}{\sqrt[q]{\sum_k y_k^q}} \leqslant \frac{x_i^p}{p \sum_k x_k^p} + \frac{y_i^q}{q \sum_k y_k^q}$$

Taking sum over i gives

$$\sum_i \frac{x_i}{\sqrt[p]{\sum_k x_k^p}} \cdot \frac{y_i}{\sqrt[q]{\sum_k y_k^q}} \leqslant \frac{1}{p} + \frac{1}{q} = 1$$

This proves Hölder's inequality.

Notice that $pq = p + q$. Let $z_i = x_i + y_i$. By Hölder's inequality, we have

$$x_1 z_1^{p-1} + \cdots + x_n z_n^{p-1} \leqslant \Big(\sum_k x_k^p\Big)^{\frac{1}{p}} \cdot \Big(\sum_k z_k^{(p-1)q}\Big)^{\frac{1}{q}} = \Big(\sum_k x_k^p\Big)^{\frac{1}{p}} \cdot \Big(\sum_k z_k^p\Big)^{\frac{1}{q}}$$

and similarly

$$y_1 z_1^{p-1} + \cdots + y_n z_n^{p-1} \leqslant \Big(\sum_k y_k^p\Big)^{\frac{1}{p}} \cdot \Big(\sum_k z_k^p\Big)^{\frac{1}{q}}$$

Adding up these two inequalities, we get

$$\sum_k z_k^p \leqslant \bigg(\Big(\sum_k x_k^p\Big)^{\frac{1}{p}} + \Big(\sum_k y_k^p\Big)^{\frac{1}{p}}\bigg) \cdot \Big(\sum_k z_k^p\Big)^{\frac{1}{q}}$$

Dividing both sides by $\Big(\sum_k z_k^p\Big)^{\frac{1}{q}}$ proves Minkowski's inequality. $\square$

Minkowski's inequality is often used in the following way:

Theorem 12.32. *Let X be a set, and let V be a normed vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $1 \leqslant p < +\infty$. For each $f \in V^X$, define the* ***l^p-norm***

$$\|f\|_p \equiv \|f\|_{l^p} = \Big(\sum_{x\in X} \|f(x)\|^p\Big)^{\frac{1}{p}} \tag{12.30}$$

Then for each $f, g \in V^X$ and $\lambda \in \mathbb{F}$ we have

$$\|f+g\|_p \leqslant \|f\|_p + \|g\|_p \qquad \|\lambda f\|_p = |\lambda| \cdot \|f\|_p \tag{12.31}$$

In particular, $\|\cdot\|_p$ is a norm on the ***l^p-space***

$$l^p(X,V) = \{f \in V^X : \|f\|_{l^p} < +\infty\} \tag{12.32}$$

If V is Banach space, then so is $l^p(X,V)$.

Recall $0 \cdot (\pm\infty) = 0$.

Proof. Choose any $A \in \mathrm{fin}(2^X)$. By Minkowski's and triangle inequality, we have

$$\begin{aligned}&\sum_A |f+g|^p \leqslant \sum_A (|f|+|g|)^p \leqslant \bigg(\Big(\sum_A |f|^p\Big)^{\frac{1}{p}} + \Big(\sum_B |f|^p\Big)^{\frac{1}{p}}\bigg)^p\\ \leqslant&\bigg(\Big(\sum_X |f|^p\Big)^{\frac{1}{p}} + \Big(\sum_X |f|^p\Big)^{\frac{1}{p}}\bigg)^p = (\|f\|_p + \|g\|_p)^p\end{aligned}$$

Taking $\lim_{A\in\mathrm{fin}(2^X)}$ gives $\|f+g\|_p^p \leqslant (\|f\|_p + \|g\|_p)^p$, proving the first inequality of (12.31). The second of (12.31) clearly holds by taking $\lim_{A\in\mathrm{fin}(2^X)}$ of

$$\sum_A |\lambda f|^p = |\lambda| \sum_A |f|^p$$

and noting that the multiplication map $t \in \overline{\mathbb{R}}_{\geqslant 0} \mapsto |\lambda| t \in \overline{\mathbb{R}}_{\geqslant 0}$ is continuous.

Suppose that V is complete. Let (f_n) be a Cauchy sequence in $l^p(X,V)$. Then for each $x \in X$, $(f_n(x))$ is a Cauchy sequence in V, converging to some element $f(x) \in V$. By Cauchyness, for each $\varepsilon > 0$ there is $N \in \mathbb{Z}_+$ such that for all $n, k \geqslant N$ we have $\|f_n - f_k\|_p \leqslant \varepsilon$, and hence

$$\sum_{x\in A} \|f_n(x) - f_k(x)\|^p \leqslant \varepsilon^p$$

for all $A \in \mathrm{fin}(2^X)$. Taking $\lim_k$ gives $\sum_A |f_n - f|^p \leqslant \varepsilon^p$. Taking $\lim_{A\in\mathrm{fin}(2^X)}$ gives $\|f_n - f\|_p \leqslant \varepsilon$ for all $n \in\geqslant N$. (In particular, $\|f\|_p \leqslant \|f_N\| + \varepsilon < +\infty$, and hence $f \in l^p(X,V)$.) This proves that (f_n) converges to f under the l^p-norm. So $l^p(X,V)$ is complete. $\square$

Theorem 12.33. *Let X be a set. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $p, q \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then there is a linear isometry*

$$\Psi : l^p(X, \mathbb{F}) \to l^q(X, \mathbb{F})^* \qquad f \mapsto \Psi(f) \tag{12.33a}$$

where for each $g \in l^q(X, \mathbb{F})$, the value of $\Psi(f)$ at g is

$$\langle \Psi(f), g \rangle = \sum_{x \in X} f(x) g(x) \tag{12.33b}$$

where the RHS converges.

In fact, we will see in Thm. 17.30 that if $q < +\infty$ (and hence $p > 1$), Ψ is surjective, and hence is an isomorphism of normed vector spaces. (This is not a difficult fact. So you can prove it yourself.)

Proof. We treat the case $1 < p, q < +\infty$, and leave the case that $p \in \{1, +\infty\}$ to the readers. Assume $f \in l^p$ and $g \in l^q$, then for each $A \in \mathrm{fin}(2^X)$ we have by Hölder's inequality that

$$\Big\| \sum_A fg \Big\| \leqslant \sum_A |fg| \leqslant \Big(\sum_A |f|^p \Big)^{\frac{1}{p}} \Big(\sum_A |g|^q \Big)^{\frac{1}{q}} \leqslant \|f\|_p \cdot \|g\|_q$$

Applying $\lim_{A \in \mathrm{fin}(2^X)}$ to the second term above implies that $\sum_X |fg| < +\infty$. So $\sum_X fg$ converges absolutely, and hence converges. So (12.33b) makes sense, and hence Ψ is a well-defined linear map. Applying $\lim_{A \in \mathrm{fin}(2^X)}$ to the first term above implies that

$$|\langle \Psi(f), g \rangle| \leqslant \|f\|_p \cdot \|g\|_q$$

Thus, by Rem. 10.24, we obtain $\|\Psi(f)\| \leqslant \|f\|_p$.

To show $\|\Psi(f)\| = \|f\|_p$, assume WLOG that $f \neq 0$, and define $g : X \to \mathbb{F}$ to be $g = \frac{\overline{f}}{|f|} \cdot |f|^{p-1}$, where $g(x)$ is understood as 0 if $f(x) = 0$. Then

$$\langle \Psi(f), g \rangle = \sum_X |f|^p = \|f\|_p^p$$

and $\|g\|_q^q = \sum_X |f|^{pq-q} = \sum_X |f|^p = \|f\|_p^p$. So

$$\|f\|_p \cdot \|g\|_q = \|f\|_p^{1+p/q} = \|f\|_p^p = \langle \Psi(f), g \rangle$$

This proves $\|\Psi(f)\| = \|f\|_p$ by Rem. 10.24. So Ψ is a linear isometry. □

I do not want to deviate too much from the main purpose of this section, which is to show some important applications of convex functions. I will therefore stop my discussion about l^p spaces, and continue this topic in the future. The crucial role of l^p spaces and their continuous versions (namely L^p spaces) in modern analysis is a long story. The study of these objects constitutes a major part of the second half of this course. Let me mention just one point: the compactness of $\overline{B}_{l^2(\mathbb{Z},\mathbb{C})}(0,1)$ under the pointwise convergence topology (by viewing it as a subspace of $\mathbb{C}^{\mathbb{Z}}$) was the most important reason for Hilbert and Schmidt to study the Hilbert space $l^2(\mathbb{Z},\mathbb{C})$, and it was the crucial property that allowed them to fully solve the eigenvalue problem in integral equations (cf. Subsec. 10.4.1).

12.7 Problems and supplementary material

Let $-\infty < a < b < +\infty$. Let V be a Banach space over $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$.

Problem 12.1. Let f be the function in Exp. 12.19. Prove that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

Problem 12.2. Let $n \in \mathbb{N}$. Let $f : [a,b] \to V$ such that $f, f', \dots, f^{(n)}$ exist everywhere on $[a,b]$. Use the higher order finite-increment theorem to prove that

$$\Big\Vert f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \Big\Vert \leqslant \frac{(x-a)^n}{n!} \cdot \sup_{a<t<x} \Vert f^{(n)}(t) - f^{(n)}(a) \Vert \tag{12.34}$$

Remark 12.34. Setting $n = 1$ and dividing both sides by $x - a$, we obtain an especially useful formula for any differentiable $f : [a,b] \to V$ as follows.

$$\Big\Vert \frac{f(x)-f(a)}{x-a} - f'(a) \Big\Vert \leqslant \sup_{a<t<x} \Vert f'(t) - f'(a) \Vert \tag{12.35}$$

Problem 12.3. Use Thm. 11.33 to prove the following theorem. (Formula (12.35) might be helpful.)

Theorem 12.35. *Let $I = [a,b]$ and $J = [c,d]$ be intervals in $\mathbb{R}$ with at least two points. Let $f : I \times J \to V$ be a function such that $\partial_1 f, \partial_2 f, \partial_2\partial_1 f$ exist on $I \times J$, and that $\partial_2\partial_1 f$ is continuous. Then $\partial_1\partial_2 f$ exists on $I \times J$ and equals $\partial_2\partial_1 f$.*

⋆ **Problem 12.4.** Prove that, for $x \in (-1,1]$,

$$\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + - \cdots, \tag{12.36}$$

and hence (**Leibniz formula**)

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots$$

Hint. Use the method in the second proof of Exp. 12.20 to prove (12.36) for $x \in (-1, 1)$. To prove (12.36) for $x = 1$, show that the series on the RHS of (12.36) converges uniformly on $[0, 1]$ to a continuous function using Dirichlet's test for uniform convergence (Thm. 4.29). □

⋆ **Problem 12.5.** Let $n \in \mathbb{N}$. Find the radius of convergence of

$$f(z) = \sum_{k=n}^{\infty} \binom{k}{n} z^k \tag{12.37}$$

Find the explicit formula of $f(z)$.

Problem 12.6. Prove the following higher order finite-increment theorem for complex variables. (For simplicity, it suffices prove the case that $z_0 = 0$.)

Theorem 12.36 (Higher order finite-increment theorem). *Let V be a Banach space over $\mathbb{C}$. Let $R > 0$ and $z_0 \in \mathbb{C}$. Let $f : \Omega \to V$ where*

$$\Omega = B_{\mathbb{C}}(z_0, R) = \{z \in \mathbb{C} : |z - z_0| < R\}$$

Assume that $f', f'', \dots, f^{(n+1)}$ exist everywhere on Ω. Then for every $z \in \Omega$ we have

$$\Big\| f(z) - \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \Big\| \leqslant \frac{|z - z_0|^{n+1}}{(n+1)!} \cdot \sup_{\xi \in [z_0, z]} \|f^{(n+1)}(\xi)\| \tag{12.38}$$

where $[z_0, z] = \{tz + (1-t)z_0 : 0 \leqslant t \leqslant 1\}$.

⋆ **Problem 12.7.** Let V be a Banach space over $\mathbb{C}$. Assume that the power series $f(z) = \sum_{k=0}^{\infty} v_k z^k$ (where $v_k \in V$) has radius of convergence $R > 0$. Choose any $z_0 \in B_{\mathbb{C}}(0, R)$. Prove that there exists a neighborhood $\Omega \in \mathrm{Nbh}_{\mathbb{C}}(z_0)$ contained inside $B_{\mathbb{C}}(0, R)$, such that the Taylor series of f at z_0 converges uniformly on Ω to f. Namely, prove that the series $g(z)$ converges locally uniformly to f on Ω where

$$g(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \tag{12.39}$$

Hint. Use Thm. 12.36. Pb. 12.5 might also be helpful. □

⋆ **Problem 12.8.** Let W be a real vector space. Let A be a nonempty subset of W. Define the **convex hull** of A to be the set of all convex combinations (recall Lem. 12.28) of elements of A, i.e.

$$\mathrm{Cvh}(A) = \big\{\text{convex combinations of } v_1, \dots, v_n : n \in \mathbb{Z}_+ \text{ and } v_1, \dots, v_n \in A\big\} \tag{12.40}$$

Prove that $\mathrm{Cvh}(A)$ is the smallest convex set containing A. In other words, prove that $\mathrm{Cvh}(A)$ is a convex set containing A, and that $\mathrm{Cvh}(A) \subset B$ if B is a convex subset of W containing A.

Problem 12.9. Prove Thm. 12.33 for the case $p = 1$ and $p = +\infty$.

13 Riemann integrals

13.1 Introduction: the origin of integral theory in Fourier series

13.1.1 Antiderivatives VS. approximation by areas of rectangles

Modern people can easily appreciate the importance of giving a rigorous definition to the integral $\int_a^b f(x)dx$ where f is a general (say, continuous) function. And you may already know that this integral is defined by taking the limit of Riemann sums, which are "areas of some rectangles". This idea sounds so natural to you, because you know that by the time Calculus was invented, people knew very well that $\int_a^b f$ means the area of the region between the graph of f and the x-axis, and that this area can be approximated by the areas of some rectangles or triangles. So why did Riemann integral not appear until 19th century, more than a hundred years after Newton's and Leibniz's invention of calculus? And why was the inadequacy of defining integrals by means of antiderivatives not recognized until 19th century?

Beyond the superficial reason that 19th century is the century in which people began to pay attention to the foundations of calculus, there is a deeper reason: the study of **partial differential equations (PDE)** and the introduction of **Fourier series** caused a drastic change in the general view of what functions are. This change of view motivated people to search for a general definition of integrals, in particular one that does not use antiderivatives.

Before the systematic study of PDEs (i.e., before 19th century), functions only mean analytic functions, which means that they can be approximated by their Taylor series. But if f is such a function, then $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. So the values of f are all determined by $f(a), f'(a), f''(a), \dots$, and hence are determined by $f|_I$ where I is an arbitrary nonempty open interval. (Technically, such functions are called **(real) analytic functions**.) Therefore, periodic functions (except trigonometric functions) were not accepted, because they are not determined by their values on a small interval. (A function with expression $f(x) = x$ on $[0,1]$ should also have the same expression on $\mathbb{R}$. So it must not be the periodic function $f(x) = x - n$ where $x \in (n, n+1]$.) The function Exp. 12.19 was not accepted, because it was not determined by $f(0), f'(0), f''(0), \dots$.

Therefore, in these old days, people did not worry about the rigorous definition of integrals, because they can simply define integrals using antiderivatives instead of using the areas of rectangles to approximate the integrals, which is practically less convenient than finding the antiderivatives. In particular, one understood

$$\int_a^b \sum a_n(x-c)^n dx \xlongequal{\text{def}} F(b) - F(a) \qquad \text{where} \qquad F(x) = \sum \frac{a_n}{n+1}(x-c)^{n+1}$$

13.1.2 PDEs and periodic functions

The wonderful dream was shattered when people started working on PDEs. The simplest such example is **wave equation**

$$\partial_x^2 f(x,t) - \partial_t^2 f(x,t) = 0 \tag{13.1}$$

This equation describes the vibration of a string: at time t, the shape of the string is the graph of the function $x \in [a,b] \mapsto f(x,t) \in \mathbb{R}$. (Here, we assume that the two end points of the string are $(a,0)$ and $(b,0)$.)

D'Alambert solved the wave equation in the following way. Using the trick

$$\partial_x^2 - \partial_t^2 = 4\partial_u\partial_v \qquad \text{where} \qquad u = x+t, v = x-t$$

it is not hard to find the general solution of the wave equation:

$$f(x,t) = \frac{1}{2}\big(g(x+t) + g(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} h(s)ds \tag{13.2}$$

where $g(x) = f(x,0)$ and $h(x) = \partial_t f(x,0)$. In particular, if we assume that $h = 0$ (i.e., at time $t = 0$ the string is held in place and does not vibrate), then the solution is

$$f(x,t) = \frac{1}{2}\big(g(x+t) + g(x-t)\big) \tag{13.3}$$

Here comes the trouble. If we assume that the two end points of the string are always pinned at $(0,0)$ and $(1,0)$ respectively, then we have $f(0,t) = f(1,t) = 0$ for all $t \in \mathbb{R}$. Translating this condition to g, we get $g(t) = -g(-t) = -g(2-t)$ and hence g is a function with period 2, totally unacceptable! Worse still, the derivatives of periodic functions might have points of discontinuities.

13.1.3 Fourier series

The next important progress was made by Fourier in the study of **heat equation**

$$\partial_t f(x,t) - \partial_x^2 f(x,t) = 0 \tag{13.4}$$

where x is defined on a closed interval (say $[-\pi,\pi]$) representing a thin rod, $f(x,t)$ is the temperature of the point x of this rod at time t. Fourier solved the problem by **separation of variables**: He first assumed $f(x,t) = u(x)v(t)$. Then the heat equation implies $u''(x)v(t) = u(x)v'(t)$, and hence

$$\frac{u''(x)}{u(x)} = \frac{v'(t)}{v(t)} \tag{13.5}$$

The LHS is independent of t, and the RHS is independ of x. So (13.5) should be a constant $-\lambda$. The solution is then $u(x) = e^{\mathbf{i}\sqrt{\lambda}x}$ and $v(t) = e^{-\lambda t}$. If one assumes that the temperature at the two end points $-\pi, \pi$ are equal, then $\sqrt{\lambda}$ must be an integer $n \in \mathbb{Z}$. So $f(x,t) = e^{\mathbf{i}nx-n^2t}$. Taking infinite linear combinations, Fourier found the general solution

$$f(x,t) = \sum_{n=-\infty}^{+\infty} a_n e^{\mathbf{i}nx-n^2t} \tag{13.6}$$

The initial temperature $g(x) = f(x,0)$ is a **Fourier series**

$$g(x) = \sum_{n=-\infty}^{+\infty} a_n e^{\mathbf{i}nx} \tag{13.7}$$

and, in particular, a function with period 2π (since each $e^{\mathbf{i}nx}$ is so).

In order to use (13.6) to determine the solution $f(x,t)$ when the initial condition $f(x,t) = g(x)$ is given, one must first find the values of these **Fourier coefficients** a_n in terms of g. In fact, since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathbf{i}kx} \cdot e^{-\mathbf{i}nx} dx = \delta_{k,n} \tag{13.8}$$

(recall (0.2)), it is not hard to guess the formula

$$\boxed{a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-\mathbf{i}nx} dx} \tag{13.9}$$

since this formula is true when $g(x) = e^{\mathbf{i}kx}$.

13.1.4 From Riemann integrals to Lebesgue integrals

Here comes the question that is closely related to integral theory: What is the meaning of the integral (13.9) if g is no longer a real analytic function?

To be more precise, in 19th century there was a long debate about what are (good) functions, and what functions have **Fourier expansions**, i.e., expansions of the form (13.7). Fourier himself believed that all "functions" can be approximated by trigonometric functions (equivalently, functions of the form $e^{\mathbf{i}nx}$). But Lagrange did not, partly due to his (and many people's) insistence that functions must be real analytic (cf. [Kli, Sec. 28.2]). The doubt on whether many functions have Fourier expansions is quite understandable. From the modern perspective, we know that many Lebesgue measurable functions are not approximated by their Fourier series pointwise or uniformly, but are approximated under some other norms (e.g. the L^2-norm $\|f\|_{L^2} = \sqrt{\int_{-\pi}^{\pi} |f|^2}$). Indeed, there are many continuous functions $[-\pi,\pi] \to \mathbb{R}$ whose Fourier series diverge on a dense subset of $[-\pi,\pi]$ (cf. [Rud-R, Thm. 5.12]). Therefore, Fourier's pioneering view that all "reasonable" functions can be approximated by Fourier series is incorrect unless we define what "approximation" means in a new and appropriate way.

In order to understand which functions have Fourier series expansions (in whatever sense), the first step is to understand for which function g the integral (13.9) makes sense. Therefore, the history of extending the class of integrable functions from continuous functions to Riemann integrable functions and finally to Lebesgue integrable functions is also part of the history of understanding which functions can have Fourier expansions and which functions are "reasonable".

The goal of this chapter is to learn Riemann integrals. The construction of Riemann integrals is much easier than Lebesgue theory. But compared to the latter, Riemann integrals have some serious drawbacks.

For example, suppose that a sequence of functions (f_n) on $[-\pi,\pi]$ converges to f in some sense. It is natural to ask whether the Fourier coefficients of (f_n) converge to those of f. (A typical example is $f_n(x) = \sum_{k=-n}^{n} a_k e^{\mathbf{i}kx}$. Then this question asks whether the n-th Fourier coefficient of the series $\sum_{k=-\infty}^{+\infty} a_k e^{\mathbf{i}kx}$ is a_n.) In view of (13.9), this problem is reduced to the problem of showing $\int f = \lim_n \int f_n$. If (f_n) is a sequence of Riemann integrable functions converging uniformly to f, then f is Riemann integrable, and $\int f = \lim \int f_n$. (See Cor. 13.21.) However, if (f_n) only converges pointwise to f, then f is not necessarily Riemann integrable. Even if f is Riemann integrable, one does not have a useful criterion for $\int f = \lim \int f_n$ in the framework of Riemann integrals.

But uniform convergence often does not hold in application, and especially in Fourier theory and PDEs. For example, let f be the function on $\mathbb{R}$ with period 2π defined by $f(x) = x$ if $-\pi < x < \pi$ and $f(\pi) = 0$. Then the Fourier series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2}{n} \sin(nx) \tag{13.10}$$

converges pointwise to f. (See also Pb. 4.3.) It does not converge uniformly to f, because the uniform limit of a sequence of continuous functions is continuous, but f is not continuous.

Lebesgue's integral theory will provide a more satisfying answer to the above problem. We will learn it in the second semester.

13.2 Riemann integrability and oscillation

In this section, we fix a Banach space V over $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$, and let $I = [a,b]$ where $-\infty < a < b < +\infty$. We understand $|I|$ as $b-a$.

13.2.1 Riemann integrals

Definition 13.1. A **partition** of the interval $I = [a, b]$ is defined to be an element of the form

$$\sigma = \{a_0, a_1, \ldots, a_n \in I : a_0 = a < a_1 < a_2 < \cdots < a_n = b, n \in \mathbb{Z}_+\} \tag{13.11}$$

Equivalently, this partition can be written as

$$I = I_1 \cup I_2 \cup \cdots \cup I_n \qquad I_j = [a_{j-1}, a_j] \tag{13.12}$$

If $\sigma, \sigma' \in \mathrm{fin}(2^I)$ are partitions of I, we say that σ' is a **refinement** of σ (or that σ' is **finer than** σ), if $\sigma \subset \sigma'$. In this case, we also write

$$\sigma \prec \sigma'$$

We define $\mathcal{P}(I)$ or simply $\mathcal{P}$ to be

$$\mathcal{P}(I) = \{\text{partitions of } I\}$$

Remark 13.2. If $\sigma, \sigma' \in \mathcal{P}(I)$, then clearly $\sigma \cup \sigma' \in \mathcal{P}(I)$ and $\sigma, \sigma' \prec \sigma \cup \sigma$. Therefore, $\prec$ is a partial order on $\mathcal{P}(I)$. We call $\sigma \cup \sigma'$ the **common refinement** of σ and σ'.

Definition 13.3. A **tagged partition** of I is an ordered pair

$$(\sigma, \xi_\bullet) = \big(\{a_0 = a < a_1 < \cdots < a_n = b\}, (\xi_1, \ldots, \xi_n)\big)$$

where $\sigma \in \mathcal{P}(I)$ and $\xi_j \in I_j = [a_{j-1}, a_j]$ for all $1 \leqslant j \leqslant n$. The set

$$\mathcal{Q}(I) = \{\text{tagged partitions of } I\}$$

equipped with the preorder $\prec$ defined by

$$(\sigma, \xi_\bullet) \prec (\sigma', \xi'_\bullet) \qquad \Longleftrightarrow \qquad \sigma \subset \sigma' \tag{13.13}$$

is a directed set.

Definition 13.4. Let $f : I \to V$. For each $(\sigma, \xi_\bullet) \in \mathcal{Q}(I)$, define the **Riemann sum**

$$S(f, \sigma, \xi_\bullet) = \sum_{j \geqslant 1} f(\xi_j)(a_j - a_{j-1})$$

The **Riemann integral** is defined to be the limit of the net $(S(f, \sigma, \xi_\bullet))_{(\sigma, \xi_\bullet) \in \mathcal{Q}(I)}$:

$$\int_a^b f \equiv \int_a^b f(x)dx = \lim_{(\sigma, \xi_\bullet) \in \mathcal{Q}(I)} S(f, \sigma, \xi_\bullet)$$

When the RHS exists, we way that f is **Riemann integrable** on I. We let

$$\mathscr{R}(I, V) = \{\text{Riemann integrable } f \in V^I\}$$

Convention 13.5. There are several equivalent ways to write the integrals:

$$\int_a^b f = -\int_b^a f = \int_{[a,b]} f$$

if $a < b$. Also, if $a = b$, the above terms are understood to be 0, and all functions on $[a, b]$ are considered Riemann integrable.

Remark 13.6. It is clear that $\int_a^b f = v \in V$ iff for every $\varepsilon > 0$ there exists $\sigma_0 \in \mathcal{P}(I)$ such that for any partition $\sigma = \{a_0 < \cdots < a_n\}$ finer than σ_0, and for any $\xi_j \in [a_{j-1}, a_j]$ (for all $1 \leqslant j \leqslant n$) we have

$$\Big\| v - \sum_{j\geqslant 1} f(\xi_j)(a_j - a_{j-1}) \Big\| < \varepsilon$$

There is not need to tag σ_0.

13.2.2 Riemann integrability and strong Riemann integrability

In this subsection, we give a useful criterion for Riemann integrability.

Definition 13.7. Let A be a nonempty subset of a metric space Y. The **diameter** of A is defined to be

$$\mathrm{diam}(A) = \sup_{x,y\in A} d(x, y)$$

If $f : X \to Y$ is a map where X is a set, and if $E \subset X$, the **oscillation** of f on E is defined to be $\mathrm{diam} f(E)$.

The following lemma allows us to control the difference of Riemann sums by means of the oscillation.

Lemma 13.8. *Let $f : I = [a, b] \to V$ and $M = \mathrm{diam}(f(I))$. Then for each $(\sigma, \xi_\bullet), (\sigma', \xi'_\bullet) \in \mathcal{Q}(I)$ we have*

$$\|S(f, \sigma, \xi_\bullet) - S(f, \sigma', \xi'_\bullet)\| \leqslant M(b - a) \tag{13.14}$$

Proof. Write $\sigma \cup \sigma' = \{a_0 = a < a_1 < \cdots < a_n = b\}$. Then there exist $\gamma_1, \ldots, \gamma_n \in \{\xi_1, \xi_2, \ldots\}$ such that

$$S(f, \sigma, \xi_\bullet) = \sum_{i=1}^{n} f(\gamma_i)(a_i - a_{i-1})$$

(Here is how to choose γ_i: Let k be the unique number such that the k-th subinterval of σ contains $[a_{i-1},a_i]$. Then let $\gamma_i=\xi_k$.) Similarly, there exist $\gamma'_1,\dots,\gamma'_n\in\{\xi'_1,\xi'_2,\dots\}$ such that

$$S(f,\sigma',\xi'_\bullet)=\sum_{i=1}^n f(\gamma'_i)(a_i-a_{i-1})$$

Then

$$\|S(f,\sigma,\xi_\bullet)-S(f,\sigma',\xi'_\bullet)\|\leqslant\sum_{i=1}^n\|f(\gamma_i)-f(\gamma'_i)\|\cdot(a_i-a_{i-1})$$

$$\leqslant\sum_{i=1}^n M(a_i-a_{i-1})\leqslant M(b-a)$$

□

Definition 13.9. Let $f:I\to V$. For each $\sigma=\{a_0<\cdots<a_n\}\in\mathcal{P}(I)$, write $I_i=[a_{i-1},a_i]$ and define the **oscillation of f on σ** to be

$$\omega(f,\sigma)=\sum_{j=1}^n \mathrm{diam}(f(I_j))\cdot(a_j-a_{j-1}) \tag{13.15}$$

Exercise 13.10. Show that if $\sigma'\supset\sigma$, then $\omega(f,\sigma')\leqslant\omega(f,\sigma)$. Therefore, $(\omega(f,\sigma))_{\sigma\in\mathcal{P}(I)}$ is a decreasing net in $\overline{\mathbb{R}}_{\geqslant0}$.

Now, Lem. 13.8 can be upgraded to the following version.

Lemma 13.11. *Let $f:I\to V$. Let $\sigma\in\mathcal{P}(I)$. Choose $(\sigma',\xi'_\bullet),(\sigma'',\xi''_\bullet)\in\mathcal{Q}(I)$ such that $\sigma\subset\sigma'$ and $\sigma\subset\sigma''$. Then*

$$\|S(f,\sigma',\xi'_\bullet)-S(f,\sigma'',\xi''_\bullet)\|\leqslant\omega(f,\sigma) \tag{13.16}$$

Proof. Write $\sigma=\{a_0<\cdots<a_n\}$ and $I_j=[a_{j-1},a_j]$. Let $S(f,\sigma',\xi'_\bullet)_{I_j}$ be the **restriction of $S(f,\sigma',\xi'_\bullet)$ to I_j**. Namely,

$$S(f,\sigma',\xi'_\bullet)_{I_j}=\sum_{\substack{\text{all } k \text{ such that}\\ [a_{k-1},a_k]\subset I_j}} f(\xi_k)(a_k-a_{k-1}) \tag{13.17}$$

Define $S(f,\sigma'',\xi''_\bullet)_{I_j}$ in a similar way. Then, by Lem. 13.8 we have

$$\|S(f,\sigma',\xi'_\bullet)_{I_j}-S(f,\sigma'',\xi''_\bullet)_{I_j}\|\leqslant\mathrm{diam}(f(I_j))|I_j|$$

Thus, (13.16) follows immediately from triangle inequality and

$$\sum_{j=1}^n S(f,\sigma',\xi'_\bullet)_{I_j}=S(f,\sigma',\xi'_\bullet)\qquad\sum_{j=1}^n S(f,\sigma'',\xi''_\bullet)_{I_j}=S(f,\sigma'',\xi''_\bullet)$$

□

Definition 13.12. We say that $f : I \to V$ is **strongly Riemann integrable** if

$$\inf_{\sigma \in \mathcal{P}(I)} \omega(f, \sigma) = 0$$

Theorem 13.13. *Let $f : I \to V$. Consider the following statements:*

(1) $f \in \mathscr{R}(I, V)$.

(2) f is strongly Riemann integrable.

Then (2)⇒(1). If V is $\mathbb{R}^N$ or $\mathbb{C}^N$, then (1)⇔(2).

When V is infinite-dimensional, Riemann integrable functions are not necessarily strongly Riemann integrable. See Pb. 13.3.

Proof. Assume (2). Choose any $\varepsilon > 0$. Then there exists $\sigma \in \mathcal{P}(I)$ such that $\omega(f, \sigma) < \varepsilon$. By Lem. 13.11, for every $(\sigma', \xi'_\bullet), (\sigma'', \xi''_\bullet) \in \mathcal{Q}(I)$ satisfying $\sigma \subset \sigma'$ and $\sigma \subset \sigma''$, we have $\|S(f, \sigma', \xi'_\bullet) - S(f, \sigma'', \xi''_\bullet)\| < \varepsilon$. This proves that $(S(f, \sigma, \xi_\bullet))_{(\sigma, \xi_\bullet) \in \mathcal{Q}(I)}$ is a Cauchy net in V, and hence $f \in \mathscr{R}(I, V)$.

Now assume that V is $\mathbb{R}^N$ or $\mathbb{C}^N$. Since $\mathbb{C}^N \simeq \mathbb{R}^{2N}$, it suffices to consider the case $V = \mathbb{R}^N$. Since a net in $\mathbb{R}^N$ converges iff each component of this net converge, and since the strong Riemann integrability can be checked componentwisely, it suffices to prove (1)⇒(2) for the case $V = \mathbb{R}$.

So let us assume $f \in \mathscr{R}(I, \mathbb{R})$. Then $(S(f, \sigma, \xi_\bullet))_{(\sigma, \xi_\bullet) \in \mathcal{Q}(I)}$ is a Cauchy net in $\mathbb{R}$. Thus, there exists $\sigma = \{a_0 < \cdots < a_n\} \in \mathcal{P}(I)$ such that for all $(\sigma', \xi'_\bullet), (\sigma'', \xi''_\bullet) \in \mathcal{Q}(I)$ satisfying $\sigma \subset \sigma'$ and $\sigma \subset \sigma''$, we have $\|S(f, \sigma', \xi'_\bullet) - S(f, \sigma'', \xi''_\bullet)\| < \varepsilon$. To prove (2), we only need to take $\sigma' = \sigma'' = \sigma$. Thus, for any tags $\xi'_\bullet, \xi''_\bullet$ of σ, we have

$$\|S(f, \sigma, \xi''_\bullet) - S(f, \sigma, \xi'_\bullet)\| < \varepsilon$$

Write $I_j = [a_{j-1}, a_j]$. Then

$$\mathrm{diam}(f(I_j)) = \sup f(I_j) - \inf f(I_j)$$

So there exist $\xi'_j, \xi''_j \in I_j$ such that

$$f(\xi'_j) \leqslant \inf f(I_j) + \varepsilon \qquad f(\xi''_j) \geqslant \sup f(I_j) - \varepsilon$$

It follows that $\mathrm{diam}(f(I_j)) - 2\varepsilon \leqslant f(\xi''_j) - f(\xi'_j)$. Thus

$$\begin{aligned} \omega(f, \sigma) = \sum_j \mathrm{diam}(f(I_j)) \cdot |I_j| \leqslant \sum_j (f(\xi''_j) - f(\xi'_j)) \cdot |I_j| + \sum_j 2\varepsilon \cdot |I_j| \\ = S(f, \sigma, \xi''_\bullet) - S(f, \sigma, \xi'_\bullet) + 2(b - a)\varepsilon < \varepsilon + 2(b - a)\varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\inf_{\sigma \in \mathcal{P}(I)} \omega(f, \sigma) = 0$. $\square$

Example 13.14. Every continuous function $f : I \to V$ is strongly Riemann integrable, and hence Riemann integrable.

Proof. Since $I = [a, b]$ is compact, by Thm. 10.7, $f \in C(I, V)$ is uniformly continuous. Therefore, for every $\varepsilon > 0$, there exists $n \in \mathbb{Z}_+$ such that for all $x, y \in I$ satisfying $|x - y| \leqslant 1/n$, we have $\|f(x) - f(y)\| < \varepsilon$. Let $\sigma = \{a_0 < a_1 < \cdots < a_n\}$ be the partition of I such that $|I_j| = a_j - a_{j-1} = 1/n$. Then $\mathrm{diam}(f(I_j)) \leqslant \varepsilon$, and hence

$$\omega(f, \sigma) \leqslant \sum_j \varepsilon \cdot |I_j| = \varepsilon(b - a)$$

Since ε is arbitrary, we get $\inf_{\sigma \in \mathcal{P}(I)} \omega(f, \sigma) = 0$. □

Example 13.15. The **Dirichlet function** $\chi_{\mathbb{Q}}$ is not (strongly) Riemann integrable on $I = [a, b]$, since for every $\sigma \in \mathcal{P}(I)$ we have $\omega(\chi_{\mathbb{Q}}, \sigma) = b - a$.

13.3 Basic properties of Riemann integrals

Let $I = [a, b]$ be a compact interval in $\mathbb{R}$. Let V be a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. We begin this section with the following fundamental fact, which will be used to prove Fubini's theorem, the second fundamental theorem calculus, and much more.

Theorem 13.16. *Let W be also a Banach space over $\mathbb{F}$. Let $T \in \mathfrak{L}(V, W)$. Then for every $f \in \mathscr{R}(I, V)$, we have $T \circ f \in \mathscr{R}(I, W)$ and*

$$T\Big(\int_a^b f\Big) = \int_a^b T \circ f \tag{13.18}$$

In other words, we have a commutative diagram

$$\begin{tikzcd} \mathscr{R}(I, V) \arrow[r, "T\circ"] \arrow[d, "\int_I"'] & \mathscr{R}(I, W) \arrow[d, "\int_I"] \\ V \arrow[r, "T"] & W \end{tikzcd} \tag{13.19}$$

where the top arrow denotes the composition map $f \mapsto T \circ f$.

Proof. By linearity, we have

$$S(T \circ f, \sigma, \xi_\bullet) = T\big(S(f, \sigma, \xi_\bullet)\big) \tag{13.20}$$

Since T is continuous, the limit over $(\sigma, \xi_\bullet) \in \mathcal{Q}(I)$ of (13.20) is

$$T\Big(\lim_{(\sigma,\xi_\bullet)\in\mathcal{Q}(I)} S(f, \sigma, \xi_\bullet)\Big) = T\Big(\int_a^b f\Big)$$

This proves that $\lim S(T \circ f, \sigma, \xi_\bullet)$ converges to $T(\int_a^b f)$, i.e., $\int_a^b T \circ f$ exists and equals $T(\int_a^b f)$. □

Remark 13.17. Let $f : I \to V$ and $\varepsilon > 0$. To simplify the following discussion, we say that f is ε-**dominated** by $\sigma \in \mathcal{P}(I)$, if for all $(\sigma', \xi'_\bullet), (\sigma'', \xi''_\bullet) \in \mathcal{Q}([a, b])$ satisfying $\sigma \subset \sigma', \sigma''$, we have $\|S(f, \sigma', \xi'_\bullet) - S(f, \sigma'', \xi''_\bullet)\| < \varepsilon$.

Thus, by Cauchy condition, if f is ε-dominated by some partition for every $\varepsilon > 0$, then $f \in \mathscr{R}(I, V)$. Moreover,

$$\Big\| \int_a^b f - S(f, \sigma, \xi_\bullet) \Big\| \leqslant \varepsilon \tag{13.21}$$

for every $(\sigma, \xi_\bullet) \in \mathcal{Q}(I)$ such that f is ε-dominated by σ. □

13.3.1 Integral operators as bounded linear maps

Proposition 13.18. *Let $f \in \mathscr{R}(I, V)$ and $g \in \mathscr{R}(I, \mathbb{R})$. Assume that $|f| \leqslant g$, i.e., $\|f(x)\| \leqslant g(x)$ for all $x \in I$. Then $\Big\| \int_a^b f \Big\| \leqslant \int_a^b g$.*

Proof. Apply $\lim_{(\sigma,\xi_\bullet)\in\mathcal{Q}(I)}$ to the obvious inequality $\|S(f, \sigma, \xi_\bullet)\| \leqslant S(g, \sigma, \xi_\bullet)$. □

Corollary 13.19. *Assume that $f : I \to V$ is strongly Riemann integrable. Then $|f| : I \to \mathbb{R}$ is (strongly) Riemann integrable, and $\Big\| \int_a^b f \Big\| \leqslant \int_a^b |f|$*

Proof. By triangle inequality, for every $\sigma \in \mathcal{P}(I)$ we have $\omega(|f|, \sigma) \leqslant \omega(f, \sigma)$. So $|f|$ is strongly integrable. The rest of the corollary follows from Prop. 13.18. □

Theorem 13.20. *$\mathscr{R}(I, V)$ is a closed linear subspace of $l^\infty(I, V)$. So $\mathscr{R}(I, V)$ is a Banach space under the l^∞-norm. Moreover, the map*

$$\int : \mathscr{R}(I, V) \to V \qquad f \mapsto \int_a^b f \tag{13.22}$$

is a bounded linear map with operator norm $b - a$ if we equip $\mathscr{R}(I, V)$ with the l^∞-norm.

Proof. By the basic properties of limits of nets, we know that if $f, g \in \mathscr{R}(I, V)$ and $\alpha, \beta \in \mathbb{F}$, then $\alpha f + \beta g \in \mathscr{R}(I, V)$, and

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g \tag{13.23}$$

This proves that $\mathscr{R}(I, V)$ is a linear subspace of V^I, and that (13.22) is linear.

Let us prove $\mathscr{R}(I, V) \subset l^\infty(I, V)$. Choose $f \in \mathscr{R}(I, V)$. Then f is 1-dominated by some $\sigma = \{a_0 < \cdots < a_n\} \in \mathcal{P}(I)$. Fix any tag $\xi_\bullet$ on σ. Choose any $x \in X$. Let $[a_{i-1}, a_i]$ be the subinterval containing x. Let $\eta_\bullet = (\xi_1, \ldots, \xi_{i-1}, x, \xi_{i+1}, \ldots, \xi_n)$. Then $\|S(f, \sigma, \eta_\bullet) - S(f, \sigma, \xi_\bullet)\| < 1$ implies that $\|f(x) - f(\xi_i)\| \leqslant 1/(a_i - a_{i-1})$. So

$$\|f\|_{l^\infty} \leqslant \max\big\{\|f(\xi_i)\| + (a_i - a_{i-1})^{-1} : 1 \leqslant i \leqslant n\big\} < +\infty$$

Let (f_n) be a sequence in $\mathscr{R}(I,V)$ converging to $f \in l^\infty(I,V)$. Choose any $\varepsilon > 0$. Then there is n such that $\|f - f_n\|_\infty < \varepsilon$. Since f_n is Riemann integrable, f_n is ε-dominated by some $\sigma \in \mathcal{P}(I)$. By triangle inequality, f is $(\varepsilon + (b-a)\varepsilon)$-dominated by σ. Since ε is arbitrary, we conclude that $f \in \mathscr{R}(I,V)$. This proves that $\mathscr{R}(I,V)$ is closed, and hence is Banach by Prop. 3.27.

Let us prove the claim about the operator norm. Choose any $f \in \mathscr{R}(I,V)$. Let $M = \|f\|_\infty < +\infty$. Then $|f| \leqslant M$. It is easy to see that $\int_a^b M = M(b-a)$. Thus, by Prop. 13.18, we have $\|\int_a^b f\| \leqslant M(b-a) = \|f\|_\infty \cdot (b-a)$, where "$\leqslant$" becomes "$=$" if we let f be a nonzero constant function. This proves that (13.22) has operator norm $b-a$ thanks to Rem. 10.24. $\square$

Corollary 13.21. *Let $(f_\alpha)_{\alpha\in\mathscr{I}}$ be a net in $\mathscr{R}(I,V)$ converging uniformly to $f \in V^I$. Then $f \in \mathscr{R}(I,V)$ and $\lim_{\alpha\in\mathscr{I}} \int_a^b f_\alpha = \int_a^b f$.*

Proof. This is immediate from Thm. 13.20, which implies that $\mathscr{R}(I,V)$ is closed in $l^\infty(I,V)$ (and hence closed in V^I since $l^\infty(I,V)$ is closed in V^I), and that the map (13.22) is continuous. $\square$

13.3.2 Some criteria for Riemann integrability

Proposition 13.22. *Let $f,g : I \to V$. Suppose that $\{x \in I : f(x) \neq g(x)\}$ is a finite set. Suppose that $f \in \mathscr{R}(I,V)$. Then $g \in \mathscr{R}(I,V)$, and $\int_a^b f = \int_a^b g$.*

Proof. By Thm. 13.20, it suffices to prove that $g - f$ is Riemann integrable and $\int_a^b (g-f) = 0$. This is easy to show, since $g - f$ is zero outside finitely many points. $\square$

Proposition 13.23. *Let $f : [a,b] \to V$. Let $c \in [a,b]$. Then $f \in \mathscr{R}([a,b],V)$ iff $f|_{[a,c]} \in \mathscr{R}([a,c],V)$ and $f|_{[c,b]} \in \mathscr{R}([c,b],V)$. Moreover, if $f \in \mathscr{R}([a,b],V)$, then*

$$\int_a^b f = \int_a^c f + \int_c^b f \tag{13.24}$$

Proof. This is obvious when $c = a$ or $c = b$ (recall Conv. 13.5). So we assume $a < c < b$.

First, we assume $f \in \mathscr{R}([a,b],V)$. By Cauchy condition, for each $\varepsilon > 0$, f is ε-dominated by some $\sigma \in \mathcal{P}(I)$. By enlarging σ, we assume that $c \in \sigma$. Then it is easy to see that $f|_{[a,c]}$ is ε-dominated by $\sigma \cap [a,c]$, and $f|_{[c,b]}$ is ε-dominated by $\sigma \cap [c,b]$. So $f|_{[a,c]}$ and $f|_{[c,b]}$ are Riemann integrable.

Now assume that $f|_{[a,c]}$ and $f|_{[c,b]}$ are Riemann integrable. Choose any $\varepsilon > 0$. Then $f|_{[a,c]}$ is ε-dominated by some $\tau \in \mathcal{P}([a,c])$, and $f|_{[c,b]}$ is ε-dominated by some $\varrho \in \mathcal{P}([c,b])$. Then f is 2ε-dominated by $\sigma = \tau \cup \varrho$. This proves that $f \in$

$\mathscr{R}(I,V)$. Let $\alpha_\bullet$ be a tag on $[a,c]$, and let $\beta_\bullet$ be a tag on $[c,b]$. Then $\xi_\bullet = (\alpha_\bullet,\beta_\bullet) = (\alpha_1,\alpha_2,\dots,\beta_1,\beta_2,\dots)$ is a tag on $[a,b]$, and

$$S(f,\sigma,\xi_\bullet) = S(f|_{[a,c]},\tau,\alpha_\bullet) + S(f|_{[c,b]},\varrho,\beta_\bullet)$$

By Rem. 13.17, we have

$$\Big\| \int_a^b f - S(f,\sigma,\xi_\bullet) \Big\| \leqslant 2\varepsilon$$

$$\Big\| \int_a^c f - S(f|_{[a,c]},\tau,\alpha_\bullet) \Big\| \leqslant \varepsilon$$

$$\Big\| \int_c^b f - S(f|_{[c,b]},\varrho,\beta_\bullet) \Big\| \leqslant \varepsilon$$

Therefore, the difference of the LHS and the RHS of (13.24) has norm $\leqslant 4\varepsilon$. This proves (13.24) since ε can be arbitrary. □

Example 13.24. Let $f \in l^\infty(I,V)$. Suppose that there exist $\sigma = \{a_0 < a_1 < \cdots < a_n\} \in \mathcal{P}(I)$ such that $f|_{(a_{j-1},a_j)} : (a_{j-1},a_j) \to V$ is continuous for each $1 \leqslant j \leqslant n$. Then $f \in \mathscr{R}(I,V)$.

Proof. By Prop. 13.23, it suffices to prove that each $f|_{[a_{j-1},a_j]}$ is Riemann integrable. Thus, we assume WLOG that $M := \|f\|_\infty < +\infty$, and that f is continuous when restricted to (a,b). Choose any $\varepsilon > 0$. Choose $\delta > 0$ such that $M\delta < \varepsilon$ and $a+\delta < b-\delta$. Let $J = [a+\delta, b-\delta]$. Then $f|_J$ is continuous, and hence Riemann integrable by Exp. 13.14. So $f|_J$ is ε-dominated by some $\varrho \in \mathcal{P}(J)$. Since $\mathrm{diam}(f(I)) \leqslant 2M$, by Lem. 13.8, $f|_{[a,a+\delta]}$ is 2ε-dominated by $\{a,a+\delta\}$, and $f|_{[b-\delta,b]}$ is 2ε-dominated by $\{b-\delta,b\}$. So f is 5ε-dominated by $\sigma = \varrho \cup \{a,b\}$. Since $\varepsilon > 0$ is arbitrary, we conclude $f \in \mathscr{R}(I,V)$ by Cauchy condition. □

Example 13.25. The function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \sin(1/x)$ if $0 < x \leqslant 1$ and $f(0) = 0$ is Riemann integrable, although f is not uniformly continuous on $(0,1]$.

Example 13.26. Let $I_1,\dots,I_n$ be intervals inside I. Choose $v_1,\dots,v_n \in V$ and set $f = \sum_{j=1}^n v_j \chi_{I_j}$. Then $f \in \mathscr{R}(I,V)$.

Proof. The case of arbitrary n follows from the case $n = 1$ by linearity and Thm. 13.20. Assume $n = 1$, and let $c = \inf I_1$ and $d = \sup I_1$. Then the restriction of f to $[a,c]$ (resp. $[c,d]$ and $[d,b]$) equals a constant function except possibly at the end points of the interval. So $f|_{[a,c]}, f|_{[c,d]}, f|_{[d,b]}$ are Riemann integrable by Prop. 13.22. So f is Riemann integrable by Prop. 13.23. □

Example 13.27. Let $f \in C(I,V)$. For each $n \in \mathbb{Z}_+$, choose a tag $\xi_{\bullet,n}$ for the partition $\sigma = \{a, a+|I|/n, a+2|I|/n, \dots, a+(n-1)|I|/n, b\}$ of I. Then

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^n f(\xi_{i,n}) = \int_a^b f \tag{13.25}$$

Proof. Let $f_n = \sum_{i=1}^n f(\xi_{i,n}) \cdot \chi_{J_{i,n}}$ where $J_{i,n} = \big(a + \frac{i-1}{n}|I|, a + \frac{i}{n}|I|\big]$ if $i > 1$, and $J_{1,n} = [a, a + |I|/n]$. Then $f_n \in \mathscr{R}(I, V)$ by Exp. 13.26. Moreover, by Prop. 13.23, the LHS of (13.25) equals $\lim_n \int_a^b f_n$.

By Thm. 10.7, f is uniformly continuous. So for every $\varepsilon > 0$ there exists $N \in \mathbb{Z}_+$ such that for all $n \geqslant N$ and all $x, y \in \mathrm{Cl}_{\mathbb{R}}(J_{i,n})$ we have $\|f(x) - f(y)\| < \varepsilon$. Thus, for all $n \geqslant N$ we have $\|f - f_n\|_{l^\infty} < \varepsilon$. Therefore $f_n \rightrightarrows f$. Thus, by Cor. 13.21 we have $\int_a^b f_n \to \int_a^b f$. This proves (13.25). $\square$

Example 13.28. By Thm. 13.30, we have $\int_1^2 x^{-1} dx = \log 2$. Thus, by Exp. 13.27, we have $\lim_{n\to\infty} \sum_{i=1}^n \frac{2-1}{n} \cdot (1 + i/n)^{-1} = \log 2$, namely

$$\lim_{n\to\infty} \big((n+1)^{-1} + (n+2)^{-1} + \cdots + (2n)^{-1}\big) = \log 2$$

13.4 Integrals and derivatives

Let $I = [a, b]$ be an interval in $\mathbb{R}$. Let V be a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

13.4.1 Fundamental theorems of calculus (FTC)

There are two versions of FTC. Roughly speaking, the first FTC says that integrals give antiderivatives. The second FTC says that antiderivatives give integrals. These two FTC are equivalent when the function f to be integrated is continuous. Otherwise, there is a subtle difference (which I can never remember) between these two theorems.

Theorem 13.29 (First FTC). *Let $f \in \mathscr{R}(I, V)$. Define*

$$F : I \to V \qquad F(x) = \int_a^x f \tag{13.26}$$

Then $F \in C(I, V)$. If f is continuous at x, then $F'(x) = f(x)$.

In particular, if $f \in C(I, V)$, then $F' = f$. Thus, by Cor. 11.24, the antiderivatives of f are precisely of the form $F(x) + v_0$ where $v_0 \in V$ is viewed as a constant function.

Recall Conv. 13.5.

Proof. Recall by Thm. 13.20 that $\|f\|_\infty < +\infty$. For each $x, y \in [a, b]$ we have

$$\|F(y) - F(x)\| = \Big\| \int_a^y f - \int_a^x f \Big\| = \Big\| \int_x^y f \Big\| \leqslant \|f\|_\infty \cdot |y - x|$$

So F has Lipschitz constant $\|f\|_\infty$. Now suppose that f is continuous at x. Then for every $\varepsilon > 0$, there exists $U \in \mathrm{Nbh}_I(x)$ such that $\|f(x) - f(y)\| < \varepsilon$ for every $y \in U$. Thus, for each $y \in U \backslash \{x\}$, since $\int_x^y f(x)dt = f(x)(y - x)$, we have

$$\Big\| \frac{F(y) - F(x)}{y - x} - f(x) \Big\| = |y - x|^{-1} \Big\| \int_x^y (f(t) - f(x)) dt \Big\|$$

$$\leqslant |y-x|^{-1}\int_{[x,y]}\|f(t)-f(x)\|dt \leqslant |y-x|^{-1}\int_{[x,y]}\varepsilon dt = \varepsilon$$

This proves $F'(x) = f(x)$. □

Theorem 13.30 (Second FTC)**.** *Let $f \in \mathscr{R}(I,V)$. Assume that $F : I \to V$ is differentiable and $F' = f$. Then*

$$\int_a^b f = F\Big|_a^b \xlongequal{\text{def}} F(b) - F(a) \tag{13.27}$$

This theorem is easy when $f \in C(I,V)$: In this case, by Thm. 13.29, we have $F(x) = v_0 + \int_a^x f$ for some $v_0 \in V$. Then (13.27) follows immediately from Prop. 13.23. The proof for the general case is more difficult:

Proof assuming Hahn-Banach. Since $\mathbb{R}$ is a subfield of $\mathbb{C}$, we may view V as a real Banach space. We first consider the special case that $V = \mathbb{R}$. Let $A = \int_a^b f$. Choose any $\varepsilon > 0$. Since $f \in \mathscr{R}(I,\mathbb{R})$, there exists $\sigma = \{a_0 < \cdots < a_n\} \in \mathcal{P}(I)$ such that for every tag $\xi_\bullet$ on σ, we have $|A - S(f,\sigma,\xi_\bullet)| < \varepsilon$. By Lagrange's MVT (Thm. 11.19), there exists $\xi_i \in (a_{i-1}, a_i)$ such that

$$F(a_i) - F(a_{i-1}) = f(\xi_i)(a_i - a_{i-1})$$

Thus, we have a tag $\xi_\bullet$ such that $S(f,\sigma,\xi_\bullet) = F(b)-F(a)$. Hence $|A-F(b)+F(a)| < \varepsilon$. Since ε is arbitrary, we get $A = F(b) - F(a)$.

The case $V = \mathbb{R}^N$ can be reduced to the above special case easily. We now consider the general case that V is a Banach space over $\mathbb{R}$. Similar to (12.15), for every $\varphi \in V^* = \mathfrak{L}(V,\mathbb{R})$, we have that $\varphi\circ F$ is differentiable, and that $(\varphi\circ F)' = \varphi\circ f$. Note that $\varphi \circ f \in \mathscr{R}(I,\mathbb{R})$ by Thm. 13.16. Apply the one-dimensional special case to $\varphi \circ f$. Then by Thm. 13.16, we have

$$\varphi\Big(\int_a^b f\Big) = \int_a^b \varphi\circ f = \varphi\circ F\Big|_a^b = \varphi(F(b) - F(a))$$

By Hahn-Banach theorem, V^* separates points of V. (See Rem. 11.26.) Therefore $\int_a^b f = F(b) - F(a)$. □

13.4.2 Applications of FTC: integration by parts

Proposition 13.31 (Integration by parts). *Let $f \in C^1(I,V)$ and $g \in C^1(I,\mathbb{F})$. Then*

$$\int_a^b f'g = (fg)\Big|_a^b - \int_a^b fg' \tag{13.28}$$

Proof. $(fg)\big|_a^b = \int_a^b (fg)' = \int_a^b f'g + \int_a^b fg'$. □

Theorem 13.32 (Taylor's theorem, integral form). *Let $n \in \mathbb{N}$ and $f \in C^{n+1}([a,b],V)$. Then for every $x \in [a,b]$ we have*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt \tag{13.29}$$

Proof. When $n = 0$, (13.29) is just FTC. We now prove (13.29) by induction. Suppose that case $n-1$ has been proved, then by integration by parts,

$$\begin{aligned}
&f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k = \int_a^x \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}dt \\
=&\int_a^x -\frac{f^{(n)}(t)}{n!}\partial_t(x-t)^n dt \\
=&-\frac{f^{(n)}(t)}{n!}(x-t)^n\Big|_{t=a}^{x} + \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt \\
=&\frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt
\end{aligned}$$

□

Exercise 13.33. Use the integral form of Taylor's theorem to give a quick proof higher order finite-increment Thm. 12.17 under the assumption that $f \in C^{n+1}([a,b],V)$. (This assumption is slightly stronger than that in Thm. 12.17, but is enough for applications.)

In the case that $V = \mathbb{R}$, the integral form of Taylor's theorem actually implies a slightly weaker (but useful enough) version of Lagrange form. This relies on the following easy fact:

Proposition 13.34 (Mean value theorem). *Let $f, g \in C([a,b],\mathbb{R})$ such that $g(x) \geqslant 0$ for all $x \in (a,b)$. Then there exists $\xi \in [a,b]$ such that*

$$\int_a^b fg = f(\xi)\int_a^b g \tag{13.30}$$

Moreover, ξ can be chosen to be in (a,b) if $g(x) > 0$ for all $x \in (a,b)$.

Proof. Let $m = \inf f(I)$ and $M = \sup f(I)$. Then $mg \leqslant fg \leqslant Mg$, and hence $m\int_I g \leqslant \int_I fg \leqslant \int_I Mg$. So there is $y \in [m,M]$ such that $\int_I fg = y\int_I g$. By extreme value theorem, we have $m, M \in f(I)$. Thus, by intermediate value property, we have $y \in f(I)$. So $y = f(\xi)$ for some $\xi \in I$.

Now assume $g(x) > 0$ for all $a < x < b$. Let $\varphi(x) = \int_a^x fg$ and $\psi(x) = \int_a^x g$. By Cauchy's MVT, there exists $\xi \in (a,b)$ such that $g(\xi)(\varphi(b)-\varphi(a)) = f(\xi)g(\xi)(\psi(b)-\psi(a))$. This finishes the proof. □

Remark 13.35. Let $f \in C^{n+1}([a,b],\mathbb{R})$. By the above mean value theorem, for each $x \in (a,b]$ there exists $\xi \in (a,x)$ such that

$$\int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt = f^{(n+1)}(\xi)\int_a^x \frac{(x-t)^n}{n!}dt = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

Thus, the integral form of Taylor's theorem implies Lagrange's form (Thm. 12.18) in the case that $f \in C^{n+1}(I,\mathbb{R})$.

13.4.3 Application of FTC: change of variables

Proposition 13.36 (Change of variables). *Let $\Phi \in C^1(I,\mathbb{R})$ such that $\Phi(I) \subset J = [c,d]$. Let $f \in C(J,V)$. Then*

$$\int_{\Phi(a)}^{\Phi(b)} f = \int_a^b (f\circ\Phi)\cdot\Phi' \tag{13.31}$$

Proof. Define $F : J \to V$ and $G : I \to V$ by $F(y) = \displaystyle\int_{\Phi(a)}^{y} f$ and $G(x) = F\circ\Phi(x) = \displaystyle\int_{\Phi(a)}^{\Phi(x)} f$. Then, by chain rule and FTC, $G' = (F'\circ\Phi)\cdot\Phi' = (f\circ\Phi)\cdot\Phi'$. By FTC, we have $G' = H'$ where $H : I \to V$ is defined by

$$H(x) = \int_a^x (f\circ\Phi)\cdot\Phi'$$

Thus, since $G(a) = F(a) = 0$, by Cor. 11.24 we have $G = H$. Then $G(b) = H(b)$ finishes the proof. □

The change of variable formula allows us to define the length of a curve:

Definition 13.37. Let γ be a C^1**-curve** in V, i.e. $\gamma \in C^1([a,b],V)$. Its **length** is defined to be

$$l(\gamma) = \int_a^b \|\gamma'(t)\|dt$$

Proposition 13.38. *The length of γ is invariant under a reparametrization. Namely, if $f : [c,d] \to [a,b]$ is a bijection and is in $C^1([c,d],\mathbb{R})$, then $l(\gamma) = l(\gamma\circ f)$.*

Proof. By Prop. 11.22, f is either increasing or decreasing. We prove the case that f is decreasing; the other case is similar. Then $f(c) = b$ and $f(d) = a$. By chain rule, we have

$$l(\gamma\circ f) = \int_c^d |(\gamma\circ f)'| = \int_c^d |(\gamma'\circ f)\cdot f'| = \int_c^d (|\gamma'|\circ f)\cdot|f'|$$

which equals $-\int_c^d(|\gamma'|\circ f)\cdot f' = \int_d^c(|\gamma'|\circ f)\cdot f'$ because $f' \leqslant 0$ by Cor. 11.20. This expression equals $\int_a^b|\gamma'| = l(\gamma)$ by Prop. 13.36. □

Example 13.39. The upper half circle $\mathbb{S}^1_+ = \{z \in \mathbb{C} : |z| = 1, \mathrm{Im} z \geqslant 0\}$ has a bijective C^∞-parametrization $\gamma : [0, \pi] \to \mathbb{C}, \gamma(t) = e^{\mathbf{i}t}$. Its length is π.

Proof. We have $l(\gamma) = \int_0^\pi |\gamma'| = \int_0^\pi 1 = \pi$. So it remains to prove that γ restricts to a bijection $[0, \pi] \to \mathbb{S}^1_+$. Clearly $\gamma([0, \pi]) \subset \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. Note that $\gamma(t) = \cos(t) + \mathbf{i}\sin(t)$. In Sec. 12.2, we have proved that $\sin(x) \geqslant 0$ when $x \in [0, \pi]$. So $\gamma([0, \pi]) \subset \mathbb{S}^1_+$.

We have also proved in Sec. 12.2 that $\cos : [0, \pi/2] \to \mathbb{R}$ is a decreasing (continuous) function such that $\cos(0) = 1$ and $\cos(\pi/2) = 0$, and that $\cos(x) = \cos(\pi - x)$. The last relation shows that $\cos : [0, \pi] \to \mathbb{R}$ is decreasing, and $\cos(0) = 1, \cos(\pi) = -1$. Therefore, by the intermediate value theorem, we see that $\cos$ sends $[0, \pi]$ bijectively to $[-1, 1]$. Since the projection map onto the x-axis sends $\mathbb{S}^1_+$ bijectively to $[-1, 1]$, we conclude that γ sends $[0, \pi]$ bijectively to $\mathbb{S}^1_+$. □

Remark 13.40. You may wonder if the above argument really proves that π is the length of $\mathbb{S}^1_+$: Suppose that $\lambda : [a, b] \to \mathbb{C}$ is another C^∞ map restricting to a bijection $[a, b] \to \mathbb{S}^1_+$, how can we show that $\int_0^\pi |\gamma'| = \int_a^b |\lambda'|$? Clearly, there is a bijection $f : [a, b] \to [0, \pi]$ such that $\lambda = \gamma \circ f$. Thus, by Prop. 13.38, the two integrals are equal if $f \in C^\infty$, or at least if $f \in C^1$. However, it seems that there is no general argument ensuring that $f \in C^1$.

In the next semester, we will learn that $f \in C^\infty$ if the two C^∞-parametrizations λ, γ satisfy that λ' and γ' are nowhere zero. (Clearly γ' is nowhere zero if $\gamma(t) = e^{\mathbf{i}t}$.) Such parametrizations are called **(smooth) immersions**. □

13.5 Problems and supplementary material

Let $I = [a, b]$ where $-\infty < a < b < +\infty$.

Problem 13.1. Let $f \in l^\infty(I, \mathbb{R})$. Recall from Subsec. 13.2.1 that the refinements of partitions define preoders on $\mathcal{P}(I)$ and $\mathcal{Q}(I)$ so that they are directed sets. For each $\sigma = \{a_0 < a_1 < \cdots < a_n\} \in \mathcal{P}(I)$, define the **upper Darboux sum** and the **lower Darboux sum**

$$\overline{S}(f, \sigma) = \sum_{i=1} M_i(a_i - a_{i-1}) \qquad \underline{S}(f, \sigma) = \sum_{i=1} m_i(a_i - a_{i-1})$$

$$\text{where} \qquad M_i = \sup_{\xi \in [a_{i-1}, a_i]} f(\xi) \qquad m_i = \inf_{\xi \in [a_{i-1}, a_i]} f(\xi)$$

Prove that

$$-(b - a)\|f\|_\infty \leqslant \underline{S}(f, \sigma) < \overline{S}(f, \sigma) \leqslant (b - a)\|f\|_\infty$$

Choose any tag $\xi_\bullet$ on σ. Prove that

$$\overline{S}(f, \sigma) = \sup_{(\sigma', \xi'_\bullet) > (\sigma, \xi_\bullet)} S(f, \sigma', \xi'_\bullet) \qquad \underline{S}(f, \sigma) = \inf_{(\sigma', \xi'_\bullet) > (\sigma, \xi_\bullet)} S(f, \sigma', \xi'_\bullet) \tag{13.32}$$

Pb. 13.1 immediately implies:

Theorem 13.41. *Let $f \in l^\infty(I,\mathbb{R})$. Define the* ***upper Darboux integral*** *and the* ***lower Darboux integral*** *to be*

$$\overline{\int_a^b} f = \inf_{\sigma\in\mathcal{P}(I)} \overline{S}(f,\sigma) \qquad \underline{\int_a^b} f = \sup_{\sigma\in\mathcal{P}(I)} \underline{S}(f,\sigma)$$

which are elements of $[-(b-a)\|f\|_\infty,(b-a)\|f\|_\infty]$. Then we have (recalling Pb. 8.2)

$$\overline{\int_a^b} f = \limsup_{(\sigma,\xi_\bullet)\in\mathcal{Q}(I)} S(f,\sigma,\xi_\bullet)$$

$$\underline{\int_a^b} f = \liminf_{(\sigma,\xi_\bullet)\in\mathcal{Q}(I)} S(f,\sigma,\xi_\bullet)$$

Therefore, by Cor. 8.37, we have

$$f\in\mathscr{R}(I,\mathbb{R}) \qquad \Longleftrightarrow \qquad \overline{\int_a^b} f = \underline{\int_a^b} f \tag{13.33}$$

Moreover, if $f\in\mathscr{R}(I,\mathbb{R})$, then $\int_a^b f = \overline{\int_a^b} f = \underline{\int_a^b} f$.

Problem 13.2. Let $\mathcal{V}$ be a vector space over $\mathbb{C}$. Since $\mathbb{R}$ is a subfield of $\mathbb{C}$, $\mathcal{V}$ can be viewed as a real normed vector space. Let $\Lambda:\mathcal{V}\to\mathbb{R}$ be a $\mathbb{R}$-linear map. Recall $\mathbf{i}=\sqrt{-1}$. Define the **complexification** of Λ to be

$$\Lambda_{\mathbb{C}}:\mathcal{V}\to\mathbb{C} \qquad \Lambda_{\mathbb{C}}(v)=\Lambda(v)-\mathbf{i}\Lambda(\mathbf{i}v) \tag{13.34}$$

1. Prove that $\Lambda_{\mathbb{C}}$ is $\mathbb{C}$-linear.

2. Given a $\mathbb{C}$-linear $\Phi:\mathcal{V}\to\mathbb{C}$, we define its **real part**

   $$\mathrm{Re}\Phi:\mathcal{V}\to\mathbb{R} \qquad v\mapsto \mathrm{Re}\,\big(\Phi(v)\big) \tag{13.35}$$

   Then clearly $\mathrm{Re}(\Phi)$ is $\mathbb{R}$-linear. Prove that $\Phi\mapsto\mathrm{Re}\Phi$ is a bijection from the set of $\mathbb{C}$-linear maps $\mathcal{V}\to\mathbb{C}$ to the set of $\mathbb{R}$-linear maps $\mathcal{V}\to\mathbb{R}$, and that its inverse is the map $\Lambda\mapsto\Lambda_{\mathbb{C}}$ defined by (13.34).

3. Assume that $\mathcal{V}$ is a (non-necessarily complete) normed $\mathbb{C}$-vector space. For each $\mathbb{C}$-linear $\Phi:\mathcal{V}\to\mathbb{C}$, prove the following equation about operator norms:

   $$\|\Phi\|=\|\mathrm{Re}\Phi\| \tag{13.36}$$

   (Hint: One of "$\leqslant$" and "$\geqslant$" is obvious. To prove the other one, for each $v\in\mathcal{V}$, find some $\theta\in\mathbb{R}$ such that $e^{\mathbf{i}\theta}\Phi(v)\in\mathbb{R}$.)

Remark 13.42. The above problem shows how to extend a real-valued integral to a complex-valued one. For example, suppose that we define real-valued Riemann integrals using Darboux integrals. Suppose we have proved that the map $f \in \mathscr{R}(I, \mathbb{R}) \mapsto \int_a^b f$ is $\mathbb{R}$-linear with operator norm $(b-a)$. Then, applying Pb. 13.2 to the $\mathbb{R}$-linear map

$$\Lambda : \mathscr{R}(I, \mathbb{C}) \to \mathbb{R} \qquad f \mapsto \int_a^b \mathrm{Re}f(t)dt$$

gives a $\mathbb{C}$-linear map

$$\int_a^b : \mathscr{R}(I, \mathbb{C}) \to \mathbb{C}$$

$$\int_a^b f = \int_a^b \mathrm{Re}f + \mathbf{i}\int_a^b \mathrm{Im}f$$

since $\mathrm{Re}(\mathbf{i}f) = -\mathrm{Im}f$. Moreover, this linear map has operator norm $(b-a)$. This defines the complex integral operator by means of real Darboux integrals. In the next semester, we will use the same method to extend real-valued Lebesgue integrals to complex-valued ones. Pb. 13.2 will also be used to prove the Hahn-Banach extension theorem.

Exercise 13.43. Let $u, v \in C(I, \mathbb{R})$. Find the $\mathbb{C}$-linear map $C(I, \mathbb{C}) \to \mathbb{C}$ whose real part is $f \in C(I, \mathbb{C}) \mapsto \int_I (u\mathrm{Re}f + v\mathrm{Im}f) \in \mathbb{R}$.

⋆ **Problem 13.3.** Let $V = l^\infty([0,1], \mathbb{R})$, equipped with the l^∞-norm. Define

$$f : [0,1] \to V \qquad f(x) = \chi_{[x,1]}$$

Then, for every $x \neq y$ in $[0,1]$ we have $\|f(x) - f(y)\|_{l^\infty([0,1],\mathbb{R})} = 1$. This implies $\omega(f, \sigma) = 1$ for every $\sigma \in \mathcal{P}(I)$. So f is not strongly Riemann integrable on any closed subintegral of $[a,b]$.

Define $F : [0,1] \to V$ such that for each $x \in [0,1]$,

$$F(x) : [0,1] \to \mathbb{R} \qquad t \mapsto \min\{t, x\}$$

Choose any $x \in [0,1]$. Prove that $f \in \mathscr{R}([0,x], V)$ and $\int_0^x f = F(x)$. (In particular, $\int_0^1 f = \mathrm{id}_{[0,1]}$.) Prove that $F'(x)$ does not exist. □

Problem 13.4. Let V be a Banach space. Use the fundamental theorem of calculus to give another proof that $C^1(I, V)$ is complete under the $l^{1,\infty}$-norm. (Do not use Thm. 11.33.)

⋆ **Theorem 13.44 (Gronwall's inequality).** *Let $f \in C([a,b], \mathbb{R}_{\geqslant 0})$. Let $\alpha \in \mathbb{R}_{\geqslant 0}$, and $\beta \in C([a,b], \mathbb{R}_{\geqslant 0})$. Assume that for each $x \in [a,b]$ we have*

$$f(x) \leqslant \alpha + \int_a^x \beta(t)f(t)dt \tag{13.37}$$

Then for each $x \in [a,b]$ we have

$$f(x) \leqslant \alpha \cdot \exp\Big(\int_a^x \beta(t)dt\Big) \tag{13.38}$$

In particular, if β is a constant, then Gronwall's inequality reads

$$f(x) \leqslant \alpha \cdot e^{\beta(t-a)}$$

⋆ **Problem 13.5.** Prove Gronwall's inequality. Hint: Let $g(x)$ be the RHS of (13.37). Show that $\exp(-\int_a^x \beta) \cdot g(x)$ is a decreasing function.

⋆ **Remark 13.45.** Gronwall's inequality is often used in the following way. Let V be a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $f \in C(I,V)$, $v \in V$, and $\beta \in C(I, \mathbb{R}_{\geqslant 0})$. Suppose that for all $x \in [a,b]$ we have

$$\|f(x) - v\| \leqslant \int_a^x \beta(t)\|f(t)\|dt \tag{13.39}$$

Applying Gronwall's inequality to $|f|$ and $\alpha = \|v\|$, we see that for every $x \in [a,b]$,

$$\|f(x)\| \leqslant \|v\| \cdot \exp\Big(\int_a^x \beta(t)dt\Big) \tag{13.40}$$

⋆ **Problem 13.6.** Let V be a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Assume that $\varphi \in C(I \times V, V)$ has Lipschitz constant $L \in \mathbb{R}_{\geqslant 0}$ over the second variable, i.e., for each $t \in [a,b]$ and $u, v \in V$ we have

$$\|\varphi(t,u) - \varphi(t,v)\| \leqslant L\|u - v\| \tag{13.41}$$

Use Gronwall's inequality to solve the following problems.

1. Let $f_1, f_2 : I \to V$ be differentiable and satisfying the differential equation

   $$f_i'(t) = \varphi(t, f_i(t)) \qquad (\forall t \in I)$$

   with the same initial condition $f_1(a) = f_2(a)$. Prove that $f_1 = f_2$ on I.

2. Let X be a topological space. Let $f : I \times X \to V$ be a function such that $\partial_1 f$ exists everywhere, and that

   $$\partial_1 f(t,x) = \varphi(t, f(t,x)) \qquad (\forall t \in I, x \in X)$$

   Assume that the function $f(a,\cdot) : X \to V$ (sending x to $f(a,x)$) is continuous. Prove that $f \in C(I \times X, V)$.

Remark 13.46. In practice, V is often $\mathbb{R}^N$, and φ is a "smooth function", i.e. a function whose (mixed) partial derivatives of all orders exist and are continuous. It is also common that φ is independent of $t \in I$. (Indeed, a t-dependent differential equation $f' = \varphi(t, f)$ can be transformed into a t-independent one $(t, f)' = (1, \varphi(t, f))$.) However, sometimes f is not defined on $\mathbb{R}^N$, but on a closed subset of $\mathbb{R}^N$, for example, on a closed ball. In this case, the uniqueness and the continuity of the solutions of differential equations can be treated by extending φ to a smooth function on $\mathbb{R}^N$ that is zero outside a compact set. (Then the Lipschitz continuity of this extended function will follow automatically.)

In fact, by the **smooth Tietze extension theorem** (cf. [Lee, Lem. 2.26]), every smooth function $A \to \mathbb{R}^k$ (where A is a compact subset of a smooth real manifold M (such as a Euclidean space, an n-dimensional sphere, etc.)) can be extended to a smooth function $M \to \mathbb{R}^k$ vanishing outside a compact set. We will study this theorem in the second semester. (The case $A \subset \mathbb{R}$ will be proved in Exp. 15.30.) $\square$

14 More on Riemann integrals

14.1 Commutativity of integrals and other limit processes

Let $I = [a, b]$ and $J = [c, d]$ where $-\infty < a < b < +\infty$ and $-\infty < c < d < +\infty$. Let V be a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

14.1.1 Fubini's theorem

Theorem 14.1 (Fubini's theorem for Riemann integrals). *Let $f \in C(I \times J, V)$. Then $\int_I \int_J f = \int_J \int_I f$. More precisely,*

$$\int_I \Big(\int_J f(x, y) dy \Big) dx = \int_J \Big(\int_I f(x, y) dx \Big) dy \tag{14.1}$$

Our strategy is to view $\int_I f(x, y)dx$ as the integral of the function $I \to C(J, V)$ sending x to $f(x, \cdot)$. Then Fubini's theorem follows from the commutativity of integrals and the bounded linear map $\int_J : C(J, V) \to V$ (cf. Thm. 13.16). We first make some general discussion before giving the rigorous proof.

Let Y be a compact topological space. By Thm. 9.3, we have a canonical equivalence $C(I \times Y, V) \simeq C(I, C(Y, V))$ by viewing $f \in C(I \times Y, V)$ as a map

$$\Phi(f) : I \to C(Y, V) \tag{14.2a}$$

where

$$\Phi(f)(x) = f(x, \cdot) : Y \to V \qquad y \mapsto f(x, y) \tag{14.2b}$$

Recall that the space of continuous functions on a compact space is equipped with the l^∞-norm, and $C(Y, V)$ is complete since V is complete (Cor. 3.50).

Lemma 14.2. *The integral $\int_I \Phi(f)$, which is an element of $C(Y, V)$, is the function*

$$\int_I f(x, \cdot)dx : Y \to V \qquad y \mapsto \int_I f(x, y)dx \tag{14.3}$$

In other words, for every $y \in Y$ we have

$$\Big(\int_I \Phi(f) \Big)(y) = \int_I f(x, y)dx \tag{14.4}$$

Consequently, the function $\int_I f(x, \cdot)dx$ is continuous.

It is easy to show that $\int_I f(x, \cdot)dx$ is continuous without assuming that Y is compact. See Exe. 14.8.

Proof. For each $y \in Y$, define linear map

$$\varphi_y : C(Y, V) \to V \qquad g \mapsto g(y)$$

Then this linear map is clear bounded (with operator norm 1). Thus, by Thm. 13.16, we have

$$\Big(\int_I \Phi(f)(x)dx\Big)(y) = \varphi_y\Big(\int_I \Phi(f)(x)dx\Big) = \int_I \varphi_y\big(\Phi(f)(x)\big)dx = \int_I f(x,y)dx$$

□

Proof of Thm. 14.1. By Thm. 13.20, the integral operator $\int_J : C(J, V) \to V$ is a bounded linear map. Therefore, by Thm. 13.16 (and in particular (13.19)), we have a commutative diagram

$$\begin{tikzcd} C(I, C(J, V)) \arrow[r, "\int_J \circ"] \arrow[d, "\int_I"'] & C(I, V) \arrow[d, "\int_I"] \\ C(J, V) \arrow[r, "\int_J"] & V \end{tikzcd}$$

where the top arrow is the map sending $\Phi(f)$ to $\int_J \circ \Phi(f)$, i.e., sending $x \mapsto f(x, \cdot)$ to $x \mapsto \int_J f(x, y)dy$. Thus, the direction $^{\rightarrow}\!\!\downarrow$ sends $\Phi(f)$ to $\int_I \int_J f(x, y)dydx$. By Lem. 14.2, the left downward arrow sends $\Phi(f)$ to the function $y \in J \mapsto \int_I f(x, y)dx$. So $\downarrow_{\rightarrow}$ sends $\Phi(f)$ to $\int_J \int_I f(x, y)dxdy$. Thus, the commutativity of the above diagram proves Fubini's theorem. □

Definition 14.3. Let $I_1, \dots, I_N$ be compact intervals in $\mathbb{R}$. Let $B = I_1 \times \cdots \times I_N$ and $f \in C(B, V)$. We define the **Riemann integral** of f to be

$$\int_B f = \int_{I_1} \cdots \int_{I_N} f(x_1, \dots, x_N)dx_N \cdots dx_1$$

Then, by Fubini's theorem, for any bijection $\sigma : \{1, \dots, N\} \to \{1, \dots, N\}$ we have

$$\int_B f = \int_{I_{\sigma(1)}} \cdots \int_{I_{\sigma(N)}} f(x_1, \dots, x_N)dx_{\sigma(N)} \cdots dx_{\sigma(1)}$$

14.1.2 Commutativity of integrals and derivatives

Recall from Cor. 11.35 that $l^{1,\infty}(J, V)$ is a Banach space. So its closed linear subspace $C^1(J, V)$ (cf. Pb. 11.1) is a Banach space under the $l^{1,\infty}$-norm $\|g\|_{1,\infty} = \|g\|_\infty + \|g'\|_\infty$.

Let $f : I \times J \to V$. Consider f as a map $\Psi(f) : I \to V^J$ sending x to

$$\Psi(f)(x) = f(x, \cdot) : J \to V \qquad y \mapsto f(x, y) \tag{14.5}$$

Let ∂_J be the (partial) derivative with respect the variable $y \in J$. Clearly

$$\partial_J\big(\Psi(f)(x)\big) = \partial_J f(x, \cdot) \tag{14.6}$$

The linear map of derivative

$$\partial_J : C^1(J, V) \to C(J, V) \qquad g \mapsto g' = \partial_J g \tag{14.7}$$

is clearly bounded (with operator norm $\leqslant 1$).

Proposition 14.4. *Let X be a topological space. Let $f : X \times J \to V$, and define $\Psi(f) : X \to V^J$ by (14.5). Then the following are equivalent.*

(1) $\Psi(f)$ is an element of $C(X, C^1(J, V))$. In other words:

  (1a) For each $x \in X$ we have $\Psi(f)(x) \in C^1(J, V)$.

  (1b) The function $x \in X \mapsto \Psi(f)(x) \in C^1(J, V)$ is continuous.

(2) $\partial_J f$ exists everywhere on $X \times J$. Moreover, we have $f, \partial_J f \in C(X \times J, V)$.

Proof. (1a) means that the functions $y \mapsto f(x, y)$ and $y \mapsto \partial_J f(x, y)$ exist and are continuous. (1b) is equivalent to that the maps

$$x \in X \mapsto \Psi(f)(x) = f(x, \cdot) \in C(J, V)$$
$$x \in X \mapsto \partial_J \Psi(f)(x) = \partial_J f(x, \cdot) \in C(J, V)$$

are continuous (under the l^∞-norm). By Thm. 9.3, this is equivalent to that f and $\partial_J f$ are continuous maps $X \times J \to V$. So (1)⇔(2). □

We return to the setting of $f : I \times J \to V$.

Lemma 14.5. *The integral $\int_I \Psi(f)$, which is an element of $C^1(J, V)$, is the function*

$$\int_I f(x, \cdot)dx : J \to V \qquad y \mapsto \int_I f(x, y)dx \tag{14.8}$$

Consequently, the function $\int_I f(x, \cdot)dx$ is in $C^1(J, V)$.

Proof. This lemma can be proved in the same way as Lem. 14.2, using the fact that for every $y \in J$, the linear map $g \in C^1(J, V) \to g(y)$ is bounded. □

Theorem 14.6. *Let $f : I \times J \to V$. Assume that $\partial_J f$ exists everywhere on $I \times J$. Assume moreover that $f, \partial_J f \in C(I \times J, V)$. Then for each $y \in J$, the LHS of (14.9) exists and equals the RHS of (14.9), where*

$$\partial_J \int_I f(x, y)dx = \int_I \partial_J f(x, y)dx \tag{14.9}$$

First proof. Again, by Thm. 13.16, we have a commutative diagram

$$\begin{tikzcd}
C(I, C^1(J,V)) \arrow[r, "\partial_J\circ"] \arrow[d, "\int_I"'] & C(I, C(J,V)) \arrow[d, "\int_I"] \\
C^1(J,V) \arrow[r, "\partial_J"] & C(J,V)
\end{tikzcd}$$

By Prop. 14.4, $\Psi(f)$ is an element of $C(I, C^1(J,V))$. The map $\Psi(f) : I \to C^1(J,V)$, composed with ∂_J, gives $x \in I \mapsto \partial_J f(x,\cdot)$. By Lem. 14.2, the direction $^{\to}\downarrow$ sends $\Psi(f)$ to $\int_I \partial_J f(x,\cdot)dx$. By Lem. 14.5, $\int_I \Psi(f)$ equals $\int_I f(x,\cdot)$. In particular, $\int_I f(x,\cdot)$ is a C^1-function. So $\downarrow_{\to}$ sends $\Psi(f)$ to $\partial_J \int_I f(x,\cdot)dx$. This finishes the proof. $\square$

Second proof. Fix any $y \in J$. In view of Cor. 13.21, it suffices to prove that the limit of the net of functions $(\varphi_p)_{p\in J\setminus\{y\}}$ from I to V converges uniformly to $\partial_J f(\cdot, y)$ under $\lim_{p\to y}$, where

$$\varphi_p(x) = \frac{f(x,p) - f(x,y)}{p-y}$$

By Rem. 12.34, we have

$$\|\varphi_p(x) - \partial_J f(x,y)\| \leqslant A(x,p) := \sup_{q\in[p,y]\cup[y,p]} \|\partial_J f(x,q) - \partial_J f(x,y)\|$$

Since $\partial_J f$ is continuous, it can be viewed as a continuous map $J \to C(I,V)$ by Thm. 9.3. Thus, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $p \in J$ satisfying $|p-y| \leqslant \delta$, we have $\sup_{x\in I} \|\partial_J f(x,q) - \partial_J f(x,y)\| \leqslant \varepsilon$ for all $q \in [p,y] \cup [y,p]$, and hence $\sup_{x\in I} A(x,p) \leqslant \varepsilon$. This proves that $A(\cdot,p)$ converges uniformly to 0 (as a net of functions $I \to \mathbb{R}$) as $p \to y$, finishing the proof. $\square$

14.1.3 Commutativity of partial derivatives

We write $\partial_I f(x,y)$ as $\partial_1 f(x,y)$ and $\partial_J f(x,y)$ as $\partial_2 f(x,y)$. In the following, we use Thm. 14.6 to give a new proof of a slightly weaker version of Thm. 12.35 on the commutativity of ∂_1 and ∂_2. The idea is as follows. Suppose we know that A, B are linear operators on a vector space such that A is invertible and $A^{-1}B = BA^{-1}$. Then one deduces $A^{-1}BA = BA^{-1}A = B$ and hence $BA = AA^{-1}BA = AB$. Now, Thm. 14.6 says that ∂_J commutes with $\int_I$, the inverse of ∂_I (in a vague sense). So one can use a similar algebraic argument to prove that ∂_J commutes with ∂_I.

Theorem 14.7. *Let $f : I \times J \to V$ be a function such that $\partial_1 f, \partial_2 f, \partial_2\partial_1 f$ exist and are continuous on $I \times J$. Then $\partial_1\partial_2 f$ exists on $I \times J$ and equals $\partial_2\partial_1 f$. (So $\partial_1\partial_2 f$ is also continuous.)*

Thm. 14.7 is weaker than Thm. 12.35 in that we assume $\partial_1 f, \partial_2 f$ to be continuous. Indeed, as we shall see in the proof, the continuity of $\partial_2 f$ is not used. However, in concrete examples, it is fairly easy to check the continuity of $\partial_1 f, \partial_2 f$.

Proof. Let $F(x,y) = \int_a^x \partial_2\partial_1 f(u,y)du$, which can be defined because $\partial_2\partial_1 f$ is continuous. By FTC, we have $\partial_1 F = \partial_2\partial_1 f$. On the other hand, by Thm. 14.6 and the continuity of $\partial_1 f, \partial_2\partial_1 f$, we have

$$F(x,y) = \partial_2 \int_a^x \partial_1 f(u,y)du = \partial_2(f(x,y) - f(a,y))$$

Therefore $\partial_1\partial_2 f$ exists and equals $\partial_1 F = \partial_2\partial_1 f$. □

Exercise 14.8. Let Y be a topological space. Let $f \in C(I \times Y, V)$. Prove that the following function is continuous:

$$I \times Y \to V \qquad (x,y) \mapsto \int_a^x f(u,y)du \tag{14.10}$$

Exercise 14.9. Use Fubini's Thm 14.1 to prove Thm. 14.6.

14.2 Lebesgue's criterion for Riemann integrability

Fix $I = [a,b]$ where $-\infty < a < b < +\infty$. Let V be a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. The goal of this section is to prove:

Theorem 14.10 (Lebesgue's criterion). *Let $f : I \to V$. Then the following are equivalent.*

(1) f is strongly Riemann integrable.

(2) f is bounded (i.e. $\|f\|_{l^\infty} < +\infty$). Moreover, the set of discontinuities

$$\{x \in I : f \text{ is not continuous at } x\} \tag{14.11}$$

is a null set.

Definition 14.11. A subset E of $\mathbb{R}$ is called a **(Lebesgue) null set** (or a set of **(Lebesgue) measure zero**) if for every $\varepsilon > 0$ there exist countably many closed intervals $I_1, I_2, \dots$ such that $E \subset \bigcup_i I_i$, and that $\sum_i |I_i| < \varepsilon$. Here, $|I_i|$ is the length of I_i.

The word "countably many" can be omitted, because the sum of uncountably many strictly positive reals numbers must be $+\infty$ due to Pb. 5.3.

Remark 14.12. Covering E by open intervals instead of closed ones does not change the definition of null sets. This is because any open/closed interval can be stretched by a factor of $1 + \delta$ to a larger closed/open interval, where δ is any given positive number.

Proposition 14.13. *A countable union of null subsets of* $\mathbb{R}$ *is a null set.*

Proof. If $E = E_1 \cup E_2 \cup \cdots$ where each E_i is a null set, then for each ε, E_i can be covered by countably many closed intervals of total length $< 2^{-i}\varepsilon$. So E can be covered by by countably many closed intervals of total length $< \varepsilon$. □

Example 14.14. Every interval with at least two points is not a null set.

Proof. Since every such interval contains a closed invertal with at least two points, it suffices to prove that the latter is not null. Thus, let us prove that I is not null. Suppose I is covered by some open intervals. Then, since I is compact, I is covered by finitely many of these open intervals, say $U_1, \ldots, U_n$. It is easy to see that $\sum_{i=1}^n |U_i| \geqslant |I| = b - a$. (For example, let $f = \sum_i \chi_{U_i}$. Then $f \geqslant \chi_I$. So $\sum_i |U_i| = \sum_i \int \chi_{U_i} = \int f \geqslant \int_a^b 1 = b - a$.) □

Before proving Lebesgue's criterion, let us first see some useful applications. Recall that if $V = \mathbb{R}^N$ then Riemann integrability is equivalent to strong Riemann integrability (Thm. 13.13).

Corollary 14.15. *Let* $f \in \mathscr{R}(I, \mathbb{R}^n)$. *Let* Ω *be a subset of* $\mathbb{R}^n$ *containing* $f(I)$. *Let* $g \in C(\Omega, \mathbb{R}^m)$ *such that* $\|g\|_\infty < +\infty$. *Then* $g \circ f \in \mathscr{R}(I, \mathbb{R}^m)$

Note that if Ω is compact, we automatically have $\|g\|_\infty < +\infty$.

Proof. Clearly $g \circ f$ is bounded. The set of discontinuities of $g \circ f$ is null since it is a subset of the set of discontinuities of f, where the latter is a null set. □

Corollary 14.16. *Let* $f \in \mathscr{R}(I, \mathbb{F}^{m\times n})$ *and* $g \in \mathscr{R}(I, \mathbb{F}^{n\times k})$. *Then* $fg \in \mathscr{R}(I, \mathbb{F}^{m\times k})$.

Proof. This is immediate from Lebesgue's criterion. □

14.2.1 Proof of Lebesgue's criterion

The first step of proving Lebesgue's criterion is to express the set of discontinuities by the oscillation.

Definition 14.17. Let X be a topological space. Let Y be a metric space. The **oscillation** of a function $f : X \to Y$ at $x \in X$ is defined to be

$$\omega(f, x) = \inf_{U \in \mathrm{Nbh}_X(x)} \mathrm{diam}(f(U))$$

Proposition 14.18. *Let X be a topological space, let Y be a metric space, and let $f : X \to Y$. Then f is continuous at $x \in X$ iff $\omega(f, x) = 0$.*

Proof. Assume that f is continuous at x. Then for every $\varepsilon > 0$, there exists $U \in \mathrm{Nbh}(x)$ such that $d(f(p), f(x)) < \varepsilon/2$ for every $p \in X$. Then clearly $\mathrm{diam}(f(U)) \leqslant \varepsilon$. So $\omega(f, x) \leqslant \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\omega(f, x) = 0$.

Conversely, suppose $\omega(f, x) = 0$. Then for every $\varepsilon > 0$ there exists $U \in \mathrm{Nbh}(x)$ such that $\mathrm{diam}(f(U)) < \varepsilon$. So for every $p \in U$ we have $d(f(p), f(x)) < \varepsilon$. □

Proof of Thm. 14.10, part 1. Let us prove (1)⇒(2). Assume that f is strongly Riemann integrable. Choose any $\varepsilon > 0$. Let us prove that

$$\Omega_\varepsilon(f) = \{x \in I : \omega(f, x) \geqslant \varepsilon\} \tag{14.12}$$

is a null set. Then the set of discontinuities, which is $\bigcup_{n \in \mathbb{Z}_+} \Omega_{1/n}(f)$, is a null set by Prop. 14.13.

Since $\inf_{\sigma \in \mathcal{P}(I)} \omega(f, \sigma) = 0$, for every $\delta > 0$, there exists $\sigma = \{a_0 < \cdots < a_n\} \in \mathcal{P}(I)$ such that, with $I_i = [a_{i-1}, a_i]$, we have

$$\sum_{i=1}^{n} \mathrm{diam}(f(I_i)) \cdot |I_i| < \delta\varepsilon \tag{14.13}$$

Note that if $x \in I \backslash \sigma$ is in some I_i where $\mathrm{diam}(f(I_i)) < \varepsilon$, then clearly $\omega(f, x) < \varepsilon$. Thus, if $\omega(f, x) \geqslant \varepsilon$ (i.e., if $x \in \Omega_\varepsilon(f)$), then either $x \in \sigma$, or $x \in I_i$ for some I_i such that $\mathrm{diam}(f(I_i)) \geqslant \varepsilon$. We conclude

$$\Omega_\varepsilon(f) \subset \{a_0, \dots, a_n\} \cup \Big(\bigcup_{k \in K} I_k\Big)$$

where $K = \{1 \leqslant k \leqslant n : \mathrm{diam}(f(I_k)) \geqslant \varepsilon\}$. Clearly $\{a_0, \dots, a_n\}$ can be covered by some intervals with total length $< \delta$. But (14.13) implies that $\sum_{k \in K} \varepsilon |I_k| < \delta\varepsilon$ and hence $\sum_{k \in K} |I_k| < \delta$. So $\Omega_\varepsilon(f)$ can be covered by intervals with total length $< 2\delta$ for every $\delta > 0$. Thus $\Omega_\varepsilon(f)$ is a null set. □

To prove the other direction, we need some preparation.

Lemma 14.19. *Let $f : X \to Y$ where X is a topological space and Y is a metric space. Then for every $\varepsilon > 0$, $\Omega_\varepsilon(f) = \{x \in X : \omega(f, x) \geqslant \varepsilon\}$ is a closed subset of X.*

Proof. Let us prove that each $x \in X \backslash \Omega_\varepsilon(f)$ is an interior point. (Recall Prop. 77.43.) Indeed, since $\inf_{U \in \mathrm{Nbh}(x)} \mathrm{diam}(f(U)) < \varepsilon$, there exists $U \in \mathrm{Nbh}(x)$ such that $\mathrm{diam}(f(U)) < \varepsilon$. Then for each $p \in U$ we have $\inf_{V \in \mathrm{Nbh}(p)} \mathrm{diam}(f(V)) < \varepsilon$ since $U \in \mathrm{Nbh}(p)$. So $U \subset X \backslash \Omega_\varepsilon(f)$. □

Lemma 14.20. *Let $\varepsilon > 0$. Suppose that for every $x \in I$ we have $\omega(f, x) < \varepsilon$. (Namely, suppose $\Omega_\varepsilon(f) = \varnothing$.) Then there exists a partition $I = I_1 \cup \cdots \cup I_n$ satisfying $\mathrm{diam}(f(I_i)) < \varepsilon$ for all i.*

Proof. For every $x \in I$, there exists $U_x \in \mathrm{Nbh}_I(x)$ such that $\mathrm{diam}(f(U_x)) < \varepsilon$. Choose $n \in \mathbb{Z}_+$ such that $1/n$ is a Lebesgue number of the open cover $\mathcal{U} = \{U_x : x \in I\}$ of I. (Recall Thm. 10.8.) Dividing I into $2n$ subintervals with the same length gives the desired partition. □

Proof of Thm. 14.10, part 2. Let us prove (2)⇒(1). Suppose that $M = \|f\|_\infty$ is $< +\infty$, and that the set of discontinuities is a null set. Then for each $\varepsilon > 0$, $\Omega_\varepsilon(f)$ is a null set. By Lem. 14.19 and Heine-Borel, $\Omega_\varepsilon(f)$ is compact. Thus, for every $\delta > 0$, $\Omega_\varepsilon(f)$ can be covered by finitely many closed intervals with total length $< \delta$. Let Δ be the union of these closed intervals. Then $\Omega_\varepsilon(f) \subset \Delta$. Clearly, both Δ and $J = I \backslash \mathrm{Int}(\Delta)$ can be written as disjoint unions of finitely many closed intervals, which are their connected components. And Δ has length $|\Delta| < \delta$.

Since $J \cap \Omega_\varepsilon(f) = \varnothing$, applying Lem. 14.20 to each of the components $J_1, J_2, \dots$ of J, we see that J_i has a partition ϱ_i such that the oscillation of f (recall Def. 13.7) on each subinterval cut out by ϱ_i is $< \varepsilon$. Let $\sigma = \varrho_1 \cup \varrho_2 \cup \cdots \cup \{a, b\}$, which is a partition of I. Then σ divides I into subintervals $I_1, I_2, \dots$ such that either $I_j \subset J$ or $I_j \subset \Delta$.

- If $I_j \subset J$, then I_j belongs to a subinterval cut out by one of $\varrho_1, \varrho_2, \dots$ in $J_1, J_2, \dots$. This implies $\mathrm{diam}(f(I_j)) < \varepsilon$ by the construction of $\varrho_1, \varrho_2, \dots$.
- If $I_j \subset \Delta$, then I_j is a component of Δ. We have $\mathrm{diam}(f(I_j)) \leqslant 2M$. Moreover, the total length of such I_j is equal to $|\Delta|$, which is $< \delta$.

From the discussion of these two cases, we see that

$$\omega(f, \sigma) = \sum_j \mathrm{diam}(f(I_j)) \cdot |I_j| \leqslant \varepsilon \cdot (b - a) + 2M \cdot \delta$$

Since $b-a$ and M are fixed numbers and ε, δ are arbitrary, we see that $\inf_\sigma \omega(f, \sigma) = 0$. So f is strongly Riemann integrable. □

14.3 Improper integrals

In this section, all intervals in $\mathbb{R}$ are assumed to contain at least two points. Let I be an interval in $\mathbb{R}$ with $a = \inf I$ and $b = \sup I$. So $-\infty \leqslant a < b \leqslant +\infty$. Let V be a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 14.21. We define

$$\mathscr{R}(I, V) = \{f \in V^I : f|_J \in \mathscr{R}(J, V) \text{ for every compact interval } J \subset I\}$$

Define the **improper integral**

$$\int_I f \equiv \int_a^b f \xlongequal{\text{def}} \lim_{\substack{u \to a \\ v \to b}} \int_u^v f = \lim_J \int_J f \tag{14.14}$$

Here, the last limit is over the directed set {compact intervals in I} with preorder "$\subset$". If the above limit exists, we say that $\int_I f$ **exists** or **converges**.

When $f \in \mathscr{R}(I, \mathbb{R})$ and takes values in $\mathbb{R}_{\geqslant 0}$, we write $f \in \mathscr{R}(I, \mathbb{R}_{\geqslant 0})$. Then $\int_I f$ clearly exists in $\overline{\mathbb{R}}_{\geqslant 0}$. We write $\displaystyle\int_I f < +\infty$ if $\int_I f$ converges in $\mathbb{R}$. □

In the case that a or b is in I, when taking the limit over $u \to a$ and $v \to b$ in (14.14), it is immaterial whether u, v can take values a, b or not, due to the following easy observation:

Lemma 14.22. *Let $f \in \mathscr{R}(I, V)$. If $I = [a, b]$, then the meanings of $\mathscr{R}(I, V)$ and $\int_I f$ are the same as before. If $I = [a, b)$ resp. $I = (a, b]$, then*

$$\int_I f = \lim_{v \to b} \int_a^v f \qquad \textit{resp.} \qquad \int_I f = \lim_{u \to a} \int_u^b f$$

where the convergence of the LHS is equivalent to that of the RHS.

Proof. Assume $I = [a, b]$. Then I is a compact interval. The new and old meanings of $\mathscr{R}(I, V)$ are the same by Prop. 13.23. Let $f \in \mathscr{R}(I, V)$. Then by Thm. 13.20, we have $\|f\|_\infty < +\infty$. If $a \leqslant u < v \leqslant b$, then Thm. 13.20 and Prop. 13.23 show that

$$\Big\| \int_a^b f - \int_u^v f \Big\| = \Big\| \int_a^u f + \int_v^b f \Big\| \leqslant \|f\|_\infty \cdot ((u - a) + (b - v))$$

which converges to 0 as $u \to a$ and $v \to b$, whether u, v take values a, b or not.

Similarly, if $I = [a, b)$, then since f is Riemann integrabe on the compact subinterval $J = [a, (a + b)/2]$, we have $M := \|f|_J\|_\infty < +\infty$. This implies that for all u, v such that $a \leqslant u < v < b$ and $u \leqslant (a + b)/2$,

$$\Big\| \int_a^v f - \int_u^v f \Big\| = \Big\| \int_a^u f \Big\| \leqslant M(u - a)$$

which converges to 0 as $u \to a$ and $v \to b$. So $(\int_a^v f)_{u,v}$ and $(\int_u^v f)_{u,v}$ are Cauchy-equivalent nets. So their convergences and values are equivalent. The case $I = (a, b]$ is similar. □

Remark 14.23. Let $f \in \mathscr{R}(I, V)$. The Cauchy condition for the convergence of $\int_a^b f$ is easy to describe. In view of $\int_u^v - \int_{u'}^{v'} = \int_u^{u'} + \int_v^{v'}$ (Prop. 13.23), we have:

- For every $\varepsilon > 0$, there exist $u_0 < v_0$ in I such that for all $u, u', v, v' \in I$ satisfying $a \leqslant u, u' < u_0$ and $v_0 < v, v' \leqslant b$ we have

$$\Big\| \int_u^{u'} f \Big\| < \varepsilon \qquad \Big\| \int_v^{v'} f \Big\| < \varepsilon$$

Definition 14.24. Let $f \in \mathscr{R}(I,V)$. We say that $\int_I f$ **converges absolutely** (or that f is **absolutely integrable on** I) if there exists $g \in \mathscr{R}(I,\mathbb{R})$ such that $|f| \leqslant g$ (i.e., $\|f(x)\| \leqslant g(x)$ for all $x \in I$, in particular $g(x) \geqslant 0$) and that $\int_I g < +\infty$. We let

$$\mathscr{R}^1(I,V) = \{f \in \mathscr{R}(I,V) : \textstyle\int_I f \text{ converges absolutely}\} \tag{14.15}$$

which is clearly a linear subspace of V^I. By the Cauchy condition in Rem. 14.23, it is clear that $\int_I f$ converges if $\int_I f$ converges absolutely. Though $\mathbb{R}_{\geqslant 0}$ is not a Banach space, we still write

$$\mathscr{R}^1(I,\mathbb{R}_{\geqslant 0}) = \{f \in \mathscr{R}(I,\mathbb{R}_{\geqslant 0}) : \textstyle\int_I f \text{ converges absolutely}\}$$

which is clearly the set of all $f \in \mathscr{R}(I,\mathbb{R}_{\geqslant 0})$ satisfying $\int_I f < +\infty$.

The superscript 1 of $\mathscr{R}^1$ has a similar meaning as that of l^1, but is different from that of C^1.

Remark 14.25. Assume that $f : I \to V$ is strongly integrable on each compact subinterval of I. (This is the case, for example, when $f \in C(I,V)$, or when $f \in \mathscr{R}(I,V)$ and $V = \mathbb{F}^N$.) Then $|f| : x \in I \mapsto \|f(x)\| \in \mathbb{R}_{\geqslant 0}$ is an element of $\mathscr{R}(I,\mathbb{R}_{\geqslant 0})$ by Cor. 13.19. Thus, in this case,

$$f \in \mathscr{R}^1(I,V) \qquad \Longleftrightarrow \qquad \int_I |f| < +\infty$$

Example 14.26. We have

$$\int_1^{+\infty} x^{-2}dx = \lim_{v\to+\infty}\int_1^v x^{-2}dx = \lim_{v\to+\infty}(-x^{-1})\Big|_1^v = 1 < +\infty$$

Therefore $\displaystyle\int_1^{+\infty}\frac{e^{\mathrm{i}x}}{x^2}dx$ converges absolutely, and hence converges.

The following proposition generalizes Prop. 13.18.

Proposition 14.27. *Let $f \in \mathscr{R}^1(I,V)$ and $g \in \mathscr{R}^1(I,\mathbb{R}_{\geqslant 0})$ such that $|f| \leqslant g$. Then $\displaystyle\Big\|\int_I f\Big\| \leqslant \int_I g$.*

Proof. Apply the limit over $u \to a, v \to b$ to $\|\int_u^v f\| \leqslant \int_u^v g$. □

The next proposition shows that improper integrals are helpful for studying series:

Proposition 14.28. *Let $f \in \mathscr{R}([1,+\infty),\mathbb{R}_{\geqslant 0})$ be decreasing. Then we have*

$$\int_1^{+\infty} f < +\infty \qquad \Longleftrightarrow \qquad \sum_{n=1}^{+\infty} f(n) < +\infty \tag{14.16}$$

Proof. Since f is decreasing, we clearly have $g \leqslant f \leqslant h$ where g, h are defined by the series in $l^\infty([0,+\infty),\mathbb{R})$:

$$g = \sum_{n=1}^{+\infty} f(n+1)\cdot\chi_{[n,n+1)} \qquad h = \sum_{n=1}^{+\infty} f(n)\cdot\chi_{[n,n+1)}$$

One computes easily that $\int_1^{+\infty} g = \sum_{n=2}^{+\infty} f(n)$ and $\int_1^{+\infty} h = \sum_{n=1}^{+\infty} f(n)$ in $\overline{\mathbb{R}}_{\geqslant 0}$. Thus

$$\sum_{n=2}^{+\infty} f(n) \leqslant \int_1^{+\infty} f \leqslant \sum_{n=1}^{+\infty} f(n)$$

The proposition follows easily. □

Example 14.29. Let $s > 0$. The function $f : [2,+\infty) \to +\infty$ defined by $f(x) = \dfrac{1}{x(\log x)^s}$ is decreasing, and

$$\int_2^x f = \begin{cases} (1-x)^{-1}\big((\log x)^{1-s} - (\log 2)^{1-s}\big) & \text{if } s \neq 1 \\ \log(\log x) - \log(\log 2) & \text{if } s = 1 \end{cases}$$

So $\sum_{n=2}^{\infty} f(n) < +\infty$ iff $\int_2^{+\infty} f < +\infty$ iff $s > 1$.

14.4 Commutativity of improper integrals and other limit processes

In this section, all intervals in $\mathbb{R}$ are assumed to contain at least two points. Let I be an interval in $\mathbb{R}$ with $a = \inf I$ and $b = \sup I$. Let $J = [c,d]$ be a compact interval. Let V be a Banach space over $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$.

The goal of this section is to generalize the main results of Sec. 14.1 to improper integrals. Namely, we shall prove the commutativity of $\int_I$ with $\int_J$ and with ∂_J under reasonable assumptions. There are three ways to achieve this goal:

- We know that $\int_I$ is the limit of integrals over compact intervals. Therefore, the problem is reduced to that of proving that $\lim_{u\to a, v\to b}$ can be moved inside in $\lim \int_J \int_u^v$ and in $\lim \partial_J \int_u^v$.
- We generalize Thm. 13.16 to improper integrals, and use this generalized version to prove the commutativity in a similar way as in Sec. 14.1.
- The commutativity of $\int_I$ with ∂_J can also be studied by generalizing Cor. 13.21 to improper integrals, similar to the second proof of Thm. 14.6.

We will use the second approach because its proof is more conceptual and involves fewer technical calculations, thus making it easier for us to remember the conditions of the theorems to be proved. Nevertheless, we will also give the appropriate generalization of Cor. 13.21, which is helpful for future application. Recall Def. 14.24 for the meaning of $\mathscr{R}^1$.

14.4.1 Commutativity of improper integrals and bounded linear maps

Theorem 14.30. *Let W be also a Banach space over $\mathbb{F}$. Let $T \in \mathfrak{L}(V, W)$. Then for every $f \in \mathscr{R}^1(I, V)$ we have $T \circ f \in \mathscr{R}^1(I, W)$ and*

$$T\Big(\int_a^b f\Big) = \int_a^b T \circ f \tag{14.17}$$

In other words, we have a commutative diagram

$$\begin{tikzcd} \mathscr{R}^1(I,V) \arrow[r, "T\circ"] \arrow[d, "\int_I"'] & \mathscr{R}^1(I,W) \arrow[d, "\int_I"] \\ V \arrow[r, "T"] & W \end{tikzcd} \tag{14.18}$$

Proof. By Thm. 13.16, $T \circ f$ is Riemann integrable on compact subintervals of I. Since $f \in \mathscr{R}^1$, there exists $g \in \mathscr{R}(I, \mathbb{R}_{\geqslant 0})$ such that $|f| \leqslant g$ and $\int_I g < +\infty$. Let $M = \|T\|$ be the operator norm, which is a finite number. By Rem. 10.24, for each $x \in I$ we have $\|T \circ f(x)\| \leqslant M\|f(x)\| \leqslant Mg(x)$, and hence $|T \circ f| \leqslant Mg$. This proves $T \circ f \in \mathscr{R}^1(I, W)$. In particular, the integrals of $f, T \circ f$ over I converge. Thus, by the continuity of T and the commutativity of T with $\int_u^v$ (when $a < u < v < b$) due to Thm. 13.16, we have

$$T\Big(\int_I f\Big) = T\Big(\lim_{\substack{u\to a\\ v\to b}} \int_u^v f\Big) = \lim_{\substack{u\to a\\ v\to b}} T\Big(\int_u^v f\Big) = \lim_{\substack{u\to a\\ v\to b}} \int_u^v T \circ f = \int_I T \circ f$$

□

14.4.2 Fubini's theorem

Recall that $J = [c, d]$ is a compact interval but I is not necessarily compact.

Lemma 14.31. *Lem. 14.2 holds verbatim to the current case that I is not necessarily compact.*

Proof. We can prove this general case in the same way as Lem. 14.2, using the commutativity of $\int_I$ and the bounded map $g \in C(Y, V) \mapsto g(y) \in V$, which is available thanks to Thm. 14.30. □

Theorem 14.32 (Fubini's theorem for improper integrals). *Let $f \in C(I \times J, V)$. Assume that there exists $h \in \mathscr{R}^1(I, \mathbb{R}_{\geqslant 0})$ satisfying*

$$\|f(x, y)\| \leqslant h(x) \qquad (\forall x \in I, y \in J) \tag{14.19}$$

Then the functions $\int_J f(\cdot, y)dy : I \to V$ and $\int_I f(x, \cdot)dx : J \to V$ are continuous, and the equation

$$\int_I \Big(\int_J f(x, y)dy\Big)dx = \int_J \Big(\int_I f(x, y)dx\Big)dy \tag{14.20}$$

holds where the integral over I on the LHS converges absolutely.

Proof. Let $T : C(J, V) \to V$ be the bounded linear map $\int_J$. By Thm. 14.30, we have a commutative diagram

$$\begin{tikzcd} \mathscr{R}^1(I, C(J,V)) \arrow[r, "T\circ"] \arrow[d, "\int_I"'] & \mathscr{R}^1(I, V) \arrow[d, "\int_I"] \\ C(J,V) \arrow[r, "T"] & V \end{tikzcd} \tag{14.21}$$

Since J is compact, by Thm. 9.3, we can view f as a continuous function $\Phi(f) : I \to C(J, V)$ sending x to $f(x, \cdot)$. Then (14.19) says that $\Phi(f) \leqslant g$. Since $g \in \mathscr{R}^1$, we conclude $\Phi(f) \in \mathscr{R}^1(I, C(J, V))$.

By Lem. 14.31, $\int_I \Phi(f)$ equals the function $\int_I f(x, \cdot)dx$. In particular, $\int_I f(x, \cdot)dx$ is continuous since $\int_I \Phi(f) \in C(J, V)$ by (14.21). The continuity of $\int_J f(\cdot, y)dy$ is similar, or is even easier because J is compact.

Clearly $T(\int_I \Phi(f))$ is $\int_J \int_I f$. By the top arrow of (14.21), $T \circ \Phi(f)$ belongs to $\mathscr{R}^1(I, V)$. Since $T \circ \Phi(f)$ is the function sending x to $\int_J f(x, \cdot)dy$, the integral of this function over I is absolutely convergent, and $\int_I T \circ \Phi(f) = \int_I \int_J f$. Thus, the commutativity of (14.21) proves (14.20). $\square$

The continuity of $\int_I f(x, \cdot)dx$ can be generalized: see Cor. 14.36.

14.4.3 Commutativity of improper integrals and partial derivatives

Lemma 14.33. *Lemma 14.5 holds verbatim to the current case that I is not necessarily compact.*

Proof. Again, this is proved in the same way as Lem. 14.2 by applying Thm. 14.30 to the bounded linear functional $g \in C^1(J, V) \to g(y)$ (where $y \in J$). $\square$

Theorem 14.34. *Let $f : I \times J \to V$. Assume that $\partial_J f$ exists everywhere on $I \times J$. Assume that $f, \partial_J f \in C(I \times J, V)$. Assume moreover that there exists $h \in \mathscr{R}^1(I, \mathbb{R}_{\geqslant 0})$ satisfying*

$$\|f(x, y)\| \leqslant h(x) \quad \text{and} \quad \|\partial_J f(x, y)\| \leqslant h(x) \qquad (\forall x \in I, y \in J) \tag{14.22}$$

Then for each $y \in J$, the LHS of (14.23) exists and equals the RHS of (14.23), where

$$\partial_J \int_I f(x, y)dx = \int_I \partial_J f(x, y)dx \tag{14.23}$$

Proof. Let $T : C^1(J, V) \to C(J, V)$ be the bounded linear map ∂_J. By Thm. 14.30, we have a commutative diagram

$$\begin{tikzcd} \mathscr{R}^1(I, C^1(J,V)) \arrow[r, "T\circ"] \arrow[d, "\int_I"'] & \mathscr{R}^1(I, C(J,V)) \arrow[d, "\int_I"] \\ C^1(J,V) \arrow[r, "T"] & C(J,V) \end{tikzcd} \tag{14.24}$$

Define $\Psi(f) : X \to V^J$ sending x to $f(x,\cdot)$. By Prop. 14.4, $\Psi(f)$ belongs to $C(I, C^1(J,V))$. By (14.22), $\Psi(f)$ belongs to $\mathscr{R}^1(I, C^1(J,V))$. As in the first proof of Thm. 14.6, one can use Lem. 14.31 and 14.33 (the improper version of Lem. 14.2 and 14.5) to show that $\int_I T\circ\Psi(f) = T\int_I \Psi(f)$ (which is a consequence of the commutativity of (14.24)) implies (14.23). □

14.4.4 Commutativity of improper integrals and net limits

Theorem 14.35. *Let $(f_\alpha)_{\alpha\in\mathscr{I}}$ be a net in $\mathscr{R}^1(I,V)$. Let $f\in V^I$. Assume that the following conditions are true:*

(1) On every compact subinterval of I, the net (f_α) converges uniformly to f.

(2) There exists $g\in\mathscr{R}^1(I,\mathbb{R}_{\geqslant 0})$ such that $|f_\alpha|\leqslant g$ for all $\alpha\in\mathscr{I}$.

Then $f\in\mathscr{R}^1(I,V)$, and $\displaystyle\int_I f = \lim_{\alpha\in\mathscr{I}}\int_I f_\alpha$.

Proof. By Cor. 13.21, $f\in\mathscr{R}(I,V)$. Since (f_α) converges pointwise to f, we clearly have $|f|\leqslant g$. So $f\in\mathscr{R}^1(I,V)$. Choose any $\varepsilon>0$. Since $\lim_{u\to a,v\to b}\int_I g$ converges, there exists u,v such that $a<u<v<b$ and

$$\int_a^u g+\int_v^b g=\int_I g-\int_u^v g<\varepsilon$$

Thus, by Prop. 14.27, for each α we have $\Vert\int_a^u f_\alpha\Vert+\Vert\int_v^b f_\alpha\Vert<\varepsilon$ and $\Vert\int_a^u f\Vert+\Vert\int_v^b f\Vert<\varepsilon$. Since f_α converges uniformly to f on $[u,v]$, by Cor. 13.21, we get $\lim_\alpha\Vert\int_u^v f_\alpha-\int_u^v f\Vert=0$. Thus

$$\begin{aligned}&\Big\Vert\int_I f-\int_I f_\alpha\Big\Vert=\Big\Vert\int_a^u f+\int_v^b f-\int_a^u f_\alpha-\int_v^b f_\alpha+\int_u^v(f-f_\alpha)\Big\Vert\\ \leqslant&2\varepsilon+\Big\Vert\int_u^v(f-f_\alpha)\Big\Vert\end{aligned}$$

where the $\limsup_\alpha$ of the RHS is 2ε. Thus $\Vert\int_I f-\int_I f_\alpha\Vert$ converges to 0 under $\limsup_\alpha$, and hence under $\lim_\alpha$. □

It is a good practice to prove Thm. 14.34 using Thm. 14.35. (See Pb. 14.2.)

Corollary 14.36. *Let Y be a topological space. Let $f\in C(I\times Y,V)$. Assume that there exists $h\in\mathscr{R}^1(I,\mathbb{R}_{\geqslant 0})$ such that*

$$\Vert f(x,y)\Vert\leqslant h(x)\qquad(\forall x\in I,y\in Y)\tag{14.25}$$

Then the map $\int_I f(x,\cdot)dx$ (sending $y\in Y$ to $\int_I f(x,y)dx$) is continuous.

Proof. By Def. 7.56-(1), we need to prove that for every net (y_α) in Y converging to y, we have $\int_I f(x,y)dx = \lim_\alpha \int_I f(x,y_\alpha)dx$. By Thm. 14.35, it suffices to prove that $\lim_\alpha f(\cdot,y_\alpha)$ converges uniformly on any compact subinterval $[u,v] \subset I$ to $f(\cdot,y)$. This is true because, by Thm. 9.3, the restriction of f to $[u,v]\times Y$ can be viewed as an element of $C(Y, C([u,v],V))$. $\square$

Example 14.37. Compute $F(t) = \displaystyle\int_0^{+\infty} e^{-tx}\cdot\frac{\sin x}{x}dx$ for all $t>0$.

Solution. We first compute $F'(t)$. Choose any $\delta>0$. Then for any $t \geqslant \delta$, the norms of $e^{-tx}\frac{\sin x}{x}$ and $\partial_t(e^{-tx}\frac{\sin x}{x}) = -e^{-tx}\sin x$ are both bounded by $e^{-\delta x}$, where the latter is absolutely integrable since $\int_0^{+\infty} e^{-\delta x}dx = \delta^{-1} < +\infty$. Thus, by Thm. 14.34, we have

$$\begin{aligned}F'(t) &= \int_0^{+\infty} -e^{-tx}\sin x dx = \int_0^{+\infty}\frac{e^{(-t+\mathbf{i})x}-e^{(-t-\mathbf{i})x}}{2\mathbf{i}}dx\\ &= \frac{t\sin x+\cos x}{1+t^2}e^{-tx}\Big|_{x=0}^{+\infty} = -\frac{1}{1+t^2}\end{aligned}$$

for all $t\geqslant\delta$ and $\delta>0$, and hence for all $t>0$. Therefore $F(t) = C - \arctan t$ for some $C\in\mathbb{R}$. Let us determine C using $C = \lim_{t\to+\infty}F(t)+\frac{\pi}{2}$.

To compute $\lim_{t\to+\infty}F(t)$, it suffices to assume $t\geqslant 1$. Then $|e^{-tx}\frac{\sin x}{x}| \leqslant e^{-x}$ where $\int_0^{+\infty} e^{-x} < +\infty$. Moreover, on every compact interval $[a,b]$ in $(0,+\infty)$ (where $0<a<b<+\infty$), $\lim_{t\to+\infty} e^{-tx}\frac{\sin x}{x}$ converges uniformly to 0 since $|e^{-tx}\frac{\sin x}{x}| \leqslant e^{-ta}$. Thus, the assumptions in Thm. 14.35 are satisfied, and hence

$$\lim_{t\to+\infty}F(t) = \int_0^{+\infty}\lim_{t\to+\infty}e^{-tx}\cdot\frac{\sin x}{x}dx = 0$$

This proves $C=\frac{\pi}{2}$, and hence $F(t) = \frac{\pi}{2}-\arctan t$. $\square$

14.5 Convolutions and smooth/polynomial approximation

Fix a Banach space V over $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$.

Definition 14.38. Let X be a topological space. The **support** of a function $f\in V^X$ is defined to be the closure

$$\mathrm{Supp}(f) = \overline{\{x\in X: f(x)\neq 0\}}$$

Define $C_c(X,V)$ to be the set of continuous functions with **compact support**

$$C_c(X,V) = \{f\in C(X,V): \mathrm{Supp}(f) \text{ is compact}\}$$

Unless otherwise stated, $C_c(X,V)$ is equipped with the l^∞-norm.

14.5.1 Convolutions and approximation of identity

A goal of this section is to show that the elements of $C_c(\mathbb{R},V)$ can be approximated by smooth functions with compact supports, i.e., elements in

$$C_c^\infty(\mathbb{R},V)=C^\infty(\mathbb{R},V)\cap C_c(\mathbb{R},V)$$

Indeed, we will do more. We shall prove the celebrated **Weierstrass approximation theorem**, which implies that elements in $C_c(\mathbb{R},V)$ can be approximated uniformly by polynomials on compact intervals. The proof we will give is also due to Weierstrass, which relies on an important construction called convolution: If $f:\mathbb{R}\to V$ and $g:\mathbb{R}\to\mathbb{F}$, then their **convolution** is a functions $f*g:\mathbb{R}\to V$ defined by

$$(f*g)(x)\equiv(g*f)(x)=\int_{\mathbb{R}}f(x-y)g(y)dy \tag{14.26a}$$

whenever the above integral converges for every $x\in\mathbb{R}$. Taking $y=x-t$, we get $\int_u^v f(x-y)g(y)dy=-\int_{x-u}^{x-v}f(t)g(x-t)dt=\int_{x-v}^{x-u}f(t)g(x-t)dt$. Letting $u\to-\infty,v\to+\infty$, we get

$$(f*g)(x)=\int_{\mathbb{R}}f(y)g(x-y)dy \tag{14.26b}$$

The main idea of doing approximation via convolutions is as follows. Suppose that g is smooth, then by (14.26b), one should expect that $(f*g)^{(n)}(x)=\int_{\mathbb{R}}f(y)(\partial_x)^n g(x-y)dy=\int_{\mathbb{R}}f(y)g^{(n)}(x-y)dy$ to be true. Thus $f*g$ is expected to be smooth. Moreover, if g can be approximated by polynomials, for example, if $g(x)=\sum a_nx^n$ on $\mathbb{R}$, then it is expected that $f*g$ can be approximated by $f*g_n$ where $g_n(x)=a_nx^n$, and it is easy to see that $f*g_n=a_n\int_{\mathbb{R}}f(y)(x-y)^ndy$ is a polynomial of x.

To summarize, one advantage of convolution is that whenever g has a good property, the same is in general true for $f*g$. Another key property of convolution is that f can be approximated by $f*g$ in some sense. The meaning of "approximation" will depend on the analytic property of f, e.g. whether f is continuous, continuous with compact support, or only integrable. In this section, we will only be interested in the case that $f\in C_c(\mathbb{R},V)$. In this case, the integrals in (14.26) are actually over compact intervals, which makes our lives easier.

We shall show that $f\in C_c(\mathbb{R},V)$ can be approximated uniformly by $f*g$. More precisely, we shall show that for every absolutely convergent $g\in C(\mathbb{R},\mathbb{R}_{\geqslant 0})$ satisfying $\int_{\mathbb{R}}g=1$, if we define $g_\varepsilon\in C(\mathbb{R},\mathbb{R})$ (where $\varepsilon\in\mathbb{R}_{>0}$) by

$$g_\varepsilon(x)=\frac{1}{\varepsilon}g\Big(\frac{x}{\varepsilon}\Big) \tag{14.27}$$

then $(f*g_\varepsilon)$ converges uniformly to f as $\varepsilon\to 0$. This method is extremely useful and is used widely in analysis. (The assumption that $g\geqslant 0$ is not necessary. We assume $g\geqslant 0$ only for simplifying discussions.)

Lemma 14.39. *Choose $g \in C(\mathbb{R}, \mathbb{R}_{\geqslant 0})$ satisfying $\int_{\mathbb{R}} g = 1$, and define g_ε by (14.27). Then $\int_{\mathbb{R}} g_\varepsilon = 1$ for each $\varepsilon > 0$. Moreover, for every $\delta > 0$ we have*

$$\lim_{\varepsilon \to 0} \int_{\delta}^{+\infty} g_\varepsilon = \lim_{\varepsilon \to 0} \int_{-\infty}^{-\delta} g_\varepsilon = 0 \tag{14.28}$$

Proof. By the change of variable formula we have $\int_u^v \varepsilon^{-1} g(\varepsilon^{-1}x)dx = \int_{u/\varepsilon}^{v/\varepsilon} g(y)dy$. This gives $\int_{\mathbb{R}} g_\varepsilon = 1$. Another change of variable shows $\int_\delta^{+\infty} g_\varepsilon = \int_{\delta/\varepsilon}^{+\infty} g = \int_0^{+\infty} g - \int_0^{\delta/\varepsilon} g$, which clearly converges to 0 as $\varepsilon \to 0$. This proves the first half of (14.28). The second half is similar. □

Proposition 14.40. *Let $f \in C_c(\mathbb{R}, V)$. Choose $g \in C(\mathbb{R}, \mathbb{R}_{\geqslant 0})$ satisfying $\int_{\mathbb{R}} g = 1$, and define g_ε by (14.27) for each $\varepsilon > 0$. Then $(f * g_\varepsilon)$ converges uniformly on $\mathbb{R}$ to f as $\varepsilon \to 0$.*

In other words, the convolution operator $f \mapsto f * g_\varepsilon$ converges pointwise to the identity map when $\varepsilon \to 0$.

Proof. Let $M = \|f\|_\infty$, which is $< +\infty$. Since $\int_{\mathbb{R}} g_\varepsilon = 1$, for each $x \in \mathbb{R}$ we have $\int_{\mathbb{R}} f(x)g_\varepsilon(y)dy = f(x)$. Thus

$$\begin{aligned}\|(f * g_\varepsilon)(x) - f(x)\| &= \Big\| \int_{\mathbb{R}} (f(x-y) - f(x)) g_\varepsilon(y)dy \Big\| \\ &\leqslant \int_{\mathbb{R}} \|f(x-y) - f(x)\| \cdot g_\varepsilon(y)dy \end{aligned} \tag{14.29}$$

By Thm. 10.7 and the compactness of $\mathrm{Supp}(f)$, f is uniformly continuous on $\mathbb{R}$. Therefore, for every $e > 0$, there exists $0 < \delta < 1$ such that $\|f(x-y) - f(x)\| \leqslant e$ for all $x \in \mathbb{R}$ and $y \in (-\delta, \delta)$. Thus, if we let $J_\delta = \mathbb{R}\backslash(-\delta, \delta)$, then

$$\begin{aligned}(14.29) &\leqslant \int_{J_\delta} \|f(x-y) - f(x)\| \cdot g_\varepsilon(y)dy + e \cdot \int_{-\delta}^{\delta} g_\varepsilon(y)dy \\ &\leqslant 2M \cdot \int_{J_\delta} g_\varepsilon(y)dy + e\end{aligned}$$

which converges to e under $\limsup_{\varepsilon \to 0}$ by Lem. 14.39. Since e is arbitrary, we conclude that $\lim_{\varepsilon \to 0} (14.29) = 0$. □

Remark 14.41. The uniform continuity of a function $f : \mathbb{R} \to V$ is equivalent to the continuity of $F : \mathbb{R} \to C(\mathbb{R}, V)$ defined by $F(t)(x) = f(x-t)$. Note that the continuity of F follows from Thm. 9.3. This means that Prop. 14.40 can also be proved by Thm. 9.3.

Remark 14.42. In analysis, there are two especially important classes of $g \in C(\mathbb{R}, \mathbb{R}_{\geqslant 0})$ satisfying $\int_{\mathbb{R}} g = 1$. The first class consists of $g \in C_c(\mathbb{R}, \mathbb{R}_{\geqslant 0})$ satisfying $\int_{\mathbb{R}} g = 1$. Although functions with compact supports are often easy to use, they cannot be approximated by their Taylor series on $\mathbb{R}$. (See Exp. 12.19 for a related example.) Instead, we should consider another type of function:

14.5.2 Polynomial approximation

Example 14.43. Define the **Gauss function** $g(x) = \frac{1}{\sqrt{\pi}}e^{-x^2}$. Then $g \in C(\mathbb{R}, \mathbb{R}_{\geqslant 0})$ and $\int_{\mathbb{R}} g = 1$. Thus, g satisfies the assumptions in Prop. 14.40.

Although g does not have compact support, it is a real analytic function. So we can use $f * g_\varepsilon$ to prove Weierstrass approximation theorem.

Proof. It is easy to see that $\int_1^{+\infty} 1/x^2 dx < +\infty$. Thus, since $e^{x^2} \geqslant x^2$, we get $\int_{\mathbb{R}} e^{-x^2} dx < +\infty$.

It is not so easy to show that

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi} \tag{14.30}$$

This integral is called **Gauss integral**. In the next semester, we will use the change of variable formula for double integrals to prove (14.30). A more elementary (but also more complicated) proof is given in Pb. 14.3. For the purpose of this section, knowing $\int_{\mathbb{R}} e^{-x^2} dx < +\infty$ is enough, since we can define g to be e^{-x^2} divided by its integral on $\mathbb{R}$. □

We now show that if g is the Gauss function, then $f * g_\varepsilon$ can be approximated uniformly by polynomials on every compact interval.

Lemma 14.44. *Let $h(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $+\infty$, where $a_n \in \mathbb{F}$ for each a_n. Let $h_n(x) = \sum_{j=0}^{n} a_j x^j$ be the partial sum. Let $f \in C_c(\mathbb{R}, V)$. Then $\lim_{n\to\infty} f * h_n$ converges uniformly on compact intervals to $f * h$, and each $f * h_n$ is in*

$$V[x] = \{v_0 + v_1 x + \cdots + v_k x^k : k \in \mathbb{N}, v_0, v_1, \ldots, v_k \in V\} \tag{14.31}$$

Note that we do not need the assumption $\int_{\mathbb{R}} |h| < +\infty$ in this Lemma. So letting $h(x) = e^x$ is also OK. Clearly, if g is the Gauss functions, then g_ε satisfies the assumptions on h. This implies that if $f \in C_c(\mathbb{R}, V)$, then $f * g_\varepsilon$ can be approximated uniformly on compact intervals by polynomials with coefficients in V. Combining this fact with Prop. 14.40, we see that f can be approximated uniformly by polynomials on compact intervals.

Proof. Note that $(f * h)(x) = \int_{\mathbb{R}} h(x-y) f(y) dy$ is well-defined since it is equals $\int_{-a}^{a} h(x-y) f(y) dy$ if $a > 0$ and if $[-a, a]$ contains $\mathrm{Supp}(f)$. (So this is a usual Riemann integral.) Similarly, $(f * h_n)(x) = \int_{-a}^{a} h_n(x-y) f(y) dy$. Since h_n is a polynomial, it is obvious that $f * h_n$ is also a polynomial.

We now show that $f * h_n$ converges uniformly to $f * h$ on $[-b, b]$ for any $b > 0$. Let $c = a + b$. Then

$$\sup_{x\in[-b,b]} \|(f * h)(x) - (f * h_n)(x)\| \leqslant \sup_{x\in[-b,b]} \int_{-a}^{a} |h(x-y) - h_n(x-y)| \cdot \|f(y)\| dy$$

$$\leqslant A_n \int_{-a}^{a} \|f(y)\| dy$$

where $A_n = \sup_{t \in [-c,c]} \|h(t) - h_n(t)\|$. By Thm. 4.27, h_n converges uniformly on $[-c, c]$ to h. So $\lim_{n \to \infty} A_n = 0$. □

Theorem 14.45 (Weierstrass approximation theorem). *Let $I = [a, b]$ where $-\infty < a < b < +\infty$. Then $V[x]$ is dense in $C(I, V)$ under the l^∞-norm.*

Proof. Choose any $f \in C(I, V)$. Then f can be extended to a continuous function $\mathbb{R} \to V$ with compact support: For example, we let $f(x) = (x - a + 1)f(a)$ if $x \in [a-1, a]$, let $f(x) = (b+1-x)f(b)$ if $b \in [b, b+1]$, and let $f(x) = 0$ if $x < a-1$ or $x > b+1$. Let g be the Gauss function. For every $e > 0$, by Prop. 14.40, there exists $\varepsilon > 0$ such that $\|f - f * g_\varepsilon\|_{l^\infty(\mathbb{R},V)} < e/2$. By Lem. 14.44, there exists a polynomial $p \in V[x]$ such that $\|f * g_\varepsilon - p\|_{l^\infty(I,V)} < e/2$. So $\|f - p\|_{l^\infty(I,V)} < e$. □

Corollary 14.46. *Let $I = [a, b]$ where $-\infty < a < b < +\infty$. Then $C([0, 1], \mathbb{R})$ is separable.*

Proof. $\mathbb{Q}[x]$, the set of polynomials with coefficients in $\mathbb{Q}$, is countable. By Thm. 14.45, $\mathbb{Q}[x]$ is dense in $C([0, 1], \mathbb{R})$. □

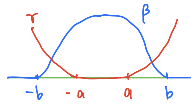
The Weierstrass approximation theorem will be generalized to **Stone-Weierstrass theorem** (cf. Thm. 15.9). Accordingly, Cor. 14.46 will be substantially generalized as an application of (the proof of) Stone-Weierstrass theorem: we will show that $C(X, \mathbb{R})$ is separable if X is a compact metric space. (See Thm. 15.37)

14.5.3 Smooth approximation

By the Weierstrass approximation theorem, we know that any $f \in C_c(\mathbb{R}, V)$ can be approximated uniformly by polynomials on compact intervals. However, unless $f = 0$, f cannot be approximated by polynomials uniformly on $\mathbb{R}$. (If $p, q \in V[x]$ are different, then $\|p - q\|_{l^\infty(X,V)} = +\infty$. So any Cauchy sequence in $V[x]$ under the $l^\infty(X, V)$-norm is eventually constant. So its limit is a polynomial, which does not have compact support unless when it is zero.) Nevertheless, we shall show that f can be approximated by smooth compactly supported functions uniformly on $\mathbb{R}$. This task is not difficult: one simply pick a nonzero $g \in C_c^\infty(\mathbb{R}, \mathbb{R})$ satisfying $g \geqslant 0$. Dividing g by $g / \int_{\mathbb{R}} g$, we may assume that $\int_{\mathbb{R}} g = 1$. Then it can be shown that $f * g_\varepsilon \in C_c^\infty(\mathbb{R}, V)$. By Prop. 14.40, $\lim_{\varepsilon \to 0} f * g_\varepsilon$ converges uniformly on $\mathbb{R}$ to f. This finishes the proof. However, we must first prove the existence of such g:

Proposition 14.47. *Let $0 < a < b$. Then there exists $g \in C_c^\infty(\mathbb{R}, \mathbb{R})$ such that $g(\mathbb{R}) = [0, 1]$, that $g^{-1}(1) = [-a, a]$, and that $g^{-1}(0) = (-\infty, -b] \cup [b, +\infty)$.*

Proof. Let $\alpha : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ be the smooth function in Exp. 12.19. Then α is increasing and $\alpha^{-1}(0) = (-\infty, 0]$. Define $\beta : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ by $\beta(x) = \alpha(x+b)\alpha(-x+b)$. Then $\beta^{-1}(0) = (-\infty, -b] \cup [b, +\infty)$. In particular, $\beta \in C_c^\infty(\mathbb{R}, \mathbb{R})$. Define $\gamma : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ by $\gamma(x) = \alpha(x-a) + \alpha(-x-a)$. Then $\gamma^{-1}(0) = [-a, a]$. Note that $\beta(x) + \gamma(x) > 0$ for all $x \in \mathbb{R}$, since $\beta^{-1}(0) \cap \gamma^{-1}(0) = \varnothing$.



Then $g = \beta/(\beta + \gamma)$ is a desired function. $\square$

Corollary 14.48. *Let I be an interval in $\mathbb{R}$ with $a = \inf I$ and $b = \sup I$ satisfying $-\infty \leqslant a < b \leqslant +\infty$. Then $C_c^\infty(I, V)$ is dense in $C_c(I, V)$ under the l^∞-norm.*

In the following, we only prove the case that $I = (a, b)$. The other cases can be reduced to this case: For example, the case $[a, b)$ is implied by the case $(a-1, b)$.

First proof. Let $f \in C_c(\mathbb{R}, V)$ supported in $[u, v]$ where $a < u < v < b$. By Prop. 14.47, there exists a nonzero smooth $g \in C_c(\mathbb{R}, \mathbb{R}_{\geqslant 0})$ supported in $[-1, 1]$ such that $\int_{\mathbb{R}} g = 1$. Since g_ε is supported in $[-\varepsilon, \varepsilon]$, it is easy to see that $f * g_\varepsilon$ is supported in $[u-\varepsilon, v+\varepsilon]$. By Lem. 14.2, $f * g_\varepsilon$ is continuous . Moreover, one can check that $f * g_\varepsilon$ is smooth (cf. Pb. 14.4). So $f * g_\varepsilon \in C_c^\infty(\mathbb{R}, V)$, and $f * g$ is supported in I when $\varepsilon < \min\{u-a, b-v\}$. By Prop. 14.40, $\lim_{\varepsilon \to 0} f * g_\varepsilon$ converges uniformly on $\mathbb{R}$ to f. $\square$

Second proof. Let $f \in C_c(\mathbb{R}, V)$ supported in $[u, v]$ where $a < u < v < b$. Choose any $e > 0$. Choose any positive $\varepsilon < \min\{u-a, b-v\}$. By Weierstrass approximation Thm. 14.45, there exists $p \in V[x]$ such that $\|f(x) - p(x)\| \leqslant e$ for all $x \in [u-\varepsilon, v+\varepsilon]$. In particular, since $f = 0$ on $J = [u-\varepsilon, u] \cup [v, v+\varepsilon]$, we have $\|p(x)\| \leqslant e$ for all $x \in J$.

By Prop. 14.47, there exists $h \in C_c^\infty(\mathbb{R}, \mathbb{R})$ such that $h(\mathbb{R}) = [0, 1]$, that $h^{-1}(1) = [u, v]$, and that $h^{-1}(0) = (-\infty, u-\varepsilon] \cup [v+\varepsilon, +\infty)$. Then $hp \in C_c^\infty(\mathbb{R}, V)$ is supported in $[u-\varepsilon, v+\varepsilon]$ and hence in $(-a, a)$. One checks easily that $\|f(x) - h(x)p(x)\| \leqslant e$ for all $x \in (a, b)$. $\square$

Remark 14.49. It should be noted that if I is an open interval, then $C_c^\infty(I, V)$ and $C_c(I, V)$ are naturally subspaces of $C_c(\mathbb{R}, V)$. This is not true when I is a closed or an half-open-half-closed interval. Therefore, Cor. 14.48 implies that any $f \in C(\mathbb{R}, V)$ supported in an open interval I can be uniformly approximated by smooth functions $\mathbb{R} \to V$ supported in I.

Remark 14.50. The above observation can be generalized: Let X be an LCH space. Assume that Ω is a nonempty *open* subset of X. Recall that Ω is LCH by Prop. 8.41. Then an element of $C_c(\Omega, V)$ is equivalently an element of $C_c(X, V)$ supported in U. The former gives the latter by "extension by zero"; the latter gives the former by restriction to Ω. We will say more about this in Sec. 15.4.

14.6 L^1-approximation; Riemann-Lebesgue lemma

A non-continuous integrable function cannot be approximated uniformly by smooth functions. However, it can be approximated by the latter under the L^1-norm. To avoid distraction, in this section we consider $\mathbb{C}$-valued functions. Fix $I \subset \mathbb{R}$ to be an interval containing at least two points. Then on $\mathscr{R}^1(I, \mathbb{C})$ one can define the L^1**-seminorm** to be

$$\|f\|_{L^1} \equiv \|f\|_{L^1(I,\mathbb{C})} = \int_I |f| \tag{14.32}$$

It is easy to check that this is a **seminorm**, i.e., it satisfies the definition of a norm, except the assumption that $\|f\|_{L^1} = 0$ implies $f = 0$.

Proposition 14.51. *Let $f \in \mathscr{R}^1(I, \mathbb{C})$. Choose any $\varepsilon > 0$. Then there exists $g \in C_c^\infty(\mathbb{R}, \mathbb{C})$ supported in I such that $\|f - g\|_{L^1} < \varepsilon$. And there exists a* ***step function*** *h supported in I (i.e., a linear combination of functions of the form χ_E where E is a compact interval in I) such that $\|f - h\|_{L^1} < \varepsilon$.*

Note that if E is a bounded interval, then χ_E is clearly a linear combination of characteristic functions over compact intervals (including the single point sets). Thus, in the above definition of step functions, one can just assume that E is a bounded interval whose closure is in I.

The following proof shows that Prop. 14.51 can be easily generalized to the case that $f : I \to V$ is strongly integrable on compact subintervals of I and $\int_I |f| < +\infty$. (V is a Banach space.)

Proof. Let $a = \inf I, b = \sup I$. Since $\int_I |f| = \lim_{u \to a, v \to b} \int_a^b |f|$, there exist u, v such that $a < u < v < b$ and that $\int_a^u |f| + \int_v^b |f| < \varepsilon/2$. We claim that there exists a step function h supported in $J = [u, v]$ such that $\int_J |f - h| < \varepsilon/2$. Then $\int_I |f - h| < \varepsilon$, finishing the proof that f can be L^1-approximated by step functions supported in I.

Since $f|_J$ is strongly Riemann integrable, there exists $\sigma = \{a_0 < \cdots < a_n\} \in \mathcal{P}(J)$ such that $\omega(f, \sigma) < \frac{\varepsilon}{2}$. Let $J_i = [a_{i-1}, a_i]$. Pick any $\lambda_i \in f(J_i)$. Then $|f - \lambda_i \cdot \chi_{J_i}| \leqslant \operatorname{diam}(f(J_i))$ on J_i, and hence $\int_{J_i} |f - \lambda_i \cdot \chi_{J_i}| \leqslant \operatorname{diam}(f(J_i)) \cdot |J_i|$. Let $h = \sum_{i=1}^n \lambda_i \cdot \chi_{J_i}$. It follows that

$$\Big| \int_J f - \int_J h \Big| \leqslant \sum_{i=1}^n \int_{J_i} |f - \lambda_i \cdot \chi_{J_i}| \leqslant \sum_{i=1}^n \operatorname{diam}(f(J_i)) \cdot |J_i| = \omega(f, \sigma) < \frac{\varepsilon}{2}$$

finishing the proof.

Finally, we show that f can be L^1-approximated by smooth functions supported in I. It suffices to prove that any step function supported in I is so. By linearity and triangle inequality, it suffices to prove that χ_E is so, if E is a nonempty compact interval in I. For each $\varepsilon > 0$, it is easy to construct a piecewise linear (and hence continuous) function $g_0 : \mathbb{R} \to \mathbb{R}$ supported in $(a, b) = \mathrm{Int}(I)$ such that $\int_I |\chi_E - g_0| < \varepsilon$. Choose a bounded open interval $(c, d) \subset (a, b)$ containing $\mathrm{Supp}(g_0)$. By Cor. 14.48, there exists $g \in C_c^\infty(\mathbb{R}, \mathbb{R})$ supported in (c, d) such that $|g(x) - g_0(x)| < \varepsilon/(d - c)$ for all $x \in (c, d)$. It follows that

$$\int_I |g - g_0| = \int_c^d |g - g_0| \leqslant \varepsilon$$

Thus $\int_I |\chi_e - g| < 2\varepsilon$, finishing the proof. □

We give an application of Prop. 14.51 by proving Riemann-Lebesgue lemma. The RL lemma plays a fundamental role in Fourier analysis, see for instance Pb. 14.6 and Cor. 14.56.

Theorem 14.52 (Riemann-Lebesgue lemma). *Let $f \in \mathscr{R}^1(\mathbb{R}, \mathbb{C})$. Then*

$$\lim_{t\to+\infty} \int_{\mathbb{R}} f(x)e^{\mathbf{i}tx}dx = \lim_{t\to-\infty} \int_{\mathbb{R}} f(x)e^{\mathbf{i}tx}dx = 0$$

The idea of the proof is the following: We L^1-approximate f by some compactly supported step function $g \in C_c(\mathbb{R}, \mathbb{C})$. Then it suffices to prove the RL lemma for g. By linearity, it suffices to assume that $g = \chi_{[a,b]}$. This special case can be proved easily.

Proof. By Prop. 14.51, for every $\varepsilon > 0$ there exists a compactly supported step function $g : \mathbb{R} \to \mathbb{C}$ such that $\int_{\mathbb{R}} |f - g| < \varepsilon$. So $|\int_{\mathbb{R}} f(x)e^{\mathbf{i}tx} - g(x)e^{\mathbf{i}tx}dx| < \varepsilon$ by Prop. 14.27. Thus

$$\limsup_{t\to\pm\infty} \Big|\int_{\mathbb{R}} f(x)e^{\mathbf{i}tx}dx\Big| \leqslant \limsup_{t\to\pm\infty} \Big|\int_{\mathbb{R}} g(x)e^{\mathbf{i}tx}dx\Big| + \varepsilon$$

Since ε is arbitrary, it suffices to prove that $\lim_{t\to\pm\infty} \int_{\mathbb{R}} g(x)e^{\mathbf{i}tx}dx = 0$. Since g is a linear combination of functions of the form $\chi_{[a,b]}$ where $-\infty < a < b < +\infty$, it suffices to prove the RL lemma for this function:

$$\int_{\mathbb{R}} \chi_{[a,b]} \cdot e^{\mathbf{i}tx}dx = \int_a^b e^{\mathbf{i}tx}dx = \frac{e^{\mathbf{i}nb} - e^{\mathbf{i}na}}{\mathbf{i}t}$$

converges to 0 as $t \to \pm\infty$. □

14.7 Problems and supplementary material

Let V be a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Problem 14.1. Solve Exe. 14.8.

Problem 14.2. Prove Thm. 14.34 using Thm. 14.35.

⋆ **Problem 14.3.** Define $f(x) = \Big(\int_0^x e^{-t^2}dt\Big)^2$ and $g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1}dt$

1. Show that $f' + g' = 0$, and conclude that $f + g = \frac{\pi}{4}$.
2. Use part 1 to prove

$$\int_{-\infty}^{+\infty} e^{-t^2}dt = \sqrt{\pi} \tag{14.33}$$

Problem 14.4. Let $f : \mathbb{R} \to V$ be strongly integrable on compact intervals, and assume $\int_{\mathbb{R}} |f| < +\infty$. Let $g \in C^1(\mathbb{R}, \mathbb{R})$. Assume that $\|g\|_\infty, \|g'\|_\infty < +\infty$. (This is automatic when g has compact support.) Use Thm. 14.35 to prove that $f * g$ is well-defined, that $f * g \in C^1(\mathbb{R}, \mathbb{R})$, and that $(f * g)' = f * g'$.

Note. Since f is not assumed to be continuous, you cannot use Thm. 14.34 *directly* to compute $(f * g)'$ and to show $f * g \in C^1$. You have two options: (1) Use Thm. 14.35. (2) Use Prop. 14.51 to L^1-approximate f by a continuous compactly-supported function. Then apply Thm. 14.34 (together with Cor. 14.36) to that continuous function. Whichever method you use, I suggest you think about how you can use the other method to solve the problem. □

Remark 14.53. In particular, if $g \in C_c^\infty(\mathbb{R}, \mathbb{R})$, then the above problem shows that $f * g \in C^\infty(\mathbb{R}, \mathbb{R})$ and $(f * g)^{(n)} = f * g^{(n)}$ for all $n \in \mathbb{N}$.

⋆ **Problem 14.5.** The translation of each $f \in \mathscr{R}^1(\mathbb{R}, \mathbb{C})$ by $t \in \mathbb{R}$ is defined to be

$$f_t : \mathscr{R}^1(\mathbb{R}, \mathbb{C}) \to \mathscr{R}^1(\mathbb{R}, \mathbb{C}) \qquad f_t(x) = f(x - t)$$

Prove that the translation map is continuous under the L^1-seminorm, in the sense that $\lim_{t\to 0} \|f - f_t\|_{L^1} = 0$. Namely, prove that

$$\lim_{t\to 0} \int_{\mathbb{R}} |f - f_t| = 0$$

Hint. Use Prop. 14.51 to L^1-approximate f by compactly supported continuous functions or step functions. (Both types of functions will work.) □

Definition 14.54. Let $f \in \mathscr{R}([-\pi, \pi], \mathbb{C})$. For each $n \in \mathbb{Z}$, define the n-th **Fourier coefficient** to be

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\mathbf{i}nx}dx$$

We call $\sum_{n=-\infty}^{+\infty} \widehat{f}(n)e^{\mathbf{i}nx}$ the **Fourier series** of f.

Definition 14.55. For each $N \in \mathbb{N}$, define the **Dirichlet kernel**

$$D_N(x) = \sum_{n=-N}^{N} e^{\mathbf{i}nx} = \frac{\sin(N+\frac{1}{2})x}{\sin(x/2)}$$

(When $x \in 2\pi\mathbb{Z}$, the RHS above should be $2N+1$.) From $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathbf{i}nx}dx = \delta_{n,0}$ we easily see $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} D_N(x)dx = 1$.

Problem 14.6. Let $f \in \mathscr{R}(\mathbb{R},\mathbb{C})$ have period 2π. Define $s_N(f;x) = \displaystyle\sum_{n=-N}^{N} \widehat{f}(n)e^{\mathbf{i}nx}$.

1. Prove that $s_N(f;x) = (f * D_N)(x)$ where the convolution is defined with respect to the integral $\frac{1}{2\pi}\int_I$ where I is any interval of length 2π. (Note that both f and D_N have period 2π. So translating I does not affect the result.) In other words, prove that

$$s_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_N(x-t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)dt \tag{14.34}$$

2. Fix $x \in \mathbb{R}$. Assume that there exist $A, B \in \mathbb{C}$ such that

$$\limsup_{t\to 0^+}\left|\frac{f(x-t)-A}{t}\right| < +\infty \qquad \limsup_{t\to 0^+}\left|\frac{f(x+t)-B}{t}\right| < +\infty \tag{14.35}$$

(In other words, assume that there exist $\delta, M > 0$ such that $|(f(x-t)-A)/t| \leqslant M$ and $|(f(x+t)-B)/t| \leqslant M$ for all $0 < t < \delta$.) Prove that

$$\lim_{N\to+\infty} s_N(f;x) = \frac{A+B}{2} \tag{14.36}$$

Hint for part 2. Choose $g : [-\pi,\pi] \to \mathbb{C}$ such that $g(t) = B$ if $t < 0$, and $g(t) = A$ if $t > 0$. Prove that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)D_N(t)dt = \frac{A+B}{2} \tag{14.37}$$

Let $\varphi : [-\pi,\pi] \to \mathbb{R}$ such that $\varphi(t) = \dfrac{f(x-t)-g(t)}{\sin(t/2)}$ if $t \neq 0$. Use Lebesgue's criterion to show that $\varphi \in \mathscr{R}([-\pi,\pi],\mathbb{C})$. Use Riemann-Lebesgue lemma to prove $\displaystyle\lim_{N\to+\infty}\int_{-\pi}^{\pi}\varphi(t)\sin\big(N+\frac{1}{2}\big)t\cdot dt = 0$. □

Corollary 14.56. *Let $g \in C^1([a-\pi,a+\pi],\mathbb{C})$. Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function such that $f(x) = g(x)$ if $a-\pi < x < a+\pi$, and that $f(a+\pi) = \dfrac{g(a-\pi)+g(a+\pi)}{2}$. Clearly $f \in \mathscr{R}(\mathbb{R},\mathbb{C})$. Then the Fourier series of f converges pointwise to f.*

Proof. Immediate from part 2 of Pb. 14.6. □

Example 14.57. If we let $a = 0$ and $g(x) = x$, then one can compute that the Fourier series of f is (13.10). It converges pointwise but not uniformly to f.

15 A topological proof of the Stone-Weierstrass theorem

15.1 *-algebras and subalgebras

Fix $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 15.1. An **$\mathbb{F}$-algebra** is defined to be a ring $\mathscr{A}$ (not necessarily having 1) which is at the same time also an $\mathbb{F}$-vector space (where the vector addition is equal to the ring addition) such that the ring multiplication and the scalar multiplication satisfy the associativity: For every $\lambda \in \mathbb{F}$ and $x, y \in \mathscr{A}$, we have

$$\lambda(xy) = (\lambda x)y = x(\lambda y) \tag{15.1}$$

An $\mathbb{F}$-algebra is called **unital** if $\mathscr{A}$, as a ring, has the identity 1. In this case, we write $\lambda \cdot 1$ as λ if $\lambda \in \mathbb{F}$.

An $\mathbb{F}$-algebra is called **commutative** or **abelian** if $xy = yx$ for all $x, y \in \mathscr{A}$.

If $\mathscr{A}$ is an $\mathbb{F}$-algebra, then an **($\mathbb{F}$-)subalgebra** is a subset $\mathscr{B}$ which is invariant under the ring addition, ring multiplication, and scalar multiplication. (Namely, $\mathscr{B}$ is a subring and also a subspace of $\mathscr{A}$.) If $\mathscr{A}$ is unital, then a **unital ($\mathbb{F}$-)subalgebra** of $\mathscr{A}$ is an $\mathbb{F}$-subalgebra containing the identity of $\mathscr{A}$. □

Remark 15.2. A unital $\mathbb{F}$-algebra $\mathscr{A}$ is equivalently a ring with unit 1, together with a ring homomorphism $\mathbb{C} \to Z(\mathscr{A})$ where $Z(\mathscr{A})$ is the **center** of $\mathscr{A}$, i.e.

$$Z(\mathscr{A}) = \{x \in \mathscr{A} : xy = yx \text{ for every } y \in \mathscr{A}\}$$

We leave it to the readers to check the equivalence.

Example 15.3. If V is a $\mathbb{F}$-vector space, then $\mathrm{End}(V)$, the set of $\mathbb{F}$ linear maps $V \to V$, is naturally an $\mathbb{F}$-algebra. If V is a normed vector space, then $\mathfrak{L}(V)$ is an $\mathbb{F}$-algebra.

Definition 15.4. A **(complex) *-algebra** is defined to be a $\mathbb{C}$-algebra together with an **antilinear map** $* : \mathscr{A} \to \mathscr{A}$ sending x to x^* (where "antilinear" means that for every $a, b \in \mathbb{C}$ and $x, y \in \mathscr{A}$ we have $(ax + by)^* = \overline{a}x^* + \overline{b}y^*$) such that for every $x, y \in \mathscr{A}$, we have

$$(x^*)^* = x \qquad (xy)^* = y^*x^*$$

Note that $*$ must be bijective. We call $*$ an **involution**. A ***-subalgebra** $\mathscr{B}$ is defined to be a subalgebra satisfying $x \in \mathscr{B}$ iff $x^* \in \mathscr{B}$. If $\mathscr{A}$ is a unital algebra with unit $\mathbf{1}$, we say that $\mathscr{A}$ is a **unital *-algebra** if $\mathscr{A}$ is equipped with an involution $* : \mathscr{A} \to \mathscr{A}$ such that $\mathscr{A}$ is a *-algebra, and that

$$\mathbf{1}^* = \mathbf{1}$$

A unital *-subalgebra is a unital subalegbra and also a *-subalgebra.

Example 15.5. The set of complex $n \times n$ matrices $\mathbb{C}^{n\times n}$ is naturally a unital $*$-algebra if for every $A \in \mathbb{C}^{n\times n}$ we define $A^* = \overline{A}^{\mathrm{t}}$, the complex conjugate of the transpose of A.

In this chapter, we are mainly interested in abelian algebras.

Example 15.6. Let X be a set. Then $\mathbb{F}^X$ is naturally a unital $\mathbb{F}$-algebra, and $l^\infty(X,\mathbb{C})$ is its unital $\mathbb{F}$-subalgebra. If X is a topological space, then $C(X,\mathbb{F})$ is a unital $\mathbb{F}$-subalgebra of $\mathbb{F}^X$. If X is compact, then $C(X,\mathbb{F})$ is a unital $\mathbb{F}$-subalgebra of $l^\infty(X,\mathbb{F})$.

The following is our main example of this chapter.

Example 15.7. Let X be a set. Then $\mathbb{C}^X$ is a unital *-algebra if for every $f \in \mathbb{C}^X$ we define

$$f^* : X \to \mathbb{C} \qquad f^*(x) = \overline{f(x)} \tag{15.2}$$

Then $l^\infty(X,\mathbb{C})$ is a unital *-subalgebra of $\mathbb{C}^X$. Assume that X is a compact topological space. Then $C(X,\mathbb{C})$ is a unital *-subalgebra of $l^\infty(X,\mathbb{C})$. If $f_1,\dots,f_n \in C(X,\mathbb{C})$, then $\mathbb{C}[f_1,\dots,f_n]$, the set of polynomials of $f_1,\dots,f_n$ with coefficients in $\mathbb{C}$, is a unital subalgebra of $C(X,\mathbb{C})$. And $\mathbb{C}[f_1,f_1^*,\dots,f_n,f_n^*]$ is a unital *-subalgebra of $C(X,\mathbb{C})$.

More generally, we have:

Example 15.8. Let $\mathscr{A}$ be an abelian unital $\mathbb{F}$-algebra. Let $\mathfrak{S} \subset \mathscr{A}$. Then

$$\mathbb{F}[\mathfrak{S}] = \mathrm{Span}_{\mathbb{F}}\{x_1^{n_1}\cdots x_k^{n_k} : k \in \mathbb{Z}_+, x_i \in \mathfrak{S}, n_i \in \mathbb{N}\} \tag{15.3}$$

the set of polynomials of elements in $\mathfrak{S}$, is the smallest unital $\mathbb{F}$-subalgebra containing $\mathfrak{S}$, called the **unital $\mathbb{F}$-subalgebra generated by** $\mathfrak{S}$. (Here, we understand $x^0 = 1$ if $x \in \mathscr{A}$.) Thus, if $\mathscr{A}$ is an abelian unital *-algebra, then $\mathbb{C}[\mathfrak{S} \cup \mathfrak{S}^*]$ (where $\mathfrak{S}^* = \{x^* : x \in \mathfrak{S}\}$) is the smallest unital *-algebra containing $\mathfrak{S}$, called the **unital *-subalgebra generated by** $\mathfrak{S}$.

15.2 The Stone-Weierstrass (SW) theorems

The main goal of this chapter is to prove the following theorem.

Theorem 15.9 (SW theorem, compact real version). *Let X be a compact Hausdorff space. Let $\mathscr{A}$ be unital subalgebra of $C(X,\mathbb{R})$ separating points of X. Then $\mathscr{A}$ is dense in $C(X,\mathbb{R})$ (under the l^∞-norm).*

Example 15.10. Let $X = [a,b]$ be a compact interval in $\mathbb{R}$. Then $\mathbb{R}[x]$ is a unital subalgebra of $C(X,\mathbb{R})$. It separates points of $[a,b]$ because it contains id. Thus, by SW theorem, $\mathbb{R}[a,b]$ is dense in $C([a,b],\mathbb{R})$. This special case was proved in Thm. 14.45.

More generally, let $X = I_1 \times \cdots \times I_N$ where each $I_i \subset \mathbb{R}$ is a compact interval. Let $\pi_i : X \to I_i$ be the projection onto the i-th component, i.e., the i-th coordinate function. Then $\pi_1, \dots, \pi_N$ separate points of X, since $\pi_1 \times \cdots \times \pi_N$ is the identity map of X. Thus, by SW theorem, $\mathbb{R}[\pi_1, \dots, \pi_N]$ is dense in $C(X,\mathbb{R})$. In fact, we will first prove this special case before we prove the general SW theorem. $\square$

From the above version of SW theorem it is easy to prove:

Theorem 15.11 (SW theorem, compact complex version). *Let X be a compact Hausdorff space. Let $\mathscr{A}$ be a unital *-subalgebra of $C(X,\mathbb{C})$ separating points of X. Then $\mathscr{A}$ is dense in $C(X,\mathbb{R})$ (under the l^∞-norm).*

Proof. Let $\mathrm{Re}\mathscr{A} = \{\mathrm{Re}f = (f + f^*)/2 : f \in \mathscr{A}\}$. Since for each $f \in \mathscr{A}$ we have $f^* \in \mathscr{A}$, we know that $\mathrm{Re}f \in \mathscr{A}$. This proves that $\mathrm{Re}\mathscr{A} \subset \mathscr{A}$. Since $f = \mathrm{Re}(f) - \mathbf{i}\mathrm{Re}(\mathbf{i}f)$, we conclude

$$\mathscr{A} = \mathrm{Re}\mathscr{A} + \mathbf{i}\mathrm{Re}\mathscr{A} \tag{15.4}$$

In other words, elements of $\mathscr{A}$ are precisely of the form $\alpha + \mathbf{i}\beta$ where $\alpha, \beta \in \mathrm{Re}\mathscr{A}$. From this, it is clear that $\mathrm{Re}\mathscr{A}$ is a unital subalgebra of $C(X,\mathbb{R})$ separating points of X. Thus, by Thm. 15.9, $\mathrm{Re}\mathscr{A}$ is dense in $C(X,\mathbb{R})$. It is clear from (15.4) that $\mathscr{A}$ is dense in $C(X,\mathbb{C})$. $\square$

Example 15.12. Let $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. For each $n \in \mathbb{Z}$, let $e_n : \mathbb{S}^1 \to \mathbb{C}$ be defined by $e_n(e^{\mathbf{i}x}) = e^{\mathbf{i}nx}$. In other words, $e_n(z) = z^n$. Then $\mathbb{C}[e_1, e_{-1}] = \mathrm{Span}_{\mathbb{C}}\{e_n : n \in \mathbb{Z}\}$ is a unital *-subalgebra of $C(\mathbb{S}^1,\mathbb{C})$. It separates points of $\mathbb{S}^1$ since e_1 does. Therefore, by SW theorem, $\mathbb{C}[e_1, e_{-1}]$ is dense in $\mathbb{C}(\mathbb{S}^1,\mathbb{C})$.

One often views

$$C(\mathbb{S}^1,\mathbb{C}) = \{g \in C([-\pi,\pi],\mathbb{C}) : g(-\pi) = g(\pi)\} \tag{15.5}$$

since any g in the RHS corrsponds bijectively to f in the LHS by setting $f(e^{\mathbf{i}t}) = g(t)$. Therefore, the above conclusion is that *any $g \in C([-\pi,\pi],\mathbb{C})$ satisfying $g(-\pi) = g(\pi)$ can be approximated uniformly by functions of the form $\sum_{n=-N}^N a_n \cdot e^{\mathbf{i}nx}$ where $a_n \in \mathbb{C}$.* This property is fundamental to the theory of Fourier series. $\square$

Example 15.13. We continue the above discussion. $\mathbb{C}[e_1] = \mathrm{Span}_{\mathbb{C}}\{e_n : n \in \mathbb{N}\}$ is a unital subalgebra (though not a *-subalgebra) of $C(\mathbb{S}^1,\mathbb{C})$ separating points of $\mathbb{S}^1$. Let us prove that it is not dense in $C(\mathbb{S}^1,\mathbb{C})$.

Proof. We view functions on $\mathbb{S}^1$ as those on $[-\pi,\pi]$ having the same values at π and at $-\pi$. We claim that e_{-1} is not in the closure of $\mathbb{C}[e_1]$. Indeed, it can be checked that $\int_{-\pi}^{\pi} e_{-1}e_1 = 2\pi$, and that $\int_{-\pi}^{\pi} p \cdot e_1 = 0$ for every $p \in \mathbb{C}[e_1]$ (since this is true when $p = e_n$ and $n \geqslant 0$). If e_{-1} is in the closure of $\mathbb{C}[e_1]$, then there is a sequence $(p_k)_{k\in\mathbb{Z}_+}$ in $\mathbb{C}[e_1]$ converging uniformly on $[-\pi,\pi]$ to e_{-1}. By Cor. 13.21, $0 = \int_{-\pi}^{\pi} p_k e_1$ converges to $\int_{-\pi}^{\pi} e_{-1}e_1 = 2\pi$ as $k \to \infty$, impossible. □

Example 15.14. Let X, Y be compact metric spaces spaces. Let $\mathscr{A}$ be the subspace of $C(X \times Y, \mathbb{R})$ spanned by elements of the form fg where $f \in C(X,\mathbb{R})$ and $g \in C(Y,\mathbb{R})$. (More precisely, one should understand fg as $(f \circ \pi_X) \cdot (g \circ \pi_Y)$ where π_X, π_Y are the projections of $X \times Y$ onto X and Y respectively.) By the Urysohn functions (cf. Rem. 7.118), we see that $C(X,\mathbb{R})$ resp. $C(Y,\mathbb{R})$ separates points of X resp. Y. (This is in fact also true when X, Y are compact Hausdorff spaces.) Thus $\mathscr{A}$ is a unital subalgebra of $C(X \times Y, \mathbb{R})$ separating points of $X \times Y$. By SW theorem, $\mathscr{A}$ is dense in $C(X \times Y, \mathbb{R})$. Nevertheless, we will prove SW theorem by first proving this special case: see Cor. 15.33.[1]

15.3 Proof of SW, I: polynomial approximation on $[0,1]^{\mathscr{I}}$

Starting from this section, we begin our proof of SW Thm. 15.9. We first explain our strategy of the proof. Let X be a compact Hausdorff space.

We first consider the special case that X is metrizable. Then by Thm. 8.45, X is homeomorphic to a closed (and hence compact) subset of $[0,1]^{\mathbb{Z}_+}$. Therefore, we may assume that X is a closed subset of $[0,1]^{\mathbb{Z}_+}$. As we will see, any continuous real-valued function on X can be extended to a continuous function on $[0,1]^{\mathbb{Z}_+}$. (This is due to **Tietze extension theorem**, which will be proved in the next section.) Thus, it suffices to prove SW theorem for $[0,1]^{\mathbb{Z}_+}$. In fact, as we will see, it suffices to prove that any element of $C([0,1]^{\mathbb{Z}_+}, \mathbb{R})$ can approximated uniformly by polynomials, i.e., by elements of $\mathbb{R}[\pi_1, \pi_2, \dots]$ where $\pi_n : [0,1]^{\mathbb{Z}_+} \to [0,1]$ is the projection onto the n-th component. This task will be achieved in this section.

The above method can be easily genearlized to the case that X is not necessarily metrizable. In fact, since $\mathscr{A} \subset C(X,\mathbb{R})$ separates points of X, as in the proof of Thm. 8.45, $\mathscr{A}$ enables us to embed X onto a compact subset of $[0,1]^{\mathscr{I}}$ where $\mathscr{I}$ is an index set. Therefore, we need to show that $[0,1]^{\mathscr{I}}$ is compact. This is indeed due to:

Theorem 15.15 (Tychonoff theorem). *Let $(X_\alpha)_{\alpha\in\mathscr{I}}$ be a family of compact topological spaces. Then the product space $S = \prod_{\alpha\in\mathscr{I}} X_\alpha$ (equipped with the product topology) is compact.*

[1]More precisely, we don't need Exp. 15.14 to prove SW. But our method of proving SW immediately implies Exp. 15.14. To put it differently, we don't need the full power of the SW theorem to prove Exp. 15.14.

Note that S is obviously Hausdorff if each X_α is Hausdorff. We will prove Tychonoff theorem in Sec. 16.3 using Zorn's lemma. Assuming Tychonoff theorem, we will show that elements of $C([0,1]^{\mathscr{I}},\mathbb{R})$ can be approximated uniformly by polynomials. This is the goal of this section:

Proposition 15.16. *Let $(I_\alpha)_{\alpha\in\mathscr{I}}$ be a family of nonempty compact intervals in $\mathbb{R}$. Let $S=\prod_{\alpha\in\mathscr{I}} I_\alpha$. For each α, define the* ***coordinate function***

$$\pi_\alpha : S \to I_\alpha \qquad x_\bullet = (x_\mu)_{\mu\in\mathscr{I}} \mapsto x_\alpha$$

Then $\mathbb{R}[\{\pi_\alpha : \alpha\in\mathscr{I}\}]$, the unital subalgebra of $C(S,\mathbb{R})$ generated by all coordinate functions, is dense in $C(S,\mathbb{R})$.

Since the coordinate functions separate points of S, Prop. 15.16 is clearly a special case of SW theorem.

Lemma 15.17. *Prop. 15.16 holds when $\mathscr{I}$ is a finite set.*

Proof. We prove by induction on N that elements of $C(S,\mathbb{R})$, where $S=I_1\times\cdots\times I_N$, can be approximated uniformly by polynomials (i.e. by elements of $\mathbb{R}[\pi_1,\dots,\pi_N]$). The case $N=1$ follows from Weierstrass approximation Thm. 14.45. Assume that case $N-1$ has been proved. Let use prove case N where $N>1$.

Write $S=I_1\times Y$ where $Y=I_2\times\cdots\times I_N$. Then by Thm. 9.3, we have a canonical equivalence of normed vector spaces

$$C(S,\mathbb{R})\simeq C(I_1,\mathcal{V}) \qquad \text{where} \qquad \mathcal{V}=C(Y,\mathbb{R})$$

Choose any $f\in C(S,\mathbb{R})$, viewed as an element of $C(I_1,\mathcal{V})$. Then by Thm. 14.45, f can be approximated uniformly on I_1 by elements of $\mathcal{V}[\pi_1]$. Thus, for any $\varepsilon>0$, there exist $n\in\mathbb{Z}_+$ and $g_0,\dots,g_n\in C(Y,\mathbb{R})$ such that $\|f-\sum_{i=0}^n g_i\cdot\pi_1^i\|_{l^\infty(I_1,\mathcal{V})}<\varepsilon$. Equivalently,

$$\Big\|f-\sum_{i=0}^n g_i\cdot\pi_1^i\Big\|_{l^\infty(S,\mathbb{R})}<\varepsilon$$

where g_i actually means the composition of g_i with the projection $S\to Y$. By case $N-1$, each g_i can be approximated uniformly on Y by elements of $\mathbb{R}[\pi_2,\dots,\pi_N]$. So by triangle inequality, f can be uniformly approximated by elements of the form $\sum_{i=1}^n h_i\cdot\pi_1^i$ where each h_i is a polynomial of $\pi_2,\dots,\pi_N$. This finishes the proof of case N. $\square$

The key to the transition from finite to general index sets is the following lemma.

Lemma 15.18. *Let $(X_\alpha)_{\alpha\in\mathscr{I}}$ be a family of nonempty topological spaces. Let $S = \prod_{\alpha\in\mathscr{I}} X_\alpha$. Let $f \in C(S,\mathbb{R})$. Let $p_\bullet = (p_\alpha)_{\alpha\in\mathscr{I}} \in S$. For each $A \in \mathrm{fin}(2^{\mathscr{I}})$, define a map $\varphi_A : S \to S$ such that for each $x_\bullet = (x_\alpha)_{\alpha\in\mathscr{I}}$,*

$$\varphi_A(x_\bullet)_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \in A \\ p_\alpha & \text{if } \alpha \notin A \end{cases}$$

Then φ_A is continuous, and hence $f \circ \varphi_A$ is continuous. Moreover, for every $x_\bullet \in S$ we have

$$\lim_{\substack{(A,y_\bullet)\in\mathrm{fin}(2^{\mathscr{I}})\times S \\ (A,y_\bullet)\to(\infty,x_\bullet)}} f \circ \varphi_A(y_\bullet) = f(x_\bullet) \tag{15.6}$$

in the sense of Prop. 9.16. Thus, if each X_α is compact, then S is compact by Tychonoff Thm. 15.15, and hence $\lim\limits_{A\in\mathrm{fin}(2^{\mathscr{I}})} f \circ \varphi_A$ converges uniformly to f by Thm. 9.12.

In other words, φ_A fixes the α-th component if $\alpha \in A$, and changes the α-th component to p_α if $\alpha \notin A$. (15.6) means that for every $\varepsilon > 0$ there exist $U \in \mathrm{Nbh}(x_\bullet)$ and a finite $A \subset \mathscr{I}$ such that for every $y_\bullet \in U$ and for every finite set B satisfying $A \subset B \subset \mathscr{I}$, we have $|f(x_\bullet) - f \circ \varphi_B(y_\bullet)| < \varepsilon$.

Proof. The proof is similar to that of Pb. 9.6. The continuity of φ_A follows easily from the net-convergence description of product topology in Thm. 7.73. Choose any $x_\bullet \in S$ and $\varepsilon > 0$. Since f is continuous, there exists $U \in \mathrm{Nbh}(x_\bullet)$ such that $|f(x_\bullet) - f(y_\bullet)| < \varepsilon$ for all $y_\bullet \in U$. By the definition of product topology by means of basis (Def. 7.71), we can shrink U to a smaller neighborhood of $x_\bullet$ of the form

$$U = \prod_{\alpha\in\mathscr{I}} V_\alpha$$

where $V_\alpha \in \mathrm{Nbh}_{X_\alpha}(x_\alpha)$ for each $\alpha \in \mathscr{I}$, and $V_\alpha = X_\alpha$ for all α outside some $A \in \mathrm{fin}(2^{\mathscr{I}})$. Thus, $y_\bullet \in S$ belongs to U iff $y_\alpha \in V_\alpha$ for each $\alpha \in A$. Therefore, if $y_\bullet \in U$, then for each finite B satisfying $A \subset B \subset \mathscr{I}$, we have that

$$\varphi_B(y_\bullet)_\alpha = y_\alpha \in V_\alpha \qquad (\forall \alpha \in A)$$

and hence that $\varphi_B(y_\bullet) \in U$. It follows that $|f(x_\bullet) - f \circ \varphi_B(y_\bullet)| < \varepsilon$. □

Proof of Prop. 15.16. Choose any continuous $f : S = \prod_{\alpha\in\mathscr{I}} I_\alpha \to \mathbb{R}$. By Lem. 15.18, f can be approximated uniformly by functions depending on finitely many variables. In other words, for any $\varepsilon > 0$, there exist $A \in \mathrm{fin}(2^{\mathscr{I}})$ and a continuous $g : S_A = \prod_{\alpha\in A} I_\alpha \to \mathbb{R}$ such that $\|f - g \circ \pi_A\|_{l^\infty(S,\mathbb{R})} < \varepsilon$, where $\pi_A : S \to S_A$ is the natural projection. By Lem. 15.17, g can be approximated uniformly by polynomials of $\{\pi_\alpha : \alpha \in A\}$. This finishes the proof. □

15.4 Partition of unity and the Tietze extension theorem

In this section, we fix a Banach space $\mathcal{V}$ over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and fix a nonempty LCH space X. Recall from Prop. 8.41 that every open subset of X is LCH.

Remark 15.19. Let U be an open subset of X. Let $f \in C_c(U, \mathcal{V})$ supported in U. Then by zero-extension, f can be viewed as an element of $C_c(X, \mathcal{V})$ supported in U.

Proof. Let f take value 0 outside U. Let $A = \{x \in U : f(x) \neq 0\}$ and $K = \mathrm{Cl}_U(A)$, which is compact by assumption. In particular, K is closed in X by Cor. 8.22. Then $X = U \cup K^c$ is an open cover on X. By assumption, $f|_U$ is continuous. Also $f|_{K^c} = 0$ is continuous. So f is continuous by Exe. 7.119. To show that $f \in C_c(X, \mathcal{V})$ and $\mathrm{Supp}(f) \subset U$, it remains to prove that $\overline{A} = \mathrm{Cl}_X(A)$ is compact and is contained in U. This fact follows from Rem. 15.20. □

Remark 15.20. Let W be a subset of a Hausdorff space Y. (In this section, we are mainly interested in the case that Y is the LCH space X and W is open. In this case, W is LCH by Prop. 8.41.) Let $A \subset W$, and recall $\overline{A} = \mathrm{Cl}_Y(A)$. Then

$$A \text{ is precompact in } W \quad \Longleftrightarrow \quad \overline{A} \text{ is compact, and } \overline{A} \subset W \tag{15.7}$$

Moreover, if A is precompact in W, then $\overline{A}$ equals $\mathrm{Cl}_W(A)$.

Note that if W is open, but if A is not precompact in W, then $\mathrm{Cl}_Y(A)$ and $\mathrm{Cl}_W(A)$ are not necessarily equal: take $Y = \mathbb{R}$ and $A = W = \mathbb{R}_{>0}$.

Proof. "$\Leftarrow$" is obvious from the definition of precompactness (recall Def. 8.39).

"$\Rightarrow$": Clearly $\mathrm{Cl}_W(A) \subset \overline{A}$ in general. Assume that A is a precompact subset of W. By Def. 8.39, we have $A \subset \mathrm{Cl}_W(A) \subset W$ where $\mathrm{Cl}_W(A)$ is compact. In particular, $\mathrm{Cl}_W(A)$ is closed in Y (by Cor. 8.22). So $\overline{A} \subset \mathrm{Cl}_W(A)$. So $\overline{A} = \mathrm{Cl}_W(A)$. Thus $\overline{A}$ is a compact subset of W since $\mathrm{Cl}_W(A)$ is so. □

Definition 15.21. Let W be a subset of a Hausdorff space Y. We write

$$A \Subset W \tag{15.8}$$

whenever A is a precompact subset of W, or equivalently, whenever $A \subset Y$ satisfies that $\mathrm{Cl}_Y(A)$ is a compact subset of W.

15.4.1 Tietze extension theorem

The goal of this section is to prove the celebrated

Theorem 15.22 (Tietze extension theorem). *Let K be a compact subset of X. Let $f \in C(K, \mathcal{V})$. Then there exists $\widetilde{f} \in C_c(X, \mathcal{V})$ such that $\widetilde{f}|_K = f$, and that $\|\widetilde{f}\|_{l^\infty(X,\mathcal{V})} = \|f\|_{l^\infty(K,\mathcal{V})}$.*

The reader can first assume this theorem and read Sec. 15.5 about the proof of SW theorem, and then return to this section to read the proof of Tietze extension theorem.

Remark 15.23. Tietze extension theorem is often used in the following form: Suppose that U is an open subset of X containing K. Applying Tietze extension to U (which is LCH by Prop. 8.41) and noticing Rem. 15.19, we see that every $f \in C(K, \mathcal{V})$ can be extended to some $\widetilde{f} \in C_c(X, \mathcal{V})$ supported in U such that $\|\widetilde{f}\|_{l^\infty(X,\mathcal{V})} = \|f\|_{l^\infty(K,\mathcal{V})}$. In other words, in Tietze extension theorem, we can assume that the extended function is compactly supported in a given open subset.

Corollary 15.24. *$C_c(X, \mathbb{R})$ separates points of X.*

Proof. Choose distinct points $x, y \in X$. Let $K = \{x, y\}$. Let $f : K \to \mathbb{R}$ such that $f(x) = 1$ and $f(y) = 0$. By Thm. 15.22, f can be extended to some $\widetilde{f} \in C_c(X, \mathbb{R})$ which clearly separates x and y. □

There is another version of Tietze extension theorem: If A is a closed subset of a normal topological space Y, then any $f \in C(A, \mathbb{R})$ can be extended to some $\widetilde{f} \in C(Y, \mathbb{R})$ without increasing the l^∞-norm. Its proof is not quite the same as the LCH version. See [Mun, Sec. 35]. We will not use this version in our course.

15.4.2 Urysohn's lemma

The proof of Tietze extension theorem involves several steps. The first step is to prove a special case: If $K \subset X$ is compact, then the characteristic function $\chi_K : X \to \mathbb{R}$ can be extended to a continuous $f \in C_c(X, \mathbb{R})$. Then $f|_K = 1$. Replacing f by $\max\{f, 0\}$, we may assume that $f \geqslant 0$. Replacing f by $\min\{f, 1\}$, we may assume $0 \leqslant f \leqslant 1$. This special case is called

Theorem 15.25 (Urysohn's lemma). *Let K be a compact subset of X. Then there exists f such that*

$$K \prec f \prec X \tag{15.9}$$

The meaning of the notations in (15.9) is explained below. Note that since any open subset $U \subset X$ is LCH (Prop. 8.41), Urysohn's lemma can be applied to U and any compact subset $K \subset U$, which shows that there exists $f \in C_c(X, [0, 1])$ such that $K \prec f \prec U$.

Definition 15.26. Let U be an open subset of X. Let K be a compact subset of X.

- $f \prec U$ means that $f \in C_c(X, [0, 1])$ (i.e. $f \in C_c(X, \mathbb{R})$ and $f(X) \subset [0, 1]$) and that $\mathrm{Supp}(f) \subset U$.
- $K \prec f$ means that $f \in C_c(X, [0, 1])$ and that $f|_K = 1$.

The symbol "$<$" is chosen for the following reason. Assume $f \in C_c(X, [0,1])$. Then $K < f$ means that $\chi_K \leqslant f$. However, the meaning of $f < U$ is slightly stronger than that of $f \leqslant \chi_U$: the latter means that U contains $\{x : f(x) \neq 0\}$, but not that U contains its closure $\mathrm{Supp}(f)$.

To prove Urysohn's lemma we need some elementary observations:

Lemma 15.27. *Let W be an open subset of X. Let K be a compact subset of W. Then there exists an open set U of X such that $K \subset U \Subset W$.*

By Rem. 15.20, this lemma simply means that every compact $K \subset W$ is contained in a precompact open subset of W. To prove it, it suffices to assume $X = W$.

Proof. We assume $X = W$. Since X is LCH, every $x \in K$ is contained in a precompact open subset U_x. Since K is compact, K is contained in $U = U_{x_1} \cup \cdots \cup U_{x_n}$ for some $x_1, \dots, x_n \in K$. Clearly U is open and precompact. $\square$

Recall Def. 14.17 for the meaning of $\omega(f, x)$, the oscillation of a function f at x.

Lemma 15.28. *Let Y be a topological space and Z a metric space. Let (f_n) be a sequence in Z^Y converging uniformly to $f \in Z^Y$. Assume that for each $x \in Y$, $\lim_{n\to\infty} \omega(f_n, x) = 0$. Then f is continuous.*

Note that Thm. 7.79 can be viewed as a special case of this lemma. However, this lemma is not used as often as Thm. 7.79. This is why we call this result only a lemma.

Proof. Choose any $\varepsilon > 0$. Then there is $N \in \mathbb{Z}_+$ such that for all $n \geqslant N$ we have $\|f - f_n\|_\infty < \varepsilon$. Since $\lim_{n\to\infty} \omega(f_n, x) = 0$, there is $n \geqslant N$ such that $\omega(f_n, x) < \varepsilon$. Thus there exist $n \geqslant N$ and $U \in \mathrm{Nbh}_X(x)$ such that $\mathrm{diam}(f_n(U)) < \varepsilon$. Then by triangle inequality, we have $\mathrm{diam}(f(U)) \leqslant 3\varepsilon$. So $\omega(f, x) \leqslant 3\varepsilon$. Since ε is arbitrary, we conclude $\omega(f, x) = 0$. So f is continuous at x by Prop. 14.18. $\square$

$\star$ ***Proof of Urysohn's lemma.*** By Lem. 15.27, we can choose $U_1 \Subset X$ containing K. In the case that X is metrizable, $f(x) = d(x, U_1^c)/(d(x, K) + d(x, U_1^c))$ gives a desired function. However, in the general case, we need to construct f in a different way.

We shall construct inductively a sequence of functions $f_n : X \to [0,1]$ such that the following conditions are satisfied:

(a) $f_n|_K = 1$ and $f_n|_{X \setminus U_1} = 0$.

(b) $\omega(f_n, x) \leqslant \frac{1}{2^n}$ for all $x \in X$.

(c) $\|f_{n+1} - f_n\|_{l^\infty} \leqslant \frac{1}{2^{n+1}}$ for all n.

Then $\|f_{n+k} - f_n\|_\infty \leqslant 1/2^n$ for all $n, k > 0$. Thus $(f_n)_{n\in\mathbb{Z}_+}$ is a Cauchy sequence in $l^\infty(X, \mathbb{R})$, converging uniformly to some $f \in l^\infty(X, \mathbb{R})$. Clearly $f(X) \subset [0, 1]$, $f|_K = 1$, and $f|_{X\setminus U_1} = 0$. So f is compactly supported since $\overline{U}_1$ is compact. By Lem. 15.28, $f \in C_c(X, [0, 1])$, finishing the proof.

In fact, our construction of f_n relies on

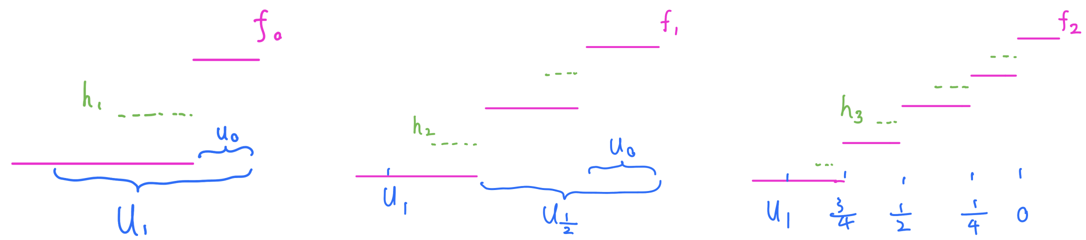
$$K \subset U_0 \Subset U_{\frac{1}{2^n}} \Subset U_{\frac{2}{2^n}} \Subset \cdots \Subset U_{\frac{2^n-1}{2^n}} \Subset U_1$$

f_0 is simply defined to be χ_{U_0} where U_0 is a precompact open subset of U_1 containing K (which exists due to Lem. 15.27).

Suppose that f_n and $U_{\frac{j}{2^n}}$ (where $0 \leqslant j < 2^n$) have been constructed. Clearly $U_{\frac{j}{2^{n+1}}}$ already exists when j is even. Suppose that j is odd, let $U_{\frac{j}{2^{n+1}}}$ be a precompact open subset of $U_{\frac{j+1}{2^{n+1}}}$ containing the closure of $U_{\frac{j-1}{2^{n+1}}}$. Then $K \subset U_0 \Subset U_{\frac{1}{2^{n+1}}} \Subset U_{\frac{2}{2^{n+1}}} \Subset \cdots \Subset U_1$. Let

$$h_{n+1} = \sum_{\substack{0<j<2^{n+1}\\ j \text{ is odd}}} 2^{-n-1} \cdot \chi_{\Delta_j} \qquad \text{where} \qquad \Delta_j = U_{\frac{j}{2^{n+1}}} \backslash U_{\frac{j-1}{2^{n+1}}}$$

and let $f_{n+1} = f_n + h_{n+1}$. The best way to understand this construction is to look at the pictures:



Clearly (a) and (c) are satisfied. Choose any $x \in X$. For each n, let $U_{\frac{2^n+1}{2^n}} = X$ and $U_{-\frac{1}{2^n}} = \emptyset$. Then X is a disjoint union of $\Delta_j = U_{\frac{j}{2^{n+1}}} \backslash U_{\frac{j-1}{2^{n+1}}}$ over $0 \leqslant j \leqslant 2^n + 1$, and f_n can be described by $f_n|_{\Delta_j} = \max\{1 - \frac{j}{2^n}, 0\}$. For each $x \in \Delta_j$ (where $1 \leqslant j \leqslant 2^n + 1$), let

$$W = U_{\frac{j}{2^n}} \backslash \overline{U_{\frac{j-2}{2^n}}}$$

Then $W \in \mathrm{Nbh}_X(x)$, and $\mathrm{diam}(f_n(W)) \leqslant \frac{1}{2^n}$ since $W \subset \Delta_j \cup \Delta_{j-1}$. If $x \in \Delta_0$, then $\Delta_0 \in \mathrm{Nbh}(x)$ and $\mathrm{diam}(f_n(\Delta_0)) = 0$. This proves (b). □

15.4.3 Partition of unity

In the second step of the proof of Tietze extension theorem, we prove the theorem on partition of unity.

Recall that we defined Riemann integrals by partitioning a compact interval into small intervals. Thus, in order to define the integral of a function on $\mathbb{R}^2$, one can partition a rectangle into smaller ones. However, it is more difficult to integrate a function defined on a more complicated space (a complicated compact surface M in $\mathbb{R}^3$, for example) by dividing M into small pieces, since these small pieces may have complicated shapes.

Instead of partitioning M, a better way to define $\int_M f$ is by partitioning f. Let $f: M \to \mathbb{R}$ be continuous. Suppose first of all that $\mathrm{Supp}(f)$ is contained in a small enough neighborhood U which can be "parametrized by a rectangle". (More precisely: we can find a bijection $\varphi: R \to U$, where $R = (a,b) \times (c,d)$ is an open rectangle in $\mathbb{R}^2$, such that φ and φ^{-1} are both smooth. Such φ is called a **diffeomorphism**.) Then we can use integrals on rectangles to define $\int_M f$ by "pulling back f to R". Now, in the general case, one can define $\int_M f$ by writting f as $f_1 + \cdots + f_n$ where each $f_n \in C(M, \mathbb{R})$ has a small enough support such that $\int_M f_i$ can be defined. Then $\sum_i \int_M f_i$ gives the formula of $\int_M f$.

Notice that it suffices to write the constant function 1 on M as $h_1 + \cdots + h_n$ where each $\mathrm{Supp}(h_i)$ is small enough. Then $f = fh_1 + \cdots + fh_n$ gives a desired partition of f. Thus, $1 = h_1 + \cdots + h_n$ is called a **partition of unity** (where "unity" means the constant function 1).

Theorem 15.29. *Let K be a compact subset of X. Let $\mathfrak{U} = (U_1, \dots, U_n)$ be a finite set of open subsets of X covering K (i.e. $K \subset U_1 \cup \cdots \cup U_n$). Then there exist $h_1, \dots, h_n \in C_c(X, \mathbb{R})$ such that the following conditions hold:*

(1) For each $1 \leqslant i \leqslant n$, we have $h_i \geqslant 0$ and $\mathrm{Supp}(h_i) \subset U_i$.

(2) $\displaystyle\sum_{i=1}^n h_i\big|_K = 1$.

(3) $0 \leqslant \displaystyle\sum_{i=1}^n h_i \leqslant 1$.

*Such $h_1, \dots, h_n$ are called a **partition of unity of K subordinate to** $\mathfrak{U}$.*

Many people do not assume (3) in the definition of partition of unity. We assume (3) since it is useful. In fact, conditions (1)-(3) imply that $\sum_i h_i$ is an Urysohn function for K and $\bigcup_i U_i$. Therefore, Urysohn's lemma is a special case of Thm. 15.29. However, we shall prove Thm. 15.29 using Urysohn's lemma. In fact, $h_1, \dots, h_n$ should be viewed as a *partition of the Urysohn function* $\sum_i h_i$.

Proof. Step 1. Let us construct $G_i \in C_c(U_i, \mathbb{R}_{\geqslant 0})$ for each $1 \leqslant i \leqslant n$ such that $G := \sum_i G_i$ is > 0 on K (i.e., $G(K) \subset \mathbb{R}_{>0}$).

By Urysohn's lemma, for each $x \in X$ there exists $g_x \in C_c(X, [0,1])$ supported in U_i such that $g_x(x) \neq 0$. Since K is compact and is contained in $\bigcup_{x \in K} g_x^{-1}(\mathbb{R}_{>0})$, there exists a finite subset $E \subset K$ such that $K \subset \bigcup_{x \in E} g_x^{-1}(\mathbb{R}_{>0})$. So $\sum_{x \in E} g_x\big|_K > 0$.

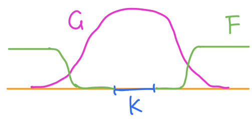
Now, for each $1 \leqslant i \leqslant n$, define

$$G_i = \sum_{\substack{x \in E \\ \mathrm{Supp}(g_x) \subset U_i}} g_x \tag{15.10}$$

Since for each $x \in E$ there is some U_i containing $\mathrm{Supp}(g_x)$, when summing up (15.10) over all i, each g_x must appear at least once in the summand. So $G := \sum_i G_i$ is $\geqslant \sum_i g_i$, and hence $G|_K > 0$. Clearly each G_i is $\geqslant 0$ and is compactly supported in U_i.

Step 2. $G_1, \dots, G_n$ satisfy (1) but not necessarily (2) or (3). It is tempting to define $h_i = G_i/G$. Then $h_1, \dots, h_n$ satisfy all the desired conditions except the continuity. To remedy this issue, we define $h_i = G_i/\widetilde{G}$, where $\widetilde{G} \in C_c(X, \mathbb{R})$ satisfies that $\widetilde{G} > 0$ on X, that $\widetilde{G} \geqslant G$ (so that $0 \leqslant \sum_i h_i \leqslant 1$), and that $\widetilde{G}|_K = G|_K$ (so that $\sum_i h_i|_K = 1$). Then $h_1, \dots, h_n$ are the desired functions.

Let us prove the existence of such $\widetilde{G}$. Let $W = \{x \in X : G(x) > 0\}$. Let $F \in C(X, [0,1])$ such that $F|_K = 0$ and that $F|_{X \setminus W} = 1$. The existence of such $1 - F$ is ensured by Urysohn's lemma. Then one can let $\widetilde{G} = G + F$.



□

15.4.4 How to use partition of unity

In Sec. 8.2, we have discussed how to use the condition of compactness: Suppose that K is compact and f is a function on K, for instance, a continuous one. To prove that f satisfies a global finiteness condition, we first prove that each $x \in K$ is contained in a neighborhood U_x on which f satisfies this finiteness condition. Then we pick finitely many $U_{x_1}, U_{x_2}, \dots$ covering K, and show that f satisfies the finiteness condition globally.

To summarize, we can use compactness to prove many finiteness properties by a local-to-global argument. As pointed out in Sec. 8.2, usually, these finiteness properties can also be proved by contradiction using net-compactness or sequential compactness.

Now assume that K is a compact subset of the LCH space X, and let f be a function on K. With the help of partition of unity, one can construct new objects from f using a local-to-global argument. Moreover, these constructions are usually very difficult to obtain by using net-compactness or sequential compactness. The following are two typical examples:

1. (**Integral problems**) Construction of an integral $\int_K f$. This was already mentioned in Subsec. 15.4.3.

2. (**Extension problems**) Construct a "good" function $\widetilde{f}$ on X extending (or "approximately extending") f. The idea is simple: Suppose that the extension exists locally, i.e., suppose that for each $x \in K$ there exists $U_x \in \mathrm{Nbh}_X(x)$ such that $f|_{U_x \cap K}$ can be extended to a good function g_x on U_x. By compactness, K is covered by $\bigcup_{x \in E} U_x$ where E is a finite subset of K. Let $(h_x)_{x \in E}$ be a partition of unity of K subordinate to this open cover. Then $\widetilde{f} = \sum_{x \in E} h_x \cdot g_x$ gives a good extension.

In the study of measure theory in the second semester, we will use partition of unity extensively. Readers who want to get a head start can do the problems in Subsec. 15.8.2, 15.8.3, and 15.8.4 to see how to build a theory of multivariable Riemann integrals using partitions of unity. For the moment, let's look at a simple example of extension problem before we prove the Tietze extension theorem. This example is not used elsewhere in this chapter, but it serves as a good illustration of how to use partitions of unity.

Example 15.30. Let I be an open interval in $\mathbb{R}$, and let K be a nonempty compact subset of I. Let $r \in \mathbb{Z}_+ \cup \{\infty\}$. Assume that $f : K \to \mathbb{R}$ is C^r, which means that for each $x \in K$ there exist an open interval $U_x \subset I$ containing x and $g_x \in C^r(U_x, \mathbb{R})$ extending $f|_{U_x \cap K}$. Then there exists $\widetilde{f} \in C^r(I, \mathbb{R})$ extending f and is compactly supported in I.

This example is the **smooth Tietze extension theorem** in dimension 1, as mentioned in Rem. 13.46.

Proof. Let U_x and g_x be as in the example. Since K is compact, it can be covered by $U_{x_1}, \dots, U_{x_n}$. Call this open cover $\mathfrak{U}$. Similar to the proof of Thm. 15.29, one can find a set of C^r-partition of unities $h_1, \dots, h_n$ of K subordinate to $\mathcal{U}$. In other words, $h_1, \dots, h_n$ are C^r functions and form a partition of unity. (Similar to the proof of Thm. 15.29, in order to find such $h_1, \dots, h_n$, it suffices to prove the C^r-version of Urysohn's lemma. But this has been done in Prop. 14.47, noting that K is contained in a compact subinterval of I.) Now each $h_i g_{x_i}$ is an C^r-function on I compactly supported in U_{x_i}. Then $\widetilde{f} = \sum_i h_i g_{x_i}$ is a desired extension. □

You will see many more examples in the future when you study differential manifolds and sheaf theory. It is no exaggeration to say that partition of unity is one of the most important techniques in modern mathematics.

15.4.5 Proof of Tietze extension theorem

Now you may wonder: Under the assumption of Tietze extension Thm. 15.22, we don't even know how to extend $f \in C(K, \mathcal{V})$ locally. Then how can we use the local-to-global argument? Here is the answer: you can find an *approximate* extension locally. Therefore, you can first find an approximate global extension of f. Then, passing to the limit, you get the desired extension. This will be our strategy of the proof of Thm. 15.22.

Lemma 15.31. *Let K be a compact subset of X, and let $f \in C(K,\mathcal{V})$. Then for every ε, there exists $\varphi \in \mathrm{Span}(C_c(X,\mathbb{R})\mathcal{V})$ such that*

$$\|f - \varphi\|_{l^\infty(K,\mathcal{V})} \leqslant \varepsilon \qquad \|\varphi\|_{l^\infty(X,\mathcal{V})} \leqslant \|f\|_{l^\infty(K,\mathcal{V})}$$

The meaning of $\mathrm{Span}(C_c(X,\mathbb{R})\mathcal{V})$ is clear: the smallest linear subspace of $C_c(X,\mathcal{V})$ containing $C_c(X,\mathbb{R})\mathcal{V}$. Thus

$$\mathrm{Span}(C_c(X,\mathbb{R})\mathcal{V}) = \{g_1v_1 + \cdots + g_nv_n : n \in \mathbb{Z}_+, g_i \in C_c(X,\mathbb{R}), v_i \in \mathcal{V}\} \qquad (15.11)$$

Proof. Let $M = \|f\|_{l^\infty(K,\mathcal{V})}$. For each $p \in K$, since f is continuous at p, there exists $U_p \in \mathrm{Nbh}_X(p)$ such that $\mathrm{diam}(f(U_p \cap K)) \leqslant \varepsilon$ (cf. Prop. 14.18). Then the contant function $f(p)$ gives a local approximate extension of $f|_{U_p\cap K}$.

Since K is compact, there exists a finite subset $E \subset K$ such that K is covered by $\mathfrak{U} = \{U_p : p \in E\}$. By Thm. 15.29, there is a partition of unity $(h_p)_{p\in E}$ of K subordinate to $\mathfrak{U}$. Since $h_p \in C_c(U_p,\mathbb{R})$ can be viewed as a compactly supported continuous function on X (Rem. 15.19), the function $\varphi = \sum_{p\in E} h_p f(p)$ is an element of $C_c(X,\mathcal{V})$.

If $x \in K$, then since $h_p \geqslant 0$, we have

$$\|f(x) - \varphi(x)\| = \Big\|\sum_{p\in E} h_p(x)\cdot\big(f(x) - f(p)\big)\Big\| \leqslant \sum_{p\in E} h_p(x)\|f(x) - f(p)\|$$

In the RHS, if $h_p(x) \neq 0$, then $x \in U_p$, and hence $\|f(x) - f(p)\| \leqslant \varepsilon$. So the RHS is no greater than $\sum_{p\in E} h_p(x)\varepsilon$, and hence no greater than ε since $\sum_{p\in E} h_p(x) = 1$. Finally, for every $x \in X$, we have

$$\|\varphi(x)\| \leqslant \sum_{p\in E} h_p(x)\|f(p)\| \leqslant \sum_{p\in E} h_p(x)M = M$$

finishing the proof. $\square$

The following special case of Lem. 15.31 is more useful for application.

Proposition 15.32. *Assume that X is a compact Hausdorff space, and let $f \in C(X,\mathcal{V})$. Then for every $\varepsilon > 0$, there exists $\varphi \in \mathrm{Span}(C_c(X,\mathbb{F})\mathcal{V})$ such that $\|f - \varphi\|_\infty < \varepsilon$.*

Corollary 15.33. *Let X, Y be compact Hausdorff spaces. Then for every $f \in C(X \times Y,\mathbb{F})$ and $\varepsilon > 0$, there exist $g_1,\dots,g_n \in C(X,\mathbb{F})$ and $h_1,\dots,h_n \in C(Y,\mathbb{F})$ such that $\|f - g_1h_1 - \cdots - g_nh_n\|_{l^\infty(X\times Y,\mathbb{F})} < \varepsilon$.*

Proof. By Thm. 9.3, we view f as an element of $C(X,\mathcal{V})$ where $\mathcal{V} = C(Y,\mathbb{R})$. Then the corollary follows immediately from Prop. 15.32. From this proof, we see that it is not necessarily to assume that Y is Hausdorff. (This is not an important fact anyway.) $\square$

Now we are ready to finish the

Proof of Tietze extension Thm. 15.22. Recall that K is compact in X and $f \in C(K,\mathcal{V})$, and that our goal is to extend f to $\widetilde{f} \in C_c(X,\mathcal{V})$. Moreover, we need that $\|\widetilde{f}\|_{l^\infty(X,\mathcal{V})}$ equals $M = \|f\|_{l^\infty(K,\mathcal{V})}$. We first note that the last requirement is easy to meet. Assume WLOG that $M > 0$. Let $\widetilde{f} \in C_c(X,\mathcal{V})$ extend f, and define $g : X \to \mathbb{R}_{\geqslant 0}$ by $g(x) = \max\{M, \|\widetilde{f}(x)\|\}$. Then g is continuous, $g \geqslant M$, and $g|_K = M$. Then $M\widetilde{f}/g$ is an element of $C_c(X,\mathcal{V})$ extending f and is l^∞-bounded by M, finishing the proof.

Second, note that it suffices to extend f to $\widetilde{f} \in C(X,\mathcal{V})$. By Urysohn lemma, there exists h such that $K \prec h \prec X$. Then $\widetilde{f}h \in C_c(X,\mathcal{V})$ extends f.

We now construct $\widetilde{f} \in C(X,\mathcal{V})$ extending f. By Lem. 15.31, there exist $\varphi_1, \varphi_2, \dots \in C_c(X,\mathcal{V})$ such that

$$\begin{gathered}
\|f-\varphi_1\|_{l^\infty(K,\mathcal{V})} \leqslant \frac{M}{2} \qquad \|\varphi_1\|_{l^\infty(X,\mathcal{V})} \leqslant M \\
\|f-\varphi_1-\varphi_2\|_{l^\infty(K,\mathcal{V})} \leqslant \frac{M}{4} \qquad \|\varphi_2\|_{l^\infty(X,\mathcal{V})} \leqslant \frac{M}{2} \\
\|f-\varphi_1-\varphi_2-\varphi_3\|_{l^\infty(K,\mathcal{V})} \leqslant \frac{M}{8} \qquad \|\varphi_3\|_{l^\infty(X,\mathcal{V})} \leqslant \frac{M}{4} \\
\vdots
\end{gathered}$$

Then $\sum_{n=1}^\infty \|\varphi_n\|_{l^\infty(X,\mathcal{V})} \leqslant 2M$, and hence $\sum_{n=1}^\infty \varphi_n$ converges to some $\widetilde{f}$ in the Banach space $C(X,\mathcal{V}) \cap l^\infty(X,\mathcal{V})$ (Cor. 3.50). Clearly $\widetilde{f}$ extends f. □

15.5 Proof of SW, II: embedding into $[0,1]^{\mathscr{I}}$

In this section, we shall finish the proof of SW Thm. 15.9.

Remark 15.34. Suppose that $\Phi : X \to Y$ is a homeomorphism of topological spaces. Then X and Y can be "viewed as the same space" via Φ. This means that a point $x \in X$ can be identified with $\Phi(x) \in Y$, that an open or closed subset $A \subset X$ can be identified with $\Phi(A)$, which is open or closed in Y. It also means, for example, that if $A \subset X$, then the closure of A can be identified with the closure of $\Phi(A)$ in Y. More precisely: Φ restricts to a homeomorphism $\mathrm{Cl}_X(A) \to \mathrm{Cl}_Y(\Phi(A))$.

The continuous functions of X and Y can also be identified: If $g \in C(Y,Z)$ where Z is a topological space (e.g. $Z = \mathbb{R}$), then g is equivalent to its **pullback under** Φ, which is an element of $C(X,Z)$ given by

$$\Phi^* g := g \circ \Phi \qquad \in C(X,Z) \tag{15.12}$$

This equivalence of functions can be illustrated by the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\Phi", "\simeq"'] \arrow[dr, "\Phi^* g"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd} \tag{15.13}$$

□

Proof of SW Thm. 15.9. Recall that X is a compact Hausdorff space and $\mathscr{A}$ is a unital subalgebra of $C(X, \mathbb{R})$ separating points of X. For each $f \in \mathscr{A}$, let $M_f = \|f\|_\infty$ and $I_f = [-M_f, M_f]$. Define

$$\Phi : X \to S = \prod_{f \in \mathscr{A}} I_f \qquad x \mapsto (f(x))_{f \in \mathscr{A}}$$

In other words, $\Phi = \bigvee_{f \in \mathscr{A}} f$, using the notation in Pb. 7.8. Thus, by Pb. 7.8 (or by Thm. 7.73), Φ is continuous. The fact that $\mathscr{A}$ separates points of X is equivalent to that Φ is injective. Since X is compact, by Thm. 8.23, Φ restricts to a homeomorphism $\Phi : S \to \Phi(S)$.

Recall that the coordinate function $\pi_f : S \to I_f$ is the projection onto the f-component. So $\Phi^* \pi_f = \pi_f \circ \Phi = f$. Therefore, by Rem. 15.34, X is equivalent to $\Phi(X)$ under Φ, and $\pi_f|_{\Phi(X)} \in C(\Phi(X), \mathbb{R})$ is equivalent to $f \in C(X, \mathbb{R})$ under Φ. Therefore, we can identify X with $\Phi(X)$ via Φ so that f is identified with $\pi_f|_{\Phi(X)}$. Thus, in this case, $\mathscr{A}$ is the set of all $\pi_f|_X$, and clearly $\mathscr{A}$ contains all the polynomials of the coordinate functions $\{\pi_f|_X : f \in \mathscr{A}\}$ since $\mathscr{A}$ is a unital subalgebra. Thus, it suffices to prove that the polynomials of coordinate functions are dense in $C(X, \mathbb{R})$.

Choose any $g \in C(X, \mathbb{R})$. Since X is compact, and since S is a compact Hausdorff space (by Tychonoff Thm. 15.15), by Tietze extension Thm. 15.22, g can be extended to $\widetilde{g} \in C(S, \mathbb{R})$. By Prop. 15.16, $\widetilde{g}$ can be approximated uniformly by the polynomials of coordinate functions of S. □

Remark 15.35. If you feel that identifying X and $\Phi(X)$ is cheating, it is easy to revise the proof without identifying them: Choose any $g \in C(X, \mathbb{R})$. One first concludes that $g \circ \Phi^{-1} \in C(\Phi(X), \mathbb{R})$ can be uniformly approximated by the polynomials of coordinate functions. Then, since the pullback of these polynomials under Φ are elements of $\mathscr{A}$, one concludes that g can be approximated uniformly by elements of $\mathscr{A}$.

15.6 Summary of the proof of SW

The key steps of the proof of SW Thm. 15.9 are as follows.

1. We first prove the case that X is a compact interval I and $\mathscr{A}$ is the polynomial algebra: Let $f \in C(I, \mathbb{R})$, and extend f to an element in $C_c(\mathbb{R}, \mathbb{R})$. Let $g(x) = \pi^{-\frac{1}{2}} e^{-x^2}$ and $g_\varepsilon(x) = \varepsilon^{-1} g(x/\varepsilon)$. On the one hand, $\lim_{\varepsilon \to 0} f * g_\varepsilon$ converges uniformly to f. On the other hand, for each ε, since g_ε is approximated by polynomials uniformly on compact intervals (consider the Taylor series of g_ε), one shows that $f * g_\varepsilon$ is also approximated by polynomials uniformly on compact intervals.

2. When $\mathcal{V}$ is a Banach space, the same method shows that $\mathcal{V}[x]$, the set of polynomials with coefficients in $\mathcal{V}$, is l^∞-dense in $C(I, \mathcal{V})$.

3. Taking $\mathcal{V} = C(J, \mathbb{R})$ where J is a compact interval, the above step shows that polynomials are dense in $C(I \times J, \mathbb{R})$.[2] Similarly, by induction, one sees that polynomials are dense in $C(I_1 \times \cdots \times I_n, \mathbb{R})$ if each I_j is a compact interval.

4. Let $S = \prod_{\alpha \in \mathscr{I}} I_\alpha$ where each I_α is a compact interval. Then S is a compact Hausdorff space by the Tychonoff theorem. One shows that any $f \in C(S, \mathbb{R})$ can be approximated by a function g depending on finitely many variables (Lem. 15.18, or Pb. 9.6 when $\mathscr{I}$ is countable). By the previous step, g can be approximated by polynomials. So f can be approximated by polynomials (of coordinate functions of S).

5. Let X be a compact subset of $S = \prod_{\alpha \in \mathscr{I}} I_\alpha$, and let $f \in C(X, \mathbb{R})$. Then by the Tietze extension theorem, f can be extended to $\widetilde{f} \in C(S, \mathbb{R})$ where $\widetilde{f}$ can be approximated by polynomials by the previous step. So f can be approximated by polynomials.

6. Now let X be a compact Hausdorff space, and let $\mathscr{A}$ be a unital subalgebra of $C(X, \mathbb{R})$ separating points of X. Then the map $\Phi = \bigvee_{f \in \mathscr{A}} f$ maps X homeomorphically to a compact subspace of $S = \prod_{\alpha \in \mathscr{A}} I_f$ where $I_f = [-\|f\|_\infty, \|f\|_\infty]$. By the previous step, continuous functions on $\Phi(X)$ can be approximated by polynomials of the coordinate functions of S. This is equivalent to the density of $\mathscr{A}$ in $C(X, \mathbb{R})$. The proof is complete.

The proof we have given, which is different from the proofs in most textbooks, has several advantages. First, it clearly shows that "$\mathscr{A}$ separates points of X" is an embedding condition, which ensures that the map $\Phi = \bigvee_{f \in \mathscr{A}} f$ is injective. The embedding of spaces is a common theme in many branches of mathematics. In differential geometry, one can show that every (second countable) smooth manifold can be smoothly embedded into a Euclidean space. (This is the **Whitney embedding theorem**.) **Projective manifolds**, the compact complex manifolds

[2] Instead of using step 2, one can also use step 1 and the fact that elements in $C(I \times J, \mathbb{R})$ can be approximated by those of the form $f_1 g_1 + \cdots + f_n g_n$ where $f_i \in C(I, \mathbb{R})$ and $g_i \in C(J, \mathbb{R})$, cf. Cor. 15.33.

that can be holomorphically embedded into complex projective spaces $\mathbb{CP}^n$, are among the most important examples in complex (algebraic) geometry.

Second, Urysohn's lemma and partition of unity are extremely important tools in differential manifolds and in measure theory, both of which will be studied next semester.

Thus, although our proof is longer than those in the other textbooks,[3] the methods we used in the proof (convolutions, embedding of spaces, partition of unity, etc.) will appear frequently in the future study. Through our proof, the SW theorem is closely and organically related to other mathematical concepts.

We close this section with an immediate consequence of the proof of SW theorem, which is parallel to Thm. 8.45.

Theorem 15.36. *Let X be a topological space. Then the following are equivalent.*

(1) X is a compact Hausdorff space.

(2) X is homeomorphic to a closed subset of $[0,1]^{\mathscr{I}}$ for some set $\mathscr{I}$.

Proof. Clearly $[0,1]^{\mathscr{I}}$ is Hausdorff. By Tychonoff Thm. 15.15, $[0,1]^{\mathscr{I}}$ is compact. So its closed subsets are compact Hausdorff. Conversely, let X be compact Hausdorff. By Cor. 15.24, $C(X,\mathbb{R})$ separates points of X. Thus there is a subset $\mathscr{I}\subset C(X,\mathbb{R})$ separating points of X such that $f(X)\subset[-1,1]$ for all $f\in\mathscr{I}$. Therefore $\Phi=\bigvee_{f\in\mathscr{I}}f:X\to[-1,1]^{\mathscr{I}}$ is a continuous injective map of X into $[-1,1]^{\mathscr{I}}$. So it reduces to a homeomorphism $X\to\Phi(X)$ by Thm. 8.23. □

15.7 Application: separability of $C(X,\mathbb{R})$

Our proof of SW theorem relies on Tychonoff Thm. 15.15, which in turn relies on Zorn's lemma, an uncountable version of mathematical induction. Though Zorn's lemma is equivalent to the axiom of choice, it is much more difficult to grasp intuitively than mathematical induction. Thus, one would like to find a proof without using Zorn's lemma if possible.

In the following, we will show that when the compact Hausdorff space X is second countable (equivalenty, metrizable), in the proof of SW theorem, it suffices to embed X into a countable product of compact intervals. The latter is compact by countable Tychonoff theorem, whose proof does not rely on Zorn's lemma (cf. Thm. 3.54 or Pb. 8.7).

[3]The proofs in most textbooks (e.g. [Fol-R, Sec. 4.7], [Rud-P, Ch. 7], [Zor-2, Sec. 16.4]) are similar and are due to M. Stone [Sto48], which used the idea of lattices. A lattice is a set L together with operations $\wedge,\vee$ satisfying $a\vee(a\wedge b)=a$ and $a\wedge(a\vee b)=a$. In analysis, one considers $L\subset C(X,\mathbb{R})$ where $f\vee g=\max\{f,g\}$ and $f\wedge g=\min\{f,g\}$. In Stone's day, lattices were relatively popular in functional analysis. But today it seems that they are a bit cold. The shorter proof in [Sto48] is actually not the original proof of SW theorem. The original proof was given in [Sto37] and is much more complicated.

We first discuss a general fact about countability in compact Hausdorff spaces. The following theorem is one of the most important general properties about separability: It tells us that for a compact Hausdorff space X, the countability property of the topology of X is equivalent to that of the uniform convergence topology of $C(X,\mathbb{R})$. This is in stark contrast to compactness, where the compactness of X does not in general imply the compactness of bounded closed subsets of $C(X,\mathbb{R})$ (cf. Exp. 17.1).

Theorem 15.37. *Let X be a compact Hausdorff space. The following are equivalent.*

(1) X is metrizable.

(2) X is second countable.

(3) $C(X,\mathbb{R})$ is separable (equivalently, second countable, cf. Prop. 8.32).

Here, as usual, $C(X,\mathbb{R})$ is equipped with the l^∞-norm. See Pb. 15.14 for a generalization of Thm. 15.37 to LCH spaces. Also, assuming (3), one can write down a metric and a countable basis explicitly; see Pb. 15.15 for details.

Proof. By Thm. 8.45, we have (1)⇔(1′) where

(1′) X is homeomorphic to a closed subset of $[0,1]^{\mathbb{Z}_+}$.

Let us prove that (2) and (3) are equivalent to (1′).

(1′)⇒(3): By Tietze extension theorem, it suffices to prove that $C(S,\mathbb{R})$ is separable where $S=[0,1]^{\mathbb{Z}_+}$. Let $\mathcal{E}=\mathbb{Q}[\{\pi_n:n\in\mathbb{Z}_+\}]$ be the set of polynomials of the coordinate functions of S with coefficients in $\mathbb{Q}$. ($\pi_n:S\to[0,1]$ is the projection onto the n-th component.) Then $\mathcal{E}$ is countable (cf. Exe. 15.38), and is clearly dense in $\mathbb{R}[\{\pi_n:n\in\mathbb{Z}_+\}]$. By Prop. 15.16, $\mathcal{E}$ is dense in $C(S,\mathbb{R})$.

(3)⇒(1′): Let $\mathcal{E}=\{f_1,f_2,\dots\}$ be a countable dense subset of $C(X,\mathbb{R})$. By enlarging $\mathcal{E}$, we assume WLOG that $\mathcal{E}$ is infinite. By Cor. 15.24, $C(X,\mathbb{R})$ separates points of X. So $\mathcal{E}$ also separates points of X. We scale $f_n\in\mathcal{E}$ by a nonzero real number such that $\|f_n\|_\infty\leqslant 1$. Then

$$\Phi:X\to[-1,1]^{\mathbb{Z}_+}\qquad x\mapsto(f_1(x),f_2(x),\dots)\tag{15.14}$$

is a continuous injective map, restricting to a homeomorphism $X\to\Phi(X)$ where $\Phi(X)$ is a compact (and hence closed) subset of $[-1,1]^{\mathbb{Z}_+}\simeq[0,1]^{\mathbb{Z}_+}$.

(1′)⇒(2): Since $[0,1]$ is separable and hence second countable, by Pb. 8.13, $[0,1]^{\mathbb{Z}_+}$ is second countable. So its subsets are second countable by Prop. 8.29.[4]

[4] Alternatively, one can also use Thm. 8.34 to prove directly (1)⇒(3). However, as the reader can feel in Rem. 8.35, I personally don't like the proof of Thm. 8.34. I would rather *define* a metrizable compact space to be a topological space homeomorphic to a closed subset of $[0,1]^{\mathbb{Z}_+}$: adopting this definition, the proof of second countability will be more intuitive and less tricky. And after all, the method in the proof of Thm. 8.34 will never be used in the future.

(2)⇒(1′): Let $(U_n)_{n\in\mathbb{Z}_+}$ be an infinite countable basis of the topology of X. For each $m,n\in\mathbb{Z}_+$, if $\overline{U}_n\subset U_m$, we choose $f_{m,n}$ such that $\overline{U}_n\prec f\prec U_m$ (which exists by Urysohn lemma); otherwise, we let $f_{m,n}=0$. Then $\{f_{m,n}:m,n\in\mathbb{Z}_+\}$ separates points of X. (Proof: Choose distinct $x,y\in X$. Since $X\backslash\{y\}\in\mathrm{Nbh}_X(x)$, there exists U_m containing x and is contained in $X\backslash\{y\}$. By Lem. 15.27, there is n such that $\{x\}\subset U_n\Subset U_m$. Then $f_{m,n}(x)=1$ and $f_{m,n}(y)=0$.) Thus, as in (15.14), the map

$$X\rightarrow[0,1]^{\mathbb{Z}_+\times\mathbb{Z}_+}\qquad x\mapsto(f_{m,n}(x))_{m,n\in\mathbb{Z}_+}$$

restricts to a homeomorphism from X to a compact subset of $[0,1]^{\mathbb{Z}_+\times\mathbb{Z}_+}$. □

Exercise 15.38. Let $\mathscr{A}$ be a unital $\mathbb{R}$-algebra (resp. unital $\mathbb{C}$-algebra). Let $\mathbb{K}$ be $\mathbb{Q}$ (resp. $\mathbb{Q}+\mathbf{i}\mathbb{Q}$). Let $x_1,x_2,\dots$ be a possibly finite sequence of elements of $\mathscr{A}$. Let $\mathcal{E}=\mathbb{K}[x_1,x_2,\dots]$ be the set of polynomials of $x_1,x_2,\dots$ with coefficients in $\mathbb{K}$. Prove that $\mathcal{E}$ is a countable set.

Remark 15.39. Note that Prop. 15.16 is used (and is only used) in the proof of (1′)⇒(3) of Thm. 15.37. Our proof of Thm. 15.37 relies on Tychonoff theorem, which in turn relies on Zorn's lemma. However, in the proof of (1′)⇒(3) we only consider countable products of integrals. In this special case, the proof of Tychonoff theorem uses mathematical induction but not Zorn's lemma. (See the proof of Thm. 3.54 or Pb. 8.7.) Therefore, our proof of Thm. 15.37 does not rely on Zorn's lemma.

Remark 15.40. Since the proof of Prop. 15.16 uses Weierstrass approximation Thm. 14.45, the proof of "⇒(3)" in Thm. 15.37 also relies on Thm. 14.45. This is as expected. In fact, even if one wants to prove that $C([0,1],\mathbb{R})$ is separable, one needs Thm. 14.45. (See Cor. 14.46.) Therefore, it is fair to say that the separability of $C(X,\mathbb{R})$ is one of the most important applications of Weierstrass approximation (or SW) theorem.

Claim 15.41. *If X is a compact Hausdorff space satisfying one of the equivalent conditions in Thm. 15.37, then the SW Thm. 15.9 can be proved without using Zorn's lemma.*

Proof. Let $\mathscr{A}$ be a unital subalgebra of $C(X,\mathbb{R})$ separating points of X. Since $C(X,\mathbb{R})$ is separable (equivalently, second countable), so is $\mathscr{A}$ (by Prop. 8.29). Therefore, $\mathscr{A}$ has a countable dense subset $\mathcal{E}=\{f_1,f_2,\dots\}$ which clearly separates points of X. By enlarging $\mathcal{E}$ we assume that it is infinite. So (15.14) restricts to a homeomorphism from X to a closed subset of $S=[0,1]^{\mathbb{Z}_+}$. As in the proof of (1′)⇒(3) of Thm. 15.37, the polynomials of the coordinate functions of S, when restricted to $\Phi(X)$, form a dense subset of $C(\Phi(X),\mathbb{R})$. Pulling back this result to X, we conclude that polynomials of $f_1,f_2,\dots$ (which are in $\mathscr{A}$) form a dense subset of $C(X,\mathbb{R})$. □

15.8 Problems and supplementary material

We always let $\mathcal{V}$ be a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

15.8.1 SW theorems for LCH spaces

Problem 15.1. Let X be an LCH space with topology $\mathcal{T}$. Define a set $X^* = X \cup \{\infty\}$ where ∞ is a new symbol not in X.

1. Prove that the set

$$\mathfrak{U} = \mathcal{T} \cup \{X^* \backslash K : K \subset X \text{ is compact}\}$$

is a basis for a topology on X^*. Let $\mathcal{T}^*$ be the topology on X^* generated by $\mathfrak{U}$. Prove that $\mathcal{T}$ is the subspace topology of $\mathcal{T}^*$. (Namely, prove that $U \subset X$ is open iff $U = X \cap V$ for some open $V \subset X^*$.)

2. Prove that $(X^*, \mathcal{T}^*)$ is a compact Hausdorff space. We call $(X^*, \mathcal{T}^*)$ (or simply call X^*) the **one-point compactification** of X.

3. Prove that if X is not compact, then X is dense in X^*.

More generally, we define:

Definition 15.42. Let X be LCH. A **one-point compactification (OPC)** of X is a compact Hausdorff space $\widehat{X}$, together with an injective map $\varphi : X \to \widehat{X}$ such that $\widehat{X} \backslash \varphi(X)$ has exactly one element, and that φ restricts to a homeomorphism $\varphi : X \to \varphi(X)$. In particular, the X^* constructed in Pb. 15.1 is a one-point compactification of X.

Problem 15.2. Prove the uniqueness of one-point compactifications in the following sense. Let X be LCH with one-point compactifications $(\widehat{X}, \varphi)$ and $(\widetilde{X}, \psi)$. Then there is a unique homeomorphism $\Phi : \widehat{X} \to \widetilde{X}$ such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ \widehat{X} \arrow[rr, "\Phi", "\simeq"'] & & \widetilde{X} \end{tikzcd} \tag{15.15}$$

Example 15.43. If X is compact Hausdorff and contains at least two points, then for every $p \in X$, the subset $X \backslash \{p\}$ has OPC X (together with the inclusion map). Thus $[0, 1]$ has OPC $[0, 1] \cup \{2\}$. $(0, 1]$ has OPC $[0, 1]$. $(0, 1)$ has OPC $\mathbb{S}^1$ (the unit circle in $\mathbb{C}$) together with the map $\varphi : x \in (0, 1) \mapsto e^{2\mathbf{i}\pi x} \in \mathbb{S}^1$.

Convention 15.44. Let X be LCH and $f : X \to \mathcal{V}$. According to part 3 of Pb. 15.1, when X is not compact, the limit of functions $\lim_{x\to\infty} f$ makes sense. In the case that X is compact, we understand $\lim_{x\to\infty} f$ as 0.

Definition 15.45. Let X be LCH with one-point compactification X^*. Define

$$C_0(X,\mathcal{V}) = \{f \in C(X,\mathcal{V}) : \lim_{x\to\infty} f(x) = 0\} \tag{15.16}$$

Since $\{X^*\backslash K : K \subset X \text{ is compact}\}$ is a neighborhood basis of $\infty \in X^*$ (recall Def. 7.115), by Def. 7.81-(2), it is clear that $f \in C(X,\mathcal{V})$ belongs to $C_0(X,\mathcal{V})$ iff

> for every $\varepsilon > 0$ there exists a compact $K \subset X$ such that for each $x \in X\backslash K$ we have $\|f(x)\| < \varepsilon$

For each $f : X \to \mathcal{V}$ define $\widetilde{f} : X^* \to \mathcal{V}$ where

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X \\ 0 & \text{if } x = \infty \end{cases} \tag{15.17}$$

Then by Def. 7.81-(1), $\widetilde{f}$ is continuous at ∞ iff $\lim_{x\to\infty} f(x) = 0$. Thus, we have

$$C_0(X,\mathcal{V}) = \{\widetilde{f}|_X : \widetilde{f} \in C(X^*,\mathcal{V}) \text{ and } \widetilde{f}(\infty) = 0\} \tag{15.18}$$

Proposition 15.46. *Let X be LCH with one-point compactification X^*. Then we have an linear isometry of Banach spaces (under the l^∞-norms)*

$$C_0(X,\mathcal{V}) \to C(X^*,\mathcal{V}) \qquad f \mapsto \widetilde{f} = (15.17) \tag{15.19}$$

Proof. Clearly $\|f\|_{l^\infty(X,\mathcal{V})} = \|\widetilde{f}\|_{l^\infty(X^*,\mathcal{V})}$. So (15.19) is a linear isometry. The range of (15.19) is $\{g \in C(X^*,\mathcal{V}) : g(\infty) = 0\}$, which is clearly a closed subset of the Banach space $C(X,\mathcal{V})$ and hence is complete by Prop. 3.27. □

Convention 15.47. According to Prop. 15.46, people often identify $C_0(X,\mathcal{V})$ with its image under (15.19), i.e., identify $f \in C_0(X,\mathcal{V})$ with $\widetilde{f}$ defined by (15.17).

Problem 15.3. Let Y be a compact Hausdorff space. Let E be a closed subspace of Y. Prove that

$$C_0(Y\backslash E,\mathcal{V}) = \{g|_{Y\backslash E} : g \in C(Y,\mathcal{V}), g|_E = 0\}$$

Theorem 15.48 (SW theorem, LCH real version). *Let X be an LCH space. Let $\mathscr{A}$ be a subalgebra of $C_0(X,\mathbb{R})$. Assume that $\mathscr{A}$ separates points of X. Assume that $\mathscr{A}$* ***vanishes nowhere on*** *X (i.e. for every $x \in X$ there is $f \in \mathscr{A}$ such that $f(x) \neq 0$). Then $\mathscr{A}$ is dense in $C_0(X,\mathbb{R})$ (under the l^∞-norm).*

Problem 15.4. Prove the above SW theorem by following the steps below. Let $X^* = X \cup \{\infty\}$ be the one-point compactification of X. By Conv. 15.47, $\mathscr{A}$ is naturally a subalgebra of $C(X^*,\mathbb{R})$. Define

$$\mathscr{B} = \{f + \lambda : f \in \mathscr{A}, \lambda \in \mathbb{R}\}$$

which is a unital subalgebra of $C(X^*,\mathbb{R})$. Use SW Thm. 15.9 to prove that $\mathscr{B}$ is dense in $C(X^*,\mathbb{R})$. Use this fact to prove that $\mathscr{A}$ is dense in $C_0(X,\mathbb{R})$.

Theorem 15.49 (SW theorem, LCH complex version). *Let X be an LCH space. Let $\mathscr{A}$ be a *-subalgebra of $C_0(X,\mathbb{C})$. Assume that $\mathscr{A}$ separates points of X. Assume that $\mathscr{A}$ vanishes nowhere on X. Then $\mathscr{A}$ is dense in $C_0(X,\mathbb{C})$ (under the l^∞-norm).*

Proof. This follows from the LCH real version, just as the compact complex version of SW theorem (Thm. 15.11) follows from the compact real version (Thm. 15.9). □

Problem 15.5. Let X be LCH. Prove that $C_c(X,\mathcal{V})$ is dense in $C_0(X,\mathcal{V})$. (This proves that $C_0(X,\mathcal{V})$ is the Banach space completion of $C_c(X,\mathcal{V})$.) Conclude that $C_0(X,\mathcal{V})$ is separable iff $C_c(X,\mathcal{V})$ is separable.

Hint. Use Urysohn's lemma or Tietze extension. □

Definition 15.50. Let $\mathscr{A}$ be an $\mathbb{F}$-algebra. A subset $J\subset\mathscr{A}$ is called an **ideal** if J is an $\mathbb{F}$-linear subspace of $\mathscr{A}$ such that $\mathscr{A}J\subset J$ (i.e. $xy\in J$ for all $x\in\mathscr{A},y\in J$).

The following problem gives an interesting application of SW Thm. 15.48.

⋆ **Problem 15.6.** (**Nullstellensatz**) Let X be a compact Hausdorff space. For each closed subset $A\subset X$, let

$$I(A)=\{f\in C(X,\mathbb{R}):f|_A=0\}$$

Clearly $I(A)$ is a closed ideal of $C(X,\mathbb{R})$. For each closed ideal J of $C(X,\mathbb{R})$, let

$$N(J)=\{x\in X:f(x)=0\text{ for all }f\in J\}$$

which is a closed subset of X. Prove that $A\mapsto I(A)$ gives a bijection between the closed subsets of A and the closed ideals of $C(X,\mathbb{R})$, and that its inverse map is $J\mapsto N(J)$. In other words, for every closed subset $A\subset X$ and every closed ideal $J\subset C(X,\mathbb{R})$, prove that

$$N(I(A))=A\qquad I(N(J))=J \tag{15.20}$$

Hint. For both parts of (15.20) it is easy to prove "$\supset$". To prove $N(I(A))=A$, use Urysohn's lemma or the Tietze extension theorem. To prove $I(N(J))=J$, identify $I(N(J))$ with $C_0(X\backslash N(J),\mathbb{R})$ (cf. Pb. 15.3) and apply SW Thm. 15.48 to the LCH space $X\backslash N(J)$. □

15.8.2 Lebesgue measures of open sets

The theory of Riemann integrals on $\mathbb{R}$ can be easily generalized to $\mathbb{R}^N$ by partitioning boxes, i.e. $I_1\times\cdots\times I_N$ where each I_j is a compact interval in $\mathbb{R}$. In the following, we establish the basic theory of Riemann integrals on bounded subsets of $\mathbb{R}^N$ using partitions of unity. Compared to partitioning boxes, the methods provided below are closer to those in measure theory.

Definition 15.51. If $f \in C_c(\mathbb{R}^N, \mathbb{R})$, define $\displaystyle\int f \equiv \int_{\mathbb{R}} f = \int_B f$ where $B \subset \mathbb{R}^N$ is any box containing $\mathrm{Supp}(f)$. (See Def. 14.3.)

Problem 15.7. Let U be an open subset of $\mathbb{R}^N$. Define the **(Lebesgue) measure** of U to be

$$\mu(U) = \sup\Big\{ \int_{\mathbb{R}^N} f : f \in C_c(U, [0,1]) \Big\} \tag{15.21}$$

which is an element of $\overline{\mathbb{R}}_{\geqslant 0}$. It is clear that if V is an open subset of U then $\mu(V) \leqslant \mu(U)$.

1. In the case that $N = 1$ and $U = (a, b)$ (where $-\infty \leqslant a < b \leqslant +\infty$), prove that $\mu(U) = b - a$.

2. Suppose that U has compact closure. Prove that $\mu(U) < +\infty$. (Hint: use Urysohn's lemma.)

3. Let $(U_\alpha)_{\alpha \in \mathscr{A}}$ be an increasing net of open subsets of $\mathbb{R}^N$. Prove

$$\mu\Big(\bigcup_{\alpha \in \mathscr{A}} U_\alpha\Big) = \sup_{\alpha \in \mathscr{A}} \mu(U_\alpha)$$

4. Let $(V_j)_{j \in \mathscr{I}}$ be a (non-necessarily increasing) family of open subsets of $\mathbb{R}^N$. Prove that

$$\mu\Big(\bigcup_{j \in \mathscr{I}} V_j\Big) \leqslant \sum_{j \in \mathscr{I}} \mu(V_j)$$

(Hint: Use part 3 to reduce to the special case $\mu(U \cup V) \leqslant \mu(U) + \mu(V)$. Prove it using partition of unity.)

5. Let $(V_j)_{j \in \mathscr{I}}$ be a family of *mutually disjoint* open subsets of $\mathbb{R}^N$. Prove

$$\mu\Big(\bigcup_{j \in \mathscr{I}} V_j\Big) = \sum_{j \in \mathscr{I}} \mu(V_j)$$

Definition 15.52. A subset $E \subset \mathbb{R}^N$ is called a **(Lebesgue) null set** if for every ε there exists an open $U \subset \mathbb{R}^N$ containing E such that $\mu(E) < \varepsilon$.

Exercise 15.53. Show that a countable union of null sets is null.

Exercise 15.54. (This exercise is not need below.) Show that when $N = 1$, the above definition of null sets agrees with Def. 14.11. More generally, for arbitrary N, show that $E \subset \mathbb{R}^N$ is null iff for every $\varepsilon > 0$, E can be covered by boxes whose total volumes are $< \varepsilon$.

15.8.3 ⋆ Multiple Riemann integral

Problem 15.8. Let $f \in C_c(\mathbb{R}^N, \mathbb{R}_{\geqslant 0})$.

1. Let $M \geqslant 0$. Assume that there is an open $U \subset \mathbb{R}^N$ such that $f|_U \leqslant M$ and that $\mathrm{Supp}(f) \subset U$. Prove
$$\int_{\mathbb{R}^N} f \leqslant \mu(U) \cdot M$$

2. Let $\varepsilon \geqslant 0$. Assume that there is an open $U \subset \mathbb{R}^N$ such that $f|_U \geqslant \varepsilon$. Prove
$$\int_{\mathbb{R}^N} f \geqslant \mu(U) \cdot \varepsilon$$

Remark 15.55. Pb. 15.8 is easy but useful. It tells us that the values of $\int f$ and $\mu(U)$ are controlled by each other. For example, part 2 implies that if $\int f$ is small, then the measure of $\{x \in \mathbb{R}^N : f(x) > \varepsilon\}$ cannot be very big, and is converging to 0 as $\varepsilon \to +\infty$.

Definition 15.56. Let $f \in l^\infty(\mathbb{R}^N, \mathbb{R})$ have compact support. Define **upper integrals** and **lower integrals** to be

$$\overline{\int} f = \inf\Big\{\int g : g \in C_c(\mathbb{R}^N, \mathbb{R}), g \geqslant f\Big\}$$
$$\underline{\int} f = \sup\Big\{\int h : h \in C_c(\mathbb{R}^N, \mathbb{R}), h \leqslant f\Big\}$$

Clearly $\underline{\int} f \leqslant \overline{\int} f$. Moreover, it is clear that if $f \in C_c(\mathbb{R}^N, \mathbb{R})$ then $\underline{\int} f = \overline{\int} f = \int f$. In general, if $\underline{\int} f = \overline{\int} f$, we say that f is **Riemann integrable**, and define its integral $\int f$ to be $\overline{\int} f$. Let

$\mathscr{R}_c(\mathbb{R}^N, \mathbb{R}) = \{f \in l^\infty(\mathbb{R}^N, \mathbb{R}) : f$ has compact support and is Riemann integrable$\}$

Remark 15.57. Let $f \in l^\infty(\mathbb{R}^N, \mathbb{R})$ be compactly supported. It is clear from the definition that the following statements are equivalent:

(1) $f \in \mathscr{R}_c(\mathbb{R}^N, \mathbb{R})$.

(2) For every $\varepsilon > 0$ there exist $g, h \in C_c(\mathbb{R}^N, \mathbb{R})$ satisfying $h \leqslant f \leqslant g$ and $\int(g - h) < \varepsilon$.

In fact, (2) is easier to use than the original definition of Riemann integrability.

Exercise 15.58. (This exercise is not needed below.) Prove that when $N = 1$, the above definition of upper and lower integrals agrees with that in Pb. 13.1.

Problem 15.9. Let $f, g \in l^\infty(\mathbb{R}^N, \mathbb{R})$ have compact supports. Prove

$$\overline{\int}(f+g) \leqslant \overline{\int} f + \overline{\int} g \qquad \underline{\int}(f+g) \geqslant \underline{\int} f + \underline{\int} g$$

Prove that $\mathscr{R}_c(\mathbb{R}^N, \mathbb{R})$ is a linear subspace of $\mathbb{R}^{\mathbb{R}^N}$. Prove that $\int : \mathscr{R}_c(\mathbb{R}^N, \mathbb{R}) \to \mathbb{R}$ is linear. Prove that if $f, g \in \mathscr{R}_c(\mathbb{R}^N, \mathbb{R})$ then

$$f \leqslant g \qquad \Longrightarrow \qquad \int f \leqslant \int g \tag{15.22}$$

Problem 15.10. (**Fubini's theorem**) Let $X = \mathbb{R}^M, Y = \mathbb{R}^N$. Let $f \in \mathscr{R}_c(X \times Y, \mathbb{R})$. Prove that $x \in X \mapsto \underline{\int}_Y f(x,y)dy$ and $x \in X \mapsto \overline{\int}_Y f(x,y)dy$ are Riemann integrable, and that $\displaystyle\int_{X\times Y} f = \int_X \underline{\int}_Y f = \int_X \overline{\int}_Y f$.

Hint. Choose $g, h \in C_c(X \times Y, \mathbb{R})$ such that $g \leqslant f \leqslant h$ and $\int_{X\times Y}(h-g) < \varepsilon$. Apply Fubini's theorem for compactly supported continuous functions (available due to Thm. 14.1) to $g, h, h-g$. ((15.22) is also useful.) □

Remark 15.59. In the next semester, we will prove Fubini's theorem for Lebesgue measurable functions using more complicated methods. The goal of Pb. 15.10 is to show you how to prove Fubini's theorem for a large class of functions (sufficient for many applications) without using those methods. When studying mathematics, it is often important to know how to simplify a proof when the objects studied are simpler.

15.8.4 ⋆ Lebesgue's criterion for multiple Riemann integrals

Definition 15.60. As in (14.12), for each $f : \mathbb{R}^N \to \mathbb{R}$ and $\varepsilon > 0$, define

$$\Omega_\varepsilon(f) = \{x \in \mathbb{R}^N : \omega(f,x) \geqslant \varepsilon\} \tag{15.23}$$

where $\omega(f,x) = \inf_{U\in\mathrm{Nbh}(x)} \mathrm{diam} f(U)$. Note that by Lem. 14.19, $\Omega_\varepsilon(f)$ is a closed subset of $\mathrm{Supp}(f)$, and hence is compact when f is compactly supported.

Problem 15.11. Let $f \in \mathscr{R}_c(\mathbb{R}^N, \mathbb{R})$. Prove that $\Omega_\varepsilon(f)$ is a null set. (Thus, $\bigcup_{n\in\mathbb{Z}_+} \Omega_{1/n}(f)$ is null. This proves that the set of discontinuities of f is null.)

Hint. For each $\delta > 0$, choose $g, h \in C_c(\mathbb{R}^N, \mathbb{R})$ such that $h \leqslant f \leqslant g$ and $\int(g-h) \leqslant \delta\varepsilon$, which exist due to Rem. 15.57. Let $U = \{x \in \mathbb{R}^N : g(x) - h(x) > \varepsilon/2\}$. Use Pb. 15.8 to give a small upper bound of $\mu(U)$. Prove that $\Omega_\varepsilon(f) \subset U$. □

The following problem can be compared with Lem. 14.20.

Problem 15.12. Let $f \in l^\infty(\mathbb{R}^N, \mathbb{R})$ be supported in a compact set K. Let $W \subset \mathbb{R}^N$ be an open set containing K. Let $\varepsilon > 0$, and assume that for every $x \in \mathbb{R}^N$ we have $\omega(f, x) < \varepsilon$. Prove that there exist $g, h \in C_c(W, \mathbb{R})$ such that

$$g \leqslant f \leqslant h \qquad 0 \leqslant g - h \leqslant \varepsilon$$

Hint. First construct the functions locally. Then pass from local to global functions using a partition of unity of K subordinate to an open cover. □

Problem 15.13. Let $f \in l^\infty(\mathbb{R}^N, \mathbb{R})$ be compactly supported. Assume that the set of discontinuities of f is a null set. Prove that $f \in \mathscr{R}_c(\mathbb{R}^N, \mathbb{R})$.

Hint. Choose any $\varepsilon, \delta > 0$. Choose an open set U containing the compact set $\Omega_\varepsilon(f)$ such that $\mu(U) < \delta$. Use a partition of unity of $K = \mathrm{Supp}(f)$ subordinate to U and $V = \mathbb{R}^N \backslash \Omega_\varepsilon(f)$ to write $f = f_V + f_U$ where $f_V, f_U \in C_c(\mathbb{R}^N, \mathbb{R})$ are supported in $K \cap V$ and U respectively. Apply Pb. 15.12 to f_V to get $g_V \leqslant f_V \leqslant h_V$ with small $\int(h_V - g_V)$. Let $M = \|f\|_\infty$. Find $g_U \in C_c(\mathbb{R}^N, [0, M])$ such that $-g_U \leqslant f_U \leqslant g_U$. Show that $\int(g_U + h_V - (-g_U + g_V))$ is small. □

Definition 15.61. Let Ω be a bounded subset of $\mathbb{R}^N$. We say that $f : \Omega \to \mathbb{R}$ is **Riemann integrable** if $\widetilde{f} \in \mathscr{R}_c(\mathbb{R}^N, \mathbb{R})$, where $\widetilde{f}$ is the zero extension of f (i.e. $\widetilde{f}|_\Omega = f$ and $\widetilde{f}|_{X\backslash\Omega} = 0$). If f is Riemann integrable, we define

$$\int_\Omega f = \int_{\mathbb{R}^N} \widetilde{f}$$

Example 15.62. Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$ and $f \in C(D, \mathbb{R})$. Then the set of discontinuities of the zero extension $\widetilde{f}$ is a subset of $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. It is not hard to show that $\mathbb{S}^1$ is a null subset of $\mathbb{R}^2$. Since $\|f\|_\infty < +\infty$ by extreme value theorem, we conclude that f is Riemann integrable thanks to Lebesgue's criterion (Pb. 15.13). Clearly $\int_\mathbb{R} f(\cdot, y)dy$ is Riemann integrable. By Fubini's theorem (Pb. 15.10), we have

$$\int_D f = \int_{\mathbb{R}^2} \widetilde{f} = \int_{-1}^1 \int_{-1}^1 \chi_D \widetilde{f}(x, y)dydx = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y)dydx$$

15.8.5 Compactness and countability

The following problem shows that the equivalence (2)⇔(3) in Thm. 15.37 can be generalized to LCH spaces.

Problem 15.14. Let X be an LCH space with one-point compactification X^*. Prove that the following are equivalent:

(1) X^* is second countable.

(2) X is second countable.

(3) $C_0(X, \mathbb{R})$ is separable (equivalently, second countable).

Here, as usual, $C_0(X, \mathbb{R})$ is equipped with the l^∞-norm. Note also that by Pb. 15.5, (3) is equivalent to that $C_c(X, \mathbb{R})$ is separable.

⋆ **Remark 15.63.** By Pb. 15.14 and Thm. 15.37, if an LCH space X is second countable, then X^* is metrizable, and hence X is metrizable. However, a metrizable LCH space is not necessarily second countable. For example, let X be an uncountable set, equipped with the discrete topology $\mathcal{T} = 2^X$. Then $\mathcal{T}$ is induced by a metric d defined by $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$. X is LCH, but not second countable.

Problem 15.15. Let X be a compact Hausdorff space. Suppose that $(f_n)_{n \in \mathbb{Z}_+}$ is a sequence in $C(X, \mathbb{R})$ separating points of X. Assume for simplicity that $\|f_n\|_{l^\infty} \leqslant 1$ for all n. Use these functions to define an explicit metric on X inducing its topology $\mathcal{T}$, and construct an explicit countable basis for $\mathcal{T}$.

Note. This problem tests whether you truly understand the method of embedding and the proof of Thm. 15.37. Hint: How do you construct a metric and a countable basis for $[-1, 1]^{\mathbb{Z}_+}$? What do their pullbacks look like under the embedding $X \to [-1, 1]^{\mathbb{Z}_+}$ defined by $f_1, f_2, \dots$? □

16 Zorn's lemma and applications

> The Axiom of Choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma?
>
> —- Jerry L. Bona

16.1 Zorn's lemma

Theorem 16.1 (Zorn's lemma). *Let $(P, \leqslant)$ be a nonempty partially ordered set. Suppose that every totally oredered subset of P has an upper bound in P. Then there is a maximal element $p \in P$.*

By a **totally ordered subset** $Q \subset P$, we mean that Q satisfies that for every $x, y \in Q$, either $x \leqslant y$ or $y \leqslant x$. An upper bound of Q (in P) means an element $p \in P$ such that $x \leqslant p$ for all $x \in Q$. A **maximal element** $p \in P$ means that p satisfies $\{x \in P : x \geqslant p\} = \{p\}$.

Zorn's lemma and the axiom of choice are equivalent. (They are both equivalent to the so called "well-ordering principle".) Although it is easier to prove the axiom of choice from Zorn's lemma than the other way round, one may prefer to take the axiom of choice as the starting point, since axiom of choice is easier to grasp intuitively. Therefore, in the following, I will give a proof of Zorn's lemma under the assumption of axiom of choice, in order to comfort those who are obsessed with building a complete and rigorous mathematical theory in their heads from a few "self-evident" axioms. However, it is recommended that you skip or only skim this proof, since it is safe enough to assume that Zorn's lemma is an axiom that needs no proof. Although Zorn's lemma is far from "self-evident", knowing how to use it is much more important than knowing how to prove it. This is because in most areas of mathematics the ideas in the proof of Zorn's lemma are never used.

$\star\star$ ***Proof of Zorn's lemma.*** Let $(P, \leqslant)$ satisfy the assumption in Zorn's lemma but has no maximal element. We shall find a contradiction. By assumption, every totally ordered subset $A \subset P$ has an upper bound $p \in P$, and p is not maximal. So there exists $x_A > p$ in P. (Here, $x_A > p$ means that $x_A \geqslant p$ and $x \neq p$.) Thus, we have a function $A \mapsto x_A$ whose existence is due to axiom of choice.

Step 1. Fix $a \in P$ throughout the proof. For an arbitrary $\mathcal{F} \subset 2^P$, consider the following conditions:

(a) $\{a\} \in \mathcal{F}$.

(b) Every $A \in \mathcal{F}$ is a totally ordered subset of P.

(c) If $A \in \mathcal{F}$ then $A \cup \{x_A\} \in \mathcal{F}$.

(d) If $\mathcal{E}$ is a nonempty totally ordered subset of $\mathcal{F}$ (under the partial order $\subset$), then $\bigcup_{A\in\mathcal{E}} A \in \mathcal{F}$. (Note that if every $A \in P$ is a totally ordered subset of P, then so is $\bigcup_{A\in\mathcal{E}} A$.)

There exists at least one $\mathcal{F}$ satisfying the above conditions. For example, one can let $\mathcal{F}$ be the set of all totally ordered subsets of P containing a.

We let $\mathcal{F}$ be the intersection of all the subsets of 2^P satisfying the above four conditions. Then $\mathcal{F}$ clearly also satisfies these conditions. (In particular, $\mathcal{F} \neq \varnothing$ because $\{a\} \in \mathcal{F}$.) So $\mathcal{F}$ is the smallest subset of 2^P satisfying these four conditions.

We claim that $\mathcal{F}$ is a totally ordered subset of 2^P. Suppose this is true. Let $B = \bigcup_{A\in\mathcal{F}} A$. Then $B \in \mathcal{F}$ by condition (d). In particular, by (b), B is a totally ordered subset of P. Then x_B is defined as at the beginning of the proof and satisfies $x_B > B$. By (c), we have $B \cup \{x_B\} \in \mathcal{F}$. By the definition of B, we get $x_B \in B$. This is impossible. So we are done with the proof.

Step 2. Let us prove that $\mathcal{F}$ is totally ordered. Let

$$\mathcal{F}_0 = \{A \in \mathcal{F} : \text{every } B \in \mathcal{F} \text{ satisfies either } A \subset B \text{ or } B \subset A\}$$

It is not hard to check that $\mathcal{F}_0$ satisfies (a,b,d). It suffices to prove that $\mathcal{F}_0$ satisfies (c). Then we will have $\mathcal{F}_0 = \mathcal{F}$ and hence that $\mathcal{F}$ is totally ordered.

Let us prove (c) for $\mathcal{F}_0$. Choose any $A \in \mathcal{F}_0$. We need to prove that $A \cup \{x_A\} \in \mathcal{F}_0$. It suffices to prove that the following set equals $\mathcal{F}$:

$$\mathcal{F}_A = \{B \in \mathcal{F} : B \subset A \text{ or } A \cup \{x_A\} \subset B\}$$

One checks easily that $\mathcal{F}_A$ satisfies (a,b,d). To check that $\mathcal{F}_A$ satisfies (c), we choose any $B \in \mathcal{F}_A$. Then there are two possible cases:

- $A \cup \{x_A\} \subset B$ or $B = A$. Then $A \cup \{x_A\} \subset B \cup \{x_B\}$.
- $B \subsetneq A$. Then, since $A \in \mathcal{F}_0$ and since $B \cup \{x_B\} \in \mathcal{F}$ (because $B \in \mathcal{F}$ and $\mathcal{F}$ satisfies (c)), we have either $A \subsetneq B \cup \{x_B\}$ or $B \cup \{x_B\} \subset A$. The former case is clearly impossible. So $B \cup \{x_B\} \subset A$.

Thus, in both cases we have $B \cup \{x_B\} \in \mathcal{F}_A$. This proves that $\mathcal{F}_A$ also satisfies (c). So $\mathcal{F}_A = \mathcal{F}$. □

16.2 Comparison of Zorn's lemma and mathematical induction

Zorn's lemma can be viewed as the uncountable version of mathematical induction. Therefore, the best way to understand Zorn's lemma is to first use it to prove some classical results traditionally proved by mathematical induction.

Example 16.2. Let $n \in \mathbb{N}$ and $s_n = 0 + 1 + 2 + \cdots + n$. Then $s_n = \frac{n(n+1)}{2}$.

Proof. Let $\mathbb{N}^* = \mathbb{N} \cup \{+\infty\}$ with the usual order. Let

$$P = \Big\{n \in \mathbb{N}^* : s_i = \frac{i(i+1)}{2} \text{ for all } i \leqslant n \text{ and } i < +\infty\Big\}$$

P is nonempty since it contains 0. (Checking that P is nonempty is important, since it corresponds to checking the base case in mathematical induction.) P is totally ordered, and clearly every nonempty subset $Q \subset P$ has an upper bound $\sup Q$. Thus, by Zorn's lemma, P has a maximal element.

If $n < +\infty$, then from $s_n = \frac{n(n+1)}{2}$ and $s_{n+1} - s_n = n+1$ we have $s_{n+1} = \frac{(n+1)(n+2)}{2}$. (This step corresponds to checking "case n implies case $n+1$" in mathematical induction.) So $n+1 \in P$, contradicting the maximality of n. So $n = +\infty$, and hence $+\infty \in P$, finishing the proof. □

Example 16.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function not continuous at 0. Assume that $f(0) = 0$. Then there exists $\varepsilon > 0$ and a sequence $(x_n)_{n \in \mathbb{Z}_+}$ in $\mathbb{R}\backslash\{0\}$ converging to 0 such that $|f(x_n)| > \varepsilon$ for all n.

Proof. By assumption, there is $\varepsilon > 0$ such that for every $\delta > 0$, there exists $x \in (-\delta, \delta)\backslash\{0\}$ such that $|f(x)| > \varepsilon$. Let

$$\begin{aligned} P = \{&\text{finite or countably infinite sequence } x_\bullet = (x_1, x_2, \dots) \\ &\text{satisfying } 0 < |x_i| < 1/i \text{ and } |f(x_i)| > \varepsilon \text{ for all } i\} \end{aligned}$$

Then P is nonempty. (This corresponds to checking the base case in mathematical induction.) View each element of P as a subset of $\mathbb{Z}_+ \times \mathbb{R}$, and let $\subset$ be the partial order on P. Then every nonempty totally ordered subset $Q \subset P$ has an upper bound (defined by taking union). Thus, by Zorn's lemma, P has a maximal element $x_\bullet$.

Suppose that $x_\bullet$ has finite length $N \in \mathbb{Z}_+$. (So $x_\bullet = (x_1, \dots, x_N)$) Then by the first sentence of the proof, there exists x_{N+1} satisfying $0 < |x_{N+1}| < 1/(N+1)$ satisfying $|f(x_{N+1})| > 0$. (This step corresponds to checking "case n implies case $n+1$" in mathematical induction.) Then $(x_1, \dots, x_N, x_{N+1})$ belongs to P and is $> x_\bullet$, contradicting the maximality of $x_\bullet$. So $x_\bullet$ must be an infinite sequence. This finishes the construction of the sequence (x_n). □

16.3 Proof of the Tychonoff theorem

We first recall Pb. 8.6: Assume for simplicity that X, Y are compact spaces. Let $(x_\alpha, y_\alpha)_{\alpha \in \mathscr{I}}$ be a net in $X \times Y$ such that $x \in X$ is a cluster point of (x_α). Then there exists $y \in Y$ such that (x, y) is a cluster point of (x_α, y_α).

The proof is easy. By the definition of cluster points (Pb. 7.2-(1)), (x_α) has a subnet $(x_{\alpha_\beta})_{\beta \in \mathscr{J}}$ converging to x. Since Y is net-compact, (y_{α_β}) has a subnet $(y_{\alpha_{\beta_\gamma}})_{\gamma \in \mathscr{K}}$ converging to some $y \in Y$. Then $(x_{\alpha_{\beta_\gamma}}, y_{\alpha_{\beta_\gamma}})$ is a subnet of (x_α, y_α) converging to (x, y).

In the following, we present a proof of Tychonoff Thm. 15.15 which was due to Chernoff [Che92]. Our method is similar to the proof of the countable Tychonoff theorem (Pb. 8.7) which uses net-compactness. It is strongly recommended that you compare the proof with the one of Pb. 8.7, and even reprove Pb. 8.7 using Zorn's lemma.

***Proof of Tychonoff Thm. 15.15**.* Recall the setting that $(X_\alpha)_{\alpha\in\mathscr{I}}$ is a family of compact topological spaces. Assume WLOG that $\mathscr{I}$ and each X_α are nonempty. We want to prove that $S=\prod_{\alpha\in\mathscr{I}}X_\alpha$ is compact.

We first introduce a few notations. For each $I\subset\mathscr{I}$, let $S_I=\prod_{\alpha\in I}X_\alpha$. If $x\in S_I$, we write x as $(x(\alpha))_{\alpha\in I}$, and view it as a function with domain I and codomain $\mathfrak{X}=\bigcup_{\alpha\in\mathscr{I}}X_\alpha$. We write $\mathrm{Dom}(x)=I$. If $x\in S_I$ and $J\subset I$, the restriction $x|_J=(x(\alpha))_{\alpha\in J}$ is clearly in S_J.

Step 1. Let $(f_\mu)_{\mu\in\mathfrak{M}}$ be a net in S. Let

$$P=\bigcup_{I\subset\mathscr{I}}\{x\in S_I: x\text{ is a cluster point of }(f_\mu|_I)_{\mu\in\mathfrak{M}}\text{ in }S_I\}$$

Let "$\subset$" be the partial order on P (defined by identifying each element of P with its graph, which is an element of $\mathscr{I}\times\mathfrak{X}$). Thus $x\subset y$ iff $\mathrm{Dom}(x)\subset\mathrm{Dom}(y)$ and $y|_{\mathrm{Dom}(x)}=x$.

Clearly P is nonempty: Choose any $\alpha\in\mathscr{I}$. Since X_α is compact, $(f_\mu(\alpha))_{\mu\in\mathfrak{M}}$ has a cluster point in X_α. This point, viewed as a function from $\{\alpha\}$ to this point, belongs to P.[1]

We claim that every totally ordered subset of P has an upper bound. Suppose this is true. Then by Zorn's lemma, there is a maximal element $x\in P$. If $\mathrm{Dom}(x)\neq\mathscr{I}$, then by Pb. 8.6 (applied to $S_{\mathrm{Dom}(x)}\times X_\beta$ where $\beta\in\mathscr{I}\backslash\mathrm{Dom}(x)$), x can be extended to a function with larger domain which is again the cluster point of the restriction of $(f_\mu)_{\mu\in\mathfrak{M}}$ to that domain.[2] This proves that P has an element strictly larger than x. This is impossible. So we must have $\mathrm{Dom}(x)=\mathscr{I}$, finishing the proof.

Step 2. Let Q be a nonempty totally ordered subset of P. Let x be the union of the elements of Q. Let $K=\mathrm{Dom}(x)$. Then Q can be written in the form

$$Q=\{x|_I: I\in\mathscr{U}\}$$

where $\mathscr{U}$ is a totally ordered subset of 2^K and $K=\bigcup_{I\in\mathscr{U}}I$.

[1]This part corresponds to checking the base case in mathematical induction.

[2]This part corresponds to checking "case n implies case $n+1$" in mathematical induction. More precisely, it corresponds to constructing $(x(1),\dots,(x(n+1)))$ from $(x(1),\dots,x(n))$ in the proof of Pb. 8.7.

To prove that $x \in P$, let us use Pb. 7.2-(2) to prove that x is a cluster point of $(f_\mu|_K)_{\mu\in\mathfrak{M}}$.[3] Let W be a neighborhood of x in S_K. By the definition of product topology (Def. 7.71), we can shrink W to a smaller neighborhood of the form

$$W = \prod_{\alpha\in K} U_\alpha$$

where each U_α is a neighborhood of $x(\alpha)$, and there exists a finite subset $E \subset K$ such that $U_\alpha = X_\alpha$ for all $\alpha \in K\backslash E$. Choose $I \in \mathscr{U}$ containing E. The fact that $x|_I$ is a cluster point of $(f_\mu|_I)_{\mu\in\mathfrak{M}}$ implies (by Pb. 7.2-(2)) that $(f_\mu|_E)_{\mu\in\mathfrak{M}}$ is frequently in $\prod_{\alpha\in E} U_\alpha$. Therefore $(f_\mu|_K)_{\mu\in\mathfrak{M}}$ is frequently in W. This finishes the proof. □

16.4 Proof of the Hahn-Banach extension theorem

Recall from Rem. 11.25 that a **linear functional** on a vector space V over a field $\mathbb{F}$ is defined to be a linear map $V \to \mathbb{F}$.

Lemma 16.4. *Let V be a normed vector space over $\mathbb{R}$, and let M be a linear subspace. Let $\varphi \in M^* = \mathfrak{L}(M,\mathbb{R})$ with operator norm $\|\varphi\| \leqslant 1$. Assume that $e \in V\backslash M$, and let $\widetilde{M} = M + \mathbb{R}e$. Then φ can be extended to a linear functional $\widetilde{\varphi} : \widetilde{M} \to \mathbb{R}$ such that $\|\widetilde{\varphi}\| \leqslant 1$.*

⋆ *Proof.* Let $A \in \mathbb{R}$ whose value will be determined later. Since any vector in $\widetilde{M}$ can be written uniquely as $x - \lambda e$ where $x \in M$ and $\lambda \in \mathbb{R}$, we can define

$$\widetilde{\varphi} : \widetilde{M} \to \mathbb{R} \qquad \widetilde{\varphi}(x - \lambda e) = \varphi(x) - \lambda A$$

It remains to prove $\|\widetilde{\varphi}\| \leqslant 1$ (for some A). This means that we want to prove $|\varphi(x) - \lambda A| \leqslant \|x - \lambda e\|$ for all $x \in M, \lambda \in \mathbb{R}$. Clearly this is true when $\lambda = 0$. Assume $\lambda \neq 0$. Then replacing x by λx and dividing both sides by λ, we see that it suffices to prove

$$|\varphi(x) - A| \leqslant \|x - e\|$$

for all $x \in V$, or equivalently,

$$\varphi(x) - \|x - e\| \leqslant A \leqslant \varphi(x) + \|x - e\|$$

To prove the existence of A satisfying the above inequalities for all $x \in V$, it suffices to prove

$$\sup_{x\in V} \big(\varphi(x) - \|x - e\|\big) \leqslant \inf_{y\in V} \big(\varphi(y) + \|y - e\|\big) \tag{16.1}$$

[3] This part corresponds to showing that x is a cluster point of (f_α) in the proof of Pb. 8.7.

namely, to prove that $\varphi(x) - \|x-e\| \leqslant \varphi(y) + \|y-e\|$ for all $x, y \in V$. Using $\|\varphi\| \leqslant 1$, we compute

$$\varphi(x) - \varphi(y) = \varphi(x-y) \leqslant \|(x-e)-(y-e)\| \leqslant \|x-e\| + \|y-e\|$$

□

Theorem 16.5 (Hahn-Banach extension theorem). *Let V be a normed vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let M be an $\mathbb{F}$-linear subspace of V. Let $\varphi \in M^* = \mathfrak{L}(M, \mathbb{F})$. Then there exists $\Phi \in V^*$ such that $\Phi|_M = \varphi$ and $\|\Phi\| = \|\varphi\|$.*

Proof. We first consider the case $\mathbb{F} = \mathbb{R}$. Assume WLOG that $\varphi \neq 0$. By scaling φ, we assume WLOG that $\|\varphi\| = 1$. Let

$$\begin{aligned} P = \big\{(W, \Phi) :& W \text{ is a linear subspace of } V \text{ and contains } M \\ & \Phi \in \mathbb{R}^W \text{ is linear and satisfies } \Phi|_M = \varphi, \|\Phi\| = 1\big\} \end{aligned}$$

Then P is nonempty since it contains (M, φ). Define a partial order on P by setting $(W, \Phi) \leqslant (W', \Phi')$ whenever $W \subset W'$ and $\Phi = \Phi'|_W$.

Suppose that Q is a totally ordered subset of P. Let $\widetilde{W} = \bigcup_{(W,\Phi) \in Q} W$, and let $\widetilde{\Phi}$ be the union of the functions Φ over all $(W, \Phi) \in Q$ (by taking the union of the graphs of the functions). So $\widetilde{\Phi} : W \rightarrow \mathbb{R}$ is the unique functions satisfying $\widetilde{\Phi}|_W = \Phi$ for all $(W, \Phi) \in Q$. Then it is easy to see that $(\widetilde{W}, \widetilde{\Phi})$ belongs to P and is an upper bound of Q.

Thus, we can use Zorn's lemma, which says that P has a maximal element (W, Φ). If $W \neq V$, we let $e \in V \backslash W$. Then by Lem. 16.4, Φ can be extended to $\widetilde{\Phi} \in \widetilde{W}^*$ where $\widetilde{W} = W + \mathbb{R}e$, and $\|\widetilde{\Phi}\| = 1$. So $(\widetilde{W}, \widetilde{\Phi})$ belongs to P and is strictly larger than (W, Φ), impossible. So $W = V$.

We are done with the proof for the case $\mathbb{F} = \mathbb{R}$. Now assume $\mathbb{F} = \mathbb{C}$. By Pb. 13.2, the real part $\mathrm{Re}\varphi : M \rightarrow \mathbb{R}$ sending v to $\mathrm{Re}(\varphi(v))$ is linear with operator norm $\|\mathrm{Re}\varphi\| = \|\varphi\|$. By the real Hahn-Banach, $\mathrm{Re}\varphi$ can be extended to $\Lambda \in \mathfrak{L}(V, \mathbb{R})$ with $\|\Lambda\| = \|\varphi\|$. By Pb. 13.2, there exists a unique $\Phi \in \mathfrak{L}(V, \mathbb{C})$ with real part Λ such that $\|\Phi\| = \|\varphi\|$. Since $\mathrm{Re}\Phi|_M = \Lambda|_M = \mathrm{Re}\varphi$, by Pb. 13.2, we have $\Phi|_M = \varphi$. □

In this course, we only use the following special case of Hahn-Banach theorem.

Corollary 16.6 (Hahn-Banach). *Let V be a nonzero normed vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Then for every $v \in V$, there exists a nonzero $\varphi \in V^*$ such that $\langle \varphi, v \rangle = \|\varphi\| \cdot \|v\|$. Consequently, the linear map*

$$V \rightarrow V^{**} \qquad v \mapsto \langle \cdot, v \rangle \tag{16.2}$$

is an isometry.

More precisely, the map (16.2) sends each $\varphi \in V^*$ to $\langle \varphi, v \rangle$. By scaling φ, Hahn-Banach implies that for each $v \in V$, there is $\varphi \in V^*$ with $\|\varphi\| = 1$ such that $\langle \varphi, v \rangle = \|v\|$.

Proof. Let $v \in V$, and assume WLOG that $v \neq 0$. Let $\varphi : \mathbb{F}v \to \mathbb{F}$ send λv to $\lambda \|v\|$. Then φ is linear and has operator norm 1, and $\langle \varphi, v \rangle = \|v\|$. By Hahn-Banach Thm. 16.5, φ can be extended to a bounded linear $V \to \mathbb{F}$ with operator norm 1. This is a desired linear functional.

Denote (16.2) by Ψ. To show that Ψ is an isometry, we need to prove that for every $v \in V$, $\Psi(v) : V^* \to \mathbb{F}$ has operator norm $\|v\|$. Choose any $\varphi \in V^*$. Then

$$\langle \Psi(v), \varphi \rangle = \langle \varphi, v \rangle \leqslant \|\varphi\| \cdot \|v\|$$

where "$\leqslant$" is "$=$" for some nonzero φ by the first paragraph. Therefore, by Rem. 10.24, we obtain $\|\Psi(v)\| = \|v\|$. □

16.5 Problems and supplementary material

Problem 16.1. Let V be a vector space over a field $\mathbb{F}$. Use Zorn's lemma to prove that V has a basis, i.e. a set E of linearly independent vectors such that any vector of V can be written as a (finite) linear combination of elements of E.

Problem 16.2. Let X be a set. Let $E \subset X$ be an infinite subset such that $X \backslash E$ is countable. (Recall that finite sets are also countable.) Prove that $\mathrm{card}(X) = \mathrm{card}(E)$.

Problem 16.3. Let X be an infinite set. Use Zorn's lemma to prove that X can be written as a countably infinite union of subsets $X = \bigsqcup_{n=1}^{\infty} X_n$ such that $\mathrm{card}(X_i) = \mathrm{card}(X_j)$ for each i, j.

Hint. Assume WLOG that X is uncountable. Consider

$$P = \Big\{\big((A_n)_{n\in\mathbb{Z}_+}, (\varphi_n)_{n\in\mathbb{Z}_+}\big) : A_1, A_2, \dots \text{ are mutually disjoint subsets of } X \\ \varphi_n : A_n \to A_{n+1} \text{ is a bijection (for every } n \in \mathbb{Z}_+)\Big\}$$

which is nonempty (why?). Define a suitable partial order on P. □

Theorem 16.7. *Let X be an infinite set, and let Y be a nonempty countable set. Then* $\mathrm{card}(X) = \mathrm{card}(X \times Y)$

Proof. By Pb. 16.3, we have $X \approx A \times \mathbb{N}$ for some subset $A \subset X$. Since $\mathbb{N} \times Y$ is infinite and countable, we have $\mathbb{N} \times Y \approx \mathbb{N}$. Therefore

$$X \times Y \approx A \times \mathbb{N} \times Y \approx A \times \mathbb{N} \approx X$$

□

⋆ **Problem 16.4.** Let E, F be two bases of a vector space V. Use Thm. 16.7 to prove that $\mathrm{card}(E) = \mathrm{card}(F)$. (When one of E, F is finite, this result was proved in linear algebra. You can assume this in your proof.)

Problem 16.5. Let V be a *separable* normed vector space over $\mathbb{R}$. Prove Hahn-Banach Thm. 16.5 for V using mathematical induction instead of Zorn's lemma. (You will need Prop. 10.28 in the proof.)

17 Compactness and completeness revisited

Tychonoff theorem asserts the compactness of function spaces under the pointwise convergence topology. In application, we are often interested in the compactness of function spaces satisfying certain additional condition such as the continuity.

Example 17.1. $\mathfrak{X} = C([0,1],[0,1])$ is not compact under either the pointwise convergence topology or the uniform convergence topology.

Proof. We first choose the pointwise convergence topology. Choose any sequence (f_n) in $\mathfrak{X}$ converging pointwise to a non-continuous function f. Then (f_n) has no subnet converging in $\mathfrak{X}$, since any subnet converging to $g \in \mathfrak{X}$ must satisfy $f = g$ and hence f is continuous. This is impossible. So $\mathfrak{X}$ is not compact.

Now we choose the uniform convergence topology for $\mathfrak{X}$, which is metrizable by the l^∞-norm. Let (f_n) be a sequence in $\mathfrak{X}$ such that $\|f_m - f_n\|_\infty = 1$ whenever $m \neq n$. (For example, one chooses f_n such that $\mathrm{Supp} f_n \subset I_n = (\frac{1}{2n+2}, \frac{1}{2n+1})$ and that $f_n(\xi_n) = 1$ for some $\xi_n \in I_n$.) Then every subsequence of (f_n) is not Cauchy and hence not convergent. Therefore, $\mathfrak{X}$ is not sequentially compact, and hence not compact. □

Note that this example does not contradict Tychonoff theorem. In fact, if we choose the pointwise convergence topology, then Tychonoff theorem implies that every sequence (f_n) in $C([0,1],[0,1])$ has a subnet convergent to some $f \in [0,1]^{[0,1]}$. However, one cannot deduce the continuity of f. To prove that the limit function is continuous, we need additional assumptions on the sequence (f_n). For example, Cor. 9.21 tells us that we need the equicontinuity.

17.1 Precompactness in function spaces

Since $C(X,[0,1])$ is in general not compact, one should search for compact subsets of $C(X,[0,1])$, or more generally, subsets with compact closures in $C(X,[0,1])$. You know what precompact sets mean geometrically: In $\mathbb{R}^N$, precompact subsets of $\mathbb{R}^N$ are exactly bounded subsets of $\mathbb{R}^N$. If $\Omega \subset \mathbb{R}^N$, then precompact subsets of Ω are precisely bounded subsets of $\mathbb{R}^N$ whose closures are inside Ω, cf. Rem. 15.20.

However, when studying function spaces, it is often more convenient to use another description of precompactness. As we shall see in Cor. 17.5, saying that $\mathscr{A} \subset C(X,[0,1])$ is precompact in $C(X,[0,1])$ is equivalent to saying that every net (f_α) in $\mathscr{A}$ converges to some $f \in C(X,[0,1])$.

Recall from Def. 8.39 that a subset A of a Hausdorff space X is called **precompact** if A is contained in a compact subset of X, or equivalently, if $\overline{A}$ is compact.

Proposition 17.2. *Let X be a metrizable topological space, and let $A \subset X$. Then the following are equivalent.*

(1) A is precompact.

(2) Every net in A has a cluster point in X.

(3) Every sequence in A has a cluster point in X.

From the following proof, it is clear that (1)⇒(2) holds even without assuming that X is metrizable.

Proof. (1)⇒(2): Since $\overline{A}$ is compact, every net in A has a cluster point in $\overline{A}$ and hence in X.

(2)⇒(3): Obvious.

(3)⇒(1): See Pb. 8.9. □

17.1.1 ⋆ Precompactness in regular spaces

(Since this section is a starred section, I will not use the results proved here in future sections. However, the material of this section is helpful for a better understanding of precompactness in function spaces.)

You may wonder to what general topological space the equivalence (1)⇔(2) generalizes. This equivalence is not true for an arbitrary topological space, but is true for regular spaces (recall Def. 9.20). Metrizable spaces are clearly regular. More generally, we have:

Example 17.3. Every subset of a regular space is regular. Every LCH space is regular. Products of regular spaces are regular.

Proof. Assume that X is regular and $A \subset X$. For each $x \in A$, choose a neighborhood of x in A, which must be of the form $U \cap A$ where $U \in \mathrm{Nbh}_X(x)$. Since X is regular, there is $V \in \mathrm{Nbh}_X(x)$ such that $\overline{V} \subset U$. So $\mathrm{Cl}_A(V \cap A) \subset \overline{V} \cap A \subset U \cap A$. So A is regular. That LCH spaces are regular follows from Lem. 15.27 (together with Rem. 15.20).

Let $S = \prod_{\alpha\in\mathscr{I}} X_\alpha$ where each X_α is regular. Choose $x = (x_\alpha)_{\alpha\in\mathscr{I}}$ in S. Choose a neighborhood of x which, after shrinking, is of the form $\prod_\alpha U_\alpha$ where $U_\alpha \in \mathrm{Nbh}_{X_\alpha}(x_\alpha)$, and there is a finite subset $E \subset \mathscr{I}$ such that $U_\alpha = X_\alpha$ if $\alpha \in \mathscr{I}\backslash E$. If $\alpha \in E$, let V_α be a neighborhood of x such that $\overline{V}_\alpha \subset U_\alpha$. If $\alpha \in \mathscr{I}\backslash E$, let $V_\alpha = X_\alpha$. Then one checks easily that $\prod_{\alpha\in\mathscr{I}} V_\alpha$ is a neighborhood of x with closure $\prod_{\alpha\in\mathscr{I}} \overline{V}_\alpha$, which is clearly inside $\prod_\alpha U_\alpha$. So S is regular. □

Theorem 17.4. *Let X be a regular topological space, and let $A \subset X$. Then the following are equivalent.*

(1) $\overline{A}$ is compact.

(2) A is contained in a compact subset of X.

(3) *Every net in A has a cluster point in X.*

Thus, although regular spaces are not necessarily Hausdorff, the two equivalent definitions of precompact subsets of Hausdorff spaces in Def. 8.39 are also equivalent in regular spaces. (However, most important examples of regular spaces are also Hausdorff. Regular Hausdorff spaces are called **T3 spaces**.)

Proof. "(1)⇒(2)" is obvious. "(2)⇒(3)" is also obvious: if $X \subset K$ where K is a compact subset of X, then every net in A is a net in K, which has a cluster point in K and hence in X.

Assume (3). Let us show that $\overline{A}$ is compact by showing that every nonempty net $(x_\alpha)_{\alpha\in I}$ in $\overline{A}$ has a cluster point in $\overline{A}$. For each $\gamma \in I$, let $E_\gamma = \{x_\alpha : \alpha \in I, \alpha \geqslant \gamma\}$. Define

$$J = \big\{(U,\gamma) \in 2^X \times I : U \text{ is open and contains } E_\gamma\big\}$$

It is not hard to check that J is a directed set if we define its preorder "$\leqslant$" to be

$$(U,\gamma) \leqslant (U',\gamma') \qquad \Longleftrightarrow \qquad U \supset U', \gamma \leqslant \gamma'$$

For each $(U,\gamma) \in J$, since $E_\gamma \subset U$, U intersects $\overline{A}$, and hence intersects A. Therefore, we can choose some $y_{U,\gamma} \in A \cap U$. In this way, we get a net $(y_{U,\gamma})_{(U,\gamma)\in J}$ in A. By (3), this net has a cluster point $x \in X$. So clearly $x \in \overline{A}$. Let us prove that x is also a cluster point of $(x_\alpha)_{\alpha\in I}$.

Assume that x is not a cluster point of $(x_\alpha)_{\alpha\in I}$. Then, by the definition of cluster points (Pb. 7.2-(2)), there exists $\Omega \in \mathrm{Nbh}_X(x)$ such that (x_α) is not frequently in Ω, i.e., eventually outside Ω. So there exists $\gamma \in I$ such that $E_\gamma \subset X\backslash\Omega$. Since X is regular, there exists $V \in \mathrm{Nbh}_X(x)$ such that $\overline{V} \subset \Omega$. Let $U = X\backslash\overline{V}$. Then $E_\gamma \subset U$, and hence $(U,\gamma) \in J$.

Since x is a cluster point of $y_\bullet$, for the neighborhood V of x, there exists $(U',\gamma') \geqslant (U,\gamma)$ in J such that $y_{U',\gamma'} \in V$. By the definition of the net $y_\bullet$, we have $y_{U',\gamma'} \in A \cap U' \subset U$. This is impossible, since $U \cap V = \emptyset$. □

Corollary 17.5. *Let X be a topological space, and let Y be a metric space. Equip $C(X,Y)$ with either the pointwise convergence topology or the uniform convergence topology. (Note that both topologies are Hausdorff.) Let $\mathscr{A}$ be a subset of $C(X,Y)$. Then the following are equivalent:*

(1) *$\mathscr{A}$ is precompact.*

(2) *Every net $(f_\alpha)_{\alpha\in\mathscr{I}}$ in $\mathscr{A}$ has a subnet converging to some $f \in C(X,Y)$.*

Proof. If the topology $\mathcal{T}$ on $C(X,Y)$ is the pointwise convergence topology, then $C(X,Y)$ is a subspace of Y^X (equipped with the product topology). By Exp. 17.3, $C(X,Y)$ is regular. So one can use Thm. 17.4 to prove (1)⇔(2). If $\mathcal{T}$ is the uniform convergence topology, then $C(X,Y)$ is metrizable (and hence also regular). So one can also use Thm. 17.4 (or even Prop. 17.2) to prove (1)⇔(2). □

17.2 Equicontinuity and precompactness; the Arzelà-Ascoli theorem

17.2.1 Precompactness under pointwise convergence topology

Definition 17.6. Let X be a set and Y be a metric space. A subset $\mathscr{A} \subset Y^X$ is called **pointwise bounded** if

$$\mathscr{A}(x) = \{f(x) : f \in \mathscr{A}\} \tag{17.1}$$

is a bounded subset of Y for every $x \in X$.

Theorem 17.7. *Let X be a topological space, and equip $C(X, \mathbb{R}^N)$ with the pointwise convergence topology. Let $\mathscr{A}$ be an equicontinuous and pointwise bounded subset of $C(X, \mathbb{R}^N)$. Then $\mathscr{A}$ is precompact in $C(X, \mathbb{R}^N)$, and $\overline{\mathscr{A}}$ is equicontinuous.*

Proof. Write $Y = \mathbb{R}^N$. Let us show that $\overline{\mathscr{A}} = \mathrm{Cl}_{C(X,Y)}(\mathscr{A})$ is equicontinuous. Since $\mathscr{A}$ is equicontinuous, for every $x \in X$ and $\varepsilon > 0$, there exists $U \in \mathrm{Nbh}(x)$ such that $\mathrm{diam}(f(U)) \leqslant \varepsilon$ for all $f \in \mathscr{F}$. Since each $g \in \overline{\mathscr{A}}$ is the pointwise limit of a net in $\mathscr{A}$, we also have $\mathrm{diam}(g(U)) \leqslant \varepsilon$. This proves that $\overline{\mathscr{A}}$ is equicontinuous at x.

Now let us show that $\overline{\mathscr{A}}$ is compact. Choose any net (f_α) in $\overline{\mathscr{A}}$. It is clear that $\overline{\mathscr{A}}(x)$ is pointwise bounded. By Heine-Borel, for each $x \in X$, $\overline{\mathscr{A}}(x)$ is contained in a compact subset $K_x \subset Y$. So $\overline{\mathscr{A}}$ is contained in $S = \prod_{x \in X} K_x$ where S is compact by Tychonoff theorem. Therefore, (f_α) has a subnet (f_β) converging pointwise to some $f : X \to Y$. Since (f_β) is equicontinuous (as $\overline{\mathscr{A}}$ is equicontinuous), by Cor. 9.21, $f \in C(X, Y)$.[1] So $f \in \mathrm{Cl}_{C(X,Y)}(\overline{\mathscr{A}}) = \overline{\mathscr{A}}$ since f can be approximated by elements of $\overline{\mathscr{A}}$. This proves that $\overline{\mathscr{A}}$ is compact. □

Remark 17.8. Theorem 17.7 is of fundamental importance because most compactness results about function spaces (such as Arzelà-Ascoli Thm. 17.15, Banach-Alaoglu Thm. 17.21) are derived from this theorem. As we will see in Claim 17.11, when X is separable, the proof of Thm. 17.7 uses only the countable version of Tychonoff theorem, and hence not using Zorn's lemma. This is in line with the history that Thm. 17.7 (which is implicit in the proof of Arzelà-Ascoli theorem) appeared earlier than Zorn's lemma and was proved using diagonal method. (If you remember, the proof of countable Tychonoff theorem uses diagonal method, cf. Thm. 3.54. And we will use Thm. 3.54 to prove Claim 17.11.)

In the late 19th and early 20th centuries, the diagonal method was often used to derive compactness properties of function spaces. Prominent examples are Hilbert's and Schmidt's solutions of eigenvalue problems in integral equations (cf. Subset. 10.4.1) and F. Riesz's solution of moment problems (cf. Rem. 17.34). Thus, Thm. 17.7 can be viewed as a summary of this method. □

[1] Alternatively, by the argument in the first paragraph, $\{f_\beta : \text{all } \beta\} \cup \{f\}$ is equicontinuous. So f is continuous.

The following exercise is a variant of Thm. 17.7. Another variant, the Banach-Alaoglu theorem, will be discussed in the next section.

Exercise 17.9. Let X be a metric space, and equip $C(X,\mathbb{R}^N)$ with the pointwise convergence topology. Let $\mathscr{A}$ be a pointwise bounded subset of $C(X,\mathbb{R}^N)$. Assume that $\mathscr{A}$ has a uniform Lipschitz constant $L<+\infty$. (Namely, $\|f(x)-f(y)\|\leqslant L\cdot d(x,y)$ for all $f\in\mathscr{A}$ and $x,y\in X$.) Prove that $\mathscr{A}$ is precompact in $C(X,\mathbb{R}^N)$, and $\overline{\mathscr{A}}$ has a uniform Lipschitz constant L. [2]

The proof of Thm. 17.7 uses Tychonoff theorem for uncountable product spaces, and hence relies on Zorn's lemma. In the following, we shall show that Zorn's lemma is not needed when X is separable. We first need a preparatory result: the following proposition is the equicontinuous analog of Prop. 9.27.

Proposition 17.10. *Let $\mathcal{V}$ be a Banach space. Let X be a topological space. Let $(f_\alpha)_{\alpha\in I}$ be an equicontinuous net in $C(X,\mathcal{V})$ converging pointwise on a dense subset E of X. Then (f_α) converges pointwise on X to some $f\in C(X,\mathcal{V})$.*

It follows that if (f_α) also converges pointwise on E to some $g\in C(X,\mathcal{V})$, then (f_α) converges pointwise on X to g. (Indeed, since $f|_E=g|_E$, we have $f=g$ because f,g are continuous and E is dense.)

Proof. Let $x\in X$. Since $\mathcal{V}$ is complete, to show that $(f_\alpha(x))$ converges, it suffices to prove that $(f_\alpha(x))_{\alpha\in I}$ is a Cauchy net. Choose any $\varepsilon>0$. Since (f_α) is equicontinuous at x, there exists $U\in\mathrm{Nbh}_X(x)$ such that $\mathrm{diam}(f_\alpha(U))<\varepsilon$ for all α. Since E is dense in X, E intersects U. Pick $p\in E\cap U$. Since $(f_\alpha(p))$ is a Cauchy net, we have $\lim_{\alpha,\beta\in I}\|f_\alpha(p)-f_\beta(p)\|=0$. Then

$$\begin{aligned}\|f_\alpha(x)-f_\beta(x)\|&\leqslant\|f_\alpha(x)-f_\alpha(p)\|+\|f_\alpha(p)-f_\beta(p)\|+\|f_\beta(p)-f_\beta(x)\|\\ &\leqslant\|f_\alpha(p)-f_\beta(p)\|+2\varepsilon\end{aligned}$$

where the RHS converges to 2ε under $\limsup_{\alpha,\beta\in I}$. Therefore $\limsup_{\alpha,\beta\in I}\|f_\alpha(x)-f_\beta(x)\|$ is $\leqslant 2\varepsilon$. Since ε is arbitrary, we conclude $\limsup_{\alpha,\beta\in I}\|f_\alpha(x)-f_\beta(x)\|=0$.

We have proved that (f_α) has a pointwise limit $f:X\to\mathcal{V}$. Since (f_α) is equicontinuous, by Cor. 9.21, f is continuous. □

Claim 17.11. *When X is a separable topological space, Thm. 17.7 can be proved without using Zorn's lemma.*

We know that separable is equivalent to second countable when X is metrizable, but is slightly weaker in general (cf. Sec. 8.5). In practice, however, almost all separable topological spaces you will encounter are Hausdorff and second countable. So there is no need to count the nuances of separability and second countability.

[2]In fact, instead of assuming that $\mathscr{A}$ is pointwise bounded, it suffices to assume that $\mathscr{A}(x)$ is bounded for some $x\in X$. Then the pointwise boundedness will follow automatically from the uniform Lipschitz continuity. Can you see why?

Proof. Let $Y = \mathbb{R}^N$. As in the proof of Thm. 17.7, $\overline{\mathscr{A}} = \mathrm{Cl}_{C(X,Y)}(\mathscr{A})$ is equicontinuous and pointwise bounded. Let (f_α) be a net in $\overline{\mathscr{A}}$. Since X is separable, we can choose a countable dense subset $E \subset X$. For each $p \in E$, $\overline{\mathscr{A}}(p)$ is contained in a compact $K_p \subset Y$. By the countable Tychonoff theorem (whose proof does not rely on Zorn's lemma, see Thm. 3.54 or Pb. 8.7), $\prod_{p\in E} K_p$ is compact. Therefore, (f_α) has a subnet (f_β) converging pointwise on E. Since $\overline{\mathscr{A}}$ is equicontinuous, so is (f_β). Therefore, by Prop. 17.10, (f_β) converges pointwise on X to some $f \in C(X,Y)$. So $f \in \mathrm{Cl}_{C(X,Y)}(\overline{\mathscr{A}}) = \overline{\mathscr{A}}$. This proves that $\overline{\mathscr{A}}$ is compact. □

Remark 17.12. Claim 17.11 and its proof can be compared with Pb. 16.5. In particular, Prop. 10.28 plays the same role in the solution of Pb. 16.5 as Prop. 17.10 does in the proof of Claim 17.11. In both situations, if one wants to prove the separable case without using Zorn's lemma, one needs an extra analytic step to pass from a dense subset to the original space.

Remark 17.13. The readers may wonder why I often give two proofs of the same theorem, one using Zorn's lemma, which applies to a (slightly) larger setting and is somewhat simpler, and the other not using Zorn's lemma, but requires more extra steps. I have mentioned that Zorn's lemma is equivalent to the axiom of choice. However, whether or not to accept the axiom of choice is a matter of taste (or faith). After all, both the statements and the proof of Zorn's lemma are very hard to understand intuitively (at least to me).

Nowadays, most mathematicians accept it because it often simplifies proofs and theories, it often proves theorems for a larger class of examples, and it is compatible with the theorems proved without using it. Therefore, the more theorems that can be proved both with and without Zorn's lemma, the more reason there is to believe in Zorn's lemma/axiom of choice. If there is any way to understand Zorn's lemma intuitively, it is to compare a proof using Zorn's lemma with a proof of the same theorem without using it, as we did in Sec. 15.7 and Ch. 16 and continue to do in this chapter. □

17.2.2 Precompactness under uniform convergence topology

In the last subsection, we see that equicontinuity implies precompactness under the pointwise convergence topology. The converse is not necessarily true. In this section, we will see that under reasonable assumptions, equicontinuity is equivalent to precompactness under the uniform convergence topology.

Theorem 17.14. *Let X be a topological space. Let $\mathcal{V}$ be a Banach space. Equip $C(X,\mathcal{V})$ with the uniform convergence metric. Let $\mathscr{A}$ be a precompact subset of $C(X,\mathcal{V})$. Then $\mathscr{A}$ is equicontinuous.*

First proof. It suffices to prove that $\overline{\mathscr{A}}$ is equicontinuous. Thus, by replacing $\mathscr{A}$ with $\overline{\mathscr{A}}$, we assume that $\mathscr{A}$ is compact (under the uniform convergence metric).

Let us prove the equicontinuity of $\mathscr{A}$ at any $x \in X$. Choose any $\varepsilon > 0$. For each $f \in \mathscr{A}$, let U_f be the open ball centered at f with radius ε, i.e.

$$W_f = \{g \in C(X, \mathcal{V}) : \|g - f\|_{l^\infty(X,\mathcal{V})} < \varepsilon\}$$

Since $\mathscr{A}$ is compact, there is a finite subset $\mathcal{E} \subset \mathscr{A}$ such that $\mathscr{A} \subset \bigcup_{f\in\mathcal{E}} W_f$. Since each $f \in \mathcal{E}$ is continuous at x, and since $\mathcal{E}$ is finite, there exists $U \in \mathrm{Nbh}_X(x)$ such that $\mathrm{diam}(f(U)) < \varepsilon$ for all $f \in \mathcal{E}$. Therefore, for each $g \in \mathscr{A}$, choose $f \in \mathcal{E}$ such that $g \in U_f$. Then since $\mathrm{diam}(f(U)) < \varepsilon$ for that particular f, by triangle inequality, we have $\mathrm{diam}(g(U)) < 3\varepsilon$. Thus $\sup_{g\in\mathscr{A}} \mathrm{diam}(g(U)) \leqslant 3\varepsilon$. Since ε can be arbitrary, we conclude that $\mathscr{A}$ is equicontinuous at x. □

Second proof. This is a fancy proof, just for entertainment. Again, we assume WLOG that $\mathscr{A}$ is compact. Consider the inclusion map $\mathscr{A} \mapsto C(X, \mathcal{V})$ (sending f to f). By Thm. 9.3, it can be viewed as a continuous map

$$X \times \mathscr{A} \to \mathcal{V} \qquad (x, f) \mapsto f(x)$$

Since $\mathscr{A}$ is compact, by Thm. 9.3, the above map can be viewed as a continuous map

$$\Phi : X \to C(\mathscr{A}, \mathcal{V})$$

where for each $x \in X$, $\Phi(x) : \mathscr{A} \to \mathcal{V}$ sends f to $f(x)$. By enlarging the codomain of Φ, we can view Φ as a map $X \to \mathcal{V}^{\mathscr{A}}$ where $\mathcal{V}^{\mathscr{A}}$ is equipped with the uniform convergence topology. The continuity of Φ means, by the very definition of equicontinuity (cf. Def. 9.6), that $\mathscr{A}$ is equicontinuous. □

Theorem 17.15 (Arzelà-Ascoli (AA) theorem). *Let X be a compact topological space. Equip $C(X, \mathbb{R}^N)$ with the uniform convergence topology. Let $\mathscr{A}$ be a subset of $C(X, \mathbb{R}^N)$. Then the following are equivalent.*

(1) *$\mathscr{A}$ is a precompact subset of $C(X, \mathbb{R}^N)$.*

(2) *$\mathscr{A}$ is pointwise bounded and equicontinuous.*

Proof. Assume (1). For each $x \in X$, since the map $f \in \mathscr{A} \mapsto f(x)$ is continuous and $\mathscr{A}$ is compact, by the extreme value theorem, we have $\sup_{f\in\mathscr{A}} \|f(x)\| < +\infty$. So $\mathscr{A}$ is pointwise bounded. By Thm. 17.14, $\mathscr{A}$ is equicontinuous.

Assume (2). Write $Y = \mathbb{R}^N$. Note that the uniform convergence topology is metrizable. Thus, to prove (1), by Prop. 17.2 it suffices to choose an arbitrary net (f_α) in $\mathscr{A}$ and show that it has a cluster point in $C(X, Y)$. By Thm. 17.7, $\mathscr{A}$ is precompact under the pointwise convergence topology. So (f_α) has a subnet (f_β) converging pointwise to some $f \in C(X, Y)$. Since X is compact and (f_β) is equicontinuous, by Cor. 9.26, (f_β) converges uniformly to f. □

In the next chapter, we will use the AA theorem to study differential equations. See Thm. 18.6.

Remark 17.16. The proof of AA theorem relies on Thm. 17.7, and hence in turn relies on Zorn's lemma. If X is separable, then AA theorem does not rely on Zorn's lemma since Thm. 17.7 does not (cf. Claim 17.11).

$\star$ **Remark 17.17.** One may wonder if AA theorem still holds when $\mathbb{R}^N$ is replaced by an arbitrary normed vector space $\mathcal{V}$ (or even a metric space). In fact, in this case, if we assume that $\mathscr{A}$ is **pointwise precompact** (i.e., for each $x \in X$, $\mathscr{A}(x)$ is precompact in $\mathcal{V}$), then Thm. 17.7 still holds, as one can check by reading the proof of Thm. 17.7. Therefore, AA theorem also holds if "pointwise bounded" is replaced by "pointwise precompact". However, in most applications, $\mathcal{V}$ is $\mathbb{R}^N$.

17.3 Operator norms and compactness: the Banach-Alaoglu theorem

> Every interesting topological space is a metric space. Every interesting Banach space is separable. Every interesting real-valued function is Baire/Borel measurable.
>
> —- Barry Simon [Sim-R, Preface of Part 1]

In this section, we fix a normed vector space V over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

We are going to apply the results and methods in Sec. 17.2 to linear maps on V. Note that if W is a normed vector space over $\mathbb{F}$, the operator norm on $\mathfrak{L}(V, W)$ is defined to be $\|T\| = \|T\|_{l^\infty(\overline{B}_V(0,1),W)}$. Therefore, if (T_α) is a net in $\mathfrak{L}(V, W)$ and $T \in \mathfrak{L}(V, W)$, then

$$\lim_\alpha \|T_\alpha - T\| = 0 \qquad \Longleftrightarrow \qquad T_\alpha \rightrightarrows T \text{ on } \overline{B}_V(0,1) \tag{17.2}$$

In other words, operator norms describe the uniform convergence of linear maps on the closed unit balls.

To apply the results in the last section, we must let W be $\mathbb{F}^N$. Let us consider the case $W = \mathbb{F}$. Then $\mathfrak{L}(V, \mathbb{F}) = V^*$. The closed unit ball $\overline{B}_V(0,1)$ is not compact unless when V is finite dimensional. (See Thm. 17.55.) We have seen such an example in Exp. 17.1 where $V = C([0,1], \mathbb{R})$. Therefore, the Arzelà-Ascoli theorem is not available. Thus, one cannot expect general compactness in V^* if the topology on V^* is the uniform convergence topology on $\overline{B}_V(0,1)$, i.e. the topology induced by the operator norm. To get compactness in V^*, one must consider the pointwise convergence topology.

17.3.1 Weak-* topology and Banach-Alaoglu theorem

Definition 17.18. The topology on $V^* = \mathfrak{L}(V, \mathbb{F})$ inherited from the product topology on $\mathbb{F}^V$ is called the **weak-* topology**. A net (φ_α) in V^* is said to **converge weak-*** to $\varphi \in V^*$ if (φ_α) converges to φ under the weak-* topology, equivalently, if (φ_α) converges pointwise to φ when viewed as functions $V \to \mathbb{F}$. By contrast, the **norm topology** (cf. Def. 7.11) on V^* is the topology induced by the (operator) norm of V^*.

Note that since $\overline{B}_V(0,1)$ spans V, a net of linear maps converges pointwise on V iff it converges pointwise on $\overline{B}_V(0,1)$. Therefore, the weak-* topology is also the one induced by the product topology on $\mathbb{F}^{\overline{B}_V(0,1)}$.

We first prove the normed vector space version of Prop. 17.10.

Proposition 17.19. *Let V and W be normed vector spaces over $\mathbb{F}$ where W is a Banach space. Let $M \in \mathbb{R}_{\geqslant 0}$, and let (T_α) be a net in $\mathfrak{L}(V, W)$ such that $\|T_\alpha\| \leqslant M$ for all α. Let E be a subset of V spanning a dense subspace of V. Assume that (T_α) converges pointwise on E. Then (T_α) converges pointwise on V to some $T \in \mathfrak{L}(V, W)$ satisfying $\|T\| \leqslant M$.*

It follows that if (T_α) also converges pointwise on E to some $T' \in \mathfrak{L}(V, W)$, then (T_α) converges pointwise on V to T'. (This is because both T and T' are bounded linear, and they are equal on the dense subset $\mathrm{Span}_{\mathbb{F}}E$ of V. So $T = T'$.)

Proof. By assumption, $F = \mathrm{Span}_{\mathbb{F}}E$ is dense in V. By linearity, (T_α) converges pointwise on F. By Prop. 10.25, (T_α) has uniform Lipschitz constant M. So (T_α) is an equicontinuous net. By Prop. 17.10 and the completeness of W, (T_α) converges pointwise on V to $T \in C(V, W)$.

For each $u, v \in V, a, b \in \mathbb{F}$ we have

$$T(au + bv) = \lim_\alpha T_\alpha(au + bv) = \lim_\alpha (aT_\alpha(u) + bT_\alpha(v)) = aT(u) + bT(v)$$

So T is linear. For each $v \in V$, since $\|T_\alpha(v)\| \leqslant \|T_\alpha\| \cdot \|v\| \leqslant M\|v\|$, we have

$$\|T(v)\| = \lim_\alpha \|T_\alpha(v)\| \leqslant M\|v\|$$

So $\|T\| \leqslant M$ by Rem. 10.24. □

Roughly speaking, Prop. 17.19 says that if a net of bounded linear operators have a uniform upper bound for their operator norms, then pointwise convergence on a dense subset (or more generally, on a subset spanning a dense subspace) implies pointwise convergence on the whose space. The assumption on the uniform upper bound cannot be removed:

Example 17.20. For each $n \in \mathbb{Z}_+$, define $T_n : l^1(\mathbb{Z}_+, \mathbb{R}) \to \mathbb{R}$ sending each f to $n^3 f(n)$. Let $E = \{\chi_{\{k\}} : k \in \mathbb{Z}_+\}$. Then E spans a dense subspace of $l^1(\mathbb{Z}_+, \mathbb{R})$. For each $k \in \mathbb{Z}_+$ we have $\lim_n T_n\chi_k = 0$. However, $\|T_n\| = n^3$ has no uniform upper bounds. Define $f \in l^1(\mathbb{Z}_+, \mathbb{R})$ by $f(n) = n^{-2}$. Then $T_n f = n$ does not converge in $\mathbb{R}$ as $n \to \infty$. So (T_n) does not converge pointwise on $l^1(\mathbb{Z}_+, \mathbb{R})$, although it converges pointwise on E (and hence on $\mathrm{Span}E$) to 0.

The following Banach-Alaoglu theorem can be viewed as the normed vector space version of Thm. 17.7.

Theorem 17.21 (Banach-Alaoglu theorem). *$\overline{B}_{V^*}(0,1)$ is **weak-* compact**, i.e., it is compact under the weak-* topology.*

By our notations, $\overline{B}_{V^*}(0,1)$ is the set of all $\varphi \in V^*$ satisfying $\|\varphi\| \leqslant 1$. Note that weak-* topology is clearly Hausdorff.

Proof. Let $\mathscr{A} = \overline{B}_{V^*}(0,1)$. Since elements in $\mathscr{A}$ have operator norms $\leqslant 1$, they have Lipschitz constant 1 by Prop. 10.25. So $\mathscr{A}$ is equicontinuous on V. For each $v \in V$, $\mathscr{A}(v)$ is bounded since $\mathscr{A}(v) \subset \overline{B}_{\mathbb{F}}(0, \|v\|)$. Thus, by Thm. 17.7, $\mathscr{A}$ has compact closure in $C(V, \mathbb{F})$ under the pointwise convergence topology. Therefore, to show that $\mathscr{A}$ is compact, it suffices to show that $\mathscr{A}$ is closed in $C(V, \mathbb{F})$. Let (φ_α) be a net in $\mathscr{A}$ converging pointwise to $\varphi \in C(V, \mathbb{F})$. By Prop. 17.19, $\varphi \in V^*$ and $\|\varphi\| \leqslant 1$. So $\mathscr{A}$ is closed. $\square$

Remark 17.22. Similar to Arzelà-Ascoli theorem, the proof of the Banach-Alaoglu theorem relies on Thm. 17.7, and hence relies on Zorn's lemma. If V is a separable normed vector space, the Banach-Alaoglu theorem does not rely on Zorn's lemma because the proof of Thm. 17.7 does not, cf. Claim 17.11.

17.3.2 Application: embedding into $C(X, \mathbb{F})$

Recall that V is a normed vector space over $\mathbb{F}$.

Theorem 17.23. *There is a compact Hausdorff space X and a linear isometry $\Phi : V \to C(X, \mathbb{F})$. Moreover, if V is separable, then X can be chosen to be metrizable.*[3]

In other words, V is isomorphic to a linear subspace of $C(X, \mathbb{F})$ (namely, $\Phi(V)$). Clearly, if V is Banach, then $\Phi(V)$ is complete and hence closed. So each Banach space is isomorphic to a closed linear subspace of $C(X, \mathbb{F})$ for some X.

A similar embedding for metric spaces is given in Pb. 17.4.

Proof. We let $X = \overline{B}_{V^*}(0,1)$, equipped with the weak-* topology. By Banach-Alaoglu, X is a compact Hausdorff space. The linear map $\Phi : V \to C(X, \mathbb{F})$ is defined by sending each v to the function

$$\Phi(v) : X \to \mathbb{F} \qquad \varphi \mapsto \langle \varphi, v \rangle$$

[3] In fact, every separable normed vector space can be embedded into $C([0,1], \mathbb{F})$. This is called the **Banach-Mazur theorem**, whose proof is more involved. Cf. [AK, Sec. 1.4].

To check the continuity of $\Phi(v) : X \to \mathbb{F}$, we let (φ_α) be any net in X converging weak-* to $\varphi \in X$. Then $\Phi(v)(\varphi_\alpha) = \langle \varphi_\alpha, v\rangle$ converges to $\langle \varphi, v\rangle = \Phi(v)(\varphi)$, proving that $\Phi(v)$ is continuous.

Let $v \in V$. For each $\varphi \in X$, we have

$$|\langle \Phi(v), \varphi\rangle| = |\langle \varphi, v\rangle| \leqslant \|\varphi\| \cdot \|v\| = \|v\|$$

This proves that $\|\Phi(v)\|_{l^\infty(X,\mathbb{F})} \leqslant \|v\|$. (Recall Rem. 10.24.) By Hahn-Banach Cor. 16.6, there is $\varphi \in X$ with $\|\varphi\| = 1$ such that $\langle \varphi, v\rangle = \|v\|$. Thus $\|\Phi(v)\|_{l^\infty(X,\mathbb{F})} = \|v\|$. This proves that Φ is a linear isometry.

Suppose that V is separable. Then we can find a sequence $(v_n)_{n\in\mathbb{Z}_+}$ dense in V. It is clear that $\Phi(V)$ separates points of X. So $\Phi(v_1), \Phi(v_2), \ldots$ separate points of X. Therefore,

$$X \to S = \mathbb{F}^{\mathbb{Z}_+} \qquad \varphi \mapsto \big(\Phi(v_n)(\varphi)\big)_{n\in\mathbb{Z}_+} \tag{17.3}$$

is a continuous injective map of X into S. Since X is compact, X is homeomorphic to $\Phi(X)$. Since S is metrizable, so is $X = \overline{B}_{V^*}(0,1)$. □

As an immediate consequence of the above proof we have:

Theorem 17.24. *V is separable iff $\overline{B}_{V^*}(0,1)$ is metrizable (under the weak-* topology).*

Recall Thm. 15.37 for equivalent descriptions of metrizable compact Hausdorff spaces. In fact, Thm. 17.24 is closely related to Thm. 15.37, since the relationship between V and $\overline{B}_{V^*}(0,1)$ is similar to that between $C(X,\mathbb{R})$ and X, as implied by the proof of Thm. 17.23.

Proof. The proof of Thm. 17.23 shows that if V is separable then $X = \overline{B}_{V^*}(0,1)$ is metrizable. Conversely, assume that X is metrizable. The proof of Thm. 17.23 shows that V is isomorphic to a linear subspace of $C(X,\mathbb{R})$. By Thm. 15.37, $C(X,\mathbb{R})$ is second countable. So V is second countable, equivalently, separable. □

Remark 17.25. It follows from Thm. 17.24 that if V is separable then $\overline{B}_{V^*}(0,1)$ is sequentially compact. In history, at a time when sequential compactness was still the primary way for people to understand compactness, there were good reasons for studying the sequential compactness of $\overline{B}_{V^*}(0,1)$. This is because early examples of Banach spaces that people focused on were separable.

Remark 17.26. A typical example of a non-separable Banach space is $l^\infty(\mathbb{Z},\mathbb{F})$, cf. Pb. 17.7. Precisely for this reason, the dual space of $l^\infty(\mathbb{Z},\mathbb{F})$ is not much studied, and the norm topology on $l^\infty(\mathbb{Z},\mathbb{F})$ is not good enough. The weak-* topology on (the unit ball of) $l^\infty(\mathbb{Z},\mathbb{F})$ is more natural since, given the equivalence $l^1(\mathbb{Z},\mathbb{F})^* \simeq l^\infty(\mathbb{Z},\mathbb{F})$ (cf. Thm. 17.30) and the separability of $l^1(\mathbb{Z},\mathbb{F})$, the closed unit ball of

$l^\infty(\mathbb{Z},\mathbb{F})$ is weak-* metrizable (equivalently, secound countable).[4] In the study of modern analysis, it is helpful to keep in mind the following two principles:

- Metrizability and second-countability are tests for whether or not a topological space is natural (e.g. whether or not it is reasonable from a natural science point of view).
- However, proving theorems only for metrizable and second-countable spaces will actually make the theory more complicated. It is mainly for the purpose of simplifying the theory (e.g. making the assumptions in the theorems shorter) that we prove the theorems in general, regardless of whether a topological space is metrizable/second-countable or not.[5]

In Rem. 17.27, I will say more about the significance of Thm. 17.24.

17.4 Banach-Alaoglu for L^p and l^p spaces

So far, we have discussed V^* and its weak-* topology on a very abstract level. In the following, we shall understand the results proved in Sec. 17.3 in a more concrete setting.

17.4.1 Weak-* topology in context

Let $1 < p \leqslant +\infty$ and $1 \leqslant q < +\infty$ satisfy $\frac{1}{p}+\frac{1}{q}=1$. Let I be an interval in $\mathbb{R}$. Let $L^p(I)$ be the set of Lebesgue measurable functions [6] $f \to \mathbb{C}$ satisfying that the L^p**-(semi)norm** $\|f\|_p = \sqrt[p]{\int_0^1 |f|^p}$ is finite. A remarkable **representation theorem** of F. Riesz says that we have an (isometric) isomorphism of Banach spaces

$$\Phi : L^p(I) \to L^q(I)^*$$

such that for each $f \in L^p(I)$, $\Phi(f)$ is the linear map sending each $g \in L^q(I)$ to

$$\langle \Phi(f), g\rangle = \int fg$$

[4] If V is infinite dimensional, the weak-* topology of V^* is indeed not first countable, and hence is neither metrizable nor second countable. *Therefore*, the weak-* topology of $\overline{B}_{V^*}(0,1)$ is more natural than that of V^*.

[5] For example, the fact that $(l^1)^* \simeq l^\infty$ shows that separability is not closed under taking dual. Thus, if we stick to separable Banach spaces, the use of Hahn-Banach and Banach-Alaoglu will be more restricted (e.g. when discussing the relationship between V and V^{**}, see Hahn-Banach Cor. 16.6 and Goldstine's Thm. 17.54).

[6] Lebesgue measurable functions are more general than Riemann integrable functions. We will discuss them in the next semester. To understand the material of this section, it is not important to know the precise meaning of them.

Therefore, if (f_α) is a net in $L^p(I)$ and $f \in L^p(I)$, then (f_α) converges weak-* to f (more precisely, $(\Phi(f_\alpha))$ converges weak-* to $\Phi(f)$) iff $\lim_\alpha \int_I f_\alpha g = \int_I fg$ for all $g \in L^q(I)$.

In fact, when $I = [-\pi, \pi]$, using the l^∞-density of $\mathrm{Span}\{e^{\mathbf{i}nx} : n \in \mathbb{Z}\}$ in $C([-\pi, \pi])$ (Exp. 15.12), it can be proved that if (f_α) is a net in $L^p([-\pi, \pi])$ satisfying $\sup_\alpha \|f_\alpha\|_p < +\infty$, then (f_α) converges weak-* to $f \in L^p([-\pi, \pi])$ iff

$$\lim_\alpha \int_{-\pi}^{\pi} f_\alpha(x) e^{-\mathbf{i}nx} dx = \int_{-\pi}^{\pi} f(x) e^{-\mathbf{i}nx} dx$$

i.e., iff the Fourier coefficients of (f_α) converge to the corresponding ones of f.

Remark 17.27. Let me discuss the importance of Thm. 17.24 in the context of L^p spaces. As we will see in the future, Lebesgue measure (and measure theory in general) is not very compatible with net convergence. The main reason is that measure theory is countable by nature, as one can feel in Sec. 14.2. For example, the pointwise limit of a sequence of Lebesgue measurable functions is Lebesgue measurable, but the pointwise limit of a net of measurable functions is not necessarily so. Lebesgue's dominated convergence theorem, a powerful theorem about the commutativity of limits and integrals, applies only to sequences but not nets of functions.

However, it is true that $L^q(I)$ is separable. Therefore, by Thm. 17.24, the closed unit ball of $L^p(I)$ is a metrizable compact space under the weak-* topology. Therefore, to study the weak-* convergence for functions $f \in L^p(I)$ satisfying $\|f\|_p \leqslant 1$, it suffices to use sequences instead of nets, because metrizable topologies and their compactness are determined by sequential convergence. Therefore, one can use all the results in measure theory to study the weak-* topology on $\overline{B}_{L^p(I)}(0, 1)$. □

17.4.2 Weak-* topology on l^p spaces

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let X be a set. Recall that $1 < p \leqslant +\infty$ and $1 \leqslant q < +\infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

In this subsection, we prove that the linear isometry Ψ in Thm. 12.33 is surjective, thus establishing the isomorphism $l^p(X, \mathbb{F}) \simeq l^q(X, \mathbb{F})^*$. Using this isomorphism, we give a concrete (and historically important) description of weak-* topology on the norm-bounded subsets of $l^p(X, \mathbb{F})$. We introduce a temporary notation

$$\mathcal{S}(X, \mathbb{F}) = \{f \in \mathbb{F}^X : f = 0 \text{ except at finitely many points}\} \tag{17.4}$$

Then $\mathcal{S}(X, \mathbb{F})$ is clearly a subspace of $l^q(X, \mathbb{F})$.

Lemma 17.28. *$\mathcal{S}(X, \mathbb{F})$ is dense in $l^q(X, \mathbb{F})$ under the l^q-norm.*

Proof. Let $f \in l^q(X,\mathbb{F})$. Then $\lim_{A\in\mathrm{fin}(2^X)}\sum_A |f|^q = \sum_X |f|^q < +\infty$. Thus, for every $\varepsilon > 0$ there is $A \in \mathrm{fin}(2^X)$ such that $\sum_{X\setminus A} |f|^q < \varepsilon^q$, and hence $\|f - f\chi_A\|_q < \varepsilon$. This finishes the proof, since $f\chi_A \in \mathcal{S}(X,\mathbb{F})$. $\square$

Exercise 17.29. Show that $\mathcal{S}(X,\mathbb{F})$ is not dense in $l^\infty(X,\mathbb{F})$ if X is an infinite set.

Theorem 17.30. *The linear isometry*

$$\Psi : l^p(X,\mathbb{F}) \to l^q(X,\mathbb{F})^*$$
$$\langle \Psi(f), g\rangle = \sum_{x\in X} f(x)g(x)$$

in Thm. 12.33 is an isomorphism of Banach spaces.

In the special case that $p = 2$, Thm. 17.30 is called the **Riesz-Fréchet representation theorem**. Recall that we are assuming $1 < p \leqslant +\infty$ in this theorem. When X is infinite and $p = 1$, Ψ is not surjective. See Pb. 17.8 for details.

Proof. It remains to prove that Ψ is surjective. Choose a nonzero bounded linear $\Lambda : l^q(X,\mathbb{F}) \to \mathbb{F}$. We want to show that Λ is in the range of Ψ. By scaling Λ we assume for simplicity that $\|\Lambda\| = 1$. Define

$$f : X \to \mathbb{F} \qquad f(x) = \langle \Lambda, \chi_{\{x\}}\rangle \tag{17.5}$$

Let us prove that $f \in l^p(X,\mathbb{F})$. If $p = +\infty$, then $|f(x)| \leqslant \|\Lambda\| \cdot \|\chi_{\{x\}}\|_q = 1$, and hence $\|f\|_\infty \leqslant 1$. Assume $p < +\infty$. We understand $\overline{f}(x)/|f(x)|$ as 0 if $f(x) = 0$. Choose any $A \in \mathrm{fin}(2^X)$. Define $g : X \to \mathbb{F}$ to be $g = (\overline{f}/|f|) \cdot |f|^{p-1}\chi_A$. Then clearly $g \in l^q(X,\mathbb{F})$. We compute

$$\langle \Lambda, g\rangle = \Big\langle \Lambda, \sum_{x\in A} g(x)\chi_{\{x\}}\Big\rangle = \sum_{x\in A} g(x)\langle \Lambda, \chi_{\{x\}}\rangle = \sum_{x\in A} f(x)g(x) = \sum_A |f|^p$$

On the other hand,

$$\|g\|_q = \Big(\sum_A |f|^p\Big)^{1/q}$$

Since $|\langle \Lambda, g\rangle| \leqslant \|\Lambda\| \cdot \|g\|_q = \|g\|_q$, we obtain $(\sum_A |f|^p) \leqslant (\sum_A |f|^p)^{1/q}$, and hence

$$\sum_A |f|^p \leqslant 1$$

Applying $\lim_{A\in\mathrm{fin}(2^X)}$, we get $\|f\|_p \leqslant 1$.

Now choose any $g \in l^q(X,\mathbb{F})$. Since $1 \leqslant q < +\infty$, as in the proof of Lem. 17.28, it is easy to see that $\sum_{x\in X} g(x)\chi_{\{x\}} = \lim_{A\in\mathrm{fin}(2^X)} \sum_{x\in A} g(x)\chi_{\{x\}}$ converges to g (under the l^q-norm). Thus, since Λ is continuous, we have

$$\langle \Lambda, g\rangle = \Big\langle \Lambda, \lim_{A\in\mathrm{fin}(2^X)} \sum_{x\in A} g(x)\chi_{\{x\}}\Big\rangle = \lim_{A\in\mathrm{fin}(2^X)} \sum_{x\in A} \Big\langle \Lambda, g(x)\chi_{\{x\}}\Big\rangle$$

$$= \lim_{A \in \text{fin}(2^X)} \sum_{x \in A} f(x)g(x) = \sum_X fg$$

This proves that $\Psi(f) = \Lambda$. $\square$

We are now able to give an explicit characterization of the weak-* topology on norm-bounded subsets (e.g. the unit ball) of $l^p(X, \mathbb{F})$.

Theorem 17.31. *Let (f_α) be a net in $l^p(X, \mathbb{F})$ satisfying $\sup_\alpha \|f_\alpha\|_p < +\infty$, and let $f \in l^p(X, \mathbb{F})$. Then the following are equivalent.*

(1) (f_α) converges weak- to f. (More precisely, $(\Psi(f_\alpha))$ converges weak-* to $\Psi(f)$.)*

(2) (f_α) converges pointwise to f as functions $X \to \mathbb{F}$.

We warn the reader that (1) is not equivalent to (2) if $\sup_\alpha \|f_\alpha\|_p = +\infty$.

Proof. For each $x \in X$ we have $f_\alpha(x) = \langle \Psi(f_\alpha), \chi_{\{x\}} \rangle$ and $f(x) = \langle \Psi(f), \chi_{\{x\}} \rangle$. Therefore, (2) is equivalent to that $\Psi(f_\alpha)$ converges to $\Psi(f)$ when acting on each $\chi_{\{x\}}$. Since Ψ is an isometry, $\sup_\alpha \|\Psi(f_\alpha)\| = \sup_\alpha \|f_\alpha\|_p < +\infty$. Therefore, by Prop. 17.19, (2) is equivalent to that $\Psi(f_\alpha)$ converges pointwise on $l^q(X, \mathbb{F})$ to $\Psi(f)$, because functions of the form $\chi_{\{x\}}$ span $\mathcal{S}(X, \mathbb{F})$, a norm-dense subspace of $l^q(X, \mathbb{F})$ due to Lem. 17.28. $\square$

Remark 17.32. Weak-* convergence and the Banach-Alaoglu theorem were first studied for (the closed unit ball of) $l^2(\mathbb{Z}, \mathbb{F})$ by Hilbert in his study of integral equations and eigenvalue problem (cf. Ch. 9.23). Of course, Hilbert didn't have the modern definition of weak-* topology. For him, the weak-* convergence on the unit ball simply means condition (2) of Thm. 17.31. Taking Thm. 17.31-(2) as the definition of weak-* convergence, the Banach-Alaoglu theorem can be proved quite easily for $l^2(\mathbb{Z}, \mathbb{F})$ (cf. Pb. 17.5). Therefore, as with many abstract definitions, the notion of weak-* topology has its origins in very concrete forms of expression.

After Hilbert's study of l^2 spaces (a.k.a. Hilbert spaces), F. Riesz generalized weak-* topology and Banach-Alaoglu theorem to l^p and L^p spaces for arbitrary $1 < p \leqslant +\infty$, as we shall see in the next section.

Remark 17.33. If $1 < p < +\infty$, the weak-* topology on $l^p(X)$ is also called the **weak topology**. See Rem. 21.26.

17.5 The birth of operator norms: moment problems

It can be said that F. Riesz made the first crucial contribution to the notion of dual spaces of Banach spaces. According to Riesz, bounded linear functionals on $L^q(I)$ (when $1 < q < +\infty$ and I is an interval in $\mathbb{R}$) are nothing abstract. They are simply elements of $L^p(I)$. But what is the advantage of viewing $f \in L^p(I)$ not only as a function on I, but also as a linear functional on $L^q(I)$? Why was Riesz interested in characterizing $L^q(I)^*$ at all?

In fact, Riesz's study of dual spaces is related to the **moment problems** which have applications to probability. In his 1910 paper [Rie10], Riesz studied the following type of moment problem: Let $g_1, g_2, \dots$ be a sequence in $L^q(I)$, let $c_1, c_2, \dots \in \mathbb{C}$, and find some $f \in L^p(I)$ such that

$$\int f g_j = c_j \qquad \text{(for all } j\text{)} \tag{17.6}$$

(For example, in the classical moment problem, $g_n(x) = x^n$, and f is understood as a "probability distribution". Then $\int f(x)xdx = c_1$ is the mean (i.e., expected value) of f, and $\int f(x)(x-c_1)^2dx$ is the variance of f. The number $c_n = \int f(x)x^n dx$ is called the **n-th moment** of f.)

By Hölder's inequality, if such f exists, then there must be some $M \in \mathbb{R}_{\geqslant 0}$ (e.g. $M = \|f\|_{L^p}$), such that

$$|a_1c_1 + \cdots + a_nc_n| \leqslant M \cdot \|a_1g_1 + \cdots + a_ng_n\|_{L^q} \qquad (\forall n \in \mathbb{Z}_+, a_1, \cdots, a_n \in \mathbb{C}) \tag{17.7}$$

Riesz proved that the existence of M satisfying (17.7) is also a sufficient condition for the existence of $f \in L^p(I)$ satisfying (17.6).

From the modern viewpoint, Riesz's result can be proved in the following way: Let V be spanned by $g_1, g_2, \dots$. By (17.7), there exists a unique linear functional $\varphi : V \to \mathbb{C}$ with operator norm $\leqslant M$ satisfying $\varphi(g_j) = c_j$ for all j. By Hahn-Banach Thm. 16.5, φ can be extended to a bounded linear functional $\varphi : L^q(I) \to \mathbb{C}$ also with operator norm $\leqslant M$. Then, by the isomorphism $L^p(I) \simeq L^q(I)^*$, φ can be realized by some $f \in L^p(I)$, which is the desired function.

Remark 17.34. In fact, Riesz didn't find the linear functional $L^q(I) \to \mathbb{C}$ in this way. He didn't see this problem as extending linear functionals. And Hahn-Banach theorem didn't exist before Riesz solved the moment problem. (Riesz's solution is actually an important motivation for the Hahn-Banach theorem.)

Riesz found $\varphi : L^q(I) \to \mathbb{C}$ in the following way (cf. [Die-H, Sec. VI.2]). In the first step, by using complicated methods, he could find $f_n \in L^p(I)$ satisfying $\|f_n\|_{L^p} \leqslant M$ and

$$\int_I f_n g_i = c_i \qquad \text{(for every } 1 \leqslant i \leqslant n\text{)}$$

In Sec. 17.9, we will explain more about how to find these f_n.

In the second step, Riesz considered $\Phi(f_n) : L^q(I) \to \mathbb{C}$ sending each $g \in L^q(I)$ to $\int f_n g$. Then, for each j, we have $\lim_{n\to\infty}\langle \Phi(f_n), g_j \rangle = c_j$. Riesz made the following crucial steps:

- He proved the Banach-Alaoglu theorem for $L^q(I)^*$ using diagonal method. Therefore, since $\sup_n \|\Phi(f_n)\| = \sup_n \|f_n\|_p \leqslant M$, one has a subsequence $\Phi(f_{n_k})$ converging weak-* to some $\varphi \in L^q(I)^*$.[7] Then clearly $\langle \varphi, g_j \rangle = c_j$ for all j.[8]

Then φ can be represented by some $f \in L^p(I)$ thanks to $L^p(I) \simeq L^q(I)^*$. □

Therefore, Riesz's study of moment problems contributed to:

(a) The discovery of important special cases of Banach-Alaoglu theorem.

(b) The realization that operator norms play an important role in the compactness of dual Banach spaces.

(c) The discovery of $L^p(I) \simeq L^q(I)^*$.

After the work of Riesz, operator norms became a central concept in modern analysis.

We refer the readers to [Die-H, Ch. VI] and [NB97] for more details about the relevant history, and to [Sim-R, Sec. 4.17 & 5.6] for a modern treatment of moment problems.

17.6 Operator norms and completeness: functional calculus

In this section, all normed vector spaces are over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

That one can do a lot of analysis on the *linear maps* of function spaces is a remarkable fact. In fact, in the early history of functional analysis, people were more interested in *nonlinear* functionals on function spaces, for example, the expression $S(f)$ in the calculus of variations (cf. (2.1)), for which one searched for the extreme values. The problem of finding extreme values is almost trivial if the functional S is linear and defined on a vector space of functions. Even Hilbert studied the eigenvalue problem for the integral operator $(Tf)(x) = \int_0^1 K(x,y)f(x)dx$ (cf. Subsec. 10.4.1) by transforming it to the extreme value problem of the functional $S(f) =$ (2.1b) defined for all f on the closed unit ball of $L^2([0,1],\mathbb{C})$, viewing $S(f)$ as an (infinite-dimensional) **quadratic form**.

The idea of operator norms was implicit in Hilbert's study of boundedness of sesquilinear/quadratic forms. (See Subsec. 21.5.2.) But it was Riesz who gave the first systematic study of **operator norms**. Riesz used operator norms mainly in the following two cases:

[7]Since $L^q(I)$ is separable, the closed ball of $L^q(I)^*$ with radius M is compact metrizable by Thm. 17.24, and hence is sequentially compact.

[8]More precisely, the following is what Riesz did: Using the diagonal method as in the proof of Thm. 3.54, he found $\Phi(f_{n_k})$ whose evaluations with the elements in a given countable dense subset of $L^q(I)$ converge. Since $\sup_k \|\Phi(f_{n_k})\| < +\infty$, by Prop. 17.19, one concludes that $(\Phi(f_{n_k}))$ converges weak-* to some $\varphi \in L^q(I)^*$.

(1) Bounded linear maps $V \to \mathbb{F}$. By Banach-Alaoglu, one can use the operator norm to get a weak-* compact set $\overline{B}_{V^*}(0,1)$. This has been discussed in previous sections.

(2) Bounded linear maps $V \to V$. As we will see below, the operator norm makes $\mathfrak{L}(V) := \mathfrak{L}(V,V)$ a Banach space which is compatible with its $\mathbb{F}$-algebra structure. Namely, $\mathfrak{L}(V)$ is a **Banach algebra**. Here, the crucial analytic property is completeness rather than compactness.

Thus, in these two cases, the operator norms are playing different roles: compactness in the first case, and completeness in the second one. For example, as I will argue in the future, Hilbert and Schmidt noticed the importance of $l^2(\mathbb{Z},\mathbb{C})$ not so much because of its completeness, but because of the weak-* compactness of its closed unit ball.

However, these two cases have one thing in common: there is a close relationship between operator norms and equicontinuity. In (1), equicontinuity ensures (pre)compactness. In (2), equicontinuity ensures the convergence of double limits (Thm. 8.10), the importance of which will be seen in the future study of spectral theory (cf. Sec. 27.6 and especially Rem. 27.54).

17.6.1 The Banach algebra $\mathfrak{L}(V)$

Let me explain the meaning of the statement "$\mathfrak{L}(V)$ is a Banach algebra". First, we observe:

Theorem 17.35. *Assume that V is a normed vector spaces and W is a Banach space. Recall that $\mathfrak{L}(V,W)$ is a linear subspace of W^V (cf. Prop. 10.27). Then $\mathfrak{L}(V,W)$, equipped with the operator norm, is a Banach space.*

Proof. By Prop. 10.27, it remains to prove that $\mathfrak{L}(V,W)$ is complete. Let (T_n) be a Cauchy sequence in $\mathfrak{L}(V,W)$. So $\lim_{m,n\to\infty}\|T_m - T_n\| = 0$. For each $v \in V$, we have $\|T_m v - T_n v\| \leqslant \|T_m - T_n\| \cdot \|v\|$ which converges to 0 under $\lim_{m,n}$. So $(T_n v)$ is a Cauchy sequence in W, converging to a vector $Tv \in W$. Thus, we have proved that (T_n) converges pointwisely to T. Clearly $M := \sup_{n\in\mathbb{Z}_+}\|T_n\|$ is a finite number. Therefore, by Prop. 17.19, we have $T \in \mathfrak{L}(V,W)$ and $\|T\| \leqslant M$.

Note that $\lim_n \|T - T_n\| = 0$ means precisely that $T_n \rightrightarrows T$ on $B_V(0,1)$. Since (T_n) converges pointwise to T, to prove that $\lim_n \|T - T_n\| = 0$, it suffices to prove that (T_n) converges uniformly to some function on $B_V(0,1)$. This is due to the completeness of $l^\infty(B_V(0,1),W)$ and the Cauchyness of $(T_n|_{B_V(0,1)})_{n\in\mathbb{Z}_+}$. □

Definition 17.36. Let $\mathscr{A}$ be an $\mathbb{F}$-algebra. Suppose that $\mathscr{A}$, as a vector space, is equipped with a norm $\|\cdot\|$ so that $\mathscr{A}$ is a normed vector space, and that

$$\|xy\| \leqslant \|x\| \cdot \|y\| \tag{17.8}$$

for all $x, y \in \mathscr{A}$. Then we call $\mathscr{A}$ a **normed algebra** over $\mathbb{F}$. If the norm is complete, we say that $\mathscr{A}$ is a **Banach algebra** over $\mathbb{F}$. A **unital Banach algebra** is a unital algebra (with unit $\mathbf{1}$) which is also a Banach algebra and satisfies

$$\|\mathbf{1}\| = 1$$

As mentioned above, the most important reason for considering operator norms on V^* is due to the Banach-Alaoglu theorem. The most important reason for considering operator norms on $\mathfrak{L}(V)$ is due to the following elementary but important fact:

Theorem 17.37. *Let V be a Banach space. Then $\mathfrak{L}(V)$, equipped with the operator norm, is a unital Banach algebra.*

Recall that the multiplication in $\mathfrak{L}(V)$ is defined by the composition of linear operators.

Proof. This is immediate from Thm. 17.35 and Prop. 17.38. □

Proposition 17.38. *Let $S : U \to V$ and $T : V \to W$ be linear maps of normed vector spaces. Then*

$$\|TS\| \leqslant \|T\| \cdot \|S\| \tag{17.9}$$

Proof. For each $u \in U$ we have $\|TSu\| \leqslant \|T\| \cdot \|Su\| \leqslant \|T\| \cdot \|S\| \cdot \|u\|$. According to Rem. 10.24, we get (17.9). □

17.6.2 Power series functional calculus

The word "functional calculus" refers in general to the procedure of replacing the variable x or z in the $\mathbb{F}$-valued function $f(x)$ or $f(z)$ by T to get $f(T)$, where T is an element in a unital Banach algebra T. The notation T suggests that the most important case is where $T \in \mathfrak{L}(V)$ for some Banach space V. Depending on whether f is a continuous/analytic/integrable/measurable function, $f(T)$ is defined in different ways. Let us consider the simplest case. In the following, we fix a complex unital Banach algebra $\mathscr{A}$. (For example, $\mathscr{A} = \mathfrak{L}(V)$ where V is a complex Banach space.)

Functional calculus was introduced by Riesz in [Rie13] to the study of spectral theorem of bounded linear operators. See Fig. 17.1 for a summary of historical motivation. The relationship between functional calculus and equicontinuity in Fig. 17.1 will not be explained in this chapter; see Rem. 27.54 and Subsec. 27.7.1 for the related history.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series in $\mathbb{C}$ with radius of convergence R. If $T \in \mathscr{A}$ and $\|T\| < R$, we can define

$$f(T) = \sum_{n=0}^{\infty} a_n T^n \tag{17.10}$$

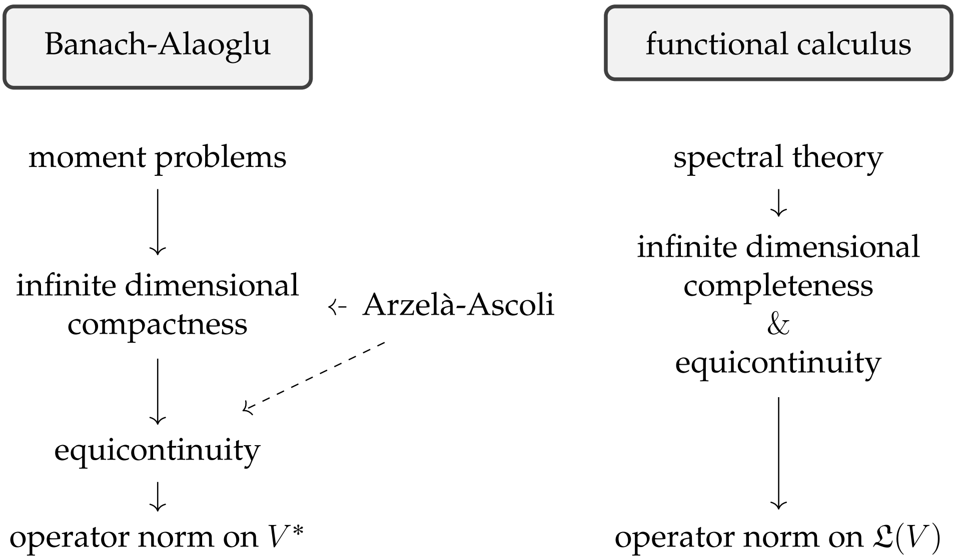


Figure 17.1 The motivation in history

By (17.8), we have $\|T^n\| \leqslant \|T\|^n$. Therefore, by root test, we have $\sum_n |a_n| \cdot \|T^n\| < +\infty$. So the RHS of (17.10) converges absolutely, and hence converges in $\mathscr{A}$. So (17.10) makes sense.

The most important general fact about functional calculus is that $f \mapsto f(T)$ is a homomorhism of unital algebras:

Proposition 17.39. *If $f(z) = \sum_n a_n z^n$ and $g(z) = \sum_n b_n z^n$ be power series in $\mathbb{C}$ with radius of convergence R_1, R_2. Let $R = \min\{R_1, R_2\}$. If $\mathscr{A}$ is a complex unital Banach algebra with unit $\mathbf{1}$, and if $T \in \mathscr{A}$ satisfies $\|T\| < R$, then for each $a, b \in \mathbb{C}$ we have*

$$1(T) = \mathbf{1} \qquad (af + bg)(T) = af(T) + bg(T) \qquad (fg)(T) = f(T)g(T)$$

Proof. The first two equations are obvious. We prove the last one. Let $c_n = \sum_{k=0}^n a_k b_{n-k}$. Then $h(z) = \sum_n c_n z^n$ equals $f(z)g(z)$ on $B_{\mathbb{C}}(0, R)$. Cor. 5.59 immediately implies $h(T) = f(T)g(T)$. □

Let me give a simple application of Prop. 17.39. Recall that one can use determinants to prove that the set of $n \times n$ complex matrices is open in $\mathbb{C}^{n \times n}$. However, in the infinite dimensional case one clearly cannot use determinants. Functional calculus provides an easy method to treat this problem.

Example 17.40. Let $T \in \mathscr{A}$ such that $\|T\| < 1$. Then $1 + T$ is invertible.

Proof. Let $f(z) = 1 + z$. Let $g(z) = \sum_{n=0}^\infty (-z)^n$, which has radius of convergence 1 and equals $(1 + z)^{-1}$ when $|z| < 1$. Since $fg = gf = 1$, by Prop. 17.39 we have $(1 + T)S = S(1 + T) = 1$ if we let $S = g(T) = \sum_{n=0}^\infty T^n$. □

Proposition 17.41. *The set $\mathscr{A}^\times$ of invertible elements is an open subset of $\mathscr{A}$.*

Proof. Let $T \in \mathscr{A}$ be invertible. Let $\delta = \|T^{-1}\|^{-1}$. Then for each $S \in \mathscr{A}$ satisfying $\|T - S\| < \delta$ we have

$$\|T^{-1}(S-T)\| \leqslant \|T^{-1}\| \cdot \|S-T\| < 1$$

Thus, by Exp. 17.40, $1 + T^{-1}(S-T)$ is invertible. So its multiplication with T (which is S) is invertible. □

Functional calculus is also useful in differential equations, even in the finite dimensional case:

Exercise 17.42. Let $A, B \in \mathscr{A}$. Suppose that $AB = BA$. Show that $e^{A+B} = e^A \cdot e^B$. Show that

$$\frac{de^{At}}{dt} = Ae^{At} = e^{At}A$$

Exercise 17.43. Let V be a complex Banach space. Let $v \in V$. Let $T \in \mathfrak{L}(V)$. Show that there is a unique differentiable $f : \mathbb{R} \to V$ satisfying the differential equation

$$f'(t) = Tf(t) \qquad f(0) = v$$

Show that $f(t) = e^{Tt}v$ satisfies this differential equation.

17.7 Contraction theorem

Definition 17.44. Let $f : X \to Y$ be a map of metric spaces. Suppose that f has Lipschitz constant $L \in [0,1)$, we say that f is a **contraction**.

Theorem 17.45 (Contraction theorem). *Let X be a nonempty complete metric space. Let $T : X \to X$ be a contraction. Then T has a unique **fixed point**, i.e., there exists a unique $x \in X$ satisfying $T(x) = x$.*

In the following proof, we let $L \in [0,1)$ be a Lipschitz constant of T.

Proof. Uniqueness: Suppose $x, y \in X$ satisfy $T(x) = x$ and $T(y) = y$. Then

$$0 \leqslant d(x,y) = d(T(x), T(y)) \leqslant Ld(x,y)$$

showing that $d(x,y)$ must be 0.

Existence: Choose $x_0 \in X$. Define $(x_n)_{n\in\mathbb{N}}$ inductively by $x_{n+1} = T(x_n)$. Then $d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leqslant Ld(x_n, x_{n-1})$. From this, we conclude

$$d(x_{n+1}, x_n) \leqslant Ld(x_n, x_{n-1}) \leqslant L^2 d(x_{n-1}, x_{n-2}) \leqslant \cdots \leqslant L^n d(x_1, x_0)$$

Therefore, for each $k \in \mathbb{Z}_+$ we have

$$\begin{aligned}&d(x_{n+k},x_n)\leqslant d(x_{n+k},x_{n+k-1})+d(x_{n+k-1},x_{n+k-2})+\cdots+d(x_{n+1},x_n)\\ \leqslant&(L^{n+k-1}+L^{n+k-2}+\cdots+L^n)d(x_1,x_0)\leqslant\frac{L^n}{1-L}d(x_1,x_0)\end{aligned}$$

This proves that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. So it converges to some $x \in X$. Since $(T(x_n)) = (x_{n+1})$ also converges to x, by the continuity of T we conclude $T(x) = x$. □

The contraction theorem is also called the **Banach fixed-point theorem**. In the next chapter, we will use the contraction theorem to study differential equations. In the next semester, we will use the contraction theorem to prove the inverse function theorem for multivariable functions.

17.8 Problems and supplementary material

Let $1 < p, q \leqslant +\infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

⋆ **Problem 17.1.** Let $1 \leqslant p < +\infty$. Let X be a set. Let $\mathscr{A}$ be a subset of $l^p(X,\mathbb{F})$ equipped with the l^p-norm. Prove that the following are equivalent:

(1) $\mathscr{A}$ is precompact.

(2) For each $x \in X$ we have $\sup_{f\in\mathscr{A}} |f(x)| < +\infty$. Moreover, for every $\varepsilon > 0$, there exists a finite subset $K \subset X$ such that for each $f \in \mathscr{A}$ we have $\|f|_{X\setminus K}\|_{l^p} < \varepsilon$; in other words,

$$\lim_{K\in\mathrm{fin}(2^X)} \sup_{f\in\mathscr{A}} \sum_{x\in X\setminus K} |f(x)|^p = 0 \tag{17.11}$$

Hint. (1)⇒(2): Mimic the first proof of Thm. 17.14. (2)⇒(1): Choose any net (f_α) in $\mathscr{A}$. First find a subnet (f_μ) converging pointwise to a function $f : X \to \mathbb{F}$. Then show that $f \in l^p(X,\mathbb{F})$ and $\lim_\mu \|f - f_\mu\|_p = 0$. □

Pb. 17.1 is the l^p-version of the **Fréchet-Kolmogorov theorem**.

⋆ **Exercise 17.46.** In Pb. 17.1, assume that $\mathscr{A}$ is precompact. Prove that $E = \bigcup_{f\in\mathscr{A}} \mathrm{Supp}(f) \equiv \bigcup_{f\in\mathscr{A}}\{x \in X : f(x) \neq 0\}$ is a countable set.

Hint. Method 1: Show that $\mathscr{A}$ is separable. Let $\mathcal{E}$ be a countable dense subset of $\mathscr{A}$. Prove $E = \bigcup_{f\in\mathcal{E}} \mathrm{Supp}(f)$.

Method 2: For each $\varepsilon = 1/n$, write the K in (2) as K_n. Prove $E \subset \bigcup_n K_n$. □

$\star$ **Problem 17.2.** Let V be a Banach space. Let $\xi \in C(\mathbb{R}, V)$ such that $\|\xi\|_{L^1} = \int_{\mathbb{R}} \|\xi(t)\|dt < +\infty$. Let $\mathfrak{X} = C(\mathbb{R},\mathbb{R}) \cap l^\infty(\mathbb{R},\mathbb{R})$, equipped with the l^∞-norm. Define a linear map

$$\Phi : \mathfrak{X} \to V \qquad f \mapsto \int_I f(t)\xi(t)dt$$

Show that Φ is a bounded linear map. Show that Φ is a **compact operator**, which means that $\Phi(\overline{B}_{\mathfrak{X}}(0,1))$ is a precompact subset of V.

Hint. By Thm. 17.23, we can view V as a closed linear subspace of $C(X,\mathbb{R})$ where X is a compact Hausdorff space. So ξ can be viewed as a function on $\mathbb{R} \times X$. Use Arzelà-Ascoli to prove that Φ is a compact operator. $\square$

Problem 17.3. Let V be a separable normed vector space over $\mathbb{F}$. Let $(v_n)_{n\in\mathbb{Z}_+}$ be a dense sequence in $\overline{B}_V(0,1)$. (The density is with respect to the norm topology.[9]) Use $(v_n)_{n\in\mathbb{Z}_+}$ to construct an explicit metric on $\overline{B}_{V^*}(0,1)$ inducing its weak-* topology, and construct an explicit countable basis for the weak-* topology on $\overline{B}_{V^*}(0,1)$.

Hint. Check the proof of Thm. 17.24 (or more precisely, Thm. 17.23). This problem is related to Pb. 15.15. $\square$

Remark 17.47. The metric you are asked to find in Pb. 17.3 can actually be found in many analysis textbooks, although their authors do not tell you how they find it. The point of this problem (together with Pb. 15.15) is to tell you that the correct geometric viewpoint (i.e., embedding into Hilbert cubes) can lead you to the formula of the metric.

Another goal of this problem is to justify the point in Rem. 8.35: For all concrete examples of compact metrizable spaces, you can explicitly construct countable bases for their topologies. Therefore, there is no need to use the indirect proof of the second countability in Thm. 8.34.[10] $\square$

Problem 17.4. Let Y be a metric space. Prove that there is a compact Hausdorff space X and an isometry $\Phi : Y \to C(X,\mathbb{F})$ following the hint below.

In particular, any metric space can be (isometrically) embedded into a Banach space. [11]

---

[9] Recall that subsets of separable (equivalently, second countable) metric spaces are separable.

[10] You can feel how much I hate the proof of Thm. 8.34 :-)

[11] If we simply want to embed Y into a Banach space V, there is a simpler method called **Kuratowski embedding**: Assume Y is nonempty and fix $a \in Y$. Let $V = l^\infty(Y,\mathbb{R})$, and define the embedding sending each $y \in Y$ to the function $p \in Y \mapsto d(y,p) - d(a,p)$.

Hint. This problem is similar to Thm. 17.23. Since $C(X,\mathbb{R}) \subset C(X,\mathbb{C})$, it suffices to assume $\mathbb{F} = \mathbb{R}$. In Thm. 17.23, X is constructed to be the set of linear functionals with operator norms $\leqslant 1$. The assumption on the operator norms ensures the equicontinuity and hence the (pre)compactness of X. Thus, in the current situation, in order to get a compact X, one should consider a pointwise bounded set of functions with a uniform Lipschitz constant (cf. Thm. 17.7 or Exe. 17.9). For example, assume Y is nonempty and fix $a \in Y$, and define

$$X = \{f \in \mathbb{R}^Y : f(a) = 0, \text{ and } f \text{ has Lipschitz constant } 1\} \tag{17.12}$$

equipped with the pointwise convergence topology. □

Remark 17.48. Similar to the proof of Thm. 17.24, one can show that $X = (17.12)$ is metrizable (equivalently, second countable) iff Y is separable.

Remark 17.49. Pb. 17.4 tells us that any metric space can be embedded into a Banach space. This fact is useful in the same way that the existence of the completions of normed vector spaces is useful, cf. Rem. 10.22. (Interestingly, Pb. 17.4 gives a new proof that every metric space Y has completion, since one can restrict $\Phi : Y \to V$ to $Y \to \overline{\Phi(Y)}$.)

For example, assume that X, Y are topological spaces, and Z is a metric space. Equip $C(Y,Z)$ with a uniform convergence metric (cf. Exp. 7.77). By Pb. 17.4, Z can be viewed as a metric subspace of a Banach space V. By Thm. 9.3, there is a canonical injection $\Psi : C(X, C(Y,V)) \to C(X \times Y, V)$ which is bijective when Y is compact. It is easy to see that Ψ restricts to an injective map $C(X, C(Y,Z)) \to C(X \times Y, Z)$, and that this restriction is surjective when Y is compact. Therefore, Thm. 9.3 can be generalized to the case that the codomain is a metric space, but not necessarily a normed vector space.[12] Similarly, Thm. 9.12 can be generalized to functions whose codomains are metric spaces, and the Moore-Osgood theorem can be generalized to functions whose codomains are complete[13] metric spaces. □

Problem 17.5. Let X be a set. By the Banach-Alaoglu theorem and Thm. 17.31, when $1 < p \leqslant +\infty$, the closed unit ball

$$B = \{f \in l^p(X,\mathbb{F}) : \|f\|_p \leqslant 1\} \tag{17.13}$$

is compact under the pointwise convergence topology (i.e. the product topology inherited from $\mathbb{F}^X$). Give a direct proof of this fact for all $1 \leqslant p \leqslant +\infty$ (including the case $p = 1$) without using Banach-Alaoglu Thm. 17.21.

Now you may wonder if the pointwise convergence topology on the closed unit ball of l^1 can be realized by a weak-* topology. The answer is yes:

[12] Of course, we can prove the case of metric spaces for Thm. 9.3 at the very beginning. We didn't do it just to avoid distraction.

[13] You need completeness because in the proof you need the fact that Y is closed in V.

⋆ **Problem 17.6.** Let X be a set. Let

$$c_0(X,\mathbb{F}) = \{f \in \mathbb{F}^X : \text{for all } \varepsilon > 0 \text{ there exists } A \in \mathrm{fin}(2^X) \text{ such that } \|f\|_{l^\infty(A^c,\mathbb{F})} < \varepsilon\}$$

equipped with the l^∞-norm. By Prop. 15.46, $c_0(X,\mathbb{F})$ is a Banach space. (You can also check it directly.) Find a natural isomorphism of Banach spaces

$$\Psi : l^1(X,\mathbb{F}) \xrightarrow{\simeq} c_0(X,\mathbb{F})^* \tag{17.14}$$

Let (f_α) be a net in $l^1(X,\mathbb{F})$ satisfying $\sup_\alpha \|f_\alpha\|_{l^1} < +\infty$, and let $f \in l^1(X,\mathbb{F})$. Prove that $\Psi(f_\alpha)$ converges weak-* to $\Psi(f)$ iff (f_α) converges pointwise to f as functions $X \to \mathbb{F}$.

⋆ **Problem 17.7.** Prove that $l^q(\mathbb{Z}_+,\mathbb{F})$ is separable (where $1 \leqslant q < +\infty$), and $l^\infty(\mathbb{Z}_+,\mathbb{F})$ is not separable.

Hint. Every subset of a second countable space is second countable and hence Lindelöf. Find an uncountable discrete (and hence non-Lindelöf) subset of $l^\infty(\mathbb{Z}_+,\mathbb{F})$. □

⋆ **Problem 17.8.** By Thm. 12.33, there is a linear isometry

$$\Psi : l^1(\mathbb{Z}_+,\mathbb{F}) \to l^\infty(\mathbb{Z}_+,\mathbb{F})^* \tag{17.15}$$

such that $\langle\Psi(f),g\rangle = \sum_n f(n)g(n)$. Let $\mathcal{B} = \overline{B}_{l^1(\mathbb{Z}_+,\mathbb{F})}(0,1)$, the closed unit ball of $l^1(\mathbb{Z}_+,\mathbb{F})$. If Ψ is surjective (and hence an isomorphism), then $\Psi(\mathcal{B})$ is the closed unit ball of $l^\infty(\mathbb{Z}_+,\mathbb{F})^*$, and hence is weak-* compact by Banach-Alaoglu.

Let $f_n = \chi_{\{n\}}$, viewed as an element of $\mathcal{B}$. Prove that the sequence $(\Psi(f_n))_{n\in\mathbb{Z}_+}$ has no weak-* convergent subnet in $\Psi(\mathcal{B})$.[14] Conclude that the map Ψ is not surjective. □

17.9 ⋆ Supplementary material: a discussion of Riesz's treatment of moment problems

Unless otherwise stated, V is a Banach space over $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$.

17.9.1 Quotient Banach spaces in moment problems

Problem 17.9. Let U be a closed linear subspace of V. Let V/U be its quotient vector space. Prove that V/U has a norm defined by

$$\|v+U\| = \inf_{u\in U} \|v+u\| \tag{17.16}$$

for all $v \in V$. Prove that V/U is complete under this norm. We call V/U the **quotient Banach space** of V by U.

[14] It is not enough to consider subsequences, since $l^\infty(\mathbb{Z}_+,\mathbb{F})$ is not separable and hence the unit ball of $l^\infty(\mathbb{Z}_+,\mathbb{F})^*$ is not weak-* metrizable, cf. Thm. 17.24.

Hint. Use Pb. 4.1 to prove that V/U is complete. $\square$

Proposition 17.50 (Abstract moment problem, finite version). *Let $\varphi_1,\dots\varphi_n\in V^*$. Let $c_1,\dots,c_n\in\mathbb{F}$. Suppose that there exists $M\in\mathbb{R}_{\geqslant 0}$ such that*

$$|a_1c_1+\cdots+a_nc_n|\leqslant M\cdot\|a_1\varphi_1+\cdots+a_n\varphi_n\| \qquad (\forall a_1,\dots,a_n\in\mathbb{F}) \tag{17.17}$$

Then for every $\varepsilon>0$ there exists $v\in V$ satisfying that $\|v\|\leqslant M+\varepsilon$ and that

$$\langle\varphi_i,v\rangle=c_i \qquad (\text{for all } 1\leqslant i\leqslant n) \tag{17.18}$$

The above proposition is credited to Helly.

Problem 17.10. Assume the setting of Prop. 17.50. Define a bounded linear map

$$\Phi:V\to\mathbb{F}^n \qquad v\mapsto(\varphi_1(v),\dots,\varphi_n(v)) \tag{17.19}$$

The following two steps will lead you to prove Prop. 17.50.

1. Consider the special case that $\mathrm{Ker}(\Phi)=0$. In particular, V is a finite dimensional normed vector space, and $\dim V=\dim\mathrm{Span}(\varphi_1,\dots,\varphi_n)$. We have $\dim V^*=\dim V<+\infty$ because all linear maps $V\to\mathbb{F}$ are bounded. Similarly $\dim V^{**}=\dim V^*$. So the canonical linear isometry

$$V\to V^{**} \qquad v\mapsto\langle\cdot,v\rangle \tag{17.20}$$

   (cf. Cor. 16.6) must be bijective. Use this observation to prove that there exists $v\in V$ satisfying $\|v\|\leqslant M$ and (17.18).

2. Reduce the general case to the special case in part 1 by considering $V/\mathrm{Ker}(\Phi)$ and its bounded linear functionals $\psi_1,\dots,\psi_n$ defined by

$$\langle\psi_i,v+\mathrm{Ker}(\Phi)\rangle=\langle\varphi_i,v\rangle \tag{17.21}$$

   (Note that you need to prove $\|\psi_i\|=\|\varphi_i\|$.)

Remark 17.51. Recall from Sec. 17.5 that Riesz wanted to solve the moment problem: Let $I=[a,b]$ be a compact interval in $\mathbb{R}$. The scalar field is chosen to be $\mathbb{C}$. Let $g_1,g_2,\dots\in L^q(I)=L^q(I,\mathbb{C})$ (where $1<q<+\infty$) and $c_1,c_2,\dots\in\mathbb{C}$ satisfying

$$|a_1c_1+\cdots+a_nc_n|\leqslant M\cdot\|a_1g_1+\cdots+a_ng_n\|_{L^q} \qquad (\forall n\in\mathbb{Z}_+,a_1,\cdots,a_n\in\mathbb{C})$$

The goal is to find $f\in L^p(I)$ (where $p^{-1}+q^{-1}=1$) satisfying $\int_I fg_j=c_j$ for all $j=1,2,\dots$. As mentioned in Rem. 17.34, Riesz's first step in solving this problem is to find $f_n\in L^p(I)$ for each $n\in\mathbb{Z}_+$ satisfying $\|f_n\|_{L^p}\leqslant M$ and

$$\int_I f_ng_i=c_i \qquad (\text{for every } 1\leqslant i\leqslant n) \tag{17.22}$$

The second step is then to find a weak-* convergent subsequence of $(f_n)_{n\in\mathbb{Z}_+}$.

In Sec. 17.5, I didn't explain how Riesz solved step 1. Some of the key ideas can now be explained. (For simplicity, the reader can replace $L^q(I), L^p(I)$ by $l^q(\mathbb{Z}), l^p(\mathbb{Z})$ and replace the integrals by the series. The main idea remains the same.)

Let us slightly weaken the above assumption to $\|f_n\|_{L^p} \leqslant M + \varepsilon$ where $\varepsilon > 0$ is fixed at the beginning. (Later, I will explain why ε can be chosen to be 0.) Then, in view of the canonical isomorphism $\Psi : L^q(I) \to L^p(I)^*$, Riesz's first step can be proved by directly applying Prop. 17.50 to the special case that $V = L^p(I)$ and $\varphi_i = \Psi(g_i)$. □

Remark 17.52. It should be noted that the proof of Prop. 17.50 relies on the fact that $V \to V^{**}$ is an isometry (cf. Pb. 17.10-1) when V is finite-dimensional. This follows from Hahn-Banach in the general case. As I said in Rem. 17.34, the Hahn-Banach theorem did not exist by the time Riesz was studying the moment problems.

In the case of moment problems, according to Pb. 17.10, one should take V to be $L^p(I)/\mathrm{Ker}(\Phi)$ for some bounded linear $\Phi : L^p(I) \to \mathbb{C}^n$. In this case, without using Hahn-Banach, one can at least show easily that $V \to V^{**}$ is bijective. But it is quite hard to give a direct (i.e. function-theoretic) proof that $V \to V^{**}$ is an isometry. Nevertheless, Riesz circumvented this problem by using a complicated method due to Schmidt (cf. [Die-H, Sec. 6.2 and 5.3]), which was rarely used after the appearance of Hahn-Banach. In some sense, one can say that Riesz proved a weak version of Hahn-Banach theorem for the finite-dimensional Banach space $L^p(I)/\mathrm{Ker}(\Phi)$. Anyway, the infinite-dimensional Hahn-Banach theorem is not needed in Riesz's method. (So Hahn-Banach is nontrivial enough in the finite-dimensional case.) □

Remark 17.53. The notion of quotient Banach spaces introduced in this section is also implicit in Riesz's work. Indeed, Riesz wanted to find $f_n \in L^p(I)$ satisfying (17.22) for small enough $\|f_n\|_{L^p}$. This amounts to the fact that in (17.16), the value of $\|v + U\|$ can be estimated by finding $u \in U$ such that $\|v + u\|$ is small enough.

In fact, Riesz was able to find f_n satisfying (17.22) *and minimizing* $\|f_n\|$; moreover, for such f_n, one has $\|f_n\| \leqslant M$ (but not just $\|f_n\| \leqslant M + \varepsilon$). This follows from Pb. 17.11, applied to $V = L^q(I)$ and $E = \{g_1, \dots, g_n\}$. □

Problem 17.11. Let E be a subset of V. Let

$$E^\perp = \{\varphi \in V^* : \varphi|_E = 0\} \tag{17.23}$$

which is clearly a weak-* closed linear subspace of V^*. Let $\varphi \in V^*$. Prove that there exists $\psi \in E^\perp$ such that $\|\varphi + \psi\|$ equals $\|\varphi + E^\perp\| \equiv \inf_{\eta\in E^\perp} \|\varphi + \eta\|$.

It is in fact true that any weak-* closed linear subspace of V^* is of the form $E^\perp$. To prove it one needs a more general version of the Hahn-Banach theorem. We will not discuss it in our notes.

Hint. Choose a sequence (ψ_n) in V^* such that $\lim_n \|\varphi+\psi_n\| = \|\varphi+E^\perp\|$. Since this sequence is clearly norm-bounded, by Banach-Alaoglu, it has a subnet (ψ_α) converging weak-* to some $\psi \in V^*$. Clearly $\psi \in E^\perp$. For each $\varepsilon > 0$, choose $v \in V$ with $\|v\| = 1$ such that $|\langle \varphi+\psi, v\rangle| \geqslant \|\varphi+\psi\| - \varepsilon$. Show that $|\langle \varphi+\psi, v\rangle| \leqslant \lim_\alpha \|\varphi+\psi_\alpha\|$. Conclude that $\|\varphi+\psi\| - \varepsilon \leqslant \|\varphi+E^\perp\|$ for all $\varepsilon > 0$. □

17.9.2 Some consequences

The goal of this subsection is to give some quick consequences of the results and the methods introduced in the previous subsections. These important properties are mainly about compactness, and are not difficult to find in many books on functional analysis. The reason I present these results is to show that they can (or should?) be understood in the light of Riesz's treatment of moment problems. (Unfortunately, this point is often not emphasized in many textbooks.)

Theorem 17.54 (Goldstine's theorem). *Let $\Gamma : V \to V^{**}$ be the linear isometry (cf. Cor. 16.6) sending v to the bounded linear functional $\varphi \in V^* \mapsto \langle \varphi, v\rangle$. Then $\Gamma(\overline{B}_V(0,1))$ is weak-* dense in $\overline{B}_{V^{**}}(0,1)$.*

Consequently, since $\overline{B}_V(0,1)$ is a compact Hausdorff space by Banach-Alaoglu, we conclude that Γ is bijective iff $\Gamma(\overline{B}_V(0,1))$ is weak-* compact.[15] A Banach space satisfying these two equivalent conditions is called **reflexive**. For example, if $1 < p < +\infty$, then Thm. 17.30 shows that $l^p(X,\mathbb{F})$ is reflexive. It is also true that $L^p([a,b],\mathbb{F})$ is reflexive.

Proof. Since $\bigcup_{0<r<1} \overline{B}_{V^{**}}(0,r)$ is norm dense and hence weak-* dense in $\overline{B}_{V^{**}}(0,1)$, it suffices to prove that for each $0 < r < 1$ and $\mathfrak{v} \in \overline{B}_{V^{**}}(0,r)$, $\mathfrak{v}$ can be approximated weak-* by elements of $\Gamma(\overline{B}_V(0,1))$.

We shall prove that for each $E \in \mathrm{fin}(2^{V^*})$ there exists $v_E \in \overline{B}_V(0,1)$ such that $\langle \mathfrak{v}, \varphi\rangle = \langle v_E, \varphi\rangle$ for all $\varphi \in E$. Then the net $(\Gamma(v_E))_{E\in\mathrm{fin}(2^{V^*})}$ converges weak-* in $\overline{B}_{V^{**}}(0,1)$ to $\mathfrak{v}$, finishing the proof. To find v_E, we write $E = \{\varphi_1,\dots,\varphi_n\}$. Let $c_i = \langle \mathfrak{v}, \varphi_i\rangle$. Since $\|\mathfrak{v}\| \leqslant r$, for each $a_1,\dots,a_n \in \mathbb{F}$ we have

$$|a_1c_1+\cdots+a_nc_n| = |\langle \mathfrak{v}, a_1\varphi_1+\cdots+a_n\varphi_n\rangle| \leqslant r\cdot\|a_1\varphi_1+\cdots+a_n\varphi_n\|$$

Therefore, by Prop. 17.50, there exists $v_E \in \overline{B}_V(0,1)$ such that $\langle v_E, \varphi_i\rangle = c_i$ for all $1 \leqslant i \leqslant n$. This finishes the construction of v_E. □

Problem 17.12. Let M be a linear subspace of V. Suppose that $e \in V$ is not in $\overline{M} = \mathrm{Cl}_V(M)$. Prove that there exists $\psi \in V^*$ such that $\psi|_M = 0$ and $\psi(e) \neq 0$.

[15] The **weak topology** on V is defined to be the pullback of the weak-* topology of V^{**} by $V \to V^{**}$. So a net (v_α) in V converges weakly to $v \in V$ iff $\lim_\alpha \langle v_\alpha, \varphi\rangle = \langle v, \varphi\rangle$ for all $\varphi \in V^*$. Thus, the conclusion is that $\Gamma : V \to V^{**}$ is bijective iff $\overline{B}_V(0,1)$ is weakly compact.

This problem is a special case of the **Hahn-Banach separation theorem**. As a consequence of this exercise, we know that M is dense in V iff every $\psi \in V^*$ vanishing on M is zero.

Hint. Use Hahn-Banach Cor. 16.6 to prove the special case that $M = 0$. Then reduce the general case to the special case by considering $V/\overline{M}$. $\square$

Problem 17.13. Let V be a reflexive Banach space. Let U be a closed linear subspace of V. Let $v \in V$. Show that there exists $u \in U$ such that $\|v + U\| = \|v + u\|$.

Hint. The canonical linear isometry $\Gamma : V \to V^{**}$ is an isomorphism of Banach spaces. Use the Hahn-Banach separation theorem (Pb. 17.12) to show that $\Gamma(U) = W^\perp$ where $W = \{\psi \in V^* : \psi|_U = 0\}$. Then apply Pb. 17.11. $\square$

Theorem 17.55 (Riesz's theorem). *Let V be a Banach space. Then the closed unit ball $\overline{B}_V(0,1)$ is compact under the norm topology iff $\dim_{\mathbb{F}} V < +\infty$.*

Proof. If $\dim_{\mathbb{F}} V < +\infty$, then the norm on V is equivalent to the Euclidean norm by Pb. 10.3. So $\overline{B}_V(0,1)$ is compact by Bolzano-Weierstrass.

Now we assume $\dim_{\mathbb{F}} V$ is not finite. We fix any $0 < \varepsilon < 1$. (For example, choose $\varepsilon = 1/2$.) Notice that for each closed linear subspace $U \subsetneq V$, there clearly exists $v \in U$ such that $\|v + U\| = 1 - \varepsilon$. By replacing v by $v + u$ for some $u \in U$, we assume moreover that $\|v\| \leqslant 1$.

Now we construct an infinite linearly-independent sequence $(v_n)_{n\in\mathbb{Z}_+}$ in $\overline{B}_V(0,1)$ as follows. v_1 is arbitrary. Suppose $v_1, \dots, v_n$ have been constructed. Let $U_n = \mathrm{Span}_{\mathbb{F}}(v_1, \dots, v_n)$. Then U_n is a closed linear subspace of V since U_n is complete. (Any finite dimensional normed vector space is equivalent as a metric space to $\mathbb{F}^n$ by Pb. 10.3, and hence is complete.) Then U_n is a closed proper linear subspace of V since $\dim_{\mathbb{F}} V < +\infty$. Therefore, by the previous paragraph, there exists $v_{n+1} \in \overline{B}_V(0,1)$ such that $\|v_{n+1} + U_n\| = 1 - \varepsilon$. So $\|v_{n+1} - v_j\| \geqslant 1 - \varepsilon$ for all $j \leqslant n$.

The sequence constructed above satisfies that $\|v_m - v_n\| \geqslant 1 - \varepsilon$ for all $m \neq n$. So $(v_n)_{n\in\mathbb{Z}_+}$ has no Cauchy subsequence, and hence no norm-convergent subsequence. Therefore $\overline{B}_V(0,1)$ is not sequentially compact under the norm topology, and hence not compact. $\square$

Remark 17.56. If V is a reflexive infinite dimensional Banach space (i.e. if V is $L^p([0,1],\mathbb{F})$ or $l^p(X,\mathbb{F})$ where $1 < p < +\infty$), the above proof that $\overline{B}_V(0,1)$ is not compact can be simplified by choosing $\varepsilon = 0$. This is due to Pb. 17.13.

18 Application to differential equations

18.1 The Picard-Lindelöf theorem

We fix a Banach space V over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let I be an interval in $\mathbb{R}$ containing at least two points.

Definition 18.1. Let T be a set. Let X, Y be metric spaces. We say that a function $f : T \times X \to Y$ is **Lipschitz continuous on** X if there exists $L \in \mathbb{R}_{\geqslant 0}$ (called the **Lipschitz constant**) such that for every $t \in T$ and $x_1, x_2 \in X$ we have

$$d(f(t, x_1), f(t, x_2)) \leqslant Ld(x_1, x_2)$$

Theorem 18.2. *Let $E \subset V$. Let $\varphi \in C(I \times E, V)$ be Lipschitz continuous on E. Fix $t_0 \in I, \xi \in E$. Then there exists at most one differentiable $f : I \to E$ satisfying*

$$f'(t) = \varphi(t, f(t)) \qquad f(t_0) = \xi \tag{18.1}$$

for all $t \in I$.

Proof. This theorem generalizes Lem. 12.22. In fact, we will prove this theorem in a similar way as we proved Lem. 12.22. Let $f, g : I \to$ be differentiable and assume that they both satisfy (18.1). Then $\Omega = \{t \in I : f(t) = g(t)\}$ is a closed subset of I since f, g are continuous. Ω is nonempty since $t_0 \in \Omega$. Since I is connected, to prove that $\Omega = I$, it suffices to prove that Ω is open.

Choose any $x \in \Omega$. We want to prove that $x \in \mathrm{Int}_I(\Omega)$. Let us prove that if $x < \sup I$ then there exists $\delta > 0$ such that $[x, x + \delta) \subset \Omega$. Then a similar argument shows that if $x > \inf I$ then there exists $\delta > 0$ such that $(x - \delta, x] \subset \Omega$. This proves $x \in \mathrm{Int}_I(\Omega)$ whether x is an endpoint of I or not.

We first choose $\delta > 0$ such that $[x, x + \delta] \subset I$. Since f, g are continuous, both $\varphi(t, f(t))$ and $\varphi(t, g(t))$ are continuous in t. Therefore, by the fundamental theorem of calculus, we have

$$f(t) = f(x) + \int_x^t \varphi(s, f(s))ds \qquad g(t) = g(x) + \int_x^t \varphi(s, g(s))ds$$

Since $f(x) = g(x)$, we have

$$\|f(t) - g(t)\| \leqslant \int_x^t \|\varphi(s, f(s)) - \varphi(s, g(s))\|ds \leqslant L \int_x^t \|f(s) - g(s)\|ds$$

where $L \in \mathbb{R}_{\geqslant 0}$ is a Lipschitz constant of φ on E. Let $A = \sup_{t \in [x, x+\delta]} \|f(t) - g(t)\|$. Then the above inequality implies $\|f(t) - g(t)\| \leqslant LA\delta$ for all $t \in [x, x + \delta]$, and hence $A \leqslant LA\delta$. Now we shrink δ so that $L\delta < 1$. Then we must have $A = 0$. So $f = g$ on $[x, x + \delta]$, and hence $[x, x + \delta) \subset \Omega$. $\square$

Example 18.3. The function $\varphi(x) = x^{\frac{1}{3}}$ is not Lipschitz continuous on any compact interval containing 0. For every $c \geqslant 0$, the function

$$f(x) = \begin{cases} \left(\frac{2}{3}(x-c)\right)^{\frac{3}{2}} & \text{if } x \geqslant c \\ 0 & \text{if } x < c \end{cases}$$

is differentiable on $\mathbb{R}$ and satisfies the differential equation

$$f'(t) = (f(t))^{\frac{1}{3}} \qquad f(0) = 0$$

Therefore, Thm. 18.2 does not hold without assume the Lipschitz continuity.

Theorem 18.4 (Picard-Lindelöf theorem). *Let $\xi \in V$ and $0 < R < +\infty$. Let $I = [a, b]$ where $-\infty < a < b < +\infty$. Assume that $\varphi \in C\big(I \times \overline{B}_V(\xi, R), V\big)$ satisfies the following conditions:*

(1) φ is Lipschitz continuous on $\overline{B}_V(\xi, R)$.

(2) $M = \|\varphi\|_{l^\infty}$ is $< +\infty$.

Assume that

$$|I| \leqslant \frac{R}{M} \tag{18.2}$$

where $|I| = b - a$. Then there exists a unique differentiable function $f : I \to \overline{B}_V(\xi, R)$ satisfying the differential equation

$$f'(t) = \varphi(t, f(t)) \qquad f(a) = \xi \tag{18.3}$$

for all $t \in I$. The same conclusion holds if $f(a) = \xi$ is replaced by $f(b) = \xi$.

Note that the assumption $M < +\infty$ is automatic when $V = \mathbb{F}^N$, since in that case $I \times \overline{B}_V(\xi, R)$ is compact.

Proof. We only prove the existence, since the uniqueness follows from Thm 18.2. We treat the case $f(a) = \xi$. The other case is similar. Also, by translating f, it suffices to assume $a = 0$. So we let $I = [0, b]$. Then $bM \leqslant R$. Let L be the Lipschitz constant of φ on $\overline{B}_V(\xi, R)$. Note that if we can find $f \in C\big(I, \overline{B}_V(\xi, R)\big)$ satisfying the integral equation

$$f(t) = \xi + \int_0^t \varphi(s, f(s))ds \tag{18.4}$$

for all $t \in I$, then f clearly satisfies (18.3).

Step 1. We first consider the special case that $bL < 1$. Define a map

$$T : C\big(I, \overline{B}_V(\xi, R)\big) \to C(I, V)$$
$$(Tf)(t) = \xi + \int_0^t \varphi(s, f(s))ds$$

Then T has range inside $X = C\big(I, \overline{B}_V(\xi, R)\big)$ since for each $f \in X$ and $t \in I = [0, b]$ we have

$$\|(Tf)(t) - \xi\| \leqslant \int_0^t \|\varphi(s, f(s))\|ds \leqslant bM \leqslant R \tag{18.5}$$

by (18.2). Therefore, T can be viewed as a map $X \to X$.

Any fixed point of T satisfies (18.4). Therefore, by the contraction Thm. 17.45, it suffices to prove that T is a contraction. Here, the metric on X is defined by the l^∞-norm on $C(I, V)$. Then for each $f, g \in X$. We compute that for each $t \in [0, b]$,

$$\|(Tf)(t) - (Tg)(t)\| = \Big\| \int_0^t \varphi(s, f(s))ds - \int_0^t \varphi(s, g(s))ds \Big\|$$
$$\leqslant \int_0^t \big\|\varphi(s, f(s)) - \varphi(s, g(s))\big\|ds \leqslant L \int_0^t \big\|f(s) - g(s)\big\|ds \leqslant bL\|f - g\|_\infty$$

Applying $\sup_{t\in[0,b]}$ to the LHS, we get $\|Tf - Tg\|_\infty \leqslant bL\|f - g\|_\infty$. Since $bL < 1$, T is a contraction.

Step 2. We consider the general case. Choose $N \in \mathbb{Z}_+$ such that $bL/N < 1$. Let us prove by induction that for each $n = 0, 1, \dots, N$ there exists

$$f_n \in C\big([0, nb/N], \overline{B}_V(\xi, nR/N)\big)$$

satisfying (18.4) for all $t \in [0, nb/N]$. Then f_N gives the desired function satisfying (18.4) for all $t \in I = [0, b]$.

The case $n = 0$ is obvious. Assume that case $n - 1$ has been proved where $1 \leqslant n \leqslant N$. Let $\xi_{n-1} = f_{n-1}((n-1)b/N)$, which is inside $\overline{B}_V(\xi, (n-1)R/N)$. Now we apply step 1, but replace the I in step 1 by $I_n = [(n-1)b/N, nb/N]$ and replace the $\overline{B}_V(\xi, R)$ by $\overline{B}_V(\xi_{n-1}, R/N)$. Note that by triangle inequality we have

$$\overline{B}_V(\xi_{n-1}, R/N) \subset \overline{B}_V(\xi, nR/N)$$

Note also that assumption (18.2) becomes $b/N \leqslant R/NM$, which is still satisfied. Thus, according to step 1, there exists $g_n \in C\big(I_n, \overline{B}_V(\xi_{n-1}, R/N)\big)$ satisfying

$$g_n(t) = \xi_{n-1} + \int_{(n-1)b/N}^t \varphi(s, g(s))ds$$

for all $t \in I_n$. Then (the graph of) f_n is defined to be the union of (the graphs of) f_{n-1} and g_n. $\square$

Thm. 18.4 still holds if we assume $R = +\infty$ and $I = \mathbb{R}$:

Corollary 18.5. *Let $\xi \in V$. Assume that $\varphi \in C(\mathbb{R} \times V, V)$ satisfies the following conditions:*

(1) φ is Lipschitz continuous on V.

(2) $M = \|\varphi\|_{l^\infty}$ is $< +\infty$.

Choose any $t_0 \in \mathbb{R}$. Then there exists a unique differentiable $f : \mathbb{R} \to V$ satisfying

$$f'(t) = \varphi(t, f(t)) \qquad f(t_0) = \xi \tag{18.6}$$

for all $t \in \mathbb{R}$.

Proof. By Thm. 18.4, for each $a > 0$ there exists a unique differentiable $f : [t_0 - a, t_0 + a] \to V$ satisfying (18.6). The union of the graphs of these f_a (for all $a > 0$) gives the graph of the desired function. □

18.2 Peano's existence theorem

Theorem 18.6. *Let $\xi \in \mathbb{R}^N$. Let $\varphi \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$ satisfy $M := \|\varphi\|_\infty < +\infty$. Let $t_0 \in \mathbb{R}$. Then there exists a differentiable $f : \mathbb{R} \to \mathbb{R}^N$ satisfying*

$$f'(t) = \varphi(t, f(t)) \qquad f(t_0) = \xi \tag{18.7}$$

for all $t \in \mathbb{R}$.

Proof. Assume WLOG that $t_0 = 0$. It suffices to find $f \in C([0,1], \mathbb{R}^N)$ satisfying

$$f(t) = \xi + \int_0^t \varphi(s, f(s))ds \tag{18.8}$$

on $[0,1]$. Then f is differentiable and satisfies (18.7). Then we can similarly find $f_1 : [1,2] \to \mathbb{R}^N$ satisfying $f_1'(t) = \varphi(t, f_1(t))$ and $f_1(1) = f(1)$. Namely, f can be extended to a function $[0,2] \to \mathbb{R}^N$ satisfying (18.7). Repeating this argument, we get $f : [0, +\infty) \to \mathbb{R}^N$ satisfying (18.7), and similarly $f : \mathbb{R} \to \mathbb{R}^N$ satisfying (18.7).

Step 1. Let $X = \overline{B}_{\mathbb{R}^N}(\xi, M)$. Since $I \times X$ is compact Hausdorff, by Stone-Weierstrass Thm. 15.9, we have a sequence of multivariable polynomials $(\varphi_n)_{n \in \mathbb{Z}_+}$ (where $\varphi_n \in \mathbb{R}^N[t, x_1, \dots, x_N]$) converging uniformly on $[0,1] \times X$ to φ. Since $M = \|\varphi\|_\infty$, we have $\lim_n \|\varphi_n\|_{l^\infty(I \times X, \mathbb{R}^N)} = M$. Therefore, by scaling each φ_n, we assume that $\|\varphi_n\|_{l^\infty(I \times X, \mathbb{R}^N)} \leqslant M$. Since polynomials are clearly Lipschitz continuous, by Picard-Lindelöf Thm. 18.4, for each n there exists a differentiable $f_n : [0,1] \to X$ satisfying $f_n'(t) = \varphi_n(t, f_n(t))$ and $f_n(0) = \xi$. Equivalently, $f_n \in C([0,1], X)$ and

$$f_n(t) = \xi + \int_0^t \varphi_n(s, f_n(s))ds \tag{18.9}$$

for all $t \in [0,1]$.

Step 2. Recall that $\|\varphi_n\| \leqslant M$. For each $0 \leqslant t_1 \leqslant t_2 \leqslant 1$, by (18.9) we have

$$\|f_n(t_2) - f_n(t_1)\| \leqslant \int_{t_1}^{t_2} \|\varphi_n(s, f_n(s))\| ds \leqslant M(t_2 - t_1)$$

Therefore $(f_n)_{n \in \mathbb{Z}_+}$ is an equicontinuous sequence in $C([0,1], \mathbb{R}^N)$ and is uniformly bounded because $f_n([0,1]) \subset X$. By Arzelà-Ascoli Thm. 17.15, $\{f_n : n \in \mathbb{Z}_+\}$ is a precompact subset of $C([0,1], \mathbb{R}^N)$. Thus, by Prop. 17.2, (f_n) has a uniformly convergent subsequence. By replacing (f_n) with this subsequence, we assume WLOG that (f_n) converges uniformly to some $f \in C([0,1], \mathbb{R}^N)$. Since $f_n \in C([0,1], X)$, we have $f \in C([0,1], X)$.

To prove (18.8), in view of (18.9), it suffices to prove for each $t \in [0,1]$ that

$$\lim_{n \to \infty} \int_0^t \varphi_n(s, f_n(s)) ds = \int_0^t \varphi(s, f(s)) ds$$

By Cor. 13.21, it suffices to prove $\lim_n \varphi_n(\cdot, f_n(\cdot)) = \varphi(\cdot, f(\cdot))$ in $C([0,1], \mathbb{R}^N)$ under the l^∞-norm. Note that

$$\|\varphi(\cdot, f(\cdot)) - \varphi_n(\cdot, f_n(\cdot))\| \leqslant \|\varphi(\cdot, f(\cdot)) - \varphi(\cdot, f_n(\cdot))\| + \|\varphi(\cdot, f_n(\cdot)) - \varphi_n(\cdot, f_n(\cdot))\|$$

The first term on the RHS converges uniformly to 0 since $f_n \rightrightarrows f$ and since φ is uniformly continuous (Thm. 10.7). The second term converges uniformly to 0 since $\varphi_n \rightrightarrows \varphi$ on $[0,1] \times X$. So the LHS converges uniformly to 0. □

Thm. 18.6 is parallel to Cor. 18.5 since φ is defined on the whole space $\mathbb{R} \times \mathbb{R}^N$. However, in applications it is common that φ is only defined on a subset of $\mathbb{R} \times \mathbb{R}^N$. Thus, we want to prove an existence theorem similar to Thm. 18.4 without assuming the Lipschitz continuity, and hence without concluding the uniqueness. We shall state this result for a finite dimensional real Banach space V. This means that we consider $V = \mathbb{R}^N$, but not necessarily equipped with the Euclidean norm. Since all norms on $\mathbb{R}^N$ are equivalent (cf. Pb. 10.3), we conclude that the closed balls under any norm of $\mathbb{R}^N$ is compact (since they are closed subsets of standard closed balls of $\mathbb{R}^N$). In practice, it is useful to consider non-Euclidean norms of $\mathbb{R}^N$. For example, under the l^∞-norm on $\mathbb{R}^N = l^\infty(\{1,\dots,N\}, \mathbb{R})$, the closed balls are actually the cubes.

Theorem 18.7 (Peano's existence theorem). *Let V be a finite-dimensional real Banach space. Let $\xi \in V$ and $0 < R < +\infty$. Let $I = [a,b]$ where $-\infty < a < b < +\infty$. Let $\varphi \in C\big(I \times \overline{B}_V(\xi, R), V\big)$. Assume*

$$|I| \leqslant \frac{R}{M} \tag{18.10}$$

where $|I| = b - a$ and $M = \|\varphi\|_\infty$. Then there exists a differentiable function $f : I \to \overline{B}_V(\xi, R)$ satisfying the differential equation

$$f'(t) = \varphi(t, f(t)) \qquad f(a) = \xi \tag{18.11}$$

for all $t \in I$. The same conclusion holds if $f(a) = \xi$ is replaced by $f(b) = \xi$.

Proof. Since $\mathbb{R} \times V \simeq \mathbb{R}^{N+1}$ is LCH and $I \times \overline{B}_V(\xi, R)$ is compact, by Tietze extension Thm. 15.22, φ can be extended to an element in $C_c(I \times V, V)$ still satisfying $\|\varphi\|_\infty = M$. Therefore, by Thm. 18.6, there exists a differentiable $\xi : \mathbb{R} \to V$ satisfying (18.11). It remains to check that $f(I) \subset \overline{B}_V(\xi, R)$: For each $t \in I = [a, b]$, since $f(t) = \xi + \int_a^t \varphi(s, f(s))dt$, we have

$$\|f(t) - \xi\| \leqslant \int_a^t \|\varphi(s, f(s))\| dt \leqslant (t - a)M \leqslant |I| \cdot M \leqslant R$$

and hence $f(t) \in \overline{B}_V(\xi, R)$. $\square$

19 Differential calculus on $\mathbb{R}^N$

In this chapter, we fix a Banach space V over $\mathbb{R}$. However, we will be mainly interested in the case that V is finite dimensional.

19.1 Differentiability and C^1

A motivating question of this short chapter is the following: Suppose that f is a function on an open subset $\Omega \subset \mathbb{R}^N$, and $\gamma : (a,b) \to \Omega$ is differentiable. How to calculate $(f \circ \gamma)'$? The key to the answer of this question is the following definition:

Definition 19.1. Let $\Omega \subset \mathbb{R}^N$ be open. Let $p \in \Omega$. Let $f : \Omega \to V$. Assume that there is an $\mathbb{R}$-linear map $A : \mathbb{R}^N \to V$ such that for all $v \in \mathbb{R}^N$ we have

$$f(p+v) = f(p) + Av + o(\|v\|)$$

(recall Def. 12.13 for the meaning of $o(\|v\|)$). Namely,

$$\lim_{v \to 0} \frac{\|f(p+v) - f(p) - Av\|}{\|v\|} = 0 \tag{19.1}$$

Then we say that f is **differentiable at** p. We write

$$A = df|_p = df(p) : \mathbb{R}^N \to V$$

and call A the **differential** of f at p. If f is differentiable at every point of Ω, we simply say that f is **differentiable on** Ω.

Remark 19.2. Every linear map $A : \mathbb{R}^N \to V$ is bounded since, if $v = a_1e_1 + \cdots + a_Ne_N$ where $e_1, \dots, e_N$ are the standard basis of $\mathbb{R}^N$, then

$$\|Av\| \leqslant \sum_i |a_i| \cdot \|Ae_i\| \leqslant \|v\| \Big(\sum_i \|Ae_i\|^2 \Big)^{\frac{1}{2}}$$

by Minkowski's inequality.

Remark 19.3. If f is differentiable at p, then f is continuous at p since, by the continuity of A, we have $\lim_{v \to 0}(Av + o(\|v\|)) = 0$.

Whenever f is differentiable at p, its differential can be computed explicitly:

Proposition 19.4. *Let $\Omega \subset \mathbb{R}^N$ be open. Let $f : \Omega \to V$ be differentiable at $p \in \Omega$. Then $\partial_1 f, \dots, \partial_N f$ exist at p. Define the **Jacobian (matrix)***

$$\mathrm{Jac} f|_p = (\partial_1 f(p), \dots, \partial_N f(p)) \tag{19.2}$$

viewed as an $1 \times N$ *matrix with entries in* V. *(When* $V = \mathbb{R}^M$, *we view* $\mathrm{Jac}f|_p$ *as an* $M \times N$ *real matrix.) Then for each* $v = (a_1, \dots, a_n)^{\mathrm{t}} \in \mathbb{R}^N$ *(viewed as an* $N \times 1$ *matrix) we have*

$$df|_p \cdot v = \mathrm{Jac}f|_p \cdot v = \sum_{i=1}^{N} a_i \partial_i f(p) \tag{19.3}$$

Note that when $V = \mathbb{R}^M$ and $f = (f^1, \dots, f^N)^{\mathrm{t}}$, (19.3) reads

$$df|_p \cdot v = \begin{pmatrix} \partial_1 f^1 & \cdots & \partial_N f^1 \\ & \vdots & \\ \partial_1 f^M & \cdots & \partial_N f^M \end{pmatrix}_p \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \tag{19.4}$$

We also define the **Jacobian (determinant)** $\mathbf{J}(f) : \Omega \to \mathbb{R}$ whose value at each $p \in \Omega$ is

$$\mathbf{J}(f)|_p = \det\big(\mathrm{Jac}(f)_p\big) \tag{19.5}$$

Thus, $\mathrm{Jac}(f)_p$ is invertible iff $\mathbf{J}(f)|_p \neq 0$.

Proof. Let $e_1, \dots, e_N \in \mathbb{R}^N$ be the standard basis of $\mathbb{R}^N$. So $v = a_1 e_1 + \cdots + a_N e_N$. Let $A = df|_p$. By (19.1), we have

$$\lim_{t \to 0} \left\| \frac{f(p + te_i) - f(p)}{t} - Ae_i \right\| = \lim_{t \to 0} \frac{\|f(p + te_i) - f(p) - tAe_i\|}{\|te_i\|} = 0$$

which shows that $\partial_i f(p) = Ae_i$. Therefore

$$Av = \sum_i a_i A e_i = \sum_i a_i \partial_i f(p)$$

□

Corollary 19.5. *f has at most one differential at a point p.*

Proof. Immediate from (19.3). □

Example 19.6. When $N = 1$, the above definition of differentiability in Def. 19.1 agrees with the one in Sec. 11.1. Moreover, we clearly have $\mathrm{Jac}f|_p = f'(p)$.

Theorem 19.7 (Chain rule). *Let* $\Gamma \subset \mathbb{R}^M$ *and* $\Omega \subset \mathbb{R}^N$ *be open. Let* $g : \Gamma \to \Omega$ *be differentiable at* $p \in \Gamma$. *Let* $f : \Omega \to V$ *be differentiable at* $g(p)$. *Then* $f \circ g : \Gamma \to V$ *is differentiable at* p, *and*

$$d(f \circ g)|_p = df|_{g(p)} \cdot dg|_p \tag{19.6}$$

Equivalently, we have

$$\mathrm{Jac}(f \circ g)|_p = \mathrm{Jac}f|_{g(p)} \cdot \mathrm{Jac}g|_p \tag{19.7}$$

Note that when $V = \mathbb{R}^L$, (19.7) is of type $L \times M = (L \times N)(N \times M)$, and reads

$$\begin{pmatrix} \partial_1 (f\circ g)^1 & \cdots & \partial_M (f\circ g)^1 \\ & \vdots & \\ \partial_1 (f\circ g)^L & \cdots & \partial_M (f\circ g)^L \end{pmatrix}_p = \begin{pmatrix} \partial_1 f^1 & \cdots & \partial_N f^1 \\ & \vdots & \\ \partial_1 f^L & \cdots & \partial_N f^L \end{pmatrix}_{g(p)} \cdot \begin{pmatrix} \partial_1 g^1 & \cdots & \partial_M g^1 \\ & \vdots & \\ \partial_1 g^N & \cdots & \partial_M g^N \end{pmatrix}_p \tag{19.8}$$

Proof. We write

$$g(p+v) = g(p) + Bv + \beta(v) \qquad f(g(p)+w) = f\circ g(p) + Aw + \alpha(w)$$

where $A : \mathbb{R}^N \to V$ and $B : \mathbb{R}^M \to \mathbb{R}^N$ are linear, and

$$\lim_{v\to 0} \beta(v)/\|v\| = \lim_{w\to 0} \alpha(w)/\|w\| = 0 \tag{19.9}$$

Then

$$\begin{aligned} &f\circ g(p+v) = f(g(p)+Bv+\beta(v)) = f\circ g(p) + A(Bv+\beta(v)) + \alpha(Bv+\beta(v)) \\ =&f\circ g(p) + ABv + A\beta(v) + \alpha(g(p+v)-g(p)) \end{aligned}$$

Note that A, B are bounded by Rem. 19.2. So $\|A\beta(v)\| \leqslant \|A\|\cdot\|\beta(v)\|$, and hence $\lim_{v\to 0} \|A\beta(v)\|/\|v\| = 0$. To finish the proof, it remains to prove

$$\lim_{v\to 0} \|\alpha(g(p+v)-g(p))\|/\|v\| = 0$$

Since g is continuous at p, there is a neighborhood Δ of $0 \in \mathbb{R}^M$ such that $g(p+v)-g(p)$ is defined and is in the domain of α (which is $\{q - g(p) : q \in \Omega\}$) for every $v \in \Delta$. Define $\gamma : \Delta \to V$ to be

$$\gamma(v) = \begin{cases} \dfrac{\alpha\big(g(p+v)-g(p)\big)}{\|g(p+v)-g(p)\|} & \text{if } g(p+v)-g(p) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then γ is continuous at $v = 0$ and $\gamma(0) = 0$ by (19.9). (This part is similar to the construction (11.5).) Thus

$$\begin{aligned} &\frac{\|\alpha(g(p+v)-g(p))\|}{\|v\|} = \|\gamma(v)\| \cdot \frac{\|g(p+v)-g(p)\|}{\|v\|} \\ =&\|\gamma(v)\| \cdot \frac{\|Bv+\beta(v)\|}{\|v\|} \leqslant \|\gamma(v)\| \cdot \|B\| + \|\gamma(v)\| \cdot \frac{\|\beta(v)\|}{\|v\|} \end{aligned}$$

where the RHS converges to 0 as $v \to 0$. $\square$

Corollary 19.8 (Chain rule). *Let I be an interval. Let $\Omega \subset \mathbb{R}^N$ be open. Let $t_0 \in I$. Let $\gamma : I \to \Omega$ be differentiable at t_0. Let $f : \Omega \to V$ be differentiable at $p = \gamma(t_0)$. Then $f \circ \gamma$ is differentiable at t_0, and*

$$(f \circ \gamma)'(t_0) = \mathrm{Jac} f|_{\gamma(t_0)} \cdot \gamma'(t_0) = \sum_{i=1}^{N} (\gamma^i)'(t_0) \cdot \partial_i f(\gamma(t_0)) \tag{19.10}$$

Proof. This follows easily from Thm. 19.7 when t_0 is an interior point of I. Suppose that t_0 is an end point of I. For example, consider $t_0 = b$ where I is $[a, b]$ or $(a, b]$. Then we can easily extend γ to a function on $[a, b+1)$ or $(a, b+1)$ which is differentiable at t_0. (For example, if $t \in (b, b+1)$, we can define $\gamma(t) = \gamma(b) + (t-b)\gamma'(b)$.) Then we can apply Thm. 19.7 again to finish the proof. $\square$

Example 19.9. Let $\Omega \subset \mathbb{R}^N$ be open. Let $f : \Omega \to V$ be differentiable at $p \in \Omega$. Then for each $v \in \mathbb{R}^N$, the **directional derivative**

$$(\nabla_v f)(p) = \lim_{t \to 0} \frac{f(p + tv) - f(p)}{t}$$

exists and equals $df|_p \cdot v = \mathrm{Jac} f|_p \cdot v$.

Proof. Apply the chain rule to $\gamma(t) = p + tv$. $\square$

We have thus proved the chain rule, one of the most important reasons for introducing differentiability on $\mathbb{R}^N$. We know that differentiable functions have partial derivatives of first order. However, having first order partial derivatives does not imply differentiability or even continuity:

Example 19.10. Let $f(x, y) = \dfrac{xy}{x^2 + y^2}$ when $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Then $\lim_{r \to 0} f(r, r) = \frac{1}{2}$ and $\lim_{r \to 0} f(r, 0) = 0$, showing that f has no limit and is not continuous $(0, 0)$ at. However, $f(x, 0) = f(0, y) = 0$. So $\partial_1 f$ and $\partial_2 f$ are both equal to 0 at $(0, 0)$.

Example 19.11. Let $f(x, y) = \dfrac{x^3}{x^2 + y^2}$ when $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Since $|f(x, y)| \leqslant |x|$ and $\lim_{(x,y) \to (0,0)} |x| = 0$, we conclude that f is continuous at $(0, 0)$ (and hence is continuous everywhere). We have $\partial_1 f|_{(0,0)} = \frac{d}{dx} x|_{x=0} = 1$ and $\partial_2 f|_{(0,0)} = 0$. Let $\gamma(t) = (at, bt)$ where $(a, b) \in \mathbb{R}^2 \backslash \{(0, 0)\}$. Then

$$(\nabla_{(a,b)} f)(0) = (f \circ \gamma)'(0) = \frac{a^3}{a^2 + b^2}$$

is not equal to $\mathrm{Jac} f|_{(0,0)} \cdot (a, b)^{\mathrm{t}} = a$. Thus, by Exp. 19.9, f is not differentiable at $(0, 0)$.

To have differentiability, we need the continuity of partial derivatives of first order.

Definition 19.12. Let $\Omega \subset \mathbb{R}^N$ be open. For each $r \in \mathbb{N}$, we let

$$C^r(\Omega, V) = \big\{f \in C(\Omega, V) : \partial_{i_1} \cdots \partial_{i_k} f \text{ exists and is in } C(\Omega, V) \\ \text{for all } 0 \leqslant k \leqslant r \text{ and } 1 \leqslant i_1, \ldots, i_k \leqslant N\big\}$$

In particular, we understand $C^0(\Omega, V)$ as $C(\Omega, V)$, and

$$C^\infty(\Omega, V) = \bigcap_{r \in \mathbb{N}} C^r(\Omega, V)$$

Elements in $C^\infty(\Omega, V)$ are called **smooth functions**.

It is clear that $C^r(\Omega, V) \subset C^q(\Omega, V)$ if $r \geqslant q$.

Remark 19.13. Suppose that $r \in \mathbb{Z}_+$ and $f \in C^r(\Omega, V)$. Let $1 \leqslant k \leqslant r$, and let $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$ be a bijection. Then by Thm. 12.35 or 14.7, for each $1 \leqslant i_1, \ldots, i_k \leqslant N$ we have

$$\partial_{i_1} \cdots \partial_{i_k} f = \partial_{i_{\sigma(1)}} \cdots \partial_{i_{\sigma(k)}} f$$

Proposition 19.14. *Let $\Gamma \subset \mathbb{R}^M$ and $\Omega \subset \mathbb{R}^N$ be open. Let $g : \Gamma \to \Omega$ and $f : \Omega \to V$ be C^r-functions (where $r \in \mathbb{N} \cup \{\infty\}$). Then $f \circ g$ is C^r.*

Proof. This is obvious when $r = 0$. Assume $r > 0$. By induction on k (where $1 \leqslant k \leqslant r$), and by using the (one-variable) chain rule and the Leibniz product rule, for each $1 \leqslant i_1, \ldots, i_k \leqslant M$ one sees that $\partial_{i_1} \cdots \partial_{i_k}(f \circ g)$ is a linear combination of products of (possibly more than two) functions of the form

$$\big((\partial_{j_1} \cdots \partial_{j_{k'}} f) \circ g\big) \qquad \text{or} \qquad \partial_{l_1} \cdots \partial_{l_{k''}} g \tag{19.11}$$

where $0 \leqslant k', k'' \leqslant k$ and $0 \leqslant j_1, \ldots, j_{k'} \leqslant N$ and $0 \leqslant l_1, \ldots, l_{k''} \leqslant M$. So $f \circ g$ is C^r since the functions in (19.11) are continuous. $\square$

Theorem 19.15. *Let Ω be an open subset of $\mathbb{R}^N$. Let $f \in C^1(\Omega, V)$. Then f is differentiable on Ω.*

Proof. We prove this by induction. The case $N = 1$ is obvious. Assume that the case N is true where $N \in \mathbb{Z}_+$. Let $\Omega \subset \mathbb{R}^{N+1}$ be open, and let $(p_0, p_1, \ldots, p_N) = (p_0, p_\bullet) \in \Omega$. Let $v = (a_0, a_\bullet) = (a_0, a_1, \ldots, a_N) \in \Omega$. Then

$$\begin{aligned}
&f(p_0 + a_0, p_\bullet + a_\bullet) - f(p_0, p_\bullet) \\
=&f(p_0 + a_0, p_\bullet + a_\bullet) - f(p_0, p_\bullet + a_\bullet) + f(p_0, p_\bullet + a_\bullet) - f(p_0, p_\bullet) \\
=&\int_0^{a_0} \partial_0 f(p_0 + t, p_\bullet + a_\bullet)dt + f(p_0, p_\bullet + a_\bullet) - f(p_0, p_\bullet)
\end{aligned}$$

By case N, there exist $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$ such that

$$f(p_0, p_\bullet + a_\bullet) - f(p_0, p_\bullet) = \sum_{i=1}^N \lambda_i a_i + o(\|a_\bullet\|)$$

(In fact, $\lambda_i = \partial_i f(p_0, p_\bullet)$ by Prop. 19.4.) Let

$$\begin{aligned} h(a_0, a_\bullet) &= \int_0^{a_0} \partial_0 f(p_0 + t, p_\bullet + a_\bullet) dt - \partial_0 f(p_0, p_\bullet) \cdot a_0 \\ &= \int_0^{a_0} \big(\partial_0 f(p_0 + t, p_\bullet + a_\bullet) - \partial_0 f(p_0, p_\bullet)\big) dt \end{aligned}$$

Since $\partial_1 f$ is continuous, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $(t, a_\bullet) \in \mathbb{R}^N$ with norm $\leqslant \delta$ we have

$$\|\partial_0 f(p_0 + t, p_\bullet + a_\bullet) - \partial_0 f(p_0, p_\bullet)\| \leqslant \varepsilon$$

and hence $\|h(a_0, a_\bullet)\| \leqslant \varepsilon |a_0|$ for all $(a_0, a_\bullet)$ with norm $\leqslant \delta$. Thus $h(a_0, a_\bullet) = o(|a_0|)$. Therefore

$$\begin{aligned} & f(p_0 + a_0, p_\bullet + a_\bullet) - f(p_0, p_\bullet) - \partial_0 f(p_0, p_\bullet) \cdot a_0 - \sum_{i=1}^N \lambda_i a_i \\ =& o(|a_0|) + o(\|a_\bullet\|) = o(\|(a_0, a_\bullet)\|) \end{aligned}$$

□

19.2 Applications of the chain rule

Example 19.16. Let I, J be nonempty open intervals in $\mathbb{R}$. Choose $f \in C(I \times J, V)$ such that $\partial_2 f$ exists and is in $C(I \times J, V)$. Let $\alpha, \beta \in C^1(J, \mathbb{R})$ such that $\alpha(J) \subset I, \beta(J) \subset I$. Then

$$\frac{d}{dy} \int_{\alpha(y)}^{\beta(y)} f(x, y) dx = \int_{\alpha(y)}^{\beta(y)} \partial_2 f(x, y) dx + f(\beta(y), y)\beta'(y) - f(\alpha(y), y)\alpha'(y) \quad (19.12)$$

Proof. Define $F : I \times I \times J \to V$ by

$$F(r, s, t) = \int_r^s f(x, t) dx = \int_e^s f(x, t) dx - \int_e^r f(x, t) dx$$

where e is any element of I. By Exe. 14.8, F is continuous. Since $\partial_2 f$ is assumed to be continuous, by Thm. 14.6, we have

$$\partial_3 F(r, s, t) = \int_r^s \partial_2 f(x, t) dx$$

By the fundamental theorem of calculus, we have

$$\partial_1 F(r,s,t) = -f(r,t) \qquad \partial_2 F(r,s,t) = f(s,t)$$

By Exe. 14.8, $\partial_1 F, \partial_2 F, \partial_3 F$ are continuous, and hence F is a C^1-function. Therefore, by the chain rule (Cor. 19.8), the LHS of (19.12) can be calculated by setting $(r,s,t) = (\alpha(y), \beta(y), y)$:

$$\begin{aligned}&\frac{d}{dy}F(\alpha(y),\beta(y),y)\\ =&\partial_1 F(\alpha(y),\beta(y),y)\alpha'(y) + \partial_2 F(\alpha(y),\beta(y),y)\beta'(y) + \partial_3 F(\alpha(y),\beta(y),y)\end{aligned}$$

which equals the RHS of (19.12). □

Example 19.17. Fix $v \in V$ and $\varphi \in C(\mathbb{R}, V)$. Let $A \in \mathfrak{L}(V)$. Then $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$ by Exe. 17.42. By Exp. 19.16 and Thm. 13.16, we have

$$\frac{d}{dt}\int_0^t e^{A(t-s)}\varphi(s)ds = \varphi(t) + \int_0^t Ae^{A(t-s)}\varphi(s)ds = \varphi(t) + A\int_0^t e^{A(t-s)}\varphi(s)ds$$

It follows that

$$f(t) = e^{At}v + \int_0^t e^{A(t-s)}\varphi(s)ds \tag{19.13}$$

is a solution of the differential equation

$$f'(t) = Af(t) + \varphi(t) \qquad f(0) = v$$

It is the unique solution by Picard-Lindelöf Cor. 18.5.

Theorem 19.18 (Finite-increment theorem). *Let Ω be an open subset of $\mathbb{R}^N$. Assume that $f : \Omega \to V$ is differentiable. Assume that $x, y \in \Omega$ satisfy $[x,y] \subset \Omega$ where*

$$[x,y] = \{(1-t)x + ty : 0 \leqslant t \leqslant 1\}$$

Then

$$\|f(y) - f(x)\| \leqslant \Big(\sup_{z\in[x,y]} \|df|_z\|\Big) \cdot \|y - x\| \tag{19.14}$$

where $\|df|_z\|$ is the operator norm of $df|_z$.

Proof. This can be proved in a similar way as Cor. 11.31. Let $\gamma : [0,1] \to \Omega$ be $\gamma(t) = (1-t)x + ty$. Then by chain rule we have $(f\circ\gamma)'(t) = df|_{\gamma(t)} \cdot \gamma'(t) = df|_{\gamma(t)} \cdot (y-x)$. Thus $\|(f\circ\gamma)'(t)\|$ is $\leqslant$ the RHS of (19.14). Therefore, by the single-variable finite-increment Thm. 11.29, we have

$$\|f(y) - f(x)\| = \|f\circ\gamma(1) - f\circ\gamma(0)\| \leqslant \sup_{t\in[0,1]} \|(f\circ\gamma)'(t)\|$$

which is $\leqslant$ the RHS of (19.14). □

The following corollary provides a lot of examples satisfying the assumptions on φ in Picard-Lindelöf Thm. 18.4. Recall Def. 11.30 for the meaning of convex sets.

Corollary 19.19. *Let Ω be an open subset of $\mathbb{R}^N$. Let $f \in C^1(\Omega, V)$. Let K be a compact convex subset of Ω. Then $f|_K$ is Lipschitz continuous.*

Proof. Since $df : \Omega \to \mathfrak{L}(\mathbb{R}^N, V)$ is continuous (because it is essentially $\mathrm{Jac} f$ by Prop. 19.4), there exists $L \in \mathbb{R}_{\geqslant 0}$ such that $\|df|_x\| \leqslant L$ for all $x \in K$. By Thm. 19.18, L is a Lipschitz constant of $f|_K$. □

Corollary 19.20. *Let Ω be a nonempty open connected subset of $\mathbb{R}^N$. Assume that $f : \Omega \to V$ satisfies that $df = 0$ everywhere on Ω. Then f is a constant.*

Proof. Fix $p \in \Omega$, and let $v = f(p)$. Let us prove that $f = v$ on Ω. Let $\Delta = \{x \in \Omega : f(x) = v\}$, which is a closed subset of Ω by the continuity of f (Rem. 19.3). Δ is nonempty since $p \in \Delta$. Thus, if we can show that Δ is open in Ω, then $\Delta = \Omega$ because Ω is connected, and hence the proof is complete.

To see that Δ is open, we choose any $x \in \Delta$, and choose $r > 0$ such that $U = B_{\mathbb{R}^N}(x, r)$ is contained in Ω. Since U is path-connected and hence connected, by Thm. 19.18 and the fact that $df = 0$, for every $y \in U$ we have $\|f(x) - f(y)\| = 0$ and hence $f(y) = v$. So $U \subset \Delta$. This proves that x is an interior point of Δ. □

20 Inner product spaces

Hilbert spaces and measure theory are two parallel but deeply connected theories that arose in the study of Fourier series and differential equations. We will spend half the semester learning these two theories in turn. The climax of this entire story is the Riesz-Fischer theorem, which establishes a connection between these two theories.

In the following three chapters, we develop the basic theory of Hilbert spaces. A Hilbert space is defined to be a complete inner product space. This definition indicates that Hilbert spaces have two important aspects:

- Geometry: orthogonal and orthonormal vectors.
- Analysis: completeness, and more.

We will focus on the geometric aspect in this chapter, and leave the discussion of the analytic aspect to the next chapter. As we shall see, the geometry of orthogonal vectors (together with the powerful **Gram-Schmidt process**) provides a uniform understanding of many identities and inequalities: Bessel's inequality (and its special case, Cauchy-Schwarz inequality), Parserval's identity. We will apply this geometric understanding to Fourier series. Surprisingly, the geometry of inner product spaces provides an elegant proof of the following analytic result: Every Riemann integrable function on $[-\pi,\pi]$ is the limit of its Fourier series under the L^2-norm.

When referring to Hilbert spaces and inner product spaces, it is usually assumed that the field is $\mathbb{C}$, since complex Hilbert spaces are more useful than real ones. We will present the theory only for complex Hilbert spaces, although it can be easily adapted to real ones.

Starting from this chapter, we adopt the notations

$$l^p(X)=l^p(X,\mathbb{C})\qquad C(X)=C(X,\mathbb{C})\qquad C_c(X)=C_c(X,\mathbb{C}) \tag{20.1}$$

20.1 Inner product spaces

We fix a complex vector space V.

Definition 20.1. A map of complex vector spaces $T:V\to W$ is called **antilinear** or **conjugate linear** if for every $a,b\in\mathbb{C}$ and $u,v\in V$ we have

$$T(au+bv)=\overline{a}u+\overline{b}v$$

where $\overline{a},\overline{b}$ are the complex conjugates of a,b.

For example, the involution $*:\mathscr{A}\to\mathscr{A}$ of a $*$-algebra is antilinear (recall Def. 15.4).

Definition 20.2. A function $\langle\cdot|\cdot\rangle : V \times V \to \mathbb{C}$ (sending $u \times v \in V^2$ to $\langle u|v\rangle$) is called a **sesquilinear form** if it is linear on the first variable, and antilinear on the second one.[1] Namely, for each $a, b \in \mathbb{C}$ and $u, v, w \in V$ we have

$$\langle au + bv|w\rangle = a\langle u|w\rangle + b\langle v|w\rangle \qquad \langle w|au + bv\rangle = \overline{a}\langle w|u\rangle + \overline{b}\langle w|v\rangle$$

More generally, if V, W are complex vector spaces, a map $V \times W \to \mathbb{C}$ is also called **sesquilinear** if it is linear on the V-component and antilinear on the W-component.

Notice the difference between the notations $\langle u|v\rangle$ and $\langle u, v\rangle$: the latter always means a bilinear form, i.e., a function which is linear on both variables.

Proposition 20.3. *Suppose that $\langle\cdot|\cdot\rangle$ and $(\cdot|\cdot)$ are sesquilinear forms on V satisfying $\langle v|v\rangle = (v|v)$ for all $v \in V$. Then $\langle u|v\rangle = (u|v)$ for all $u, v \in V$.*

Proof. Let us prove that $\langle u|v\rangle$ can be written in terms of expressions of the form $\langle \xi|\xi\rangle$ where $\xi \in V$. Choose any $t \in [-\pi, \pi]$. Let

$$f(t) := \langle u + e^{\mathbf{i}t}v|u + e^{\mathbf{i}t}v\rangle = \langle u|u\rangle + \langle v|v\rangle + e^{-\mathbf{i}t}\langle u|v\rangle + e^{\mathbf{i}t}\langle v|u\rangle$$

Then $\langle u|v\rangle$ is a Fourier coefficient of f. Namely, using the fact that $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathbf{i}nt}dt = \delta_{n,0}$ (where $n \in \mathbb{Z}$), we have

$$\langle u|v\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \langle u + e^{\mathbf{i}t}v|u + e^{\mathbf{i}t}v\rangle e^{\mathbf{i}t}dt \tag{20.2}$$

This finishes the proof. □

Remark 20.4. In practice, it is sometimes more convenient to have a discrete version of (20.2): We view $f(t)$ as a function on the finite abelian group $\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\} \simeq \mathbb{Z}/4\mathbb{Z}$. Then $\langle u|v\rangle$ is a coefficient of the "discrete Fourier transform" of f:

$$\begin{aligned}\langle u|v\rangle &= \frac{1}{4}\sum_{t=0,\frac{\pi}{2},\pi,\frac{3\pi}{2}} \langle u + e^{\mathbf{i}t}v|u + e^{\mathbf{i}t}v\rangle e^{\mathbf{i}t} \\ &= \frac{1}{4}\Big(\langle u + v|u + v\rangle - \langle u - v|u - v\rangle + \mathbf{i}\langle u + \mathbf{i}v|u + \mathbf{i}v\rangle - \mathbf{i}\langle u - \mathbf{i}v|u - \mathbf{i}v\rangle\Big)\end{aligned} \tag{20.3}$$

We call (20.3) the **polarization identity**.

Definition 20.5. A function $\langle\cdot|\cdot\rangle : V \times V \to \mathbb{C}$ is called a **Hermitian form** if it is linear on the first variable and satisfies $\overline{\langle u|v\rangle} = \langle v|u\rangle$ for all $u, v \in V$. Then $\langle\cdot|\cdot\rangle$ is automatically antilinear on the second variable, i.e., a Hermitian form is automatically a sesquilinear form.

[1] Physicists prefer the opposite convention, i.e., their sesquilinear forms are antilinear on the first variables.

Example 20.6. Let $A \in \mathbb{C}^{N\times N}$ be a complex $N \times N$ matrix. Define $\langle\cdot|\cdot\rangle : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}$ by $\langle u|v\rangle = \overline{v^{\mathrm{t}}} \cdot A \cdot u$ where u, v are viewed as column vectors. Then $\langle\cdot|\cdot\rangle$ is a sesquilinear form on $\mathbb{C}^N$. (It is an easy linear algebra exercise that every sesquilinear form on $\mathbb{C}^N$ is of this form.) Moreover, $\langle\cdot|\cdot\rangle$ is a Hermitian form iff A is a **Hermitian matrix**, i.e., $A = \overline{A^{\mathrm{t}}}$.

The following is our first application of the polarization identity:

Proposition 20.7. *Let $\langle\cdot|\cdot\rangle$ be a sesquilinear form on V. The following are equivalent:*

(1) $\langle\cdot|\cdot\rangle$ is a Hermitian form.

(2) For each $v \in V$ we have $\langle v|v\rangle \in \mathbb{R}$.

Proof. (1)⇒(2): Obvious. (2)⇒(1): Let $(u|v) = \overline{\langle v|u\rangle}$. Then $(\cdot|\cdot)$ is a sesquilinear form on V. Assuming (2), we have $\langle v|v\rangle = (v|v)$ for all $v \in \mathbb{R}$. Therefore, by Prop. 20.3, we have $\langle u|v\rangle = (u|v) = \overline{\langle v|u\rangle}$ for all u, v. So (1) is true. □

Definition 20.8. A sesquilinear form $\langle\cdot|\cdot\rangle$ on V is called **positive semi-definite** (or simply **positive**), if $\langle v|v\rangle \geqslant 0$ for all $v \in V$. If a positive sesquilinear form $\langle\cdot|\cdot\rangle$ on V is fixed, we define

$$\|v\| = \sqrt{\langle v|v\rangle} \qquad \text{for all } v \in V \tag{20.4}$$

A vector $v \in V$ satisfying $\|v\| = 1$ is called a **unit vector**.

By Prop. 20.7, a positive sesquilinear form is Hermitian.

Definition 20.9. Let $\langle\cdot|\cdot\rangle$ be a positive sesquilinear form on V. By sesquilinearity, we clearly have $\langle 0|0\rangle = 0$. We say that $\langle\cdot|\cdot\rangle$ is an **inner product** if it is also **non-degenerate**, i.e., if the only $v \in V$ satisfying $\langle v|v\rangle = 0$ is $v = 0$. We call the pair $(V, \langle\cdot|\cdot\rangle)$ (or simply call V) an **inner product space** or a **pre-Hilbert space** .

Example 20.10. $\mathbb{C}^N$, equipped with the Euclidean inner product $\langle u|v\rangle = \overline{v^{\mathrm{t}}}u$, is an inner product space.

Example 20.11. Let X be a set. Then $l^2(X, \mathbb{C})$, together with the standard inner product

$$\langle f|g\rangle = \sum_{x\in X} f(x)\overline{g(x)} \tag{20.5}$$

(where the RHS converges by Thm. 12.33), is an inner product space.

Example 20.12. Let $-\infty < a < b < +\infty$. Then $C([a,b], \mathbb{C})$, together with the inner product

$$\langle f|g\rangle = \int_a^b fg^* = \int_a^b f(x)\overline{g(x)}dx \tag{20.6}$$

is an inner product space.

The gap between positive forms and inner products is not very big:

Proposition 20.13. *Let $\langle\cdot|\cdot\rangle$ be a positive sesquilinear form on V. Let*

$$\mathscr{N}=\{v\in V:\|v\|=0\}\tag{20.7}$$

Then $\mathscr{N}$ is a linear subspace of V. Moreover, there is an inner product $(\cdot|\cdot)$ on the quotient space $V/\mathscr{N}$ satisfying

$$(u+\mathscr{N}|v+\mathscr{N})=\langle u|v\rangle\qquad\textit{for all } u,v\in V\tag{20.8}$$

Proof. (20.8) suggests that if $\|v\|=0$, then $\langle u|v\rangle=(u+\mathscr{N}|v+\mathscr{N})=(u+\mathscr{N}|\mathscr{N})=\langle u|0\rangle=0$ for all $u\in V$. Motivated by this observation, let us prove

$$\mathscr{N}=\{v\in V:(u|v)=0\text{ for all }u\in V\}\tag{20.9}$$

Then the linearity of $\mathscr{N}$ follows immediately from (20.9). Moreover, if we define a function $(\cdot|\cdot)$ on $V/\mathscr{N}$ by (20.8), then it is well-defined: If $v+\mathscr{N}=v'+\mathscr{N}$, then $v-v'\in\mathscr{N}$. Thus, by (20.9), we have $\langle u|v-v'\rangle=0$. So

$$(u+\mathscr{N}|v+\mathscr{N})=\langle u|v\rangle=\langle u|v'\rangle=(u+\mathscr{N}|v'+\mathscr{N})$$

A similar argument shows that if $u+\mathscr{N}=u'+\mathscr{N}$ then $(u+\mathscr{N}|v+\mathscr{N})=(u'+\mathscr{N}|v+\mathscr{N})$. This proves the well-definedness. It is easy to check that $(\cdot|\cdot)$ is sesquilinear and positive. If $(v+\mathscr{N}|v+\mathscr{N})=0$, then $\langle v|v\rangle=0$ and hence $v\in\mathscr{N}$. This proves that $(\cdot|\cdot)$ is non-degenerate.

Let us prove (20.9). Clearly $\|v\|=\sqrt{\langle v|v\rangle}=0$ holds whenever $\langle u|v\rangle=0$ for all u. Conversely, suppose that $\|v\|=0$, i.e., $\langle v|v\rangle=0$. Choose any $u\in V$. Then for each $t\in\mathbb{R}$ we have

$$0\leqslant\langle u+tv|u+tv\rangle=\|u\|^2+2t\cdot\mathrm{Re}\langle u|v\rangle$$

where the RHS is a linear function of t. Any linear function which is always $\geqslant 0$ must be zero. So $\mathrm{Re}\langle u|v\rangle=0$. Similarly, $\mathrm{Im}\langle u|v\rangle=-\mathrm{Re}\langle\mathbf{i}u|v\rangle=0$. So $\langle u|v\rangle=0$. $\square$

Example 20.14. Let $-\infty<a<b<+\infty$ and $\mathscr{V}=\mathscr{R}[a,b]=\mathscr{R}([a,b],\mathbb{C})$ (the space of Riemann integrable complex functions on $[a,b]$). Then $\mathscr{V}$ has a positive sesquilinear form defined by (20.6). Let $\mathscr{N}=\{f\in V:\langle f|f\rangle=0\}$. Then

$$\mathscr{N}=\{f\in\mathscr{V}:\ f\text{ is zero outside a null subset of }[a,b]\}\tag{20.10}$$

Thus, $V=\mathscr{V}/\mathscr{N}$ is the set of all $f\in\mathscr{R}[a,b]$ such that $f,g\in\mathscr{R}[a,b]$ are viewed as the same element of V iff $f=g$ almost everywhere (i.e. $\{x:f(x)\neq g(x)\}$ is null).

Proof. Let $g = |f|^2$. We want to show that $\int g = 0$ iff $\Delta = \{x \in [a,b] : g(x) > 0\}$ is null. Assume that $\int g > 0$. Then f has a strictly positive lower Darboux sum (recall Thm. 13.41). This implies that there exist c, d satisfying $a \leqslant c < d \leqslant b$ such that $(d-c) \cdot \inf_{\xi \in [c,d]} g(\xi) > 0$. So $g > 0$ on $[c,d]$. So Δ contains $[c,d]$, and hence is not null.

Assume that $\int g = 0$. By Lebesgue's criterion 14.10, g is continuous outside a null subset of $[a,b]$. Thus, it suffices to prove that $g(p) = 0$ for any $p \in [a,b]$ at which g is continuous. Suppose that $\varepsilon := g(p) > 0$. Then by the continuity, there is an interval $I \subset [a,b]$ containing p with $|I| > 0$ such that $g > \varepsilon/2$ on I. Thus $g \geqslant \frac{\varepsilon}{2} \cdot \chi_I$, and hence $0 = \int g \geqslant \frac{\varepsilon}{2} \cdot |I|$, impossible. □

We close this section with an elementary but important fact. It says that linear maps are determined by their associated sesquilinear forms.

Exercise 20.15. Suppose that $S, T : U \to V$ are linear maps of inner product spaces.

1. Prove that $S = T$ iff $\langle Su|v\rangle = \langle Tu|v\rangle$ for all $u \in U, v \in V$.

2. Assume that $U = V$. Prove that $S = T$ iff $\langle Sv|v\rangle = \langle Tv|v\rangle$ for all $v \in V$.

20.2 Pythagorean and Gram-Schmidt

Unless otherwise stated, we fix an inner product $\langle \cdot|\cdot \rangle$ on a complex vector space V so that V is an inner product space.

Definition 20.16. A set $\mathfrak{S}$ of vectors of V are called **orthogonal** if $\langle u|v\rangle = 0$ for any distinct $u, v \in V$. An orthogonal set $\mathfrak{S}$ is called **orthonormal** if $\|v\| = 1$ for all $v \in V$.

Remark 20.17. We will also talk about an **orthogonal** resp. **orthonormal family of vectors** $(e_i)_{i \in I}$. This means that $\langle e_i|e_j\rangle = 0$ for any distinct $i, j \in I$ (resp. $\langle e_i|e_j\rangle = \delta_{i,j}$ for any $i, j \in I$).

In particular, two vectors $u, v \in V$ are called orthogonal when $\langle u|v\rangle = 0$. A fundamental fact about orthogonal vectors is

Proposition 20.18 (Pythagorean identity). *Suppose that $u, v \in V$ are orthogonal. Then*

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 \tag{20.11}$$

In particular,

$$\|v\| \leqslant \|u + v\| \tag{20.12}$$

Proof. $\|u+v\|^2 = \langle u+v|u+v\rangle = \langle u|u\rangle + \langle v|v\rangle + 2\mathrm{Re}\langle u|v\rangle = \langle u|u\rangle + \langle v|v\rangle$. □

Note that by applying (20.11) repeatedly, we see that if $v_1,\dots,v_n \in V$ are orthogonal, then

$$\|v_1+\cdots+v_n\|^2 = \|v_1\|^2+\cdots+\|v_n\|^2 \tag{20.13}$$

Remark 20.19. Suppose that $\mathfrak{S}$ is an orthonormal set of vectors of V. Then $\mathfrak{S}$ is clearly linearly independent. (If $e_1,\dots,e_n \in \mathfrak{S}$ and $\sum_i a_i e_i = 0$, then $a_j = \sum_i \langle a_i e_i|e_j\rangle = \langle 0|e_j\rangle = 0$.) Thus, by linear algebra, if $\mathfrak{S} = \{e_1,\dots,e_n\}$ is finite, then one can find uniquely $a_1,\dots,a_n \in \mathbb{C}$ and $u \in V$ such that $v = a_1e_1+\cdots+a_ne_n+u$ and that u is orthogonal to $e_1,\dots,e_n$. The expressions of $a_1,\dots,a_n,u$ can be expressed explicitly:

Proposition 20.20 (Gram-Schmidt). *Let $e_1,\dots,e_n$ be orthonormal vectors in V. Let $v \in V$. Then*

$$v - \sum_{i=1}^n \langle v|e_i\rangle \cdot e_i \tag{20.14}$$

is orthogonal to $e_1,\dots,e_n$.

Proof. This is a direct calculation and is left to the readers. □

Remark 20.21. "Gram-Schmidt" usually refers to the following process. Let $v_1,\dots,v_n$ be a set of linearly independent vectors of V. Then there is an algorithm of finding an orthonormal basis of $U = \mathrm{Span}\{v_1,\dots,v_n\}$: Let $e_1 = v_1/\|v_1\|$. Suppose that a set of orthonormal vectors $e_1,\dots,e_k$ in U have been found. Then e_{k+1} is defined by $\widetilde{v}_{k+1}/\|\widetilde{v}_{k+1}\|$ where $\widetilde{v}_{k+1} = v_{k+1} - \sum_{i=1}^k \langle v_{k+1}|e_i\rangle \cdot e_i$.

Combining Pythagorean with Gram-Schmidt, we have:

Corollary 20.22 (Bessel's inequality). *Let $(e_i)_{i\in I}$ be a family of orthonormal vectors of V. Then for each $v \in V$ we have*

$$\sum_{i\in I} |\langle v|e_i\rangle|^2 \leqslant \|v\|^2 \tag{20.15}$$

In particular, the set $\{i \in I : \langle v|e_i\rangle \neq 0\}$ is countable.

Proof. The LHS of (20.15) is $\lim_{J\in\mathrm{fin}(2^I)} \sum_{j\in J} |\langle v|e_j\rangle|^2$. Thus, it suffices to that for each $J \in \mathrm{fin}(2^I)$ we have $\sum_{j\in J} |\langle v|e_j\rangle|^2 \leqslant \|v\|^2$. Let

$$u_1 = \sum_{j\in J} \langle v|e_j\rangle \cdot e_j \qquad u_2 = v - u_1$$

(Namely, $v = u_1 + u_2$ is the orthogonal decomposition of v with respect to $\mathrm{Span}\{e_j : j \in J\}$.) By Gram-Schmidt, we have $\langle u_1|u_2\rangle = 0$. By Pythagorean, we have $\|u_1\|^2 \leqslant \|v\|^2$. But Pythagorean (20.13) also implies

$$\|u_1\|^2 = \sum_{j\in J} |\langle v|e_j\rangle|^2$$

The last statement about countability follow from Pb. 5.3. $\square$

Theorem 20.23 (Cauchy-Schwarz inequality). *For each $u, v \in V$ we have*

$$|\langle u|v\rangle| \leqslant \|u\| \cdot \|v\| \tag{20.16}$$

Proof. If $v = 0$, then the inequality trivially holds. Assume $v \neq 0$. Then $\|v\| \neq 0$. By dividing v by $\|v\|$, we assume $\|v\| = 1$. Then $\{v\}$ is a set of orthonormal vector. By Bessel's inequality, we have $|\langle u|v\rangle| \leqslant \|u\|$. $\square$

Remark 20.24. In the general case that $\langle\cdot|\cdot\rangle$ is a positive sesquilinear form, the Cauchy-Schwarz inequality $|\langle u|v\rangle|^2 \leqslant \langle u|u\rangle \cdot \langle v|v\rangle$ still holds.

Proof. We use the notations in Prop. 20.13. Applying Thm. 20.23 to $V/\mathscr{N}$, we have $|(u+\mathscr{N}|v+\mathscr{N})|^2 \leqslant (u+\mathscr{N}|u+\mathscr{N}) \cdot (v+\mathscr{N}|v+\mathscr{N})$. This proves $|\langle u|v\rangle|^2 \leqslant \langle u|u\rangle \cdot \langle v|v\rangle$. $\square$

Corollary 20.25. *V is a normed vector space if we define $\|v\| = \sqrt{\langle v|v\rangle}$.*

Proof. By Cauchy-Schwarz, we have

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2\mathrm{Re}\langle u|v\rangle \leqslant \|u\|^2 + \|v\|^2 + 2\|u\| \cdot \|v\| = (\|u\| + \|v\|)^2$$

This proves the triangle inequality. The other conditions are obvious. $\square$

Corollary 20.26. *The map $\langle\cdot|\cdot\rangle : V \times V \to \mathbb{C}$ is continuous if V is equipped with the norm topology.*

Proof. For each $u, u_0, v, v_0 \in V$ such that $\|u - u_0\| \leqslant \varepsilon$, $\|v - v_0\| \leqslant \varepsilon$ where $0 < \varepsilon < 1$, we have by Cauchy-Schwarz that

$$|\langle u|v\rangle - \langle u_0|v_0\rangle| \leqslant |\langle u - u_0|v\rangle| + |\langle u_0|v - v_0\rangle| \leqslant \varepsilon(\|v_0\| + 1) + \varepsilon\|u_0\|$$

$\square$

Remark 20.27. Since V is a normed vector space, V is also a metric space with $d(u,v) = \|u - v\| = \sqrt{\langle u - v|u - v\rangle}$. Now, the polarisation identity (Prop. 20.3) says that for inner product spaces, norms determine inner produts, and hence metrics determine inner products. Therefore, if W is an inner product space, and if $T : V \to W$ is a linear isometry, then

$$\langle Tu|Tv\rangle = \langle u|v\rangle$$

for all $u, v \in V$. In particular, if T is an isomorphism of normed vector sapces (i.e., T is a linear surjective isometry, cf. Def. 9.2), then T is an equivalence of inner product spaces. In this case, we say that T is a **unitary map**, and say that V, W are **isomorphic inner product spaces** (or that V, W are **unitarily equivalent**). $\square$

20.3 Orthogonal decompositions

Fix an inner product space V. In the last section, we used Gram-Schmidt process and Pythagorean inequality (20.12) to derive many useful inequalities. In this section, we will have a deeper understanding of the geometry behind Gram-Schmidt process and orthogonal projections.

Definition 20.28. Let U be a linear subspace of V. Let $v \in V$. An **orthogonal decomposition** of v with respect to U is an expression of the form $v = u + w$ where $u \in U$ and $w \perp U$ (i.e. w is orthogonal to every vector of U). Orthogonal decompositions of v are unique if exist. We call u the **orthogonal projection** of v onto U.

Proof of uniqueness. Suppose that $v = u' + w'$ is another orthogonal decomposition. Then $u - u'$ equals $w' - w$. Let $\xi = u - u'$. Then $\xi \in U$ and $\xi \perp U$. So $\langle \xi|\xi \rangle = 0$, and hence $\xi = 0$. So $u = u'$ and $w = w'$. □

Example 20.29. Let $e_1, \dots, e_n$ be orthonormal vectors of V. Let $U = \mathrm{Span}\{e_1, \dots, e_n\}$. Choose any $v \in V$. Then by Gram-Schmidt, $v = u + w$ is the orthogonal decomposition if we let $u = \sum_{i=1}^n \langle v|e_i \rangle e_i$ and $w = v - u$.

The inequalities in the last section relies on the Pythagorean inequality $\|u\| \leqslant \|v\|$ for an orthogonal decomposition $v = u + w$. In this section, we need an optimization property about orthogonal decompositions:

Proposition 20.30. *Let U be a linear subspace of V. Suppose that $v \in V$ has orthogonal decomposition $v = u + w$ with respect to U. Then*

$$\|v - u\| = \inf_{\xi \in U} \|v - \xi\| \tag{20.17}$$

Proof. Clearly "$\geqslant$" holds. Choose any $\xi \in U$. Then $v - \xi = v - u + u - \xi = w + (u - \xi)$. Since $u - \xi \in U$, we have $w \perp u - \xi$. Thus, by Pythagorean, we have $\|w\| \leqslant \|v - \xi\|$. □

Corollary 20.31. *Let $e_1, \dots, e_n$ be orthonormal vectors of V. For each $v \in V$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ we have*

$$\|v\|^2 - \sum_{i=1}^n |\langle v|e_i \rangle|^2 = \Big\|v - \sum_{i=1}^n \langle v|e_i \rangle e_i\Big\|^2 \leqslant \Big\|v - \sum_{i=1}^n \lambda_i e_i\Big\|^2 \tag{20.18}$$

Proof. By Gram-Schmidt, we have orthogonal decomposition $v = u + w$ where $w = \sum_i \langle v|e_i \rangle e_i$. The Pythagorean identity $\|v\|^2 - \|u\|^2 = \|w\|^2$ proves the first equality. Prop. 20.30 proves the "$\leqslant$". □

We now give several applications of Cor. 20.31.

Definition 20.32. A set $\mathfrak{S}$ (or a family $(e_i)_{i\in I}$) of orthonormal vectors of V is called an **orthonormal basis** of V if it spans a dense subspace of V.

Example 20.33. If X is a set, by Lem. 17.28, $l^2(X)$ has an orthonormal basis $(\chi_{\{x\}})_{x\in X}$.

Example 20.34. If V is separable, then V has a countable orthonormal basis.

Proof. Let $\{v_1, v_2, \dots\}$ be a dense subset of V where $v_1 \neq 0$. Then by Gram-Schmidt (Rem. 20.21), we can find $e_1, e_2, \dots \in V$ such that the set $\{e_1, e_2, \dots\}$ is orthnormal (after removing the duplicated terms), and that $\mathrm{Span}\{v_1, \dots, v_n\} = \mathrm{Span}\{e_1, \dots, e_n\}$ for each n. Then $\{e_1, e_2, \dots\}$ clearly spans a dense subspace of V. □

We remark that there are non-separable and non-complete inner product spaces that do not have orthonormal bases. See [Gud74].

Theorem 20.35. *Suppose that $(e_i)_{i\in I}$ is an orthonormal basis of V. Then for each $v \in V$, the RHS of the following converges (under the norm of V) to the LHS:*

$$v = \sum_{i\in I} \langle v|e_i\rangle \cdot e_i \tag{20.19}$$

Proof. Note that for $J \in \mathrm{fin}(2^I)$, the expression

$$\Big\| v - \sum_{j\in J} \langle v|e_j\rangle e_j \Big\|^2 = \|v\|^2 - \sum_{j\in J} |\langle v|e_j\rangle|^2$$

decreases when J increases. Thus, it suffices to prove that the $\inf_{J\in\mathrm{fin}(2^I)}$ of this expression is 0. This follows immediately from Cor. 20.31 and the fact that we can find J and $(\lambda_j)_{j\in J}$ in $\mathbb{C}$ such that $\|v - \sum_{j\in J} \lambda_j e_j\|$ is small enough. □

Corollary 20.36 (Parseval's identity). *Suppose that $(e_i)_{i\in I}$ is an orthonormal basis of V. Then for each $u, v \in V$ we have*

$$\langle u|v\rangle = \sum_{i\in I} \langle u|e_i\rangle \cdot \langle e_i|v\rangle \tag{20.20}$$

In particular,

$$\|v\|^2 = \sum_{i\in I} |\langle v|e_i\rangle|^2 \tag{20.21}$$

Proof. By Thm. 20.35, $u = \lim_{J\in\mathrm{fin}(2^I)} u_J$ where $u_J = \sum_{j\in J} \langle u|e_j\rangle \cdot e_j$. By the continuity of $\langle\cdot|\cdot\rangle : V \times V \to \mathbb{C}$ (Cor. 20.26), we have

$$\langle u|v\rangle = \lim_{J\in\mathrm{fin}(2^I)} \langle u_J|v\rangle = \lim_{J\in\mathrm{fin}(2^I)} \sum_{j\in J} \langle u|e_j\rangle \cdot \langle e_j|v\rangle = \sum_{i\in I} \langle u|e_i\rangle \cdot \langle e_i|v\rangle$$

□

Remark 20.37. In Hilbert's original definition, an orthonormal basis $(e_i)_{i\in I}$ (called by Hilbert a complete orthogonal system of functions) is a set of orthonormal vectors satisfying Parseval's identity (20.20) for all $u, v \in V$. Hilbert did not have the topological understanding of orthonormal basis as in Def. 20.32 (i.e., a set of orthonormal vectors spanning a *dense* subspace of V). See [BK84, Sec. 8].

Corollary 20.38. *Suppose that $(e_x)_{x\in X}$ is an orthonormal basis of V. Then there is a linear isometry*

$$\Phi : V \to l^2(X) \qquad v \mapsto (\langle v|e_x\rangle)_{x\in X} \tag{20.22}$$

whose range is dense in $l^2(X)$.

Since $l^2(X)$ is complete (Thm. 12.32), it follows that Φ gives a Banach space completion of V, cf. Def. 10.18.

Proof. Parseval's identity shows that $(\langle v|e_x\rangle)_{x\in X}$ has finite l^2-norm $\|v\|$. So the map Φ defined by (20.22) is clearly a linear isometry. $\square$

20.4 Application to Fourier series

In this section, we apply the results about inner product spaces to the study of Fourier series.

Let $\mathscr{V} = \mathscr{R}[-\pi, \pi] = \mathscr{R}([-\pi, \pi], \mathbb{C})$, the vector space of (strongly) Riemann integrable complex valued functions on $[-\pi, \pi]$. For each $f \in \mathscr{V}$ and $n \in \mathbb{Z}$, define its n-th **Fourier coefficient** to be

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\mathbf{i}nx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} fe_{-n} \tag{20.23}$$

where $e_n(x) = e^{\mathbf{i}nx}$. Then the **Fourier series** of f is

$$\sum_{n\in\mathbb{Z}} \widehat{f}(n)e_n \tag{20.24}$$

We view $\widehat{f}$ as a function on $\mathbb{Z}$.

In Fourier analysis, one asks whether the Fourier series of f converges to f, and if so, in which sense does it converge? We have seen in Subsec. 13.1.4 that (20.24) might not converge uniformly to f. We have also mentioned there that (20.24) might be divergent at many points of $[-\pi, \pi]$, although in many cases it converges pointwise to f (cf. Pb. 14.6). In this section, we will see that (20.24) converges to f under the L^2-norm. Another important result of this section is the classification of all f whose Fourier modes are all 0.

Let V be the vector space of all $f \in \mathscr{R}[-\pi,\pi]$, where $f,g \in \mathscr{R}[-\pi,\pi]$ are equal elements in V iff $\{x \in [-\pi,\pi] : f(x) \neq g(x)\}$ is null. By Exp. 20.14, V is an inner product space whose inner product $\langle\cdot|\cdot\rangle$ is defined by

$$\langle f|g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} fg^* = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)} \tag{20.25}$$

It follows that

$$\widehat{f}(n) = \langle f|e_n\rangle \tag{20.26}$$

Proposition 20.39. *$\{e_n : n \in \mathbb{Z}\}$ is an orthonormal basis of V.*

Proof. Clearly $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal set of vectors. By Stone-Weierstrass, $\mathrm{Span}\mathfrak{S}$ is l^∞-dense in $C(\mathbb{S}^1)$, the space of 2π-periodic continuous functions (cf. Exp. 15.12). So $\mathrm{Span}\mathfrak{S}$ is L^2-dense in $C(\mathbb{S}^1)$. Indeed, for any $f \in C(\mathbb{S}^1)$, pick a sequence $(f_k)_{k\in\mathbb{Z}_+}$ of elements in $\mathrm{Span}\mathfrak{S}$ converging uniformly on $[-\pi,\pi]$ to f. Therefore $\lim_{k\to\infty}\int|f-f_k|^2 = 0$ by Cor. 13.21.

To show that $\mathrm{Span}\mathfrak{S}$ is dense in V, it remains to prove that $C(\mathbb{S}^1)$ is L^2-dense in V. Let $\mathcal{S} = \mathrm{Span}\{\chi_I : I \text{ is an interval in } [-\pi,\pi]\}$. It is easy to see that each χ_I can be L^2-approximated by elements of $C(\mathbb{S}^1)$. Therefore, it suffices to prove that $\mathcal{S}$ is L^2-dense in V.

Let $f \in \mathscr{R}[-\pi,\pi]$. We want to show that f can be L^2-approximated by elements of $\mathcal{S}$. By considering the real part and the imaginary part separately, we assume that f is real. Let $M = \|f\|_{l^\infty}$, which is finite. By the proof of Prop. 14.51 (or by approximating $\int f$ by upper Darboux sums, cf. Thm. 13.41), we can find a sequence (f_n) in $\mathcal{S}$ such that $\lim_n \int_{-\pi}^{\pi}|f-f_n| = 0$, and that $|f_n| \leqslant M$ for all n. Therefore

$$\int_{-\pi}^{\pi}|f-f_n|^2 \leqslant 2M\int_{-\pi}^{\pi}|f-f_n|$$

where the RHS converges to 0 as $n \to \infty$. □

Corollary 20.40. *Let $f \in \mathscr{R}[-\pi,\pi]$. Then the following are equivalent*

(1) Each Fourier coefficient $\widehat{f}(n)$ is 0.

(2) f is zero outside a null set.

Proof. This is immediate from Prop. 20.39 and the obvious fact that a vector in an inner product space (equipped with an orthonormal basis) is zero iff its inner product with any element in the orthonormal basis (or more generally, any element in a densely-spanning subset) is zero. □

Theorem 20.41. *Let $f, g \in \mathscr{R}[a,b]$. Then we have* ***Parseval's identity***

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} fg^* = \sum_{n=-\infty}^{+\infty} \widehat{f}(n)\overline{\widehat{g}(n)} \tag{20.27}$$

where the RHS converges absolutely. In particular,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 = \sum_{n=-\infty}^{+\infty} |\widehat{f}(n)|^2 \tag{20.28}$$

Moreover, we have

$$\lim_{m,n\to+\infty}\int_{-\pi}^{\pi}\Big|f - \sum_{k=-m}^{n}\widehat{f}(k)e_k\Big|^2 = 0 \tag{20.29}$$

Proof. Immediate from Cor. 20.36 and Thm. 20.35. □

The following result was mentioned in Subsec. 10.4.1.

Corollary 20.42. *The space $C[-\pi,\pi]$, under the inner product (20.25), has a Banach space completion*

$$C[-\pi,\pi] \to l^2(\mathbb{Z}) \qquad f \mapsto \widehat{f} \tag{20.30}$$

The same is true if $C[-\pi,\pi]$ is replaced by V.

Proof. Prop. 20.39 implies that $\{e_n : n \in \mathbb{Z}\}$ is also an orthonormal basis of $C[-\pi,\pi]$. Thus, the corollary follows from Cor. 20.38. □

20.5 Problems and supplementary material

20.5.1 ⋆ The Sobolev space $H^s(\mathbb{S}^1)$

For each $s \geqslant 0$, and for each $\varphi : \mathbb{Z} \to \mathbb{C}$, let

$$\|\varphi\|_{h^s} = \sum_{n\in\mathbb{Z}}(1+n^2)^s|\varphi(n)|^2 \tag{20.31}$$

It is clear that $\|\varphi\|_{h^0} = \|\varphi\|_2$, and that

$$s \leqslant t \qquad \Longrightarrow \qquad \|\varphi\|_{h^s} \leqslant \|\varphi\|_{h^t}$$

Define

$$h^s(\mathbb{Z}) = \{\varphi \in \mathbb{C}^{\mathbb{Z}} : \|\varphi\|_{h^s} < +\infty\} \tag{20.32}$$

which is clearly a subset of $l^2(\mathbb{Z})$. Clearly $h^s(\mathbb{Z}) \supset h^t(\mathbb{Z})$ if $s \leqslant t$.

Problem 20.1. Prove that $h^s(\mathbb{Z})$ is a linear subspace of $l^2(\mathbb{Z})$, that $h^s(\mathbb{Z})$ has a well-defined inner product described by

$$\langle\varphi|\psi\rangle_{h^s} = \sum_{n\in\mathbb{Z}}(1+n^2)^s\varphi(n)\overline{\psi(n)} \tag{20.33}$$

and that $h^s(\mathbb{Z})$ is complete under this inner product.

Let $C(\mathbb{S}^1)$ be the set of complex continuous functions on $\mathbb{S}^1$, equivalently, continuous 2π-periodic functions on $\mathbb{R}$. More generally, for each $n \in \mathbb{N} \cup \{\infty\}$, let

$$C^n(\mathbb{S}^1) = \{2\pi\text{-periodic } f \in C^n(\mathbb{R})\} \tag{20.34}$$

Equip $C(\mathbb{S}^1)$ with the inner product $\langle f|g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} fg^*$. The norm determined by this inner product is called the L^2-norm. By Cor. 20.42, we have a Hilbert space completion

$$\Phi : C(\mathbb{S}^1) \to l^2(\mathbb{Z}) \qquad f \mapsto \widehat{f} \tag{20.35}$$

Recall that $e_n(x) = e^{\mathbf{i}nx}$.

Problem 20.2. Prove that $l^1(\mathbb{Z}) \subset \Phi(C(\mathbb{S}^1))$, and that the linear injection

$$\Phi^{-1} : l^1(\mathbb{Z}) \to C(\mathbb{S}^1) \tag{20.36}$$

can be described by

$$\Phi^{-1}(\varphi) = \sum_{n\in\mathbb{Z}}\varphi(n)e_n \tag{20.37}$$

where the RHS converges under the $l^\infty(\mathbb{S}^1)$-norm (and hence under the L^2-norm).

Problem 20.3. Let $s > \frac{1}{2}$. Prove that $h^s(\mathbb{Z}) \subset l^1(\mathbb{Z})$.

Hint. Use Cauchy-Schwarz or Hölder's inequality. □

Definition 20.43. Let $s > \frac{1}{2}$. Define

$$H^s(\mathbb{S}^1) \xlongequal{\text{def}} \Phi^{-1}(h^s(\mathbb{Z})) \tag{20.38}$$

According to Pb. 20.2 and 20.3, we have

$$H^s(\mathbb{S}^1) \subset \Phi^{-1}(l^1(\mathbb{Z})) \subset C(\mathbb{S}^1) \tag{20.39}$$

and Φ restricts to a linear bijection

$$\Phi : H^s(\mathbb{S}^1) \xrightarrow{\simeq} h^s(\mathbb{Z}) \tag{20.40}$$

whose inverse sends

$$\varphi \mapsto \sum_{n\in\mathbb{Z}} \varphi(n)e_n$$

where the RHS converges uniformly. Define inner product $\langle\cdot|\cdot\rangle_{H^s}$ on $H^s(\mathbb{S}^1)$ to be the pullback of h^s, i.e.

$$\langle f|g\rangle_{H^s} = \langle \widehat{f}|\widehat{g}\rangle_{h^s} = \sum_{n=-\infty}^{+\infty} (1+n^2)^s \widehat{f}(n)\overline{\widehat{g}(n)} \tag{20.41}$$

Then $H^s(\mathbb{S}^1)$ is a Hilbert space since $h^s(\mathbb{Z})$ is so. We call $H^s(\mathbb{S}^1)$ a **Sobolev space** of $\mathbb{S}^1$. Thus Φ is a unitary map.

Problem 20.4. Let $k \in \mathbb{Z}_+$. Prove that $C^k(\mathbb{S}^1) \subset H^k(\mathbb{S}^1)$. Prove for each $f \in C^{2k}(\mathbb{S}^1), g \in H^k(\mathbb{S}^1)$ that

$$\langle f|g\rangle_{H^k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} (1-\partial^2)^k f \cdot g^* \tag{20.42}$$

where $\partial f = f'$.

Hint. For each $f \in C^k(\mathbb{S}^1)$, prove by induction on k that

$$\widehat{f^{(k)}}(n) = (\mathbf{i}n)^k \widehat{f}(n) \tag{20.43}$$

Apply Parserval's identity to $\int |f^{(k)}|^2$ and to the RHS of (20.42). □

As an application, we obtain a useful criterion for the uniform convergence of Fourier series:

Corollary 20.44. *If $f \in C^1(\mathbb{S}^1)$, then $\sum_{n\in\mathbb{Z}} \widehat{f}(n)e_n$ converges uniformly to f.*

Proof. By Pb. 20.4 and (20.39), we have $f \in C^1(\mathbb{S}^1) \subset H^1(\mathbb{S}^1) \subset \Phi^{-1}(l^1(\mathbb{Z}))$. Since $\widehat{f} = \Phi(f)$ and hence $f = \Phi^{-1}(\widehat{f})$, by Pb. 20.2 (the description of Φ^{-1} on $l^1(\mathbb{Z})$), $\sum_{n\in\mathbb{Z}} \widehat{f}(n)e_n$ converges uniformly to f. □

Theorem 20.45. *Let $k \in \mathbb{Z}_+$. Then $H^k(\mathbb{S}^1)$ is the Hilbert space completion of $C^\infty(\mathbb{S}^1)$ under the inner product defined by (20.42) for all $f, g \in C^\infty(\mathbb{S}^1)$.*

Proof. Equip $C^\infty(\mathbb{S}^1)$ with the inner product defined by (20.42). We know that

$$\Phi : C^\infty(\mathbb{S}^1) \to h^k(\mathbb{Z}) \qquad f \mapsto \widehat{f} \tag{20.44}$$

is a linear isometry. Since $\Phi(e_n) = \delta_{\{n\}}$, the range of Φ contains $\mathrm{Span}\{\chi_{\{n\}} : n \in \mathbb{Z}\}$. From this it follows easily that the range is dense in $h^k(\mathbb{Z})$. Thus, Φ gives a completion of $C^\infty(\mathbb{S}^1)$. This is equivalent to saying that the inclusion map $C^\infty(\mathbb{S}^1) \hookrightarrow H^k(\mathbb{S}^1)$ is a completion of $C^\infty(\mathbb{S}^1)$ (since (20.44) extends to a unitary map $\Phi : H^k(\mathbb{S}^1) \to h^k(\mathbb{Z})$). □

Problem 20.5. Suppose that $s > \frac{3}{2}$. Prove that the map of derivative $\partial : C^1(\mathbb{S}^1) \to C(\mathbb{S}^1)$, $f \mapsto f'$ restricts to

$$\partial : H^s(\mathbb{S}^1) \to H^{s-1}(\mathbb{S}^1) \tag{20.45}$$

(Namely, each $f \in H^s(\mathbb{S}^1)$ is differentiable, and has derivative in $H^{s-1}(\mathbb{S}^1)$.) Prove for each $f \in H^s(\mathbb{S}^1)$ that

$$\Phi(f')(n) = \mathbf{i}n\widehat{f}(n) \tag{20.46}$$

Hint. By Pb. 20.2, we have uniform convergence $f = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e_n$ where $\widehat{f} \in h^s(\mathbb{Z})$. Show that $\sum_{n\in\mathbb{Z}} \mathbf{i}n\widehat{f}(n)e_n$ converges uniformly. Then use Thm. 11.33. $\square$

Theorem 20.46 (Sobolev embedding). *If $s > \frac{1}{2}$, then $H^s(\mathbb{S}^1) \subset C^{\lceil s-\frac{3}{2}\rceil}(\mathbb{S}^1)$ where $\lceil s - \frac{3}{2}\rceil$ is the smallest integer $\geqslant s - \frac{3}{2}$.*

Proof. Choose $f \in H^s(\mathbb{S}^1)$. By (20.39) we have $f \in C(\mathbb{S}^1)$. Thus the theorem is proved when $\frac{1}{2} < s \leqslant \frac{3}{2}$. Suppose that $s > \frac{3}{2}$. Let $k = \lceil s - \frac{3}{2}\rceil$. Then $\frac{1}{2} < s - k \leqslant \frac{3}{2}$. By Pb. 20.5, we have $f' \in H^{s-1}(\mathbb{S}^1)$, $f'' \in H^{s-2}(\mathbb{S}^1), \dots$, and hence $f^{(k)} \in H^{s-k}(\mathbb{S}^1) \subset C(\mathbb{S}^1)$. This proves $f \in C^k(\mathbb{S}^1)$. $\square$

Corollary 20.47. *We have $C^\infty(\mathbb{S}^1) = \bigcap_{k\in\mathbb{Z}_+} H^k(\mathbb{S}^1) = \bigcap_{s>\frac{1}{2}} H^s(\mathbb{S}^1)$.*

Proof. By Pb. 20.4, Thm. 20.46, and the fact that $H^s(\mathbb{S}^1)$ decreases as s increases. $\square$

21 Hilbert spaces

Definition 21.1. An inner product space $\mathcal{H}$ is called a **Hilbert space** if it is complete under the norm defined by $\|\xi\| = \sqrt{\langle\xi|\xi\rangle}$.

Remark 21.2. Let $\mathcal{H}$ be a Hilbert space. Since a subset of a complete metric space is complete iff it is closed (Prop. 3.27), the phrases "closed linear subspaces of $\mathcal{H}$" and "Hilbert subspaces of $\mathcal{H}$" are synonymous.

Example 21.3. If $\mathcal{H}$ is a finite-dimensional inner product space, then $\mathcal{H}$ is a Hilbert space.

Proof. Since $\mathcal{H}$ is spanned by finitely many vectors, by Gram-Schmidt process, $\mathcal{H}$ has an orthonormal basis $(e_x)_{x\in X}$ where X is a finite set. The canonical linear isometry $\Phi : \mathcal{H} \to l^2(X)$ in Cor. 20.38 must be surjective. So $\mathcal{H} \simeq l^2(X)$. Hence $\mathcal{H}$ is complete. $\square$

Theorem 21.4. *Let V be an inner product space. Then V has a* ***Hilbert space completion****, i.e., a Hilbert space $\mathcal{H}$ and a linear isometry $\Phi : V \to \mathcal{H}$ with dense range.*

Moreover, Hilbert space completions are unique up to unitary equivalences: If $\Psi : V \to \mathcal{K}$ is another Hilbert space completion, then there is a unitary $\Gamma : \mathcal{H} \to \mathcal{K}$ such that the following diagram commutes:

$$\begin{tikzcd} & V \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & \\ \mathcal{H} \arrow[rr, "\Gamma", "\simeq"'] & & \mathcal{K} \end{tikzcd} \tag{21.1}$$

When no confusion arises, we identify V with $\Phi(V)$ so that V can be viewed as a dense inner product subspace of $\mathcal{H}$.

Proof. Since V is a normed vector space, by Thm. 10.19, we have a Banach space $\mathcal{H}$ (with norm $\|\cdot\|_{\mathcal{H}}$) and a linear isometry $\Phi : V \to \mathcal{H}$ with dense range. We assume WLOG that V is a normed vector subspace of $\mathcal{H}$ so that the norm $\|\cdot\|_{\mathcal{H}}$ restricts to that of V. Define a function $\langle\cdot|\cdot\rangle_{\mathcal{H}} : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ using the polarization identity (20.3), i.e., for each $\xi,\eta \in \mathcal{H}$ we set

$$\langle\xi|\eta\rangle_{\mathcal{H}} = \frac{1}{4}\sum_{t=0,\frac{\pi}{2},\pi,\frac{3\pi}{2}} \|\xi + e^{\mathbf{i}t}\eta\|_{\mathcal{H}}^2 \cdot e^{\mathbf{i}t}$$

Then $\langle\cdot|\cdot\rangle_{\mathcal{H}}$ restricts to the inner product $\langle\cdot|\cdot\rangle$ of V. Moreover, the function $\langle\cdot|\cdot\rangle_{\mathcal{H}}$ is continuous (with respect to the norm $\|\cdot\|_{\mathcal{H}}$), and $V\times V$ is a dense subset of $\mathcal{H}\times\mathcal{H}$. Therefore, as in the proof of Thm. 10.19, we can use the fact that $\langle\cdot|\cdot\rangle$ is an inner product on V to prove that $\langle\cdot|\cdot\rangle_{\mathcal{H}}$ is a positive sesquilinear form on $\mathcal{H}$, and we can use the fact that the norm of V is defined by the inner product of V to prove that

$\|\xi\|_{\mathcal{H}}^2 = \langle\xi|\xi\rangle_{\mathcal{H}}$. Therefore, $\langle\cdot|\cdot\rangle_{\mathcal{H}}$ is non-degenerate, i.e., an inner product, and this inner product defines the complete norm $\|\cdot\|_{\mathcal{H}}$. It follows that $\mathcal{H}$ is a Hilbert space, and hence a Hilbert space completion of V.

The uniqueness of Hilbert space completions (up to unitary equivalences) follows directly from that of Banach space completions, cf. Thm. 10.19. □

21.1 Introduction: completeness, the most familiar stranger

> What is familiar is what we are used to; and what we are used to is most difficult to "know"—that is, to see as a problem; that is, to see as strange, as distant, as "outside us".
>
> —- Friedrich Nietzsche (cf. [Nie, Sec. 355])

We introduced completeness at the beginning of last semester and applied it to function spaces. It has allowed us to give a unified understanding of many analytic problems, especially those related to uniform convergence: the uniform convergence of series of functions, the commutativity of two limit processes and its relationship with uniform convergence, etc.. In fact, the vast majority of Banach spaces we considered last semester were defined by the l^∞-norm.

From this perspective, it is perfectly natural to consider completeness for inner product spaces. However, we refuse to take lightly the consideration of completeness as natural for the following reasons:

- The notion of completeness was originally applied only to $\mathbb{R}$. Historically, however, the idea of applying completeness to function spaces was not entirely inspired by the study of uniform convergence, since uniform convergence is not too far from pointwise convergence.[1] The apparently successful application of l^∞-completeness to the problems of uniform convergence does not justify the *a priori* value of completeness in the study of other norms (such as the L^2-norm). [2]

- For inner product spaces, some analytic properties are equivalent to completeness. The most important one is the weak (or weak-*) compactness of the unit ball.[3] The main reason that the Hilbert space $l^2(\mathbb{Z})$ was introduced in history by Hilbert is due to this compactness rather than completeness.

[1]As a matter of fact, most results learned in the last semester can be formulated and proved without introducing completeness to function spaces, as implied by history and by many analysis textbooks.

[2]That l^∞ and L^2 are both called "norms" is a *result*, not a starting point, of the observation in history that different types of problems can be treated in a similar fashion.

[3]We have learned that $l^2(X)$ satisfies this property due to $l^2(X) \simeq l^2(X)^*$ (cf. Thm. 17.30) and Banach-Alaoglu, or more directly, by Pb. 17.5.

21.1.1 The late-coming concept of completeness

The modern definition of Hilbert spaces as complete inner product spaces was due to von Neumann [vNeu30] in late 1920s, many years after the introduction of $l^2(\mathbb{Z})$ by Hilbert and Schmidt (same person as the Schmidt in the Gram-Schmidt process![4]) to the study of integral equations in 1900s. We will see in this chapter that all separable Hilbert spaces are isomorphic to $\mathbb{C}^N$ or $l^2(\mathbb{Z})$, and that all Hilbert spaces are isomorphic to $l^2(X)$. Therefore, one can equivalently define a Hilbert space to be an inner product space isomorphic to $l^2(X)$ for some set X. This is indeed closer to how people originally understood Hilbert spaces than the modern definition.

As opposed to 1900s, by the time von Neumann gave Def. 21.1, the importance of completeness in function spaces was fully recognized. In my opinion, there are two main reasons for this change of viewpoint. The first one, which we will not discuss in detail, is the application of Baire's category theory (or its early version, the gliding hump method) to the study of function spaces. In this course, we will focus on the second reason: The completeness of a function spaces is closely related to the viewpoint of *linear operators* as opposed to *sesquilinear/bilinear forms*. The emphasis on the linear operator perspective was due to F. Riesz:

(a) In 1913, Riesz gave a new (and improved) interpretation of the spectral theorem for bounded Hermitian forms, originally due to Hilbert. Instead of working with bounded Hermitian forms, Riesz worked with bounded self-adjoint operators. This allowed him to introduce the influential idea of **functional calculus**, which remains the standard treatment of spectral theory to this day.

(b) In 1918, Riesz studied eigenvalue problems of compact operators on $C(X)$. His method is readily applied to compact operators on any Banach space, thus generalizing Hilbert-Schmidt's results for the Hilbert space $l^2(\mathbb{Z})$.

In the following, we explain why the idea of completeness in function spaces is related to the viewpoint of linear operators.

21.1.2 Scalar-valued functions vs. vector-valued functions, compactness vs. completeness, Hilbert vs. Riesz

Given inner product spaces U, V, a bounded linear map $T : U \to V$ can also be viewed as a continuous sesquilinear map

$$\omega_T : U \times V \to \mathbb{C} \qquad (u, v) \mapsto \langle Tu|v\rangle \tag{21.2}$$

[4]Thus, the discovery of Gram-Schmidt process is not only for the purpose of solving linear algebra problems. It has a deep background in analysis.

In the special case that $U = V$, T can be viewed as a continuous **quadratic form**

$$V \times V \to \mathbb{C} \qquad v \mapsto \langle Tv|v\rangle \tag{21.3}$$

(Note that (21.3) completely determines (21.2) by the polarization identity (Prop. 20.3), and hence determines T.) This viewpoint, insisting on the study of scalar-valued functions, was hold by people before Riesz, especially by Hilbert (and his students). On the other hand, the viewpoint of vector-valued functions and linear operators was emphasized by Riesz.

In Hilbert's scalar-valued function viewpoint, completeness plays a very marginal role (with very few exceptions, see Rem. 21.42), and emphasis was put on the compactness. To empathize with this phenomenon, compare it with the equivalence $C(X, C(Y)) \simeq C(X \times Y)$ in Thm. 9.3 (where Y is compact): If we take the viewpoint of $C(X \times Y)$, we put more emphasis on the compactness of Y. However, if we take the viewpoint of $C(X, C(Y))$, we view $C(Y)$ as an abstract Banach space V. Therefore, we forget about the compactness of Y and focus on the completeness of V.

Therefore, the following three ideas are closely related:

(1) The completeness of function spaces.

(2) The study of linear maps between function spaces $U \to V$, as opposed to the study of scalar-valued functions on $U \times V$.

(3) The operator norm on $\mathfrak{L}(U, V)$.

We have mentioned the relationship between (1) and (3) in Sec. 17.6. Here, let me briefly explain why (1) and (2) are related by recalling two fundamental facts learned before:

Suppose that U, V are normed vector spaces and U_0 is a dense linear subspace of V. To extend a bounded linear map $T : U_0 \to V$ to a bounded linear $U \to V$, one needs the completeness of V. (Cf. Prop. 10.28.) Given a net of bounded linear maps (T_α) from U to V satisfying $\sup_\alpha \|T_\alpha\| < +\infty$, to show that the pointwise convergence of (T_α) on U_0 implies the pointwise convergence on U, one also needs the completeness of V. (Cf. Prop. 17.19.)

We will appreciate the close connection among (1), (2), and (3) when we study the spectral theorem for bounded self-adjoint operators in Sec. 27.5.

21.2 Key property 1: convergence of summing orthogonal vectors

In this section, we fix a Hilbert space $\mathcal{H}$.

Hilbert and Schmidt introduced $l^2(\mathbb{Z})$ to the study of integral equations. As we have mentioned, the main interesting analytic property for them is not completeness. Indeed, they mainly used the following conditions, both equivalent to the completeness:

1. If $\sum_i |a_i|^2 < +\infty$ and if $(e_i)_{i\in I}$ is orthormal, then $\sum a_i e_i$ converges.

2. The weak(-*) compactness of the closed unit ball.

(We will show the equivalence in Thm. 21.5 and Cor. 21.35.) In this section, we study the first property. The second property will be studied in the next section. The first property is indeed a direct consequence of the second one; see Thm. 21.34. Thus, one may also say that the second property is the main analytic property used by Hilbert and Schmidt. In the next chapter, we will see how these two properties are used to study integral equations.

Theorem 21.5. *Let V be an inner product space. The following are equivalent.*

(1) V is complete.

(2) For each orthonormal family $(e_i)_{i\in I}$ in V, and for each $(a_i)_{i\in I}$ in $\mathbb{C}$ satisfying $\sum_{i\in I} |a_i|^2 < +\infty$, the discrete integral $\sum_{i\in I} a_i e_i$ converges (under the norm of V).

(3) V is isomorphic to $l^2(X)$ for some X.

Proof. (3)⇒(1): This is because $l^2(X)$ is complete, cf. Thm. 12.32.

(1)⇒(2): Since $\sum_i |a_i|^2 < +\infty$, by Rem. 5.43, for each $\varepsilon > 0$ there exists $J \in \mathrm{fin}(2^I)$ such that for all finite $K \subset I\backslash J$ we have $\sum_{k\in K} |a_k|^2 < \varepsilon$, and hence, by the Pythagorean identity,

$$\Big\|\sum_{k\in K} a_k e_k\Big\|^2 = \sum_{k\in K} |a_k e_k|^2 < \varepsilon$$

By Rem. 5.43 again and the completeness of V, we see that $\sum_{i\in I} a_i e_i$ converges.

(2)⇒(3): Assume (2). We first show that V has an orthonormal basis. By Zorn's lemma, we can find a maximal (with respect to the partial order $\subset$) set of orthonormal vectors, written as a family $(e_i)_{i\in I}$. The maximality implies that every nonzero vector $\xi \in V$ is not orthogonal to some e_i. (Otherwise, $\{e_i : i \in I\}$ can be extended to $\{e_i : i \in I\} \cup \{\xi/\|\xi\|\}$.)

Let us prove that $(e_i)_{i\in I}$ is an orthonormal basis. Suppose not. Then $U = \mathrm{Span}\{e_i : i \in I\}$ is not dense in X. Let $\xi \in X\backslash\overline{U}$. By Bessel's inequality, we have

$$\sum_{i\in I} |\langle \xi|e_i\rangle|^2 < +\infty$$

Therefore, by (2),

$$\sum_{i\in I} \langle \xi|e_i\rangle \cdot e_i \tag{21.4}$$

converges to some vector $\eta \in V$. By the continuity of $\langle\cdot|\cdot\rangle$, we see that $\langle\eta|e_i\rangle = \langle\xi|e_i\rangle$ for all i, and hence

$$\langle\xi - \eta|e_i\rangle = 0 \qquad \text{for all } i \in I \tag{21.5}$$

Since $\eta \in \overline{U}$ and $\xi \notin \overline{U}$, we conclude that $\xi - \eta$ is a nonzero vector orthogonal to all e_i. This contradicts the maximality of $(e_i)_{i\in I}$.

Now we have an orthonormal basis $(e_i)_{i\in I}$. By Cor. 20.38, we have a linear isometry

$$\Phi : V \to l^2(I) \qquad \xi \mapsto \big(\langle\xi|e_i\rangle\big)_{i\in I}$$

with dense range. If $(a_i)_{i\in I}$ belongs to $l^2(I)$, by (2), the discrete integral $\sum_{i\in I} a_i e_i$ converges to some $\xi \in V$. Clearly $\Phi(\xi) = (a_i)_{i\in I}$. This proves that Φ is surjective, and hence is a unitary map. So $V \simeq l^2(I)$. □

Condition (2) of Thm. 21.5 is the main subject of this section. From the above proof, we see that this condition is an easy special case of the completeness. This special case is often sufficient for applications without fully utilizing the completeness of $l^2(\mathbb{Z})$. (To understand how nontrivial the direction (2)⇒(1) is even in the separable case, try to give a direct proof of it!)

In the following, we show that many well-known properties about Hilbert spaces follow Thm. 21.5-(2). Recall that we have fixed a Hilbert space $\mathcal{H}$.

Corollary 21.6. *$\mathcal{H}$ has an orthonormal basis. Moreover, $\mathcal{H}$ is separable iff the orthonormal basis can be chosen to be countable.*

Proof. That $\mathcal{H}$ has an orthonormal basis follows from the proof of Thm. 21.5 or from the fact that $l^2(X)$ has an orthonormal basis $(\chi_{\{x\}})_{x\in X}$. If X is countable, then $l^2(X)$ has dense subset $\mathrm{Span}_{\mathbb{Q}+\mathbf{i}\mathbb{Q}}\{\chi_{\{x\}} : x \in X\}$ (by Lem. 17.28) and hence is separable. Conversely, we have proved in Exp. 20.34 that every separable inner product space has a countable orthonormal basis. □

Theorem 21.7. *Let $(e_x)_{x\in X}$ be an orthonormal basis of $\mathcal{H}$. Then we have a unitary map*

$$\mathcal{H} \xrightarrow{\simeq} l^2(X) \qquad \xi \mapsto \big(\langle\xi|e_x\rangle\big)_{x\in X} \tag{21.6}$$

Proof. This is clear from the proof of Thm. 21.5. □

Definition 21.8. If $\mathcal{K}$ is a linear subspace of $\mathcal{H}$, we define its **orthogonal complement**

$$\mathcal{K}^\perp = \{\xi \in \mathcal{H} : \langle\xi|\eta\rangle = 0 \text{ for all } \eta \in \mathcal{K}\}$$

It is clear that

$$\mathcal{K}^\perp = \overline{\mathcal{K}}^\perp \tag{21.7}$$

Proof of (21.7). If $\xi \in \mathcal{H}$ is orthogonal to $\overline{\mathcal{K}}$, then ξ is clearly orthogonal to $\mathcal{K}$. Conversely, if ξ is orthogonal to $\mathcal{K}$, the continuity of $\eta \in \mathcal{H} \mapsto \langle \xi|\eta\rangle$ implies that $\xi \perp \overline{\mathcal{K}}$. $\square$

Definition 21.9. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. Consider the direct sum of vector spaces $\mathcal{H}_1 \oplus \mathcal{H}_2$. Namely, $\mathcal{H}_1 \oplus \mathcal{H}_2$ equals $\mathcal{H}_1 \times \mathcal{H}_2$ as a set, (ξ, η) is also written as $\xi \oplus \eta$, and the linear structure is defined by $(\xi \oplus \eta) + (\xi' \oplus \eta') = (\xi + \xi') \oplus (\eta + \eta')$ and $\lambda(\xi \oplus \eta) = \lambda\xi \oplus \lambda\eta$ (where $\xi, \xi' \in \mathcal{H}_1, \eta, \eta' \in \mathcal{H}_2, \lambda \in \mathbb{C}$), and the zero vector is $0 \oplus 0$. Equip $\mathcal{H}_1 \oplus \mathcal{H}_2$ with inner product defined by

$$\langle \xi \oplus \eta|\xi' \oplus \eta'\rangle = \langle \xi|\xi'\rangle + \langle \eta|\eta'\rangle$$

Then $\mathcal{H}_1 \oplus \mathcal{H}_2$ is clearly a Hilbert space. We call $\mathcal{H}_1 \oplus \mathcal{H}_2$ the **(Hilbert space) direct sum** of $\mathcal{H}_1, \mathcal{H}_2$.

Remark 21.10. In Def. 21.9, we clearly have linear isometries

$$\begin{aligned} \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2 \qquad & \xi \mapsto \xi \oplus 0 \\ \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2 \qquad & \eta \mapsto 0 \oplus \eta \end{aligned}$$

with ranges $\mathcal{H}_1 \oplus 0$ and $0 \oplus \mathcal{H}_2$ respectively. It is clear that $\mathcal{H}_1 \oplus 0$ and $0 \oplus \mathcal{H}_2$ are orthogonal complements of each other. We often identify $\mathcal{H}_1$ with $\mathcal{H}_1 \oplus 0$ and $\mathcal{H}_2$ with $0 \oplus \mathcal{H}_2$. Then $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert subspaces of $\mathcal{H}_1 \oplus \mathcal{H}_2$, and are orthogonal complements of each other.

Theorem 21.11. *Let $\mathcal{K}$ be a closed linear subspace of $\mathcal{H}$. Note that $\mathcal{K}$ and $\mathcal{K}^\perp$ are both Hilbert subspaces of $\mathcal{H}$. Then there is a unitary map*

$$\Psi : \mathcal{K} \oplus \mathcal{K}^\perp \xrightarrow{\simeq} \mathcal{H} \qquad \xi \oplus \eta \mapsto \xi + \eta \tag{21.8}$$

Proof. It is a routine check that Ψ is a linear isometry. It remains to prove that Ψ is surjective. This means that we need to write each $\psi \in \mathcal{H}$ in the form $\psi = \xi + \eta$ where $\xi \in \mathcal{K}, \eta \in \mathcal{K}^\perp$. Thus, the surjectivity of Ψ means that every $\psi \in \mathcal{H}$ has an orthogonal decomposition with respect to $\mathcal{K}$ (recall Def. 20.28).

Let $\psi \in \mathcal{H}$. By Cor. 21.6, $\mathcal{K}$ has an orthonormal basis $(e_i)_{i\in I}$. As in the proof of (2)⇒(3) of Thm. 21.5, Bessel's inequality implies that $\sum_{i\in I}\langle \psi|e_i\rangle \cdot e_i$ converges to some vector $\xi \in \mathcal{K}$. Then one checks easily that $\eta = \psi - \xi$ is orthogonal to all e_i, and hence is orthogonal to $\mathcal{K}_0 = \mathrm{Span}\{e_i : i \in I\}$. Thus $\eta \perp \mathcal{K}$ by (21.7). $\square$

Remark 21.12. Due to Thm. 21.11, given a Hilbert subspace $\mathcal{K}$, people often write

$$\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp \tag{21.9}$$

Corollary 21.13. *Let $\mathcal{K}$ be a closed linear subspace of $\mathcal{H}$. Then every vector $\psi \in \mathcal{H}$ has an orthogonal decomposition with respect to $\mathcal{K}$. Moreover, we have $(\mathcal{K}^\perp)^\perp = \mathcal{K}$.*

Proof. The existence of the orthogonal decomposition follows from Thm. 21.11. Clearly $\mathcal{K} \subset (\mathcal{K}^\perp)^\perp$. By Thm. 21.11 (applied to $\mathcal{K}^\perp$ instead of $\mathcal{K}$), there is a unitary $\mathcal{K}^\perp \oplus (\mathcal{K}^\perp)^\perp \to \mathcal{H}$ sending $\eta \oplus \xi \mapsto \eta + \xi$, and hence a unitary $\Gamma : (\mathcal{K}^\perp)^\perp \oplus \mathcal{K}^\perp \to \mathcal{H}$ sending $\xi \oplus \eta \mapsto \xi + \eta$. By Thm. 21.11 again, Γ restricts to the unitary map $\Psi =$ (21.8). So we must have $(\mathcal{K}^\perp)^\perp \oplus \mathcal{K}^\perp = \mathcal{K} \oplus \mathcal{K}^\perp$, and hence $\mathcal{K} = (\mathcal{K}^\perp)^\perp$. $\square$

Corollary 21.14. *Let V be a linear subspace of $\mathcal{H}$. Then V is dense iff the only vector of $\mathcal{H}$ orthogonal to V is 0.*

This corollary gives a useful criterion for the density of linear subspaces.

Proof. Cor. 21.13 implies for every closed linear subspaces $\mathcal{K}_1, \mathcal{K}_2$ that

$$\mathcal{K}_1 = \mathcal{K}_2 \qquad \Longleftrightarrow \qquad \mathcal{K}_1^\perp = \mathcal{K}_2^\perp \tag{21.10}$$

Therefore $\overline{V} = \mathcal{H}$ iff $\overline{V}^\perp = \mathcal{H}^\perp = 0$ iff (by (21.7)) $V^\perp = 0$. $\square$

Remark 21.15. The existence of orthogonal decompositions with respect to closed linear subspaces (Thm. 21.11 or Cor. 21.13) is a key feature of Hilbert spaces that is not satisfied by general inner product spaces. A different (and fancier) proof can be found in many textbooks which proves the existence of the orthogonal decomposition $\psi = \xi + \eta$ by defining ξ to be the vector in $\mathcal{K}$ such that $\|\psi - \xi\|$ attains its minimum: the existence of such ξ relies on the completeness of $\mathcal{H}$ and the parallelogram law

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2 \tag{21.11}$$

cf. [Fol-R, Thm. 5.24] or [Rud-R, Thm. 4.11] for instance.[5]

The proof we give here is closer to Schmidt's method in that we used the Gram-Schmidt process instead of the parallelogram law. With this proof, we want to convey the idea that the existence of orthogonal decompositions is a direct consequence of the convergence of summing orthogonal vectors (condition (2) of Thm. 21.5). $\square$

21.3 Key property 2: weak(-*) compactness of the closed unit balls

We fix Hilbert spaces $\mathcal{H}, \mathcal{K}$ in this section.

[5]The proof of the existence of this minimizing ξ makes use of the convexity of $\mathcal{K}$ and thus holds for $\mathcal{K}$ being an arbitrary closed convex subset. Convexity in Banach spaces is a very important topic, but its importance was not realized until long after the birth of Hilbert spaces and much of F. Riesz's work. We believe that introducing convexity (even if secretly introduced) at the first encounter with Hilbert spaces is off-topic and distracting, because it hinders the understanding of the roles played by the other central properties in Hilbert spaces, especially the analytic properties.

Lemma 21.16. *For each $\xi \in \mathcal{H}$, we have*

$$\|\xi\| = \sup_{\eta \in \overline{B}_{\mathcal{H}}(0,1)} |\langle \xi | \eta \rangle| \tag{21.12}$$

Proof. Clearly "$\geqslant$" holds by Cauchy-Schwarz. To prove "$\leqslant$", assume WLOG that $\xi \neq 0$. Then $\|\xi\| = \langle \xi | \eta \rangle$ if we choose $\eta = \xi / \|\xi\|$. □

Recall that if $T : V \to W$ is a linear map of normed vector spaces, its operator norm is $\|T\| = \sup_{v \in \overline{B}_V(0,1)} \|Tv\|$. Operator norms describe uniform convergence on $\overline{B}_V(0,1)$: If (T_α) is a net of bounded linear operators $V \to W$, then $\lim_\alpha \|T - T_\alpha\| = 0$ iff T_α converges uniformly on $\overline{B}_V(0,1)$ to T.

We now show that in the case of Hilbert spaces, the operator norms of linear maps can be translated to certain upper bounds about sesquilinear forms, which are closer in spirit to Hilbert's original understanding of "boundedness" (cf. Pb. 21.7 and Rem. 21.39).

Proposition 21.17. *Let $T : \mathcal{H} \to \mathcal{K}$ be a linear map. Then*

$$\|T\| = \sup_{\substack{\xi \in \overline{B}_{\mathcal{H}}(0,1) \\ \eta \in \overline{B}_{\mathcal{K}}(0,1)}} \big|\langle T\xi | \eta \rangle\big| \tag{21.13}$$

Proof. By Lem. 21.16, the RHS of (21.13) equals $\sup_{\xi \in \overline{B}_{\mathcal{H}}(0,1)} \|T\xi\| = \|T\|$. □

Corollary 21.18. *Let $T : \mathcal{H} \to \mathcal{K}$, and let (T_α) be a net of linear maps $\mathcal{H} \to \mathcal{K}$. Define their associated sesquilinear maps $\omega_T, \omega_{T_\alpha} : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ by*

$$\omega_T(\xi|\eta) = \langle T\xi | \eta \rangle \qquad \omega_{T_\alpha}(\xi|\eta) = \langle T_\alpha \xi | \eta \rangle \tag{21.14}$$

Then $\lim_\alpha \|T - T_\alpha\| = 0$ iff (ω_{T_α}) converges uniformly on $\overline{B}_{\mathcal{H}}(0,1) \times \overline{B}_{\mathcal{K}}(0,1)$ to ω_T.

21.3.1 Weak topology

We know that, contrary to the norm topology, the weak-* topology describes poinwise convergence. Let us recall several basic facts about weak-* topology.

Recall from Def. 17.18 that if V is a normed vector space, the weak-* topology of V^* is the unique topology such that a net (φ_α) in V^* converges to φ under this topology iff $\lim_\alpha \langle \varphi_\alpha, v \rangle = \langle \varphi, v \rangle$ for all $v \in V$. It is more important to study the weak-* topology on $\overline{B}_{V^*}(0,1)$ (or on any bounded closed ball of V^*): By Banach-Alaoglu Thm. 17.21, the closed unit ball $\overline{B}_{V^*}(0,1)$ is weak-* compact. Moreover, the following elementary fact allows for a flexible characterization of weak-* topology:

Lemma 21.19. *Let V be a normed vector space. Assume that $E \subset V$ spans a dense subspace of V. Let (φ_α) be a net in $\overline{B}_{V^*}(0,1)$, and let $\varphi \in \overline{B}_{V^*}(0,1)$. Then (φ_α) converges weak-* to φ iff $\lim_\alpha \langle \varphi_\alpha, v \rangle = \langle \varphi, v \rangle$ for all $v \in E$.*

Proof. Immediate from Prop. 17.19. □

The most relevant example of this chapter is $l^2(X)$, which can be viewed as the dual space of $l^2(X)$ through the Banach space isomorphism (cf. Thm. 17.30)

$$\Psi : l^2(X) \to l^2(X)^* \qquad f \mapsto \langle \cdot | f^* \rangle \tag{21.15}$$

(Recall that $f^*(x) = \overline{f(x)}$. So $\langle g|f^*\rangle = \sum_{x\in X} f(x)g(x)$.) In this case, one can choose $E = \{\chi_{\{x\}} : x \in X\}$. Then Lem. 21.19 says that for any net (f_α) in $l^2(X)$ satisfying $\|f_\alpha\|_{l^2} \leqslant 1$ and any $f \in l^2(X)$ satisfying $\|f\|_{l^2} \leqslant 1$,

$$f_\alpha \xrightarrow{\text{weak-*}} f \qquad \Longleftrightarrow \qquad \lim_\alpha f_\alpha(x) = f(x) \text{ for all } x \in X \tag{21.16}$$

This is the content Thm. 17.31. In view of (21.16), the Banach-Alaoglu for $l^2(X)$ says that any net (f_α) in $\overline{B}_{l^2(X)}(0,1)$ has a subnet converging pointwise on X to some $f \in \overline{B}_{l^2(X)}(0,1)$, which can be checked directly using the Tychonoff theorem. (You were asked to give this direct proof in Pb. 17.5.)

It should be kept in mind that whenever talking about the weak-* topology on a Banach space V, we should specify a normed vector space U and an isomorphism of Banach spaces $\Psi : V \to U^*$ so that V can be viewed as a dual Banach space.

Definition 21.20. Let V be a normed vector space. The data (Ψ, U) (sometimes simply written as U) is called a **predual** of V if U is a Banach space, and if $\Psi : U \to V^*$ is an isomorphism of normed vector spaces. It is customary to write a predual U of V as V_*.

Since we know that every Hilbert space $\mathcal{H}$ is isomorphic to $l^2(X)$ for some X, the isomorphism (21.15) suggests that we can talk about the weak-* topology of $\mathcal{H}$, which should be the unique topology whose convergence $\xi_\alpha \to \xi$ means that $\langle \xi_\alpha | \eta \rangle \to \langle \xi | \eta \rangle$ for all η. However, calling this topology a weak-* topology is confusing. In fact, there is no standard choice of isomorphism $\mathcal{H} \to \mathcal{H}^*$. The isomorphism (21.15) relies on the antiunitary map $f \mapsto f^*$.

Definition 21.21. A linear map of inner product spaces $T : U \to V$ is called **antiunitary** if it is an antilinear surjective isometry.

There is no canonical antiunitary operator on a Hilbert space: any antiunitary operator, composed with a unitary one, is again antiunitary. However, we do have a canonical antiunitary map $\mathcal{H} \to \mathcal{H}^*$:

Theorem 21.22 (Riesz-Fréchet representation theorem). *For any Hilbert space $\mathcal{H}$, the (operator) norm of the dual Banach space $\mathcal{H}^*$ is induced by a unique inner product. (So $\mathcal{H}^*$ is a Hilbert space.) Moreover, we have an antiunitary map*

$$\Phi : \mathcal{H} \to \mathcal{H}^* \qquad \xi \mapsto \langle \cdot | \xi \rangle \tag{21.17}$$

Proof. Define a linear map Φ by (21.17). By Thm. 21.5-(3), we assume WLOG that $\mathcal{H} = l^2(X)$. Then Φ is related to the Banach space isomorphism (21.15) by $\Phi(\xi) = \Psi(\xi^*)$. Since $\xi \mapsto \xi^*$ is antiunitary, Φ must be an antilinear surjective isometry. Define an inner product on $\mathcal{H}^*$ by

$$\langle\varphi|\mu\rangle = \langle\Phi^{-1}\mu|\Phi^{-1}\psi\rangle \qquad (\forall\varphi,\mu \in \mathcal{H}^*) \tag{21.18}$$

Then this inner product induces the operator norm $\|\cdot\|$ on $\mathcal{H}^*$ because

$$\|\varphi\|^2 = \|\Phi^{-1}\varphi\|^2 = \langle\Phi^{-1}\varphi|\Phi^{-1}\varphi\rangle = \langle\varphi|\varphi\rangle$$

where the first identity is due to the fact that Φ is an isometry. The uniqueness of the inner product follows from the polarization identity (20.3). $\square$

Recall that any linear map $T : \mathcal{H} \to \mathcal{K}$ is determined by the expressions $\langle T\xi|\eta\rangle$.

Corollary 21.23. *For every $T \in \mathfrak{L}(\mathcal{H}, \mathcal{K})$ there is a unique bounded linear map $T^* : \mathcal{K} \to \mathcal{H}$ (called the* ***adjoint*** *of T) satisfying for all $\xi \in \mathcal{H}, \eta \in \mathcal{K}$ that*

$$\langle T\xi|\eta\rangle = \langle\xi|T^*\eta\rangle \tag{21.19}$$

Moreover, we have $\|T\| = \|T^*\|$, and $(T^*)^* = T$.

Proof. For each $\eta \in \mathcal{K}$, the linear map $\xi \in \mathcal{H} \mapsto \langle T\xi|\eta\rangle \in \mathbb{C}$ is bounded since $|\langle T\xi|\eta\rangle| \leqslant \|\xi\| \cdot \|T\| \cdot \|\eta\|$. Therefore, by Riesz-Fréchet, there is a unique $\psi \in \mathcal{H}$ such that $\langle T\xi|\eta\rangle = \langle\xi|\psi\rangle$ for all $\xi \in \mathcal{H}$. We let $T^*\eta = \psi$. This gives a map $T^* : \mathcal{K} \to \mathcal{H}$ satisfying (21.19) for all $\xi \in \mathcal{H}, \eta \in \mathcal{K}$. The formula (21.19) clearly shows that T^* is linear, and that $T^{**} = T$ if T^* is bounded. That $\|T\| = \|T^*\|$ (and in particular, the boundedness of T^*) follows from Prop. 21.17. $\square$

$\star$ **Exercise 21.24.** Recall that for every normed vector space V there is a canonical bounded linear map $V \to V^{**}, v \mapsto \langle v, \cdot\rangle$ which is a linear isometry by Hahn-Banach Cor. 16.6. Now let $V = \mathcal{H}$. Show that this map $\mathcal{H} \to \mathcal{H}^{**}$ is equal to the composition of the antiunitary maps $\mathcal{H} \to \mathcal{H}^*$ and $\mathcal{H}^* \to \mathcal{H}^{**}$, both defined by Riesz-Fréchet. Conclude that the canonical linear isometry $\mathcal{H} \to \mathcal{H}^{**}$ is unitary.

Hint. You may prove it in a general fashion. But the easiest way to think about this question is to assume WLOG that $\mathcal{H} = l^2(X)$. $\square$

According to this exercise, we can view $\mathcal{H}$ as the dual space of $\mathcal{H}^*$, and talk about the weak-* topology of $\mathcal{H}$. Indeed, it is more customary to call it weak topology in this case:

Definition 21.25. For any normed vector space V, the **weak topology** of V is defined to be the pullback of the weak-* topology of V^{**} through the canonical linear isometry $V \to V^{**}$. Convergence under the weak topology is called **weak convergence**. Thus, weak topology is described by the condition that for every net (v_α) in V and every $v \in V$,

$$v_\alpha \xrightarrow{\text{weakly}} v \qquad \Longleftrightarrow \qquad \lim_\alpha \langle v_\alpha, \varphi\rangle = \langle v, \varphi\rangle \text{ for all } \varphi \in V^* \tag{21.20}$$

If (ξ_α) is a net in $\mathcal{H}$ and $\xi \in \mathcal{H}$, then by Riesz-Fréchet,

$$\xi_\alpha \xrightarrow{\text{weakly}} \xi \qquad \Longleftrightarrow \qquad \lim_\alpha \langle \xi_\alpha | \eta\rangle = \langle \xi | \eta\rangle \text{ for all } \eta \in \mathcal{H} \tag{21.21}$$

Remark 21.26. Assume that $1 < p, q < +\infty$ and $p^{-1} + q^{-1} = 1$. The weak-* topology on $l^p(X)$ defined by the isomorphism $l^q(X)^* \simeq l^p(X)$ in Thm. 17.30 is clearly equal to the weak topology of $l^p(X)$. Therefore:

> We will not distinguish between weak topology and weak-* topology for $l^p(X)$ where $1 < p < +\infty$.

Corollary 21.27. *The closed unit ball $\overline{B}_{\mathcal{H}}(0,1)$ of $\mathcal{H}$ is weakly compact. Moreover, $\mathcal{H}$ is separable iff $\overline{B}_{\mathcal{H}}(0,1)$ is weakly metrizable.*

It follows that if $\mathcal{H}$ is separable, then $\overline{B}_{\mathcal{H}}(0,1)$ is sequentially compact.

Proof. By Cor. 21.6, we may assume $\mathcal{H} = l^2(X)$ where X is countable if $\mathcal{H}$ is separable. By Banach-Alaoglu and $l^2(X) \simeq l^2(X)^*$, or by Pb. 17.5 and Thm. 17.31, $\overline{B}_{\mathcal{H}}(0,1)$ is weakly compact. By $l^2(X) \simeq l^2(X)^*$ and Thm. 17.24, $l^2(X)$ is separable iff $\overline{B}_{l^2(X)}(0,1)$ is metrizable under the weak(-*) topology. $\square$

⋆ **Remark 21.28.** Assume that X is countable, say, X is $\{1, 2, \dots, n\}$ or $\mathbb{Z}_+$. Then explicit metrics of $\overline{B}_{l^2(X)}(0,1)$ can be found, e.g., d and δ defined by

$$d(f,g) = \sup_n n^{-1}|f(n) - g(n)| \qquad \delta(f,g) = \sum_n 2^{-n}|f(n) - g(n)|$$

for each $f, g \in \overline{B}_{l^2(X)}(0,1)$. This is because, by Cor. 7.76, both d and δ induce the pointwise convergence topology, and hence the weak topology on $\overline{B}_{l^2(X)}(0,1)$ by Thm. 17.31. (Compare also Pb. 17.3 or Pb. 15.15.)

⋆ **Remark 21.29.** A normed vector space V is called **reflexive** if the canonical linear isometry $V \to V^{**}$ is surjective (and hence an isomorphism). A reflexive space must be complete since V^{**} is complete by Thm. 17.35. By Thm. 17.30, if $1 < p < +\infty$ then $l^p(X)$ is reflexive. In particular, every Hilbert space is reflexive.

If V is reflexive, the canonical isomorphism $V \simeq V^{**}$ allows us to view V as a dual space of V^*, and talk about its weak-* topology. This is clearly equal to the weak topology. If V is not reflexive, we do not view the weak topology of V as the weak-* topology, since V might have a meaningful predual such that the weak-* topology defined by this predual is different from the weak topology.

For example, consider $V = l^\infty(\mathbb{Z})$ with predual $l^1(\mathbb{Z})$. The weak-* topology on $l^\infty(\mathbb{Z})$ defined by the predual $l^1(\mathbb{Z})$ is different from the weak topology (which is defined by elements of $l^\infty(\mathbb{Z})^*$), cf. Pb. 17.8. □

21.3.2 Strong and weak convergence

The following proposition clarifies the relationship between norm and weak convergence in Hilbert spaces:

Proposition 21.30. *Let (ξ_α) be a net in an inner product space V, and let $\xi \in V$. The following are equivalent:*

(1) $\lim_\alpha \xi_\alpha = \xi$.

(2) (ξ_α) *converges weakly to ξ, and*

$$\lim_\alpha \langle \xi_\alpha | \xi_\alpha \rangle = \langle \xi | \xi \rangle \tag{21.22}$$

Proof. By the (norm-)continuity of $\langle \cdot | \cdot \rangle$ we clearly have $(1) \Rightarrow (2)$. Assume (2). Then

$$\langle \xi - \xi_\alpha | \xi - \xi_\alpha \rangle = \langle \xi | \xi \rangle + \langle \xi_\alpha | \xi_\alpha \rangle - \langle \xi | \xi_\alpha \rangle - \langle \xi | \xi_\alpha \rangle$$

converges to $\langle \xi | \xi \rangle + \langle \xi | \xi \rangle - \langle \xi | \xi \rangle - \langle \xi | \xi \rangle = 0$ by (21.22) and the weak convergence. So (1) is true. □

That (21.22) does not always hold means that the norm function $\| \cdot \|$ is not continuous under the weak topology:

Example 21.31. Suppose that $(e_n)_{n \in \mathbb{Z}_+}$ is an orthormal sequence in an inner product space V. Then $\lim_n e_n$ converges weakly to 0, for instance, by the fact that $\sum_n |\langle \xi | e_n \rangle|^2 < +\infty$ for all $\xi \in \mathcal{H}$ (which implies $\lim_n \langle \xi | e_n \rangle = 0$). However, $\lim_n \|e_n\| = 1 \neq 0$.

Prop. 21.30 will be used to prove Thm. 21.34, the main result of the next section.

21.4 ⋆ Key property 1 is a direct consequence of key property 2

In this starred section, we shall give a direct proof that an inner product space V satisfies Condition (2) of Thm. 21.5 (the convergence of summing orthogonal vectors) if $\overline{B}_V(0,1)$ is weakly compact. The word "direct" means that the proof will clearly show how weak compactness leads almost directly to the conclusion. Let us begin the discussion by an easy observation:

Remark 21.32. If W is a normed vector space and V is a linear subspace, then we have a canonical linear map $W^* \to V^*, \varphi \mapsto \varphi|_V$. It is easy to see (cf. Prop. 10.28) that this map is an isomorphism of Banach spaces if V is dense in W. In particular, if V is an inner product space with completion $\mathcal{H}$, there is a canonical isomorphism $V^* \simeq \mathcal{H}^*$.

The following lemma tells us that given an inner product space V with completion $\mathcal{H}$, the weak topology of $\overline{B}_V(0,1)$ can be described in terms of the vectors of V but not necessarily of $\mathcal{H}$ or V^*. Thus, you may take Lem. 21.33-(2) as the *definition* of the weak compactness of $\overline{B}_V(0,1)$ if you want to invoke Occam's razor.

Lemma 21.33. *Let V be an inner product space. Let (v_α) be a net in $\overline{B}_V(0,1)$ and $v \in \overline{B}_V(0,1)$. Then the following are equivalent.*

(1) (v_α) converges weakly to v.

(2) $\lim_\alpha \langle v_\alpha | w\rangle = \langle v|w\rangle$ for all $w \in V$.

Similar to Thm. 17.31, this lemma is not true if $\overline{B}_V(0,1)$ is replaced by V.

Proof. Let $\mathcal{H}$ be the Hilbert space completion of V. So $V^* \simeq \mathcal{H}^*$. By Riesz-Fréchet, (1) means that $\lim\langle v_\alpha|\xi\rangle = \langle v|\xi\rangle$ for all $\xi \in \mathcal{H}$. Thus, one checks easily that (1)⇔(2) using the density of V in $\mathcal{H}$. □

Theorem 21.34. *Let V be an inner product space. Suppose that the unit ball $\overline{B}_V(0,1)$ is weakly compact. Then V satisfies condition (2) of Thm. 21.5.*

Proof. Let $(e_i)_{i\in I}$ be an orthonormal family of vectors in V. Let $(a_i)_{i\in I}$ be in $\mathbb{C}$ satisfying $\lambda := \sum_i |a_i|^2 < +\infty$. By scaling $(a_i)_{i\in I}$, assume WLOG that $\lambda = 1$. For each $J \in \mathrm{fin}(2^I)$, let $v_J = \sum_{j\in J} a_j e_j$. Then $\|v_J\|^2 = \sum_{j\in J}|a_j|^2$. Therefore, $(v_J)_{J\in\mathrm{fin}(2^I)}$ is a net in $\overline{B}_V(0,1)$, and $\lim_{J\in\mathrm{fin}(2^I)} \|v_J\|^2 = 1$.

Since $\overline{B}_V(0,1)$ is weakly compact, (v_J) has a subnet (v_{J_β}) converging weakly to some $v \in \overline{B}_V(0,1)$. In particular, $\|v\|^2 \leqslant 1$. For each $i \in I$ we have

$$\langle v|e_i\rangle = \lim_\beta \langle v_{J_\beta}|e_i\rangle = \lim_J \langle v_J|e_i\rangle = a_i$$

Therefore, by Bessel's inequality (Cor. 20.22), we have $\|v\|^2 \geqslant \sum_{i\in I}|a_i|^2 = 1$. Thus $\|v\|^2 = 1 = \lim_\beta \|v_{J_\beta}\|^2$. Therefore, by Prop. 21.30, we have $\lim_\beta v_{J_\beta} = v$. It is now easily to see (e.g. by Prop. 5.35) that $\lim_J v_J = v$, i.e., $\sum_{i\in I} a_i e_i$ converges to v. □

Corollary 21.35. *Let V be an inner product space. Then V is a Hilbert space iff $\overline{B}_V(0,1)$ is weakly compact.*

Proof. Cor. 21.27 implies "⇒". Thm. 21.34 and 21.5 imply "⇐". □

21.5 Problems and supplementary material

Fix Hilbert spaces $\mathcal{H}, \mathcal{K}$.

21.5.1 Basic facts about Hilbert spaces

$\star$ **Problem 21.1.** Prove that any two orthonormal bases of $\mathcal{H}$ have the same cardinality.

Hint. The case $\dim \mathcal{H} < +\infty$ is obvious by linear algebra. In the case that either one of the two orthonormal bases is infinite, use Thm. 16.7. □

Problem 21.2. Assume that $\mathcal{K}$ is a closed linear subspace of $\mathcal{H}$. Define a map $P : \mathcal{H} \to \mathcal{H}$ such that for each $\xi \in \mathcal{H}$, $\xi = P\xi + (1 - P)\xi$ is the orthogonal decomposition with respect to $\mathcal{K}$. (So $P\xi \in \mathcal{K}$ and $(1-P)\xi \in \mathcal{K}^\perp$.) Prove that P is a bounded linear map, and $P^2 = P$, $P^* = P$. We call P the **projection operator** of $\mathcal{K}$.

$\star$ **Problem 21.3.** Let $P \in \mathfrak{L}(\mathcal{H})$ satisfying $P^2 = P$ and $P = P^*$. Prove that $P(\mathcal{H})$ is a closed linear subspace of $\mathcal{H}$, and P is the projection operator of $P(\mathcal{H})$.

Recall Pb. 8.2 for the basic properties of $\limsup$ and $\liminf$.

Problem 21.4. (**Fatou's lemma for Hilbert spaces**) Prove that the norm function $\xi \in \mathcal{H} \mapsto \|\xi\| \in \mathbb{R}_{\geqslant 0}$ is lower semicontinuous if $\mathcal{H}$ is equipped with the weak topology. In other words, prove that if (ξ_α) is a net in $\mathcal{H}$ converging weakly to ξ, then $\liminf_\alpha \|\xi_\alpha\| \geqslant \|\xi\|$.

Note. Give a general argument using the Cauchy-Schwarz inequality. Do not identify $\mathcal{H}$ with $l^2(X)$. □

Remark 21.36. The reader may try to prove the following Fatou's lemma for dual Banach spaces (which is not difficult to prove): Let V be a normed vector space. Then the map $\varphi \in V^* \mapsto \|\varphi\| \in \mathbb{R}_{\geqslant 0}$ is lower semicontinuous, where V^* is equipped with the weak-* topology.

Problem 21.5. Let $(e_i)_{i\in I}$ be an orthonormal basis of $\mathcal{H}$.

1. For each $A \in \mathrm{fin}(2^X)$, let $P_A \in \mathfrak{L}(\mathcal{H})$ be defined by $P_A\xi = \sum_{i\in A}\langle \xi|e_i\rangle e_i$. Prove that $\lim\limits_{A\in\mathrm{fin}(2^X)} P_A$ converges pointwise to the identity operator.

2. Let $T \in \mathfrak{L}(\mathcal{H}, \mathcal{K})$. Use part 1 to prove that for every $\xi \in \mathcal{H}$, the RHS of the following equation converges to the LHS:

$$T\xi = \sum_{i\in I}\langle \xi|e_i\rangle Te_i \tag{21.23}$$

21.5.2 Hilbert's notion of boundedness

Let X,Y be sets.

Problem 21.6. Let $T:l^2(Y)\rightarrow l^2(X)$ be a bounded linear map. Prove that T is uniquely determined by its **matrix representation**

$$K:X\times Y\rightarrow\mathbb{C}\qquad K(x,y)=\langle T\chi_{\{y\}}|\chi_{\{x\}}\rangle \tag{21.24}$$

Prove that for each $x\in X$, the RHS of the following converges to the LHS:

$$(T\xi)(x)=\sum_{y\in Y}K(x,y)\xi(y) \tag{21.25}$$

Hint. Show that the RHS of (21.25) (together with its convergence) is equal to $\lim_{B\in\mathrm{fin}(2^Y)}\langle T(\chi_B\xi)|\chi_{\{x\}}\rangle$. □

Let's study the question of which $\infty\times\infty$ matrices are the matrix representations of bounded linear maps. We first consider the special case that $l^2(X)=\mathbb{C}$, and write K as a function $f:Y\rightarrow\mathbb{C}$:

Exercise 21.37. Let $f\in\mathbb{C}^Y$. Prove that

$$\|f\|_{l^2(Y)}=\sup_{B\in\mathrm{fin}(2^Y)}\ \sup_{g\in\overline{B}_{l^2(Y)}(0,1)}\langle f\chi_B|g\rangle \tag{21.26}$$

In particular, we have $f\in l^2(Y)$ iff the RHS of (21.26) is finite.

Hint. Use Lem. 21.16 to prove $\sum_{y\in B}|f(y)|^2=\sup_{g\in\overline{B}_{l^2(B)}(0,1)}\langle f\chi_B|g\rangle$. □

⋆ **Remark 21.38.** There is indeed a stronger criterion:

$$f\in l^2(Y)\qquad\Longleftrightarrow\qquad\sup_{B\in\mathrm{fin}(2^Y)}\langle f\chi_B|g\rangle<+\infty\text{ for all }g\in l^2(Y)$$

You can think about how to prove it if you know Baire's category theorem. (If you want to know the answer directly, search for "Banach-Steinhaus theorem".)

Now, we consider the general case.

Problem 21.7. Let $K:X\times Y\rightarrow\mathbb{C}$. For each $A\in\mathrm{fin}(2^X)$, $B\in\mathrm{fin}(2^Y)$, define

$$M_{A,B}=\sup_{\substack{\psi\in\overline{B}_{l^2(X)}(0,1)\\ \xi\in\overline{B}_{l^2(Y)}(0,1)}}\Big|\sum_{x\in A,y\in B}K(x,y)\xi(y)\overline{\psi(x)}\Big| \tag{21.27}$$

Assume that $M<+\infty$ where

$$M=\sup_{A\in\mathrm{fin}(2^X),B\in\mathrm{fin}(2^Y)}M_{A,B} \tag{21.28}$$

In the following, $\lim_{A,B}$ means $\lim_{A\in\mathrm{fin}(2^X),B\in\mathrm{fin}(2^Y)}$.

1. For each $A \in \mathrm{fin}(2^X)$, $B \in \mathrm{fin}(2^Y)$, define

$$T_{A,B} : l^2(Y) \to l^2(X) \qquad \xi \mapsto \sum_{x\in A, y\in B} K(x,y)\xi(y) \cdot \chi_{\{x\}} \tag{21.29}$$

   Prove that $\|T_{A,B}\| = M_{A,B}$.

2. For each $y \in Y$, prove that $\lim_{A,B} T_{A,B}\chi_{\{y\}}$ converges in $l^2(X)$.

3. Prove that $\lim_{A,B} T_{A,B}$ converges pointwise on $l^2(Y)$ to some bounded linear

$$T : l^2(Y) \to l^2(X)$$

   satisfying $\|T\| = M$.

4. Prove that K is the matrix representation of T, i.e., (21.24) is satisfied.

Hint. Part 2: Use Exe. 21.37. Part 3: Use Prop. 17.19 and the fact that $\{\chi_y : y \in Y\}$ spans a dense subspace of $l^2(Y)$. □

Remark 21.39. If $T : l^2(Y) \to l^2(X)$ is a bounded linear map, then its matrix representation K clearly satisfies (21.28) $< +\infty$ by Prop. 21.17. Therefore, Pb. 21.7 gives a description of bounded linear maps $l^2(Y) \to l^2(X)$ in terms of an explicit analytic condition on the matrix representations, and (21.28) gives an equivalent definition of operator norms.

Indeed, the above definitions of bounded "linear maps" and their "operator norms" (21.28) were introduced by Hilbert in an influential paper published in 1906, i.e., the fourth part (vierter Abschnitt) of [Hil12] (cf. also [Die-H, Sec. 5.2]). If you compare them with the modern definitions of bounded linear maps and operator norms in Sec. 10.6 (which are due to F. Riesz), you will notice two features.

First, of course, Hilbert's definition is more explicit and easier to calculate. Through learning this definition, we know that mathematicians never invent abstract definitions (such as those of Riesz) out of thin air.

The second and more important aspect is that Hilbert's definitions are not really about linear operators (i.e. maps from $l^2(Y)$ to $l^2(X)$), but are about sesquilinear forms. To illustrate this point, let's start with some preparation. □

Definition 21.40. For each linear map $T : \mathcal{H} \to \mathcal{K}$, define a sesquilinear map

$$\omega_T : \mathcal{H} \times \mathcal{K} \to \mathbb{C} \qquad \omega_T(\xi, \eta) = \langle T\xi|\eta\rangle \tag{21.30}$$

Problem 21.8. Let $\omega(\cdot|\cdot) : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ be sesquilinear. Define its norm

$$\|\omega\| = \sup_{\substack{\xi\in\overline{B}_{\mathcal{H}}(0,1)\\ \eta\in\overline{B}_{\mathcal{K}}(0,1)}} |\omega(\xi|\eta)| \tag{21.31}$$

Clearly $\|\omega\|$ is the smallest element in $[0,+\infty]$ satisfying

$$|\omega(\xi|\eta)| \leqslant \|\omega\|\cdot\|\xi\|\cdot\|\eta\| \qquad (\forall\xi\in\mathcal{H},\forall\eta\in\mathcal{K})$$

We say that ω is **bounded** if $\|\omega\|<+\infty$. Prove that the following are equivalent.

(1) ω is bounded.

(2) There exists a (necessarily unique) $T\in\mathfrak{L}(\mathcal{H},\mathcal{K})$ such that $\omega=\omega_T$.

(3) ω is continuous.

(4) ω is continuous at $(0,0)$.

Hint. (1)⇒(2): Mimic the proof of Cor. 21.23. (4)⇒(1): Mimic the proof of Prop. 10.25. □

Remark 21.41. Pb. 21.8 suggests that Riesz's language of bounded linear maps can be translated into Hilbert's language of bounded sesquilinear forms, and vice versa. Let us now translate Pb. 21.7 into the language of sesquilinear forms.

Hilbert's goal is to find a general condition ensuring the existence of a bounded form $\omega: l^2(Y)\times l^2(X)\to\mathbb{C}$ satisfying $\omega(\chi_y|\chi_x)=K(x,y)$ for all $x\in X, y\in Y$. For each $A\in\mathrm{fin}(2^X), B\in\mathrm{fin}(2^Y)$, one defines the truncated form $\omega_{A,B}: l^2(Y)\times l^2(X)\to\mathbb{C}$ by

$$\omega_{A,B}(\xi|\psi)=\sum_{x\in A,y\in B}K(x,y)\xi(y)\overline{\psi(x)}$$

(Namely, $\omega_{A,B}$ is defined to be $\omega_{T_{A,B}}$ where $T_{A,B}=$ (21.29).) One wants to find a condition so that $\lim_{A,B}\omega_{A,B}$ converges pointwise to some continuous function ω (which is clearly sesquilinear).

The condition found by Hilbert, namely $M<+\infty$ in Pb. 21.7, is a condition of equicontinuity[6]: Clearly $M_{A,B}=\|\omega_{A,B}\|$. Thus, $M<+\infty$ means that $\sup_{A,B}\|\omega_{A,B}\|<+\infty$. This implies that $(\omega_{A,B})_{A\in\mathrm{fin}(2^X),B\in\mathrm{fin}(2^Y)}$ is an equicontinuous family of functions when restricted to $\Delta_R=\overline{B}_{l^2(Y)}(0,R)\times\overline{B}_{l^2(X)}(0,R)$ for every $R>0$. Moreover, this family clearly converges pointwise on the dense subset $\Delta_R\cap E$ where $E=\mathrm{Span}\{\chi_{\{y\}}:y\in Y\}\times\mathrm{Span}\{\chi_{\{x\}}:x\in X\}$. Therefore, by Prop. 17.10, $\lim_{A,B}\omega_{A,B}$ converges pointwise on each Δ_R to some continuous function.[7] By considering all R, we get the desired function ω on $l^2(Y)\times l^2(X)$. □

⋆ **Remark 21.42.** Soon after Hilbert's work, in 1906, Hellinger and Toeplitz simplified Hilbert's boundedness condition by proving that $\sup_{A,B}\|\omega_{A,B}\|<+\infty$ iff $\sup_{A,B}|\omega_{A,B}(\xi|\psi)|<+\infty$ for all $\xi\in l^2(Y),\psi\in l^2(X)$. Similar to Rem. 21.38, this

[6]Recall that operator norms are also related to equicontinuity!

[7]This part is similar to the proof of Pb. 21.7-3, which uses Prop. 17.19, a special case of Prop. 17.10.

result is nowadays proved using Baire's category theorem (or its consequence, the Banach-Steinhaus theorem). To my knowledge, this **Hellinger-Toeplitz theorem** (in its original form) is one of the very few theorems before Riesz's work that made full use of the *completeness* of $l^2(X)$ without adopting the viewpoint of linear operators (cf. [Die-H, Sec. 6.4]).

21.5.3 Direct sums of Hilbert spaces

A tip for solving the problems in this subsection: Use cleverly Prop. 10.28 or Prop. 17.19 or other "reduction to densely-spanning subsets" tricks to simplify the proof. (We have already used such tricks in the solution of Pb. 21.7.)

Problem 21.9. Let $(\mathcal{H}_i)_{i\in I}$ be a family of Hilbert spaces. Recall that elements of $\prod_{i\in I}\mathcal{H}_i$ are of the form $\xi_\bullet = (\xi_i)_{i\in I}$ where $\xi_i \in \mathcal{H}_i$. Then $\prod_{i\in I}\mathcal{H}_i$ is a vector space whose linear structure is defined componentwise. Define

$$\bigoplus_{i\in I}\mathcal{H}_i = \Big\{(\xi_i)_{i\in I} \in \prod_{i\in I}\mathcal{H}_i : \sum_{i\in I}\|\xi_i\|^2 < +\infty\Big\} \tag{21.32}$$

equipped with the inner product

$$\langle\xi_\bullet|\eta_\bullet\rangle = \langle(\xi_i)_{i\in I}|(\eta_i)_{i\in I}\rangle = \sum_{i\in I}\langle\xi_i|\eta_i\rangle \tag{21.33}$$

Prove that the rightmost term of (21.33) converges (absolutely). Prove that the inner product space $\bigoplus_{i\in I}\mathcal{H}_i$ is complete. (So $\bigoplus_{i\in I}\mathcal{H}_i$ is a Hilbert space, called the **(Hilbert space) direct sum** of $(\mathcal{H}_i)_{i\in I}$.) An element $(x_\bullet)$ in $\bigoplus_{i\in I}\mathcal{H}_i$ is also written as $\oplus_{i\in I}\xi_i$.

Problem 21.10. Let $(\mathcal{H}_i)_{i\in I}$ be a family of Hilbert spaces. For each $i \in I$, choose $T_i \in \mathfrak{L}(\mathcal{H}_i)$. Assume that

$$\sup_{i\in I}\|T_i\| < +\infty \tag{21.34}$$

Prove that there is a unique bounded linear operator T on $\mathcal{H} := \bigoplus_{i\in I}\mathcal{H}_i$ such that for each $\oplus_i\xi_i \in \mathcal{H}$,

$$T(\oplus_i\xi_i) = \oplus_i(T_i\xi_i) \tag{21.35}$$

We write $T = \oplus_{i\in I}T_i$ and call it the **direct sum** of $(T_i)_{i\in I}$. Prove that

$$(\oplus_i T_i)^* = \oplus_i(T_i^*) \tag{21.36}$$

Problem 21.11. Let $(\mathcal{H}_i)_{i\in I}$ be a mutually orthogonal family of closed linear subspaces of $\mathcal{H}$. Assume that this family spans a dense subspace of $\mathcal{H}$. Prove that there is a unitary map

$$\Phi : \bigoplus_{i\in I} \mathcal{H}_i \xrightarrow{\simeq} \mathcal{H} \qquad \oplus_{i\in I}\, \xi_i \mapsto \sum_{i\in I} \xi_i \tag{21.37}$$

Hint. Once you have proved that Φ is a linear isometry, to prove that Φ is surjective, you only need to show that Φ has dense range. (Why?) □

Problem 21.12. In Pb. 21.11, choose $T \in \mathfrak{L}(\mathcal{H})$ such that each $\mathcal{H}_i$ is T**-invariant**, i.e., $T\mathcal{H}_i \subset \mathcal{H}_i$. Thus, the restriction of T to each $\mathcal{H}_i$ gives $T_i \in \mathfrak{L}(\mathcal{H}_i)$. Clearly $\sup_i \|T_i\| \leqslant \|T\| < +\infty$. Prove that the following diagram commutes:

$$\begin{tikzcd} \bigoplus_{i\in I} \mathcal{H}_i \arrow[r, "\oplus_i T_i"] \arrow[d, "\Phi"', "\simeq"] & \bigoplus_{i\in I} \mathcal{H}_i \arrow[d, "\simeq"', "\Phi"] \\ \mathcal{H} \arrow[r, "T"] & \mathcal{H} \end{tikzcd} \tag{21.38}$$

In other words, Φ implements a **unitary equivalence** of $\oplus_i T_i$ and T.

22 The birth of Hilbert spaces

22.1 From the Dirichlet problems to integral equations

Around 1906, the Hilbert space $l^2(\mathbb{Z})$ was introduced by Hilbert and Schmidt to the study of integral equations. As mentioned in the previous chapter, the main analytic properties of $l^2(\mathbb{Z})$ attracting Hilbert and Schmidt are the two key properties studied in Sec. 21.2 and 21.3 respectively: 1. The norm convergence (equivalently, the weak convergence, by Prop. 21.30) of $\sum_i a_i e_i$ when $\sum_i |a_i|^2 < +\infty$ and $(e_i)_{i\in I}$ is orthonormal. 2. The weak compactness of the closed unit ball.

22.1.1 Dirichlet problems

The goal of this chapter is to learn how Hilbert and Schmidt used $l^2(\mathbb{Z})$ and the above mentioned properties to study integral equations. A main source of integral equations comes from solving the **Poisson equation with Dirichlet boundary condition**:

- Let $\overline{\Omega}$ be a compact region in $\mathbb{R}^N$ with interior Ω and smooth boundary $\partial\Omega$.[1] Given good enough (say C^r, or C^∞) functions g on $\partial\Omega$ and φ on Ω, find good enough u on Ω satisfying

$$-\Delta u|_\Omega = \varphi \qquad u|_{\partial\Omega} = g \tag{22.1}$$

  Here, Δ is the Laplacian $\partial_1^2 + \cdots + \partial_N^2$.

In the following, we abbreviate $-\Delta u|_\Omega$ to $-\Delta u$.

The Tietze type extension gives a good $\widetilde{g}$ on $\overline{\Omega}$ with $\widetilde{g}|_{\partial\Omega} = g$. Then (22.1) becomes $-\Delta(u-\widetilde{g}) = \varphi + \Delta\widetilde{g}$ with $(u-\widetilde{g})|_{\partial\Omega} = 0$. Therefore, (22.1) can be reduced to the special case

$$-\Delta u = \varphi \qquad u|_{\partial\Omega} = 0 \tag{22.2}$$

By replacing $\overline{\Omega}$ with a more general smooth compact manifold with (or without) boundary, this problem, and also the **Helmholtz equation**

$$-\Delta u = \lambda u \qquad u|_{\partial\Omega} = 0 \tag{22.3}$$

have wide applications in differential geometry and in other types of differential equations. (For example, let $v = v(t, x_1, \dots, x_N)$, then the **heat equation** $\partial_t v = \Delta v$ and the **wave equation** $\partial_t^2 v = \Delta v$ can be solved by $v = \sum_j e^{-\lambda_j t} u_j$ and $v = \sum_j e^{\pm \mathbf{i}\sqrt{\lambda_j} t} u_j$ where $-\Delta u_j = \lambda_j u_j$.)

[1] Technically speaking, $\overline{\Omega}$ is a compact smooth n-dimensional submanifold of $\mathbb{R}^N$ with boundary

22.1.2 Compact operators (i.e. completely continuous operators)

The following is roughly the modern treatment of the **Dirichlet problem** (22.2): Let $L^2(\overline{\Omega})$ be the Hilbert space of Lebesgue integrable functions $u : \overline{\Omega} \to \mathbb{C}$ satisfying $\int_{\overline{\Omega}} |u|^2 < +\infty$. The inner product is given by $\langle u|v\rangle = \int_{\overline{\Omega}} uv^*$. One shows that $-\Delta$, defined on a suitable dense linear subspace of $L^2(\overline{\Omega})$, has a bounded inverse $T \in \mathfrak{L}(L^2(\overline{\Omega}))$ satisfying $\langle T\xi|\xi\rangle \geqslant 0$ for all $\xi \in L^2(\overline{\Omega})$.[2] (Namely, T is a bounded **positive operator**.) Thus, the problem (22.2) has solution $u = T\varphi$ in $L^2(\overline{\Omega})$. An analysis of regularity (with the help of Sobolev spaces) then shows that u is good enough.

Moreover, one shows that T is a compact operator (equivalently, a completely continuous operator, cf. Def. 22.3):

Definition 22.1. A linear map $T : V \to W$ of Banach spaces is called **compact** if $T(\overline{B}_V(0,1))$ is precompact in W (under the norm topology).

According to the **Hilbert-Schmidt theorem** (to be learned in this chapter), T has countably many eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$ satisfying $\lim_n \lambda_n = 0$, and the (normalized) eigenvectors of T form an orthonormal basis of $L^2(\overline{\Omega})$. The regularity analysis also shows that the eigenvectors of T are smooth. Thus, the eigenvalue problem (22.3) can be fully solved.

We refer the interested readers to [Eva, Ch. 6], [Tay] (especially Sec. 5.1), [Yu] (especially Sec. 78-80) for a detailed study of this topic.

22.1.3 From Dirichlet problems to integral equations

In the days of Hilbert and Schmidt, the Dirichlet problem (22.1) was understood in a different way by transforming it to a problem about integral equations. It was in the process of solving these integral equations that the Hilbert space $l^2(\mathbb{Z})$ was discovered. In the following, I will sketch how to transform Dirichlet problems to integral equations using "double layer potentials". Cf. [Fol-P, Sec. 3C] (where all the statements are accompanied by detailed proofs), [Sim-O, Sec. 3.3], or [RN, Sec. 81]. A discussion of the history of this method can be found in [Die-H, Sec. 2.5].

Set $x = (x_1, \dots, x_N)$. Without imposing the boundary condition $u|_{\partial\Omega}$, and assuming that the φ in $-\Delta u = \varphi$ can be extended to a good function on $\mathbb{R}^N$ with compact support, then $-\Delta u = \varphi$ has a solution

$$u(x) = (\Phi * \varphi)(x) = \int_{\mathbb{R}^N} \Phi(x-y)\varphi(y)dy$$

[2] In the case that $\overline{\Omega}$ is a compact manifold without boundary, $-\Delta$ should be replaced by $C - \Delta$ where $C > 0$.

where Φ, called the **fundamental solution**, is defined by

$$\Phi(x)=\begin{cases} -\dfrac{1}{2\pi}\log\|x\| & \text{if } N=2 \\ \dfrac{1}{(N-2)\sigma_{N-1}\cdot\|x\|^{N-2}} & \text{if } N\geqslant 3\end{cases} \tag{22.4}$$

where $\sigma_{N-1}=2\pi^{\frac{N}{2}}\Gamma(N/2)^{-1}$ is the volume of $\mathbb{S}^{N-1}=\{x\in\mathbb{R}^N:\|x\|=1\}$. $\Delta\Phi$ is the delta function at 0. In particular, it is zero outside 0. Cf. [Eva, Sec. 2.2.1].

By replacing u with $u-\Phi*\varphi$, (22.1) is transformed to the **harmonic equation** (with Dirichlet boundary condition)

$$-\Delta u=0 \qquad u|_{\partial\Omega}=g \tag{22.5}$$

Write $\nabla v=(\partial_1 v,\dots,\partial_N v)$ for each differentiable function v on subsets of $\mathbb{R}^N$. For each $y\in\partial\Omega$ we let $\mathbf{n}_y$ be the outward pointing unit vector at y orthogonal to (the tangent space of) $\partial\Omega$ at y. Define $G(x,y)$ for each $x\in\mathbb{R}^N,y\in\partial\Omega$ by

$$G(x,y)=\big\langle(\nabla\Phi)(x-y),\mathbf{n}_y\big\rangle=\frac{\langle y-x,\mathbf{n}_y\rangle}{\sigma_{N-1}\|x-y\|^N} \tag{22.6}$$

For each $x\in\mathbb{R}^N$ and continuous function f on $\partial\Omega$, define $(\mathcal{D}f)(x)=\int_{\partial\Omega}f(y)(\nabla\Phi)(x-y)\cdot d\mathbf{S}$ where the RHS is a "surface integral of second type", i.e.

$$(\mathcal{D}f)(x)=\int_{\partial\Omega}G(x,y)f(y)dy \tag{22.7}$$

When $x\in\partial\Omega$, this is an improper integral, which converges because

$$G(x,y)\sim\|x-y\|^{2-N} \tag{22.8}$$

when $x\in\partial\Omega$ approaches y. (Note that $\int_U\|y\|^{2-N}dy$ converges if $U\subset\mathbb{R}^{N-1}$ is a bounded neighborhood of 0.)

$\mathcal{D}f$ is not continuous at the points of $\partial\Omega$. However, if we define $u:\overline{\Omega}\to\mathbb{C}$ by

$$u(x)=\begin{cases}(\mathcal{D}f)(x) & \text{if } x\in\Omega\\ f(x)+(\mathcal{D}f)(x) & \text{if } x\in\partial\Omega\end{cases} \tag{22.9}$$

then u is continuous on $\overline{\Omega}$. Moreover, $\Delta u=0$ on Ω since $\Delta G(x,y)=0$ when $x\neq y$. Therefore, the Dirichlet problem (22.5) can be solved if there exists a good function f on $\partial\Omega$ such that $f+\mathcal{D}f|_{\partial\Omega}=g$.

To summarize, define an integral operator T on the space $L^2(\partial\Omega)$ of Lebesgue square integrable functions by

$$(Tf)(x)=\int_{\partial\Omega}G(x,y)f(y)dy \tag{22.10}$$

The equation (22.5) has solution (22.9) if there is an (at least continuous) $f:\partial\Omega\to\mathbb{C}$ satisfying $f+Tf=g$. The problem is then reduced to finding such f for a given g.

22.1.4 Summary of the problem of integral equations

Given a function g on $\partial\Omega$, we want to find a function f on $\partial\Omega$ satisfying

$$f + Tf = g \tag{22.11a}$$

Moreover, we shall consider the case that $N = 2$ and hence $\dim \partial\Omega = 1$, which is in line with history. Let us assume that $\partial\Omega$ has only one connected component so that $\partial\Omega \simeq \mathbb{S}^1$. So we view f, g also as 2π-periodic functions with L^2-norms $\|f\| = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^2}$ and $\|g\| = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}|g|^2}$. Then

$$(Tf)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K(x,y)f(y)dy \tag{22.11b}$$

where $K(x,y) = G(x,y)\cdot\lambda(y)\lambda(x)$ for some $\lambda \in C^\infty(\mathbb{S}^1, \mathbb{R})$ defined by the change of variable from $\partial\Omega$ to $\mathbb{S}^1$ (similar to the function Φ' in (13.31)).

Since G takes real values, so does K. (22.8) suggests that K is uniformly bounded on $\mathbb{S}^1 \times \mathbb{S}^1$. Indeed, K is continuous (cf. [RN, Sec. 81]). However, we will only need the weaker fact that $K \in L^2(\mathbb{S}^1 \times \mathbb{S}^1)$, i.e. [3]

$$\|K\|_{L^2} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|K(x,y)|^2 dydx < +\infty \tag{22.12}$$

In finite-dimensional linear algebra, we know that a linear operator is surjective iff it is injective. If this is the case for $1+T$, then solving $(1+T)f = g$ can be reduced to the easier task of proving that -1 is not an eigenvalue of T. However, there are bounded linear operators on infinite dimensional function spaces that are injective but not surjective: Consider the right translation operator on $l^2(\mathbb{N})$, which is the bounded linear operator sending each $\chi_{\{n\}}$ to $\chi_{\{n+1\}}$.

To understand the behavior of T, and to solve other problems leading to integral equations, it turns out that we must have a good understanding of the eigenvalues and eigenvectors of T. As soon as we understand the eigenvalue problem of T very well, we can prove the theorem of **Fredholm alternative**, which says that $\mathrm{Ker}(1+T) = 0$ iff $1+T$ is surjective.

22.2 Discretizing the integral equations

Fix $K \in L^2(\mathbb{S}^1 \times \mathbb{S}^1)$. (The readers can assume for simplicity that K is continuous, which is often the case when K is defined by the 2d Dirichlet problem as in Subsec. 22.1.4.) Define the integral operator T sending each 2π-periodic function f to Tf, where

$$(Tf)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K(x,y)f(y)dy \tag{22.13}$$

[3] When $N > 2$, we do not necessarily have $\|K\|_{L^2} < +\infty$. However, T is always a completely continuous operator (equivalently, a compact operator) on $L^2(\partial\Omega)$. See [Fol-P] Sec. 3.B, Prop. 3.11.

K is called the **kernel** of T. Our goal is to understand the eigenvalues and eigenvectors of T.

Hilbert's idea is to use Fourier series. Functions on $\mathbb{S}^1 \times \mathbb{S}^1$ have Fourier series of the form $\sum_{m,n} a_{m,n} e^{\mathbf{i}(-nx+my)}$ in a similar way as functions on $\mathbb{S}^1$. Therefore, for each $m, n \in \mathbb{Z}$, define the Fourier coefficient

$$\widehat{K}(m,n) = \langle Te_n | e_m \rangle = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K(x,y) e^{\mathbf{i}(nx-my)} dxdy \tag{22.14}$$

Then, by Thm. 20.35, we get $f = \sum_n \widehat{f}_n e_n$ and $g = \sum_m \widehat{g}_m e_m$, and hence

$$\langle Tf | g \rangle = \sum_{m,n} \widehat{K}(m,n) \widehat{f}(n) \overline{\widehat{g}(m)} = \langle \widehat{T}\widehat{f} | \widehat{g} \rangle$$

if we view $\widehat{f}, \widehat{g}$ as in $l^2(\mathbb{Z})$ and define a linear map $\widehat{T} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ by

$$\boxed{(\widehat{T}\widehat{f})(m) = \sum_{n \in \mathbb{Z}} \widehat{K}(m,n) \widehat{f}(n)} \tag{22.15}$$

By applying Parseval's identity to K, we obtain $\sum_{m,n} |\widehat{K}(m,n)|^2 = (2\pi)^{-2} \iint K(x,y) dxdy$ (which is a finite number), and hence

$$\widehat{K} \in l^2(\mathbb{Z} \times \mathbb{Z}) \tag{22.16}$$

Thus, to understand the original integral equation, one must first understand the equation

$$\widehat{f} + \widehat{K}\widehat{f} = \widehat{g} \tag{22.17}$$

In fact, at this point, Hilbert did not have the notion of $l^2(\mathbb{Z})$ and L^2 yet. However, the idea of transforming the eigenvalue problem about T to that about the *matrix* $\widehat{K}$ without introducing $l^2(\mathbb{Z})$ is conceivable. After all, almost every mathematical progress leaves some questions about rigor until the end.

Remark 22.2. Parseval's identity $\frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 = \sum_n |\widehat{f}(n)|^2$ is well-known by the time Hilbert studied integral equations around 1906. However, Parseval's identity is far from enough to motivate the notion of $l^2(\mathbb{Z})$: Unlike the completeness of $l^2(\mathbb{Z})$ or the weak compactness of $\overline{B}_{l^2(\mathbb{Z})}(0,1)$, Parseval's identity is a property about individual functions, not about the set of all functions *and their interactions*.

In the remaining sections, we will forget the original function K on $\mathbb{S}^1 \times \mathbb{S}^1$, as Hilbert did, and focus on the matrix $\widehat{K} \in l^2(\mathbb{Z} \times \mathbb{Z})$. Thus, we will let K denote $\widehat{K}$, or denote a general function $X \times Y \to \mathbb{C}$ where X, Y are sets.

22.3 Hilbert's complete continuity

We fix Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Let X, Y be sets. For each linear map $T : \mathcal{H} \to \mathcal{K}$ we let

$$\omega_T : \mathcal{H} \times \mathcal{K} \to \mathbb{C} \qquad \omega_T(\xi|\eta) = \langle T\xi|\eta\rangle \tag{22.18}$$

In his important 1906 paper, Hilbert introduced two crucial analytic properties about sesquilinear forms. The first one is boundedness (i.e., the condition $M < +\infty$ in Pb. 21.7), which is equivalent to the boundedness of linear maps between Hilbert spaces. See Subsec. 21.5.2 for details. In particular, we have the obvious fact

$$T \in \mathfrak{L}(\mathcal{H}, \mathcal{K}) \qquad \Rightarrow \qquad \omega_T \text{ is continuous} \tag{22.19}$$

The second condition is stronger than the first one (cf. [Hil12, p.147]): A sesquilinear $\omega : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ is called **completely continuous** if ω is weakly continuous when restricted to $\overline{B}_{\mathcal{H}}(0,1) \times \overline{B}_{\mathcal{K}}(0,1)$. It is easy to adapt this definition to linear maps:

Definition 22.3. A linear map $T : \mathcal{H} \to \mathcal{K}$ is called **completely continuous** if ω_T is continuous on $\overline{B}_{\mathcal{H}}(0,1) \times \overline{B}_{\mathcal{K}}(0,1)$ where $\overline{B}_{\mathcal{H}}(0,1)$ and $\overline{B}_{\mathcal{K}}(0,1)$ are equipped with their weak topologies.

Thus, complete continuity means that if (ξ_α) converges weakly in $\overline{B}_{\mathcal{H}}(0,1)$ to ξ and (η_α) converges weakly in $\overline{B}_{\mathcal{K}}(0,1)$ to η, then $\lim_\alpha \langle T\xi_\alpha|\eta_\alpha\rangle = \langle T\xi|\eta\rangle$.

Completely continuous sesquilinear forms are clearly continuous (i.e., bounded, cf. Pb. 21.8). Thus, it is not hard see:

Lemma 22.4. *A completely continuous linear map $T : \mathcal{H} \to \mathcal{K}$ is bounded.*

Proof. ω_T is weakly continuous and hence continuous on $\overline{B}_{\mathcal{H}}(0,1) \times \overline{B}_{\mathcal{K}}(0,1)$. Since $\mathcal{B}_{\mathcal{H}}(0,1) \times \mathcal{B}_{\mathcal{K}}(0,1)$ is a neighborhood of $(0,0)$ in $\mathcal{H} \times \mathcal{K}$, the function $\omega_T : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ is continuous at $(0,0)$. Thus, there exists $\varepsilon > 0$ such that $|\omega_T(\xi|\eta)| \leqslant 1$ whenever $\|\xi\| \leqslant \varepsilon, \|\eta\| \leqslant \varepsilon$. Thus, whenever $\|\xi\| \leqslant 1, \|\eta\| \leqslant 1$, we have

$$|\omega_T(\xi|\eta)| = \varepsilon^{-2}|\omega_T(\varepsilon\xi|\varepsilon\eta)| \leqslant \varepsilon^{-2}$$

By Prop. 21.17, we have $\|T\| \leqslant \varepsilon^{-2} < +\infty$. □

Example 22.5. Assume that $\mathcal{H}$ is an infinite-dimensional Hilbert space. Then its identity operator $\mathbf{1}$ is not completely continuous. This is because $\omega_{\mathbf{1}}$ is the inner product function $(\xi, \eta) \in \mathcal{H} \times \mathcal{H} \mapsto \langle \xi|\eta\rangle \in \mathbb{C}$, which is not weakly continuous by Exp. 21.31.

Exercise 22.6. Show that a finite linear combination of completely continuous operators is continuous. Show that the adjoint of a completely continuous operator is completely continuous. Show that if $T \in \mathfrak{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $S \in \mathfrak{L}(\mathcal{H}_2, \mathcal{H}_3)$ where each $\mathcal{H}_i$ is a Hilbert space, and if one of T and S is completely continuous, then $ST : \mathcal{H}_1 \to \mathcal{H}_3$ is completely continuous. In particular, the set of completely continuous operators on $\mathcal{H}$ is a two-sided ideal of $\mathfrak{L}(\mathcal{H})$.

The easiest examples of completely continuous maps are finite-rank operators:

Definition 22.7. A linear map of vector spaces is said to have **finite rank** if its range is finite-dimensional.

Proposition 22.8. *Let $T \in \mathfrak{L}(\mathcal{H}, \mathcal{K})$. The following are equivalent:*

(1) T has finite rank.

(2) There exist finitely many vectors $\mu_1, \dots, \mu_n \in \mathcal{H}$ and $\eta_1, \dots, \eta_n \in \mathcal{K}$ such that for all $\xi \in \mathcal{H}$,

$$T\xi = \sum_{i=1}^{n} \langle \xi | \mu_i \rangle \eta_i \tag{22.20}$$

Note that without assume that T is bounded, (1) does not imply (2).

Proof. "(2)⇒(1)" is obvious. Assume (1). Restrict the inner product of $\mathcal{K}$ to the finite-dimensional $V = T(\mathcal{H})$. By Gram-Schmidt, V is spanned by an orthonormal set of vectors $\eta_1, \dots, \eta_n$. Then, for each $\psi \in V$ we have (by Thm. 20.35) $\psi = \sum_{i=1}^{n} \langle \psi | \eta_i \rangle \eta_i$. Therefore

$$T\xi = \sum_i \langle T\xi | \eta_i \rangle \eta_i = \sum_i \langle \xi | \mu_i \rangle \eta_i$$

where $\mu_i = T^* \eta_i$. □

Example 22.9. If $T \in \mathfrak{L}(\mathcal{H}, \mathcal{K})$ has finite rank, then T is completely continuous.

Proof. By Prop. 22.8, and by linearity, it suffices to assume that T takes the form $T\xi = \langle \xi | \mu \rangle \eta$ for some $\mu \in \mathcal{H}, \eta \in \mathcal{K}$. Then $\omega_T(\xi|\psi) = \langle \xi | \mu \rangle \langle \eta | \psi \rangle$ is clearly weakly continuous with respect to ξ and ψ. □

In order to have nontrivial examples of completely continuous operators, we need the following simple fact:

Theorem 22.10. *Let (T_α) be a net of completely continuous maps $\mathcal{H} \to \mathcal{K}$. Assume that $T : \mathcal{H} \to \mathcal{K}$ is linear and $\lim_\alpha \|T - T_\alpha\| = 0$. Then T is completely continuous.*

Proof. By Cor. 21.18, ω_{T_α} converges uniformly on $\overline{B}_{\mathcal{H}}(0,1) \times \overline{B}_{\mathcal{K}}(0,1)$ to ω_T. Since the uniform limit of a net of continuous functions is continuous, we conclude that ω_T is weakly continuous on $\overline{B}_{\mathcal{H}}(0,1) \times \overline{B}_{\mathcal{K}}(0,1)$. □

Motivated by the above theorem, we make the following definition:

Definition 22.11. We say that a linear map $T : \mathcal{H} \to \mathcal{K}$ is **approximable** if there exists a net (equivalently, a sequence) of finite-rank operators (T_α) in $\mathfrak{L}(\mathcal{H}, \mathcal{K})$ such that $\lim_\alpha \|T - T_\alpha\| = 0$. Approximable operators are clearly bounded.

Theorem 22.12. *Let $T : \mathcal{H} \to \mathcal{K}$. Then T is approximable iff T is completely continuous.*

⋆ *Proof.* Since finite-rank bounded linear operators are completely continuous, by Thm. 22.10, approximability implies complete continuity. Conversely, assume that T is completely continuous. Assume WLOG that $\mathcal{H} = l^2(Y)$ and $\mathcal{K} = l^2(X)$. For each $A \in \mathrm{fin}(2^X)$ and $B \in \mathrm{fin}(2^Y)$ we set

$$T_{A,B} : l^2(Y) \to l^2(X) \qquad T_{A,B}\xi = \chi_A \cdot T(\chi_B \xi) \tag{22.21}$$

which is a bounded linear map of finite rank. We claim that $\lim_{A,B} \|T - T_{A,B}\| = 0$.

By Cor. 21.18, we need to show that $(\omega_{T_{A,B}})_{A \in \mathrm{fin}(2^X), B \in \mathrm{fin}(2^Y)}$ converges uniformly on $\Omega = \overline{B}_{l^2(Y)}(0,1) \times \overline{B}_{l^2(X)}(0,1)$ to ω_T. We equip Ω with the product weak topology, which is compact by Cor. 21.27. Since each $\omega_{T_{A,B}}$ is (weakly) continuous, by Thm. 9.12 and Prop. 9.16, it suffices to prove that

$$\lim_{A,B,\xi',\eta'} |\omega_T(\xi'|\eta') - \omega_{T_{A,B}}(\xi'|\eta')| = |\omega_T(\xi|\eta) - \omega_T(\xi|\eta)| = 0$$

where ξ' converges weakly to ξ and η' converges weakly to η. Equivalently, it suffices to prove that for each net $(\xi_i, \eta_i)_{i \in I}$ in Ω converging to $(\xi, \eta) \in \Omega$, we have

$$\lim_{A,B,i} |\langle (T - T_{A,B})\xi_i | \eta_i \rangle| = 0$$

Clearly $\lim \langle T\xi_i | \eta_i \rangle = \langle T\xi | \eta \rangle$ since T is completely continuous. Note that $\langle T_{A,B}\xi_i | \eta_i \rangle = \langle T\chi_B \xi_i | \chi_A \eta_i \rangle$. It remains to show that this expression also converges to $\langle T\xi|\eta\rangle$. Since T is completely continuous, it suffices to prove that $\lim_{B,i} \chi_B \xi_i$ converges weakly to ξ and $\lim_{A,i} \chi_A \eta_i$ converges weakly to η.

By Thm. 17.31, we know for each $y \in Y$ that $\lim_i \xi_i(y) = \xi(y)$ and hence $\lim_{B,i} \chi_B(y)\xi_i(y) = \xi(y)$. By Thm. 17.31 again, we conclude that $\lim_{B,i} \chi_B \xi_i$ converges weakly to ξ. The second (weak) limit can be proved in the same way. □

Remark 22.13. Since the proof of the direction "⇐" in Thm. 22.12 is slightly more complicated (although the main idea is clear), we will not use this direction in the future. But see Rem. 22.39 for an alternative proof.

Remark 22.14. In the proof of Thm. 22.12 we have shown that if $T : l^2(Y) \to l^2(X)$ is completely continuous, then

$$\lim_{A\in \mathrm{fin}(2^X), B\in\mathrm{fin}(2^Y)} \|T - T_{A,B}\| = 0 \tag{22.22}$$

In contrast, if T is only bounded, then it is easy to see that $\lim_{A,B} T_{A,B}$ converges pointwise to T.

22.4 Hilbert-Schmidt operators

Let X and Y be sets. Recall Pb. 21.6 for the basic facts about matrix representations.

Theorem 22.15. *Let $K \in l^2(X \times Y)$. Then K is the matrix representation of a (necessarily unique) completely continuous $T : l^2(Y) \to l^2(X)$. Moreover, we have*

$$\|T\| \leqslant \|K\|_{l^2} \tag{22.23}$$

Such T is called a **Hilbert-Schmidt operator**. See Pb. 22.2 for the general definition of Hilbert-Schmidt operators.

Lemma 22.16. *Thm. 22.15 is true when X, Y are finite sets.*

Proof. The only nontrivial part is $\|T\| \leqslant \|K\|_{l^2}$. By Prop. 21.17, it suffices to prove for all $f \in \overline{B}_{l^2(Y)}(0,1)$ and $g \in \overline{B}_{l^2(X)}(0,1)$ that $|\langle Tf|g\rangle| \leqslant \|K\|_2$. But

$$|\langle Tf|g\rangle| = \Big|\sum_{x,y} K(x,y)f(y)\overline{g(x)}\Big| = |\langle K|\Gamma\rangle| \leqslant \|K\|_2 \cdot \|\Gamma\|_2$$

where $\Gamma : X \times Y \to \mathbb{C}$ is defined by $\Gamma(x,y) = \overline{f(y)}g(x)$. Since

$$\|\Gamma\|_2^2 = \sum_{x,y} |f(y)^2 g(x)^2| = \sum_y |f(y)|^2 \cdot \sum_x |g(x)|^2 \leqslant 1$$

we have $|\langle Tf|g\rangle| \leqslant \|K\|_2$. $\square$

Proof of Thm. 22.15. For each $\Omega \in \mathrm{fin}(2^{X\times Y})$, let $K_\Omega : X \times Y \to \mathbb{C}$ be $K_\Omega = K\chi_\Omega$. Let $T_\Omega : l^2(Y) \to l^2(X)$ be the finite rank operator whose matrix representation is K_Ω, namely,

$$T_\Omega \xi = \sum_{(x,y)\in\Omega} K(x,y)\xi(y) \cdot \chi_{\{x\}} \tag{22.24}$$

Since $K \in l^2(X \times Y)$, we know that $\lim_\Omega \|K - K_\Omega\|_2 = 0$, and hence $(K_\Omega)_{\Omega\in\mathrm{fin}(2^{X\times Y})}$ is a Cauchy net in $l^2(X \times Y)$. For each $\Omega, \Gamma \in \mathrm{fin}(2^{X\times Y})$, since $T_\Omega - T_\Gamma$ has matrix representation $K_\Omega - K_\Gamma$, by Lem. 22.16 we have

$$\|T_\Omega - T_\Gamma\| \leqslant \|K_\Omega - K_\Gamma\|_{l^2}$$

Therefore $\lim_{\Omega,\Gamma}\|T_\Omega - T_\Gamma\| = 0$, i.e., (T_Ω) is a Cauchy net in $\mathfrak{L}(l^2(Y), l^2(X))$. By Thm. 17.35, $\mathfrak{L}(l^2(Y), l^2(X))$ is complete. So $\lim_\Omega T_\Omega$ converges under the operator norm to some $T \in \mathfrak{L}(l^2(Y), l^2(X))$.

Clearly T has matrix representation K. Since each T_Ω has finite rank, by Thm. 22.12, T is completely continuous. Finally, by Lem. 22.16, $\|T_\Omega\| \leqslant \|K_\Omega\|_2 \leqslant \|K\|_2$ for all Ω. Taking $\lim_\Omega$, we get $\|T\| \leqslant \|K\|_2$. □

Remark 22.17. In the above proof, we have used the completeness of $\mathfrak{L}(l^2(Y), l^2(X))$, which relies on the completeness of $l^2(X)$. This seems contradictory to our earlier statement that Hilbert and Schmidt did not rely on the completeness of l^2 spaces to study integral equations. In fact, there is no contradiction since Hilbert took the sesquilinear form perspective: One can modify the above proof by showing that for every $R > 0$, ω_{T_Ω} converges uniformly on $\overline{B}_{l^2(X)}(0,R) \times \overline{B}_{l^2(X)}(0,R)$ to some function ω. This gives the desired completely continuous sesquilinear form for T without using the completeness of l^2 spaces.

22.5 Triumph of weak(-*) compactness: the Hilbert-Schmidt theorem

Fix a Hilbert space $\mathcal{H}$. For each $T \in \mathfrak{L}(\mathcal{H})$, let $\omega_T : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be $\omega_T(\xi|\eta) = \langle T\xi|\eta\rangle$ as usual.

Recall from linear algebra that $\lambda \in \mathbb{C}$ is called an **eigenvalue** of T if there is a nonzero $\xi \in \mathcal{H}$ such that $T\xi = \lambda\xi$. In this case, ξ is called the **λ-eigenvector** of T.

22.5.1 Self-adjoint operators and positive operators

Definition 22.18. Let $T \in \mathfrak{L}(\mathcal{H})$. We say that T is **self-adjoint** if one of the following conditions hold:

(1) $T = T^*$.

(2) $\langle T\xi|\eta\rangle = \langle \xi|T\eta\rangle$ for all $\xi, \eta \in \mathcal{H}$.

(3) The sesquilinear form ω_T is a Hermitian.

(4) $\langle T\xi|\xi\rangle \in \mathbb{R}$ for all $\xi \in \mathcal{H}$.

Proof of equivalence. Since we always have $\langle T\xi|\eta\rangle = \langle \xi|T^*\eta\rangle$, the equivalence (1)⇔(2) is obvious. Recall that ω_T is Hermitian iff $\omega_T(\xi|\eta) = \overline{\omega_T(\eta|\xi)}$. But $\overline{\omega_T(\eta|\xi)} = \overline{\langle T\eta|\xi\rangle} = \langle \xi|T\eta\rangle$, which equals $\langle T\xi|\eta\rangle$ for all ξ, η iff (2) is true. This proves (2)⇔(3). By Prop. 20.7, we have (3)⇔(4). □

Remark 22.19. Let $T \in \mathfrak{L}(\mathcal{H})$. Assume that $E \subset \mathcal{H}$ spans a dense subspace of $\mathcal{H}$. By sesquilinearity, and by the continuity of $\langle\cdot|\cdot\rangle$, T is self-adjoint iff $\langle T\xi|\eta\rangle = \langle \xi|T\eta\rangle$ for all $\xi, \eta \in E$. Therefore:

Example 22.20. Let X be a set. Let $T \in \mathfrak{L}(l^2(X))$. Let $K : X \times X \to \mathbb{C}$ be the matrix representation of T, i.e., $K(x,y) = \langle T\chi_{\{y\}}|\chi_{\{x\}}\rangle$ for all $x, y \in Y$. Then T is self-ajoint iff $K(x,y) = \overline{K(y,x)}$ for all $x, y \in Y$.

Proof. This is because $\{\chi_{\{x\}} : x \in X\}$ spans a dense subspace, and $\overline{K(y,x)} = \overline{\langle T\chi_{\{y\}}|\chi_{\{x\}}\rangle} = \langle \chi_{\{x\}}|T\chi_{\{y\}}\rangle$. $\square$

Definition 22.21. Let $T \in \mathfrak{L}(\mathcal{H})$. We say that T is **positive** and write $T \geqslant 0$ if $\langle T\xi|\xi\rangle \geqslant 0$ for all $\xi \in \mathcal{H}$. Positive operators are clearly self-adjoint. More generally, if $S, T \in \mathfrak{L}(\mathcal{H})$, we write

$$S \leqslant T \qquad \Longleftrightarrow \qquad T - S \geqslant 0 \tag{22.25}$$

In other words, $S \leqslant T$ means $\langle S\xi|\xi\rangle \leqslant \langle T\xi|\xi\rangle$ for all $\xi \in \mathcal{H}$. Clearly "$\leqslant$" is a partial order on $\mathfrak{L}(\mathcal{H})$.[4]

Self-adjoint operators and positive operators are analogous to real-valued functions and positive functions. We will make this analogy precise in the future. At this point, let us see an example of the analogy:

Example 22.22. Suppose that $T \in \mathfrak{L}(\mathcal{H})$ is self-adjoint, and $\lambda \geqslant \|T\|$. Then $\lambda + T$ and $\lambda - T$ are positive, i.e., $-\lambda \leqslant T \leqslant \lambda$.

Proof. Since $\langle T\xi|\xi\rangle \leqslant \|T\| \cdot \|\xi\|^2 \leqslant \lambda\langle\xi|\xi\rangle$, we get $T \leqslant \lambda$. Similarly, $-T \leqslant \lambda$. $\square$

The following two basic facts will be used in the proof of the Hilbert-Schmidt theorem:

Lemma 22.23. *Suppose that $T \in \mathfrak{L}(\mathcal{H})$ is positive. Assume that $\xi \in \mathcal{H}$ satisfies $\langle T\xi|\xi\rangle = 0$. Then $T\xi = 0$.*

Proof. Our assumption is $\omega_T(\xi|\xi) = 0$. Since the sesquilinear form ω_T is positive, by Cauchy-Schwarz (cf. Rem. 20.24), for each $\eta \in \mathcal{H}$ we have $|\omega_T(\xi|\eta)|^2 \leqslant \omega_T(\xi|\xi)\omega_T(\eta|\eta) = 0$ and hence $\langle T\xi|\eta\rangle = 0$. (Alternatively, one can use (20.9) to show $\omega_T(\xi|\eta) = 0$.) Therefore $T\xi = 0$. $\square$

Definition 22.24. Let $T \in \mathfrak{L}(\mathcal{H})$. We say that a linear subspace $\mathcal{K} \subset \mathcal{H}$ is **invariant under** T (or simply T**-invariant**) if $T\mathcal{H} \subset \mathcal{H}$.

Proposition 22.25. *Let $T \in \mathfrak{L}(\mathcal{H})$. Let $\mathcal{K}$ be a linear subspace of $\mathcal{H}$. Suppose that $\mathcal{K}$ is invariant under T and T^*. Then so is $\mathcal{K}^\perp$.*

In particular, if T is self-adjoint and $\mathcal{K}$ is T-invariant, then $\mathcal{K}^\perp$ is T-invariant.

Proof. Let $\eta \in \mathcal{K}^\perp$. Then $\langle T\eta|\mathcal{K}\rangle = \langle \eta|T^*\mathcal{K}\rangle \subset \langle\eta|\mathcal{K}\rangle = 0$ and $\langle T^*\eta|\mathcal{K}\rangle = \langle\eta|T\mathcal{K}\rangle \subset \langle\eta|\mathcal{K}\rangle = 0$. So $T\eta, T^*\eta \in \mathcal{K}^\perp$. $\square$

Remark 22.26. Prop. 22.25 gives a simple method of decomposing the action of T: If $\mathcal{K}$ is invariant under T, T^*, then by Pb. 21.12, T is unitarily equivalent to the "block diagonal operator" $T|_{\mathcal{K}} \oplus T|_{\mathcal{K}^\perp}$.

[4] Note that if $T \leqslant S$ and $S \leqslant T$, then $\omega_{T-S}(\xi|\xi) = 0$ for all ξ. By the polarization identity (20.3), we get $\omega_{T-S} = 0$ and hence $T = S$.

22.5.2 The Hilbert-Schmidt theorem

We first prove the Hilbert-Schmidt theorem for positive operators.

Theorem 22.27 (Hilbert-Schmidt theorem). *Assume that $T \in \mathfrak{L}(\mathcal{H})$ is positive and completely continuous. Then $\mathcal{H}$ has an orthonormal basis $(e_1, e_2, \dots) \cup (f_j)_{j\in J}$, where the countable family $(e_1, e_2, \dots)$ is possibly finite, such that:*

(a) $Te_n = \lambda_n e_n$ *for some* $\lambda_n \in \mathbb{R}$, *and* $Tf_j = 0$.

(b) $\lambda_1 \geqslant \lambda_2 \geqslant \dots > 0$.

(c) If $(e_1, e_2, \dots)$ *is infinite, then* $\lim_n \lambda_n = 0$.

Before proving this theorem, we give an interpretation:

Remark 22.28. In Thm. 22.27, write $(e_1, e_2, \dots)$ as $(e_i)_{i\in I}$ where $I = \mathbb{Z}_+$ or $I = \{1, 2, \dots, N\}$ for some $N \in \mathbb{Z}_+$. Let $X = I \sqcup J$. Let $\Phi : l^2(X) \to \mathcal{H}$ be the unitary map sending each φ to $\sum_{x\in X} \varphi(x)\chi_{\{x\}}$. Then we have a commutative diagram

$$\begin{tikzcd}
l^2(X) \arrow[r, "\widehat{T}"] \arrow[d, "\Phi"', "\simeq"] & l^2(X) \arrow[d, "\simeq"', "\Phi"] \\
\mathcal{H} \arrow[r, "T"] & \mathcal{H}
\end{tikzcd} \tag{22.26}$$

where $\widehat{T}$ has matrix representation

$$\mathrm{diag}(\lambda_1, \lambda_2, \dots, (0)_{j\in J}) \tag{22.27}$$

i.e., for each $i \in I, j \in J$, we have $\widehat{T}\chi_{\{i\}} = \lambda_i \chi_{\{i\}}$ and $T\chi_{\{j\}} = 0$. A similar description holds in Thm. 22.29.

Thm. 22.27 will be proved by finding $e_1, e_2, \dots$ inductively. $(f_j)_{j\in J}$ will be an arbitrary orthonormal basis of $\mathrm{Ker}(T)$.

Proof of Thm. 22.27. We make our first simplification by noting that (b) can be weakened to

(b′) $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant 0$.

Then, by moving those of $e_1, e_2, \dots$ with 0-eigenvalue to the list $(f_j)_{j,\in J}$, the theorem is proved. Moreover, we assume for simplicity that $\mathcal{H}$ is infinite dimensional; the finite dimensional case will follow from a similar but easier proof.

Step 1. We first explain how to find λ_1, e_1. Let $\Omega = \overline{B}_{\mathcal{H}}(0, 1)$. Since T is completely continuous, the function $g : \Omega \to \mathbb{R}_{\geqslant 0}$ defined by $g(\xi) = \omega_T(\xi|\xi) = \langle T\xi|\xi\rangle$

is weakly continuous. By Cor. 21.27, Ω is weakly compact. Therefore, by the extreme value theorem (Lem. 8.9), g attains its maximum $\lambda_1 = g(e_1) \geqslant 0$ at some $e_1 \in \Omega$. Since g attains its minimum at 0, it suffices to assume $e_1 \neq 0$. Since $g(e_1/\|e_1\|) = \|e_1\|^{-2}g(e_1) \geqslant g(e_1)$, by replacing e_1 by $e_1/\|e_1\|$, we assume $\|e_1\| = 1$.

Since $\langle T\xi|\xi\rangle \leqslant \lambda_1$ for any unit vector $\xi \in \mathcal{H}$, by (sesqui)linearity, we get

$$\langle T\xi|\xi\rangle \leqslant \lambda_1\langle\xi|\xi\rangle \qquad \text{for all } \xi \in \mathcal{H} \tag{22.28}$$

This proves $0 \leqslant T \leqslant \lambda_1$.

Since $\langle(\lambda_1 - T)e_1|e_1\rangle = 0$ and $\lambda_1 - T \geqslant 0$, by Lem. 22.23, we get $(\lambda_1 - T)e_1 = 0$. This finishes the construction of λ_1 and e_1.

Step 2. Suppose that we have found $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$ and orthonormal $e_1, \dots, e_n$ such that $Te_i = \lambda_i e_i$ for all $1 \leqslant i \leqslant n$, and that

$$\langle T\xi|\xi\rangle \leqslant \lambda_n\langle\xi|\xi\rangle \qquad \text{for all } \xi \in V_n^\perp \tag{22.29}$$

Here, $V_n = \mathrm{Span}\{e_1, \dots, e_n\}$, which is a finite-dimensional Hilbert subspace of $\mathcal{H}$.

Clearly V_n is T-invariant. Therefore, by Prop. 22.25, $V_n^\perp$ is T-invariant. (Here we have used $T = T^*$.) Clearly $T|_{V_n^\perp} \geqslant 0$. By the process in Step 1, there exists a unit vector $e_{n+1} \in V_n^\perp$ such that $Te_{n+1} = \lambda_{n+1}e_{n+1}$ for some $\lambda_{n+1} \in \overline{\mathbb{R}}_{\geqslant 0}$, and $\langle T\xi|\xi\rangle \leqslant \lambda_{n+1}\langle\xi|\xi\rangle$ for all $\xi \in V_n^\perp$. Since (22.29) holds for $\xi = e_{n+1}$, we have $0 \leqslant \lambda_{n+1} \leqslant \lambda_n$.

Step 3. By the inductive process in Step 2, we obtain an (infinite) orthonormal sequence $(e_n)_{n\in\mathbb{Z}_+}$ satisfying that $Te_n = \lambda_n e_n$, that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$, and that (22.29) holds for each n. We claim that $\lim_n \lambda_n = 0$. Suppose this is true. Let $\mathcal{K} = \overline{\mathrm{Span}\{e_1, e_2, \dots\}}$. By (22.29), for any vector $\xi \in \mathcal{K}^\perp$ we have $\langle T\xi|\xi\rangle = 0$, and hence $T\xi = 0$ by Lem. 22.23. [5] So $T|_{\mathcal{K}^\perp} = 0$. Thus, the proof is finished by choosing (f_j) to be an orthonormal basis of $\mathcal{K}^\perp$: By Thm. 21.11, an orthonormal basis of $\mathcal{H}$ can be obtained by taking the union of one of $\mathcal{K}$ and one of $\mathcal{K}^\perp$.

Let us prove the claim. Let $\lambda = \lim_n \lambda_n = \inf_n \lambda_n$. So $\lambda = \lim_n\langle Te_n|e_n\rangle$. Since $\lim_n e_n$ converges weakly to 0 (cf. Exp. 21.31), and since T is completely continuous, we have $\lim_n\langle Te_n|e_n\rangle = \langle T0|0\rangle = 0$. So $\lambda = 0$. □

Thm. 22.27 can be easily generalized to self-adjoint completely continuous operators by slightly weakening condition (b):

Theorem 22.29 (Hilbert-Schmidt theorem). *Assume that $T \in \mathfrak{L}(\mathcal{H})$ is self-adjoint and completely continuous. Then $\mathcal{H}$ has an orthonormal basis $(e_1, e_2, \dots) \cup (f_j)_{j\in J}$, where the countable family $(e_1, e_2, \dots)$ is possibly finite, such that:*

(a) $Te_n = \lambda_n e_n$ for some $\lambda_n \in \mathbb{R}$, and $Tf_j = 0$.

[5] Since $\mathcal{K}^\perp$ is T-invariant, one can also use the polarization identity (20.3) (applied to $T|_{\mathcal{K}^\perp} : \mathcal{K}^\perp \to \mathcal{K}^\perp$) instead of Lem. 22.23 to conclude $T\xi = 0$.

(b) $|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots > 0$.

(c) If $(e_1, e_2, \dots)$ *is infinite, then* $\lim_n \lambda_n = 0$.

Proof. As in the proof of Thm. 22.27, (b) can be weakened to $|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant 0$. One then produces orthonormal $e_1, e_1', e_2, e_2', \dots$ such that $Te_n = \mu_n e_n$ and $Te_k' = \mu_k' e_k'$ for each n, k, that $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant 0$ and $\mu_1' \leqslant \mu_2' \leqslant \cdots \leqslant 0$, and that

$$\begin{aligned} \langle T\xi|\xi\rangle \leqslant \mu_n\langle\xi|\xi\rangle \qquad & \text{if } \xi \perp \{e_1, \dots, e_n\} \\ \mu_k'\langle\xi|\xi\rangle \leqslant \langle T\xi|\xi\rangle \qquad & \text{if } \xi \perp \{e_1', \dots, e_k'\} \end{aligned}$$

(To see this, one first finds e_1 as in the proof of Thm. 22.27. Restricting $-T$ to $\{e_1\}^\perp$, one finds e_1'. Restricting T to $\{e_1, e_1'\}^\perp$, one finds e_2. Restricting $-T$ to $\{e_1, e_1', e_2\}^\perp$, one finds e_2'. Repeat this procedure.) □

Remark 22.30. The converse of Thm. 22.29 is also true: If $T \in \mathfrak{L}(\mathcal{H})$ has an orthonormal basis $(e_1, e_2, \dots) \cup (f_j : j \in J)$ satisfying the description in Thm. 22.29, then T is self-adjoint and completely continuous. In fact, by Rem. 22.28, it suffices to prove:

Exercise 22.31. Let $I = \mathbb{Z}_+$ or $\{1, \dots, N\}$. Let J be a set. Let $X = I \sqcup J$. Choose $(\lambda_i)_{i \in I}$ in $\mathbb{C}\backslash\{0\}$ satisfying $\lim_i \lambda_i = 0$ if $I = \mathbb{Z}_+$. Prove that there is a (necessarily unique) completely continuous $T \in \mathfrak{L}(l^2(X))$ satisfying $T\chi_{\{i\}} = \lambda_i \chi_{\{i\}}$ for all $i \in I$, and $T\chi_{\{j\}} = 0$ for all $j \in J$. Prove that $T = T^*$ iff $\lambda_i \in \mathbb{R}$ for all $i \in I$.

Hint. Define T to be the limit (under the operator norm) of a sequence of finite-rank operators. Then the complete continuity follows from Thm. 22.12. □

Remark 22.32. Not all completely continuous operators are Hilbert-Schmidt: According to Exe. 22.31, we have a completely continuous operator on $l^2(\mathbb{Z}_+)$ whose matrix representation is $\mathrm{diag}(1, 1/\sqrt{2}, 1/\sqrt{3}, \dots)$. It is not Hilbert-Schmidt, because $\sum_n n^{-1} = +\infty$.

Corollary 22.33 (Fredholm alternative). *Let* $T \in \mathfrak{L}(\mathcal{H})$ *be self-adjoint and completely continuous. Let* $\lambda \in \mathbb{R}\backslash\{0\}$. *Then one of the following two, and only one of them, is true:*

(1) λ *is an eigenvalue of* T*, i.e.,* $T\xi = \lambda\xi$ *for some nonzero* $\xi \in \mathcal{H}$.

(2) $\lambda - T$ *is surjective.*

In other words, $\lambda - T$ is surjective iff λ is not an eigenvalue.

Proof. Assume for simplicity that $\mathcal{H}$ is infinite dimensional. By Hilbert-Schmidt Thm. 22.29, we may assume that $\mathcal{H} = l^2(X)$ where $X = \mathbb{Z}_+ \sqcup J$ and J is a set. We assume that there is a sequence $(\lambda_n)_{n \in \mathbb{Z}_+}$ in $\mathbb{R}$ such that $|\lambda_n|$ decreases to 0, and that T has matrix representation $\mathrm{diag}(\lambda_1, \lambda_2, \dots, (0)_{j \in J})$.

If $\lambda = \lambda_n$ for some n, then clearly $\chi_{\{n\}}$ is not in the range of $\lambda - T$. So $\lambda - T$ is not surjective. Conversely, suppose that $\lambda \neq \lambda_n$ for all n. For each $\eta \in l^2(X)$, let $\xi : X \to \mathbb{R}$ be defined by $\xi(n) = (\lambda - \lambda_n)^{-1}\eta(n)$ if $n \in \mathbb{Z}_+$, and $\xi(j) = \lambda^{-1}\eta(j)$ if $j \in J$. Then clearly $\xi \in l^2(X)$, and $(\lambda - T)\xi = \eta$. □

22.6 Concluding remarks

22.6.1 On the proof of the Hilbert-Schmidt theorem

The proof of the Hilbert-Schmidt theorem in the last section, despite written in the modern language, is very close to the proof in Hilbert's 1906 paper, the fourth part of his work [Hil12]. (Hilbert's proof is located in p.148-150 of [Hil12].)

The two key properties in Ch. 21 are the only analytic properties[6] used in the proof of the Hilbert-Schmidt theorem. The key property 2, the weak compactness of $\overline{B}_{\mathcal{H}}(0,1)$, is the most crucial one. In the proof of Thm. 21.11, we have used this property to find the unit vector e_1 maximizing the function $g(\xi) = \langle T\xi|\xi\rangle$ defined on $\overline{B}_{\mathcal{H}}(0,1)$. This method is heavily influenced by the method of variation, and is compatible with Hilbert's sesquilinear form viewpoint (rather than Riesz's operator viewpoint). The subsequent vectors $e_2, e_3, \dots$ are constructed in the same way by restricting T to orthogonal complements of finite dimensional subspaces. Hilbert used exactly the same method in his work! The weak compactness of $\overline{B}_{l^2(\mathbb{Z})}(0,1)$ is the single most important reason that Hilbert spaces were introduced in history.

The key property 1, the convergence of summing orthogonal vectors, is used to ensure the existence of the orthogonal decompositions (cf. the proof of Thm. 21.11). In the proof of Thm. 22.27, this property is used (and only used) to show that $e_1, e_2, \dots$ and an orthonormal basis $(f_j)_{j\in J}$ of $\mathcal{K}^\perp = \{e_1, e_2, \dots\}^\perp$ form an orthonormal basis of $\mathcal{H}$. (Let us quickly recall the key point: It is obvious that $e_1, e_2, \dots$ and (f_j) form an orthonormal family. To show that they are densely-spanning, for each $\xi \in \mathcal{H}$, one sets $\eta = \sum_n \langle\xi|e_n\rangle e_n$, which converges by key property 1. Then $\xi - \eta \in \mathcal{K}^\perp$. Apply Thm. 20.35 to $\xi - \eta$. Then we have $\xi = \eta + \sum_j \langle\xi - \eta|f_j\rangle f_j$. So ξ is approximated by linear combinations of $e_1, e_2, \dots$ and (f_j).)

Indeed, Hilbert did not use key property 1 and orthogonal decomposition in his proof. He did not need the vectors $(f_j)_{j\in J}$ since his "Hilbert-Schmidt" theorem is stated in the following form (cf. [Hil12, Satz 35]): There exist orthonormal vectors $e_1, e_2, \dots$ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$ satisfying $\lim_n \lambda_n = 0$ and

$$\omega_T(\xi|\xi) = \sum_n \lambda_n \langle\xi|e_n\rangle\langle e_n|\xi\rangle \tag{22.30}$$

for all vectors ξ. That each e_n is a λ_n-eigenvector of T is a mere consequence of this formula.

22.6.2 On the applicability of the Hilbert-Schmidt theorem

The Hilbert-Schmidt theorem lies at the heart of modern partial differential equations. As mentioned in Subsec. 22.1.2, the inverse of $-\Delta$ (with boundary con-

[6]Here, by "analytic property" I mean any property about inner product spaces that is equivalent to completeness, such as those described in Thm. 21.5.

dition $u|_{\partial\Omega} = 0$) is a completely continuous *positive* operator on $L^2(\overline{\Omega})$. Therefore, by the Hilbert-Schmidt theorem, the unbounded operator $-\Delta$ have eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ converging to $+\infty$, and the eigenvectors (which can be proved to be "good enough" such as C^r or C^∞) form an orthonormal basis of $L^2(\overline{\Omega})$. With the spectral analysis of $-\Delta$, the Dirichlet problem (22.1) can be understood very well.

However, this modern theory was also developed with the help of some other theories that were not yet available at the time of Hilbert-Schmidt: distributions and Sobolev spaces, unbounded closed operators, etc.. As we have mentioned in Sec. 22.1, in the early days, the Dirichlet problem was studied in terms of their associated integral equations of functions on $\partial\Omega$. However, the Hilbert-Schmidt theorem has very limited application to these integral equations. Let me explain this in the following.

We have mentioned that the operator T in the integral equation (22.11), after discretization by taking Fourier series, gives a bounded linear operator $\widehat{T}$ on $l^2(\mathbb{Z})$ whose matrix representation $\widehat{K} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ is l^2-finite, i.e. $\sum_{m,n} |\widehat{K}(m,n)|^2 < +\infty$. (See Sec. 22.2.) Therefore, $\widehat{T}$ is a Hilbert-Schmidt operator, and hence is completely continuous by Thm. 22.15. Moreover, some elementary calculations show that -1 is not an eigenvalue of T, cf. [Sim-O, Thm. 3.3.9] or [RN, Sec. 81]. Therefore, if T is self-adjoint, the Fredholm alternative (Cor. 22.33) shows that for each $g \in L^2(\partial\Omega)$ there exists a (necessarily unique) $f \in L^2(\partial\Omega)$ satisfying (22.11). More precisely, one finds the Fourier series $\widehat{f} \in l^2(\mathbb{Z})$ solving $\widehat{f} + \widehat{T}\widehat{f} = \widehat{g}$.

Unfortunately, in many cases T is not self-adjoint (since the real-valued function G in (22.6) does not satisfy $G(x,y) = G(y,x)$). Therefore, although the Hilbert-Schmidt theorem would later be proved to be an effective tool for studying the Dirichlet problem, at the time of its inception, its relevance to the Dirichlet problem was not significant. Hilbert restricted his study to self-adjoint operators (more precisely, Hermitian forms), perhaps in view of the many other uses of integral equations, such as the Sturm-Liouville problem. (See e.g. [Sim-O, Sec. 3.2], [Tes, Ch. 5], or [CL, Ch. 7] for a detailed discussion of the application of the Hilbert-Schmidt theorem to the integral equations in Sturm-Liouville problem.)

Assuming self-adjointness, the original problem about integral equation can be fully solved: Let $K \in C^r(\mathbb{S}^1 \times \mathbb{S}^1)$ where $r \geqslant 0$. Then we have a linear operator $T : C^r(\mathbb{S}^1) \to C^r(\mathbb{S}^1)$ defined by $(Tf)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K(x,y)f(y)dy$. Assume that K is **symmetric**, i.e. $K(x,y) = \overline{K(y,x)}$. In the following, we prove the C^r-Fredholm alternative: If there is no non-zero $f \in C^r(\mathbb{S}^1)$ satisfying $f + Tf = 0$, then for each $g \in C^r(\mathbb{S}^1)$ there exists $f \in C^r(\mathbb{S}^1)$ such that $f + Tf = g$.

22.6.3 ⋆⋆ Regularity of the solutions of integral equations

Define $\widehat{K}$ by (22.14), which is an element of $l^2(\mathbb{Z}\times\mathbb{Z})$. Equip $C^r(\mathbb{S}^1)$ with the inner product $\langle f|g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} fg^*$. Then we have a commutative diagram

$$\begin{array}{ccc} C^r(\mathbb{S}^1) & \xrightarrow{\Psi} & l^2(\mathbb{Z}) \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{T}} \\ C^r(\mathbb{S}^1) & \xrightarrow{\Psi} & l^2(\mathbb{Z}) \end{array} \tag{22.31}$$

where Ψ is the map $f\mapsto \widehat{f}$ (which is a linear isometry with dense range, cf. Cor. 20.42), and $\widehat{T}$ is defined by (22.15). Then $K(x,y)=\overline{K(y,x)}$ implies that $\widehat{K}(m,n)=\overline{\widehat{K}(n,m)}$, and hence that $\widehat{T}$ is self-adjoint (cf. Exp. 22.20). Briefly speaking, (22.31) asserts $\widehat{Tf}=\widehat{T}\widehat{f}$ where $f\in C^r(\mathbb{S}^1)$.

It can be proved that

$$\widehat{T}\big(l^2(\mathbb{Z})\big)\subset\Psi\big(C^r(\mathbb{S}^1)\big) \tag{22.32}$$

To see this, choose $\varphi\in l^2(\mathbb{Z})$, and let $f_n=\sum_{k=-n}^{n}\varphi(k)e_k$. It is not hard to check (using Hölder's inequality) that the linear map $T:C^r(\mathbb{S}^1)\to C(\mathbb{S}^1)$ is bounded if the source and the target are equipped respectively with the L^2-norm and the l^∞-norm. Thus $\lim_n Tf_n$ converges uniformly on $\mathbb{S}^1$ to some $g\in C(\mathbb{S}^1)$. By a similar argument, one checks that $\lim_n(Tf_n)^{(k)}$ converges uniformly for all $k\leqslant r$. Thus, by Thm. 11.33, we see that $g\in C^r(\mathbb{S}^1)$. Using Parseval's identity for continuous functions on $\mathbb{S}^1\times\mathbb{S}^1$, one checks that $\widehat{g}=\widehat{T}\varphi$, i.e., $\Psi(g)=\widehat{T}\varphi$. This proves (22.32).

We now show that if $\lambda\in\mathbb{C}\backslash\{0\}$, a λ-eigenvector φ of $\widehat{T}$ corresponds to a λ-eigenvector $f\in C^r(\mathbb{S}^1)$ of T satisfying $\widehat{f}=\varphi$. Proof: Let $\varphi\in l^2(\mathbb{Z})$ and $\widehat{T}\varphi=\lambda\varphi$. By (22.32), there exists $f\in C^r(\mathbb{S}^1)$ such that $\widehat{T}\varphi=\lambda\widehat{f}$. So $\widehat{f}=\varphi$, and hence $\widehat{Tf}=\widehat{T}\widehat{f}=\widehat{T}\varphi=\lambda\varphi=\lambda\widehat{f}$, and hence $Tf=\lambda f$ (since Ψ is injective). QED.

We now finish the proof. Suppose that there is no non-zero $f\in C^r(\mathbb{S}^1)$ satisfying $f+Tf=0$. By the above paragraph, there is no non-zero $\varphi\in l^2(\mathbb{Z})$ satisfying $\varphi+\widehat{T}\varphi=0$. Choose any $g\in C^r(\mathbb{S}^1)$. By Fredholm alternative (Cor. 22.33), there exists $\varphi\in l^2(\mathbb{Z})$ such that $\varphi+\widehat{T}\varphi=\widehat{g}$. By (22.32), there is $f\in C^r(\mathbb{S}^1)$ such that $\widehat{T}\varphi=\widehat{g-f}=\widehat{g}-\widehat{f}$. Thus $\widehat{f}=\varphi$, and hence $\widehat{f}+\widehat{Tf}=\widehat{f}+\widehat{T}\widehat{f}=\widehat{f}+\widehat{T}\varphi=\widehat{g}$. So $f+Tf=g$.

22.6.4 Toward measure theory

We have seen that the problem of finding an C^r-solution f of an integral equation $f+Tf=g$ is solved by first finding its Fourier series $\widehat{f}$, which is an element of $l^2(\mathbb{Z})$ solving the equation $\widehat{f}+\widehat{T}\widehat{f}=\widehat{g}$. To solve the later equation, one uses only the fact that $\widehat{K} \in l^2(\mathbb{Z} \times \mathbb{Z})$, or more precisely, that $\widehat{K}$ is the matrix representation of a completely continuous (self-adjoint) operator on $l^2(\mathbb{Z})$.

The lesson here is that even if we are ultimately interested in the Fourier series of functions in $C^r(\mathbb{S}^1)$, i.e., the elements of $\Psi(C^r(\mathbb{S}^1))$, at the outset, we must consider all possible elements of $l^2(\mathbb{Z})$: If you remember, the proof of the Hilbert-Schmidt theorem uses the weak-compactness of the closed unit ball of $l^2(\mathbb{Z})$. However, $\Psi(C(\mathbb{S}^1))$ is a dense proper subspace of $l^2(\mathbb{Z})$, and its closed unit ball is therefore not weakly compact (by Thm. 21.35).

This lesson naturally leads to the question: What is the function-theoretic meaning of an element $\varphi \in l^2(\mathbb{Z})$ that is not necessarily the Fourier series of a continuous (or even Riemann-integrable) 2π-periodic function?

Lebesgue's integral theory was invented in 1902. And it was F. Riesz and Fischer who realized that Lebesgue integral gives a satisfying answer to the above question. They showed in 1907 (a year after Hilbert's seminal work [Hil12] introducing the l^2-method to integral equations) that the linear isometry $\Psi : C(\mathbb{S}^1) \to l^2(\mathbb{Z})$ can be extended to a unitary map

$$\Psi : L^2(\mathbb{S}^1) \to l^2(\mathbb{Z})$$

where $L^2(\mathbb{S}^1)$ is the space of 2π-periodic Lebesgue measurable complex functions f satisfying $\int_{-\pi}^{\pi} |f|^2 < +\infty$, and

$$(\Psi f)(n) = \widehat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f e_{-n}$$

where $e_n(x) = e^{\mathbf{i}nx}$, and the integral on the RHS is the Lebesgue integral. This result is called the **Riesz-Fischer theorem**, as already mentioned in Sec. 10.4.

With this result, the proof of (22.32) becomes more straightforward: Let $\varphi \in l^2(\mathbb{Z})$. By Riesz-Fischer, $\varphi = \widehat{f}$ for some $f \in L^2(\mathbb{S}^1)$. Using the commutativity of limits and integrals (under certain assumptions) one sees that $g \in C^r(\mathbb{S}^1)$ if g is defined by the Lebesgue integral $g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K(x,y)f(y)$. Parseval's identity implies $\widehat{g} = \widehat{T}\widehat{f}$.

It will be our task in the next few chapters to study Lebesgue integrals, the Riesz-Fischer theorem, and some of their generalizations.

22.7 ⋆ Epilogue: Riesz's compact operators

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. Recall (22.18) for the meaning of ω_T.

In the previous section, we have mentioned that many integral equations do not have self-adjoint integral operators. When a completely continuous operator T on a $\mathcal{H}$ is not self-adjoint, the Hilbert-Schmidt theorem does not hold. However, one can still expect that the Fredholm alternative is true. In fact, in Fredholm's 1900 paper there was no "self-adjointness". However, Fredholm's original approach is quite complicated: Assuming that the function $K(x,y)$ in the integral operator $T = \int_{-\pi}^{\pi} K(x,y)f(y)dy$ is continuous, Fredholm used Riemann sums to approximate the Riemann integral, studied the determinants of these summation operators, and then passed to the integral operator T by taking limit. (See [Die-H, Sec. 5.1] for a brief discussion. See [Lax, Ch. 24] and [Sim-O, Sec. 3.11] for a detailed account of Fredholm's approach in a modern language.)

Fredholm clearly did not have the idea of Hilbert spaces and completely continuous operators. Now, to study completely continuous operators that are not necessarily self-adjoint, one needs to give an equivalent but different characterization of complete continuity.

Hilbert's original definition of complete continuity (cf. Def. 22.3) is essentially about *sesquilinear forms*, not about *linear operators*. Recall that T is completely continuous iff $\omega_T : \overline{B}_{\mathcal{H}}(0,1) \times \overline{B}_{\mathcal{K}}(0,1) \to \mathbb{C}$ is weakly continuous. To describe this property in terms of the linear map T, the weak continuity on the second variable $\overline{B}_{\mathcal{K}}(0,1)$, which is a property about the source, should be raised to the target. This is achieved with the help of the following variant of Prop. 21.30.

Lemma 22.34. *Let $(\xi_\alpha)_{\alpha\in I}$ be a net in $\overline{B}_{\mathcal{H}}(0,1)$, and let $\xi \in \overline{B}_{\mathcal{H}}(0,1)$. Then the following are equivalent.*

(1) $\lim_\alpha \xi_\alpha = \xi$.

(2) *(ξ_α) converges weakly to ξ. Moreover, for every net $(\eta_\beta)_{\beta\in J}$ in $\overline{B}_{\mathcal{H}}(0,1)$ converging weakly to $\eta \in \overline{B}_{\mathcal{H}}(0,1)$ we have*

$$\lim_{\alpha\in I,\beta\in J} \langle \xi_\alpha | \eta_\beta \rangle = \langle \xi | \eta \rangle \tag{22.33}$$

Similar to Prop. 21.30, this lemma is also true when $\mathcal{H}$ is only an inner product space, as you can see from the following proof.

Proof. Assume (1). Let (η_β) converge weakly in $\overline{B}_{\mathcal{H}}(0,1)$ to η. For each $\beta \in J$, define a continuous function $g_\beta : \mathcal{H} \to \mathbb{C}$ by $g_\beta(\psi) = \langle \psi | \eta_\beta \rangle$. Then g_β converges pointwise to $g : \psi \in \mathcal{H} \mapsto \langle \psi | \eta \rangle \in \mathbb{C}$. Since $\sup_\beta \lVert \eta_\beta \rVert < +\infty$, $(g_\beta)_{\beta\in J}$ is equicontinuous. Therefore, by Thm. 9.12 (and noting Prop. 9.16), we get $\lim_{\alpha,\beta} g_\beta(\xi_\alpha) = g(\xi)$. This proves (2).[7]

Assume (2). Then (1) follows directly from Prop. 21.30 by choosing $J = I$ and $\eta_\alpha = \xi_\alpha$ for each $\alpha \in I$. □

Proposition 22.35. *Let $T \in \mathfrak{L}(\mathcal{H},\mathcal{K})$. The following are equivalent.*

(1) *T is completely continuous, i.e., ω_T is weakly continuous on $\overline{B}_{\mathcal{H}}(0,1) \times \overline{B}_{\mathcal{K}}(0,1)$.*

[7] It is not hard to prove (1)⇒(2) directly. We leave such a direct proof to the readers as an exercise.

(2) $T : \overline{B}_{\mathcal{H}}(0,1) \to \mathcal{K}$ *is continuous if* $\overline{B}_{\mathcal{H}}(0,1)$ *is equipped with the weak topology, and if* $\mathcal{K}$ *is equipped with the norm topology.*

Proof. (2) means that if (x_α) is a net converging weakly in $\overline{B}_{\mathcal{H}}(0,1)$ to ξ, then $T\xi_\alpha$ converges in norm to $T\xi$. By Lem. 22.34 (applied to $\|T\|^{-1}T\xi_\alpha$), we see that $\lim_\alpha T\xi_\alpha = T\xi$ is equivalent to that $\lim_{\alpha,\beta}\langle T\xi_\alpha|\eta_\beta\rangle = \langle T\xi|\eta\rangle$ for every net $(\eta_\beta)_{\beta\in J}$ converging weakly in $\overline{B}_{\mathcal{K}}(0,1)$ to η. This proves the equivalence. □

Definition 22.36. A linear map of normed vector spaces $T : V \to W$ is called **completely continuous** if T is bounded, and if the restriction $T : \overline{B}_V(0,1) \to W$ is continuous, where $\overline{B}_V(0,1)$ is given the weak topology, and W is given the norm topology.

Def. 22.36 was introduced by F. Riesz in 1910 and was later applied by him to l^p spaces. However, this is not a good definition for $C[0,1]$ because, unlike $l^p(X)$ (where $1 < p < +\infty$), $C[0,1]$ is not reflexive and (hence) its closed unit ball is not weakly compact (cf. Thm. 17.54). In order to study integral equations on non-necessarily reflexive function spaces such as $C[0,1]$, in 1918, Riesz introduced the modern definition of compact operators (which he still called completely continuous operators). Let us recall the definition:

Definition 22.37. Let $T : V \to W$ be a linear map of normed vector spaces. We say that T is a **compact operator** if $T(\overline{B}_V(0,1))$ is precompact in W (under the norm topology).

Note that if T is compact, then T is bounded. This is because every compact set in a metric space, in particular $\overline{T(\overline{B}_V(0,1))}$, is bounded.

Theorem 22.38. *Let* $T : \mathcal{H} \to \mathcal{K}$ *be linear. Then* T *is completely continuous iff* T *is compact.*

The following proof shows that this theorem is also true when $\mathcal{H},\mathcal{K}$ are only normed vector spaces and $\overline{B}_{\mathcal{H}}(0,1)$ is weakly compact. (By Thm. 17.54, the latter condition is equivalent to that $\mathcal{H}$ is reflexive.)

Proof. Suppose that T is completely continuous. Since $\overline{B}_{\mathcal{H}}(0,1)$ is weakly compact, and since the range of a compact set under a continuous map is compact, we conclude that $T(\overline{B}_{\mathcal{H}}(0,1))$ is norm-compact, and hence norm-precompact.[8]

Conversely, assume that T is compact. Let (ξ_α) be a net in $\overline{B}_{\mathcal{H}}(0,1)$ converging weakly to $\xi \in \overline{B}_{\mathcal{H}}(0,1)$. Since T is compact, $\Gamma = \overline{T(\overline{B}_{\mathcal{H}}(0,1))}$ is (norm-)compact. Therefore, to show that $\lim_\alpha T\xi_\alpha = T\xi$, by Pb. 8.1, it suffices to prove that every cluster point of $(T\xi_\alpha)$ is $T\xi$. This is clear, since every cluster point of $(T\xi_\alpha)$ is a weak cluster point, which must be $T\xi$ by Pb. 22.1. □

[8]Note that we have actually proved that $T(\overline{B}_{\mathcal{H}}(0,1))$ is compact rather than just precompact. In fact, for Hilbert spaces, the precompactness of $T(\overline{B}_{\mathcal{H}}(0,1))$ implies the compactness. However, for a general non-reflexive Banach space, it is too restrictive to assume compactness.

Def. 22.37 is the correct definition which allowed Riesz to prove the Fredholm alternative for compact operators on Banach spaces (including integral operators on $C[0,1]$) in 1918. A discussion of this history, as well as the main ideas in Riesz's proof of Fredholm alternative, can be found in [Die-H, Sec. 7.1]. Riesz's theory on compact operators can be found in many textbooks on functional analysis, such as [Lax, Ch. 21], [Rud-F, Ch. 4], [Sim-O, Sec. 3.3].

22.8 Problems and supplementary material

Let $\mathcal{H},\mathcal{K}$ be Hilbert spaces. Recall (22.18) for the meaning of ω_T.

Problem 22.1. Let $T \in \mathfrak{L}(\mathcal{H},\mathcal{K})$. Prove that T is continuous if both $\mathcal{H}$ and $\mathcal{K}$ are equipped with their weak topologies.

$\star$ **Problem 22.2.** Let (e_i) and (f_j) be orthonormal bases of $\mathcal{H},\mathcal{K}$ respectively. Let $T \in \mathfrak{L}(\mathcal{H},\mathcal{K})$. Define

$$\|T\|_{\mathrm{HS}} = \Big(\sum_{i,j} |\langle Te_i|f_j\rangle|^2\Big)^{\frac{1}{2}} \tag{22.34}$$

which is in $[0,+\infty]$. Prove that $\|T\|_{\mathrm{HS}}$ is independent of the choice of bases. We say that T is a **Hilbert-Schmidt operator** if $\|T\|_{\mathrm{HS}} < +\infty$.

Problem 22.3. Let $T \in \mathfrak{L}(\mathcal{H})$ be positive. Prove that $\|T\| \leqslant 1$ iff $0 \leqslant T \leqslant 1$.

Hint. To prove "$\Leftarrow$", apply Cauchy-Schwarz to ω_T. □

$\star$ **Problem 22.4.** Let $T \in \mathfrak{L}(\mathcal{H})$ satisfy $0 \leqslant T \leqslant 1$. Prove that $T^2 \leqslant T$.

Hint. Use Cauchy-Schwarz to give an upper bound of $|\omega_T(\xi|T\xi)|$. □

$\star$ **Problem 22.5.** Let $(e_n)_{n\in\mathbb{Z}_+}$ be an orthonormal sequence in $\mathcal{H}$. Let $T:\mathcal{H}\to\mathcal{K}$ be completely continuous. Prove that $\lim_{n\to\infty} Te_n = 0$.

Hint. By Pb. 22.1, $\lim_n Te_n$ converges weakly to 0. □

$\star$ **Problem 22.6.** Assume that $\mathcal{H}$ is infinite dimensional. Let $T \in \mathfrak{L}(\mathcal{H},\mathcal{K})$. Assume that for each orthonormal sequence $(e_n)_{n\in\mathbb{Z}_+}$ in $\mathcal{H}$ we have $\lim_{n\to\infty} Te_n = 0$.

1. By mimicking the proof of the Hilbert-Schmidt Thm. 22.27, find a sequence of orthonormal vectors (e_n) in $\mathcal{H}$ such that, by setting $V_0 = \{0\}$ and $V_n = \mathrm{Span}\{e_1,\dots,e_n\}$ (where $n>0$), we have
$$\|Te_{n+1}\| \geqslant \frac{1}{2}\|T|_{V_n^\perp}\|$$
for each $n\in\mathbb{N}$, where $\|T|_{V_n^\perp}\|$ is the operator norm of $T|_{V_n^\perp}: V_n^\perp \to \mathcal{K}$.

2. Let $P_n \in \mathfrak{L}(\mathcal{H})$ be the projection operator onto V_n (cf. Pb. 21.2). Prove that $\lim_n \|T - TP_n\| = 0$. Conclude that T is approximable.

Remark 22.39. Pb. 22.5 and 22.6 give an alternative proof that any completely continuous operator $T:\mathcal{H}\to\mathcal{K}$ is approximable.

23 Measure spaces

23.1 Introduction

Starting from this chapter, we will study Lebesgue integrals, and study measure theory in general. We will be able to define $\int_{\mathbb{R}^N} f$ for a large class of functions $f : \mathbb{R}^N \to \mathbb{R}$ called **Lebesgue measurable functions**. It is worth noting that when Lebesgue introduced his integral theory in 1902, he was primarily concerned with bounded measurable functions on a compact interval $[a, b] \subset \mathbb{R}$. For example, the dominated convergence theorem, one of the most important theorems in measure theory, was first proved by Lebesgue for bounded measurable functions on $[a, b]$.

Only by knowing what were the earliest primary examples in history, and only by knowing how the theory can be developed and applied to these crucial examples, can one comprehend the essence of the theory. Therefore, I will sometimes give a more straightforward proof of an important special case of a theorem after proving this theorem in full generality.

In the following, we sketch Lebesgue's main idea of the construction of integrals. A detailed account of the history can be found in [Haw, Sec. 5.1] and [Jah, Sec. 9.6].

Let $-\infty < a < b < +\infty$, and let $f : (a, b) \to \mathbb{R}$ be bounded, say $-M + 1 \leqslant f \leqslant M - 1$ for some $M > 1$.[1] We know that Riemann integrals are defined by partitioning the domain (a, b). By contrast, the Lebesgue integra $\int_a^b f dm$ is defined by partitioning the codomain: Let $\{c_0 < c_1 < \cdots < c_n\}$ be a partition of $[-M, M]$. Let $\xi_i \in (c_{i-1}, c_i]$. Then it is expected that $\int_a^b f dm$ is approximated by the **Lebesgue sum**

$$\sum_{i=1}^{n} \xi_i \cdot m(E_i) \tag{23.1}$$

where $E_i = f^{-1}(c_{i-1}, c_i]$, and $m(E_i)$ is the "measure" of E_i. Thus, one can attempt to define $\int_a^b f dm$ to be the limit of the "Lebesgue sum" (23.1). In fact, since $\int f$ should be the area of the region below the graph of f, it is expect that if $\sup_i |c_i - c_{i-1}| < \varepsilon$, then

$$\Big|\int_a^b f dm - \sum_{i=1}^{n} \xi_i \cdot m(E_i)\Big| \leqslant (b - a)\varepsilon$$

Therefore, in order to define the Lebesgue integral, one must first define the **Lebesgue measure** $m(E_i)$. Let me temporarily denote this value by $m^*(E_i)$ to reflect the fact (to be explained shortly) that not every subset of (a, b) can be assigned

[1]Lebesgue originally considered functions on $[a, b]$. For the simplicity of the following discussion, we consider functions on (a, b) instead, which makes no essential difference to the development of the theory.

a measure. The construction is as follows. Suppose that $E \subset (a,b)$. Then

$$m^*(E) = \inf\Big\{\sum_{n\in\mathbb{Z}_+} |I_n| : I_1, I_2, \dots \text{ are intervals covering } E\Big\} \tag{23.2}$$

is called the **outer Lebsgue measure** of E. Clearly $m^*(E) \leqslant m^*(F)$ if $E \subset F$.

Unfortunately, it is not necessarily true that $m^*(E \cup F) = m^*(E) + m^*(F)$ if $F \subset (a,b)$ is disjoint from E. Indeed, it is not necessarily true that $b - a = m^*(E) + m^*((a,b)\backslash E)$. Consequently, if we use the Lebesgue sum to define the integral, then it is not necessarily true that $\int_a^b 1 = \int_a^b \chi_E + \int_a^b \chi_{(a,b)\backslash E}$. In general, we only have

$$m^*(E \cup F) \leqslant m^*(E) + m^*(F) \tag{23.3}$$

if $E, F \in (a,b)$ are disjoint. Thus, we must focus on a class $\mathfrak{M}$ of subsets of (a,b) satisfying certain nice properties. The elements in $\mathfrak{M}$ will be called **Lebesgue measurable sets**.

The most remarkable property about $\mathfrak{M}$ is the **countable additivity**: If $E_1, E_2, \dots \in \mathfrak{M}$ are mutually disjoint, then $\bigcup_n E_n \in \mathfrak{M}$, and $m^*(\bigcup_n E_n) = \sum_n m^*(E_n)$. Then the Lebesgue integral will be defined for **Lebesgue measurable functions**, e.g., bounded functions $f : (a,b) \to \mathbb{R}$ satisfying $f^{-1}(c,d] \in \mathfrak{M}$ for all $c < d$. The powerful properties concerning $\lim_n \int f_n = \int \lim_n f_n$ will be proved as easy consequences of the countable additivity. Therefore, the construction of Lebesgue measure satisfying countable additivity is the most central and difficult part of the whole theory of Lebesgue's integrals.

23.2 Measurable spaces and measurable functions

Definition 23.1. Let X be a set. A subset $\mathfrak{M}$ of 2^X is called a σ**-algebra** if it satisfies the following conditions:

- $\varnothing \in \mathfrak{M}$.
- If $E \in \mathfrak{M}$ then $X\backslash E \in \mathfrak{M}$.
- If we have countably many elements $E_1, E_2, \dots \in \mathfrak{M}$, then $\bigcup_n E_n \in \mathfrak{M}$.

If $\mathfrak{M}$ is a σ-algebra, we say that $(X, \mathfrak{M})$ is a **measurable space**.

The second condition means that $\mathfrak{M}$ is closed under complements. The third condition means that $\mathfrak{M}$ is closed under *countable* unions. Since $(\bigcup_n E_n)^c = \bigcap_n E_n^c$, we see that a σ-algebra is also closed under countable intersections.

Remark 23.2. In Def. 23.1-(3) it suffices to assume that $\mathfrak{M}$ is closed under countably *infinite* unions. This is because any finite union could be enlarged to a countably infinite union by including $\varnothing$.

Exercise 23.3. If $(\mathfrak{M}_i)_{i\in I}$ is a family of σ-algebras on X, then $\bigcap_{i\in I}\mathfrak{M}_i$ is a σ-algebra on X.

Definition 23.4. Let $\mathfrak{E} \subset 2^X$. By Exe. 23.3,

$$\sigma(\mathfrak{E}) := \bigcap_{\mathfrak{M} \text{ is a } \sigma\text{-algebra containing } \mathfrak{E}} \mathfrak{M} \tag{23.4}$$

is a σ-algebra on X. We call $\sigma(\mathfrak{E})$ the **σ-algebra generated by** $\mathfrak{E}$. It is the smallest σ-algebra containing $\mathfrak{E}$.

The most important σ-algebras in this course are Borel σ-algebras:

Definition 23.5. Let $(X, \mathcal{T}_X)$ be a topological space. Recall that $\mathcal{T}_X$ is the set of open subsets of X. We let

$$\mathfrak{B}_X := \sigma(\mathcal{T}_X) \tag{23.5}$$

and call $\mathfrak{B}_X$ the **Borel σ-algebra** of X. Elements of $\mathfrak{B}_X$ are called **Borel sets**.

Example 23.6. Let X be a topological space. Every closed subset of X is Borel since it is the complement of an open set. $[a, b)$ is Borel set of $\mathbb{R}$ since it equals $(-\infty, b) \cap [a, +\infty)$

Definition 23.7. Let $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ be measurable sets. Let $f : X \to Y$ be a function. Then

$$f^{-1}(\mathfrak{N}) = \{f^{-1}(E) : E \in \mathfrak{N}\} \tag{23.6}$$

is clearly a σ-algebra on X. We say that f is **measurable** if $f^{-1}(\mathfrak{N}) \subset \mathfrak{M}$, i.e., if $f^{-1}(E) \in \mathfrak{M}$ for each $E \in \mathfrak{N}$.

Remark 23.8. If $f : X \to Y$ and $g : Y \to Z$ are measurable, then clearly $g \circ f : X \to Z$ is measurable.

Definition 23.9. Let $(X, \mathfrak{M})$ be a measurable space, and let $(Y, \mathcal{T}_Y)$ be a topological space. A map $f : X \to Y$ is called **measurable** if f is measurable as a map $(X, \mathfrak{M}) \to (Y, \mathfrak{B}_Y)$, i.e., $f^{-1}(E) \in \mathfrak{M}$ for each Borel set $E \subset Y$.

Remark 23.10. The most important measurable functions $X \to Y$, where Y is a topological space, are those such that $Y = \mathbb{R}, \overline{\mathbb{R}}_{\geqslant 0}, \mathbb{C}$. Note that $\mathbb{R}, \mathbb{C}$ are equipped with the Euclidean topologies, and $\overline{\mathbb{R}}_{\geqslant 0} = [0, +\infty]$ is equipped with its standard topology (cf. Exp. 7.17), i.e., the unique topology such that any increasing bijection $\overline{\mathbb{R}}_{\geqslant 0} \to [0, 1]$ is a homeomorphism.

Definition 23.11. Let $f : X \to Y$ be a map of topological spaces. We say that f is **Borel measurable** (or simply **Borel**) if f is measurable as a map of measurable spaces $f : (X, \mathfrak{B}_X) \to (Y, \mathfrak{B}_Y)$.

Checking that a map f is Borel using the original definition is difficult, since Borel sets of the codomain could be very complicated. In the following, we will see a very useful method of showing that a map is measurable.

Proposition 23.12. *Let $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ be measurable spaces where $\mathfrak{N} = \sigma(\mathfrak{E})$ for some $\mathfrak{E} \subset 2^Y$. Then the following are equivalent.*

(1) f is measurable, i.e., $f^{-1}(\sigma(\mathfrak{E})) \subset \mathfrak{M}$.

(2) $f^{-1}(\mathfrak{E}) \subset \mathfrak{M}$.

Proof. Clearly (1) implies (2). Assume (2). Then

$$\mathfrak{K} = \{E \in 2^Y : f^{-1}(E) \in \mathfrak{M}\}$$

is a σ-algebra containing $\mathfrak{E}$. So $\mathfrak{K}$ contains $\sigma(\mathfrak{E})$. Thus, for each $E \in \sigma(\mathfrak{E})$ we have $E \in \mathfrak{K}$, i.e., $f^{-1}(E) \in \mathfrak{M}$. This proves (1). □

Corollary 23.13. *Let $f : X \to Y$ be a map of sets. Let $\mathfrak{E} \subset 2^Y$. Then*

$$\sigma(f^{-1}(\mathfrak{E})) = f^{-1}(\sigma(\mathfrak{E})) \tag{23.7}$$

Proof. $f^{-1}(\sigma(\mathfrak{E}))$ is a σ-algebra on X, and it contains $f^{-1}(\mathfrak{E})$. Therefore $f^{-1}(\sigma(\mathfrak{E}))$ contain the smallest σ-algebra containing $f^{-1}(\mathfrak{E})$. This prove "$\subset$". Since $f^{-1}(\mathfrak{E})$ is contained in $\mathfrak{M} := \sigma(f^{-1}(\mathfrak{E}))$, by Prop. 23.12, $f^{-1}(\sigma(\mathfrak{E}))$ is contained in $\mathfrak{M}$. This proves "$\supset$". □

Corollary 23.14. *Let $(X, \mathfrak{M})$ be a measurable space. Let $(Y, \mathcal{T}_Y)$ be a topological space. Let $f : X \to Y$ be a map. The following are equivalent:*

(1) f is measurable, i.e., $f^{-1}(\mathfrak{B}_Y) \subset \mathfrak{M}$.

(2) $f^{-1}(\mathcal{T}_Y) \subset \mathfrak{M}$.

(3) $f^{-1}\{$closed subsets of $Y\} \subset \mathfrak{M}$.

Proof. This follows from Prop. 23.12 and the fact that $\mathfrak{B}_Y$ is generated by $\mathcal{T}_Y$ and also by the set of closed subsets of Y. □

Example 23.15. Every continuous map of topological spaces is Borel.

Proof. Immediate from Cor. 23.14. □

Checking $f^{-1}(\mathcal{T}_Y) \subset \mathfrak{M}$ is still not very easy. To make further simplification we observe:

Proposition 23.16. *Let $(Y, \mathcal{T}_Y)$ be a second countable topological space. Let $\mathcal{U}$ be a basis for the topology $\mathcal{T}_Y$. Then $\sigma(\mathcal{U}) = \mathfrak{B}_Y$. Consequently, a function $f : (X, \mathfrak{M}) \to Y$ is measurable iff $f^{-1}(\mathcal{U}) \subset \mathfrak{M}$.*

Proof. Since $\mathcal{U} \subset \mathcal{T}_Y$, we clearly have $\sigma(\mathcal{U}) \subset \mathfrak{B}_Y$. If we can prove that $\sigma(\mathcal{U}) \supset \mathcal{T}_Y$, then $\sigma(\mathcal{U}) \supset \sigma(\mathcal{T}_Y)$, finishing the proof.

Choose any open set $O \subset Y$. Then for each $y \in O$ there is a neighborhood $U_y \in \mathcal{U}$ of y contained inside O. Thus $O = \bigcup_{y \in O} U_y$. So U is a union of elements of $\mathcal{U}$. Since Y is second countable, the subset O is also second countable and hence (by Cor. 8.31) Lindelöf. Therefore, O is a countable union of elements of $\mathcal{U}$. This proves $O \in \sigma(\mathcal{U})$. □

Example 23.17. Let $(X, \mathfrak{M})$ be a measurable space. Let Y be a separable (equivalently, second countable) metric space. Then a map $f : X \to Y$ is measurable iff $f^{-1}(B_Y(y,r)) \in \mathfrak{M}$ for each $y \in Y, r > 0$. This is because the open balls of Y form a basis for the topology of Y.

Example 23.18. We have

$$\begin{aligned}\mathfrak{B}_{\overline{\mathbb{R}}} &= \sigma\{(a,+\infty] : a \in \overline{\mathbb{R}}\} = \sigma\{[a,+\infty] : a \in \overline{\mathbb{R}}\}\\ &=\sigma\{[-\infty,a) : a \in \overline{\mathbb{R}}\} = \sigma\{[-\infty,a] : a \in \overline{\mathbb{R}}\}\end{aligned}$$

Thus, for example, if $f : X \to \overline{\mathbb{R}}$ where X is a measurable space, then f is measurable iff $f^{-1}(a,+\infty]$ is measurable for all $a \in \overline{\mathbb{R}}$.

Proof. Let $\mathfrak{M} = \sigma\{(a,+\infty] : a \in \overline{\mathbb{R}}\}$, which is clearly a subset of $\mathfrak{B}_{\overline{\mathbb{R}}}$. Then $[a,+\infty] \in \mathfrak{M}$ since $[a,+\infty] = \bigcap_{n \in \mathbb{Z}_+}(a-1/n,+\infty]$. Taking complements and intersections, we see $[-\infty,b) \in \mathfrak{M}$ and $(a,b) \in \mathfrak{M}$. Therefore, $\mathfrak{M}$ contains a basis for $\mathcal{T}_{\overline{\mathbb{R}}}$, and hence contains $\mathfrak{B}_{\overline{\mathbb{R}}}$. This prove the first relation. The other relations can be proved in the same way. □

Remark 23.19. Let $f : (X, \mathfrak{M}) \to (Y, \mathcal{T}_Y)$. Suppose that Z is a subspace of Y containing $f(X)$. Equip Z with the subspace topology $\mathcal{T}_Z$. Then $f : X \to Y$ is measurable iff $f : X \to Z$ is measurable.

Proof. $f : X \to Y$ is measurable iff $f^{-1}(U)$ is measurable for each $U \in \mathcal{T}_Y$. But $f^{-1}(U)$ equals $f^{-1}(U \cap Z)$, and the open subsets of Z are precisely of the form $U \cap Z$. □

Example 23.20. Let $(X, \mathfrak{M})$ be a measurable space. Let $A \subset X$. Then $\chi_A : X \to \mathbb{R}$ is measurable iff $\chi_A : X \to \{0,1\}$ is measurable iff $\chi_A^{-1}(1) = A$ and $\chi_A^{-1}(0) = X \backslash A$ are measurable iff $A \in \mathfrak{M}$.

Proposition 23.21. *Let $(X, \mathfrak{M})$ be a measurable space. Let Y_1, Y_2 be second countable topological spaces. Let $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ be functions. Then $f_1 \vee f_2 : X \to Y_1 \times Y_2$ is measurable iff f_1 and f_2 are both measurable.*

Proof. Let $f = f_1 \vee f_2$. Let $\pi_i : Y_1 \times Y_2 \to Y_i$ be the projection. Then $f_i = \pi_i \circ f$. Thus, if f is measurable, since π_i is continuous and hence Borel measurable, we conclude that f_i is measurable (cf. Rem. 23.8).

Conversely, assume that f_1, f_2 are measurable. If V_1 and V_2 are open subsets of Y_1, Y_2 respectively, then $f_i^{-1}(V_i)$ belongs to $\mathfrak{M}$, and hence $f^{-1}(V_1 \times V_2) = f_1^{-1}(V_1) \cap f_2^{-1}(V_2)$ belongs to $\mathfrak{M}$. Since subsets of the form $V_1 \times V_2$ form a basis for the topology of $Y_1 \times Y_2$, by Prop. 23.16, f is measurable. $\square$

Corollary 23.22. *Let $(X, \mathfrak{M})$ be a measurable space. Let $f, g : X \to \mathbb{C}$ or $f, g : X \to \overline{\mathbb{R}}_{\geqslant 0}$ be measurable. Then $f + g$ and fg are measurable functions from X to $\mathbb{C}$ or $\overline{\mathbb{R}}_{\geqslant 0}$.*

Recall Def. 1.36 for the definition of additions and multiplications in $\overline{\mathbb{R}}_{\geqslant 0}$.

Proof. The multiplication map $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is continuous, and the multiplication map $\overline{\mathbb{R}}_{\geqslant 0} \times \overline{\mathbb{R}}_{\geqslant 0} \to \overline{\mathbb{R}}_{\geqslant 0}$ is lower semicontinuous and hence Borel measurable (cf. Exp. 23.67). By Prop. 23.21, $f \vee g$ is measurable. So its composition with the multiplication map (i.e., fg) is measurable. The same argument shows that $f + g$ is measurable. $\square$

The pointwise limit of a sequence of Riemann integrable functions is not necessarily Riemann integrable. However, the following theorem shows that the pointwise limit of a sequence of measurable functions is measurable. Thus, for example, the Dirichlet function, which is the limit of $f_n(x) = \chi_{A_n}$ where $A_n = \{a_1, \ldots, a_n\}$ and $\mathbb{Q} \cap [0,1] = \{a_1, a_2, \ldots\}$, is not Riemann integrable (cf. Exp. 13.15). But it is Borel measurable.

Theorem 23.23. *Let X be a measurable space. Let $(f_n)_{n \in \mathbb{Z}_+}$ be a sequence of measurable functions $X \to \overline{\mathbb{R}}$. Then $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, and $\liminf_n f_n$ are measurable.*

Proof. Let $F = \sup_n f_n$. Then for each $a \in \overline{\mathbb{R}}$, we have

$$F^{-1}[-\infty, a] = \bigcap_n f_n^{-1}[-\infty, a]$$

where the RHS is measurable. Therefore, by Exp. 23.18, $\sup_n f_n$ is measurable. Similarly, $\inf_n f_n$ is measurable.

For each n, let $g_n : X \to \overline{\mathbb{R}}$ be defined by $g_n(x) = \sup_{k \geqslant n} f_k(x)$. The first paragraph shows that g_n is measurable for each n, and hence $\limsup_n f_n = \inf_n g_n$ is measurable. Similarly, $\liminf_n f_n$ is measurable. $\square$

Corollary 23.24. *Let X be a measurable space. Let Y be $\overline{\mathbb{R}}$ or $\mathbb{R}^N$ or $\mathbb{C}^N$. Let (f_n) be a sequence of measurable functions $X \to Y$ converging pointwise to $f : X \to Y$. Then f is measurable.*

Proof. This is immediate from Thm. 23.23 and Prop. 23.21. $\square$

The pointwise limit of a *net* of measurable functions is not necessarily measurable:

Example 23.25. Let $(X, \mathfrak{M})$ be a measurable space such that $\mathfrak{M} \neq 2^X$, and that $\mathfrak{M}$ contains all finite subsets of X. (For example, let X be an uncountable set, and let $\mathfrak{M}$ be the set of all $E \subset X$ such that either E or $X \backslash E$ is countable.) Let $E \in 2^X \backslash \mathfrak{M}$. Then $(\chi_A)_{A \in \mathrm{fin}(2^E)}$ is a net of measurable functions $X \to \mathbb{R}$. However, its pointwise limit χ_E is not measurable.

23.3 Measures and measure spaces

Definition 23.26. Let $(X, \mathfrak{M})$ be a measurable space. A function $\mu : \mathfrak{M} \to [0, +\infty]$ is called a **measure** if it satisfies the following conditions:

- $\mu(\varnothing) = 0$.
- (**Countable additivity**) If we have countably many $E_1, E_2, \dots \in \mathfrak{M}$ that are pairwise disjoint, then $\mu\big(\bigcup_n E_n\big) = \sum_n \mu(E_n)$.

We call $(X, \mathfrak{M}, \mu)$ (or simply call (X, μ)) a **measure space**. If X is a topological space and μ is defined on $\mathfrak{M} = \mathfrak{B}_X$, we call μ a **Borel measure**. If $\mu(X) < +\infty$, we say that μ is a **finite measure**.

Example 23.27. Let X be a nonempty set. Fix $p \in X$. Define $\delta_p : 2^X \to [0, +\infty]$ by $\delta_p(E) = 1$ if $p \in E$, and $\delta_p(E) = 0$ if $p \notin E$. Then δ_p is a measure on $\mathfrak{M}$, called the **Dirac measure**.

Example 23.28. Let $(X, \mathfrak{M})$ be a measurable space. Let $(\mu_i)_{i \in I}$ be a family of measures $\mathfrak{M} \to [0, +\infty]$. Then the sum $\sum_i \mu_i$ (sending each $E \in \mathfrak{M}$ to the discrete integral $\sum_{i \in I} \mu_i(E)$) is a measure on $\mathfrak{M}$.

Proof. Clearly $\sum_i \mu_i$ sends $\varnothing$ to 0. That $\sum_i \mu_i$ satisfies the countable additivity follows from Fubini's theorem for discrete integrals (Thm. 5.50). $\square$

Example 23.29. Let X be a set. For each $E \subset X$, let $\mu(E) = \sum_{x \in E} 1$. Namely, $\mu(E)$ is the cardinality of E when E is a finite set, and $\mu(E) = +\infty$ when E is not finite. Then $(X, 2^X, \mu)$ is a measure space. μ is called the **counting measure**. It is easy to see that $\mu = \sum_{x \in X} \delta_x$.

Proposition 23.30. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. The following are true.*

*(a) (**Monotonicity**) If $E, F \in \mathfrak{M}$ and $E \subset F$, then $\mu(E) \leqslant \mu(F)$.*

(b) If $(E_n)_{n \in \mathbb{Z}_+}$ is an increasing sequence of elements of $\mathfrak{M}$, then $\mu\big(\bigcup_n E_n\big) = \lim_{n \to \infty} \mu(E_n)$.

(c) If $(E_n)_{n \in \mathbb{Z}_+}$ is a decreasing sequence of elements of $\mathfrak{M}$, and if $\mu(E_1) < +\infty$, then $\mu\big(\bigcap_n E_n\big) = \lim_{n \to \infty} \mu(E_n)$.

*(d) (**Countable subadditivity**) If $E_1, E_2, \dots \in \mathfrak{M}$, then $\mu\big(\bigcup_n E_n\big) \leqslant \sum_n \mu(E_n)$*

Proof. (a) $\mu(F) = \mu(E \sqcup (F\backslash E)) = \mu(E) + \mu(F\backslash E) \geqslant \mu(E)$.

(b) Let $F_1 = E_1$, and $F_n = E_n\backslash E_{n-1}$ if $n > 1$. Then

$$\mu\Big(\bigcup_n E_n\Big) = \mu\Big(\bigsqcup_n F_n\Big) = \sum_n \mu(F_n) = \lim_n \big(\mu(F_1) + \cdots + \mu(F_n)\big)$$
$$= \lim_n \mu(F_1 \cup \cdots \cup F_n) = \lim_n \mu(E_n)$$

(c) Let $F_n = E_1\backslash E_n$. Then E_1 is the disjoint union of $E = \bigcap_n E_n$ and $F = \bigcup_n F_n$ By (b), we have $\mu(F) = \lim_n \mu(F_n)$. Equivalently, $\mu(E_1) - \mu(E) = \lim_n \big(\mu(E_1) - \mu(E_n)\big)$.

(d) We have $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2\backslash E_1) \leqslant \mu(E_1) + \mu(E_2)$. By induction, we get $\mu(E_1 \cup \cdots \cup E_n) \leqslant \mu(E_1) + \cdots + \mu(E_n)$. Thus

$$\mu(E_1 \cup \cdots \cup E_k) \leqslant \sum_n \mu(E_n)$$

for each k. Apply $\lim_k$ to the LHS. Then (b) implies $\mu(\bigcup E_n) \leqslant \sum_n \mu(E_n)$. $\square$

Definition 23.31. Let $(X, \mathfrak{M}, \mu)$ be a measure space. A subset $E \subset X$ is called a μ**-null set** (or simply a **null set**) if $E \in \mathfrak{M}$ and $\mu(E) = 0$. If $P : X \to \{\text{true}, \text{false}\}$ is a property on X, we say that P is true μ**-almost everywhere** (or simply that P is true μ**-a.e.**) if P is true outside a μ-null set.

Remark 23.32. By the countable subadditivity, a countable union of null sets is null.

Definition 23.33. A measure space (X, μ) is called **complete** if every subset of a null set is measurable (and hence is null by the monotonicity).

A main advantage of completeness is the following property:

Proposition 23.34. *Let $(X, \mathfrak{M}, \mu)$ be a complete measure space. Let $(Y, \mathfrak{N})$ be a measurable space. If $f, g : X \to Y$ are equal a.e. (namely, there is a null set outside of which f and g are equal), and if f is measurable, then g is measurable.*

Proof. Let $X_0 \in \mathfrak{M}$ with null complement such that $f|_{X_0} = g|_{X_0}$. Then for each $E \in \mathfrak{N}$, we have $f^{-1}(E) \cap X_0 = g^{-1}(E) \cap X_0$. Since $g^{-1}(E)\backslash X_0$ is a subset of X_0^c, by the completeness, $g^{-1}(E)\backslash X_0$ is measurable. So $g^{-1}(E)$ is measurable. $\square$

Definition 23.35. Let $(X, \mathfrak{M}, \mu)$ be a measure space. If ν is a measure on a σ-algebra $\mathfrak{N} \subset 2^X$, we say that $(\mathfrak{N}, \nu)$ is an **extension of** $(\mathfrak{M}, \mu)$ and write $(\mathfrak{M}, \mu) \subset (\mathfrak{N}, \nu)$, if $\mathfrak{M} \subset \mathfrak{N}$ and $\nu|_{\mathfrak{M}} = \mu$.

Theorem 23.36. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Then there is a (necessarily unique) smallest complete extension $(\overline{\mathfrak{M}}, \overline{\mu})$ of $(\mathfrak{M}, \mu)$. Moreover, the elements of $\overline{\mathfrak{M}}$ are precisely of the form $E \cup F$ where $E \in \mathfrak{M}$ and F is a subset of a μ-null set in $\mathfrak{M}$, and $\overline{\mu}(E \cup F) = \mu(E)$. We call $(\overline{\mathfrak{M}}, \overline{\mu})$ the **completion** of $(\mathfrak{M}, \mu)$.*

The phrase "smallest complete extension" means that $(\overline{\mathfrak{M}}, \overline{\mu})$ is a complete measure on X extending $(\mathfrak{M}, \mu)$, and that every complete measure extending $(\mathfrak{M}, \mu)$ also extends $(\overline{\mathfrak{M}}, \overline{\mu})$.

We clearly have an equivalent description of $\overline{\mathfrak{M}}$: A subset $G \subset X$ belongs to $\overline{\mathfrak{M}}$ iff there exist $A, B \in \mathfrak{M}$ such that $A \subset G \subset B$ and $\mu(B \backslash A) = 0$.

Proof. Define $\overline{\mathfrak{M}}$ to be the set of subsets of X of the form $E \cup F$ where $E \in \mathfrak{M}$ and F is a subset of a μ-null set (in $\mathfrak{M}$), and let $\overline{\mu}(E \cup F) = \mu(E)$. This is well-defined: Suppose that $E \cup F = E' \cup F'$ where $E, E' \in \mathfrak{M}$, and F, F' are subsets of null sets A, A' respectively. Then $\mu(E \cup F) \leqslant \mu(E' \cup A')$. Since $A' \backslash E'$ is null, we have $\mu(E' \cup A') = \mu(E')$. So $\mu(E) \leqslant \mu(E')$. Similarly, $\mu(E') \leqslant \mu(E)$.

Given sets $E_1 \cup F_1, E_2 \cup F_2, \dots$ where $E_n \in \mathfrak{M}$ and F_n is inside a null set, then $\bigcup_n E_n \in \mathfrak{M}$, and $\bigcup F_n$ is inside a null set (Rem. 23.32). So $\overline{\mathfrak{M}}$ is closed under countable unions. Let $E \in \mathfrak{M}$ and F be inside a null set A. Then $(E \cup F)^c = E^c \backslash F$ can be written as $E^c \backslash A$ union a subset of A. So $\overline{\mathfrak{M}}$ is closed under complements. This proves that $\overline{\mathfrak{M}}$ is a σ-algebra. Using the countable additivity of μ, one checks easily that $\overline{\mu}$ is a measure.

If $(\mathfrak{N}, \nu)$ is a complete measure on X extending $(\mathfrak{M}, \mu)$, then for each $E \in \mathfrak{M}$ and F inside a null set in $\mathfrak{M}$, we have $F \in \mathfrak{N}$ by the completeness of ν. So $E \cup F \in \mathfrak{N}$. This proves $\overline{\mathfrak{M}} \subset \mathfrak{N}$. Moreover, $\nu(E \cup F) = \nu(E) + \nu(F \backslash E) = \nu(E)$ since $F \backslash E$ is ν-null. So $\nu(E \cup F) = \nu(E) = \mu(E) = \overline{\mu}(E)$. This proves that ν extends $\overline{\mu}$. $\square$

23.4 The Lebesgue measure m on $\mathbb{R}^N$

Let $N \in \mathbb{Z}_+$. In this section, we construct the Lebesgue measure m^N on $\mathbb{R}^N$, which is the completion of a Borel measure. We write m^N as m when no confusion arises.

Recall Subsec. 15.8.2 for the definition of the Lebesgue measure $m(U)$ of an open set $U \subset \mathbb{R}^N$: It is the supremum of the multiple Riemann integral $\int_{\mathbb{R}^N} f$ (cf. Def. 14.3) where f ranges over all elements of $C_c(U, [0,1]) = \{f \in C_c(U, \mathbb{R}) : f(X) \subset [0,1]\}$. It clearly satisfies the monotonicity: If $U_1 \subset U_2$ are open, then $m(U_1) \leqslant m(U_2)$.

Definition 23.37. For each $E \subset \mathbb{R}^N$, define the **outer Lebesgue measure** to be

$$m^*(E) = \inf \big\{ m(U) : U \text{ is an open set containing } E \big\} \tag{23.8a}$$

Clearly $m^*(U) = m(U)$ when U is open. Clearly $m^*(E) \leqslant m^*(F)$ if $E \subset F \subset \mathbb{R}^N$. Define the **inner Lebesgue measure** to be

$$m_*(E) = \sup \big\{ m^*(K) : K \text{ is a compact subset of } E \big\} \tag{23.8b}$$

Clearly $m_*(E) \leqslant m^*(E)$. A set $E \subset \mathbb{R}^N$ is called m-**regular** if $m^*(E) = m_*(E)$.

From the definition, it is clear that compact sets are regular.

When Lebesgue introduced his integral theory in 1902, he focused on the measures of *bounded subsets* of $\mathbb{R}$. He defined a bounded measurable set to be a bounded m-regular set. (Cf. [Haw, Sec. 5.1] and [Jah, Sec. 9.6].) This definition should be modified for unbounded sets. At this moment, let use show that open sets are regular.

Lemma 23.38. *Any open subset $U \subset \mathbb{R}^N$ is m-regular.*

Proof. Let U be open. Then $m_*(U) \leqslant m^*(U) = m(U)$. We want to show $m_*(U) \geqslant m(U)$. Since $m(U) = \sup\{\int f : f \in C_c(U,[0,1])\}$, it suffices to show for each $f \in C_c(U,[0,1])$ that $\int f \leqslant m_*(U)$. By the definition of $m_*(U)$, it suffices to find a compact $K \subset U$ such that $\int f \leqslant m^*(K)$.

Let $K = \mathrm{Supp}(f)$, which is a compact subset of U. Then $m^*(K)$ is the infimum of $m(V)$ where V is open and contains K. So we shall prove that $\int f \leqslant m(V)$ for each open $V \subset \mathbb{R}^N$ containing K. But this is obvious from the definition of $m(V)$. □

Theorem 23.39. *m^* is a measure on $\mathfrak{B}_{\mathbb{R}^N}$. Its completion will be denoted by $(\mathfrak{M}, m)$ and called the **Lebesgue measure** on $\mathbb{R}^N$. Elements in $\mathfrak{M}$ are called **Lebesgue measurable sets**. Moreover, for each bounded set $E \in \mathfrak{M}$ we have $m(E) < +\infty$.*

Proof. That m^* is a measure on $\mathfrak{B}_{\mathbb{R}^N}$ is an easy consequence of Thm. 23.53 (which can be applied to the current situation, cf. Exp. 23.44), to be proved in the next section. Suppose that $E \in \mathfrak{M}$ is bounded. Then E is contained in a bounded open box $U = (a_1,b_1) \times \cdots \times (a_N,b_N)$. For each $f \in C_c(\mathbb{R}^N,[0,1])$, we have $\int f \leqslant (b_1 - a_1)\cdots(b_N - a_N) < +\infty$. This proves $m(U) < +\infty$, and hence $m(E) < +\infty$. □

There exist bounded subsets of $\mathbb{R}^N$ that are not Lebesgue measurable. See [Rud-R, Thm. 2.22].

23.5 A general method for constructing measures

Let $(X, \mathcal{T}_X)$ be a Hausdorff topological space. Let $\mu : \mathcal{T}_X \to [0,+\infty]$ be a function. Throughout this section, we assume the following assumption.

Assumption 23.40. For each $E \subset X$, define the **outer measure** $\mu^*(E)$ and the **inner measure** $\mu_*(E)$ to be

$$\mu^*(E) = \inf\{\mu(U) : U \text{ is an open subset of } X \text{ containing } E\} \tag{23.9a}$$

$$\mu_*(E) = \sup\{\mu^*(K) : K \text{ is a compact subset of } E\} \tag{23.9b}$$

Then the following conditions are satisfied:

(a) $\mu(\varnothing) = 0$.

(b) (Monotonicity) If $U \subset V \subset X$ and U, V are open, then $\mu(U) \leqslant \mu(V)$.

(c) (Countable subadditivity) For countably many open subsets $U_1, U_2, \dots$ of X we have $\mu\big(\bigcup_n U_n\big) \leqslant \sum_n \mu(U_n)$.

(d) (Additivity) If U_1, U_2 are disjoint open subsets of X, then $\mu(U_1 \cup U_2) = \mu(U_1) + \mu(U_2)$.

(e) (Regularity on open sets) If $U \subset X$ is open, then $\mu(U) = \mu_*(U)$.

If $\mu : \mathcal{T}_X \to [0, +\infty]$ only satisfies (a,b,c,d), then μ is called a **premeasure** (cf. [RF, Sec. 17.5]). We will not use this definition.

Exercise 23.41. The reason we assume only additivity but not countable additivity in (d) is because the latter is automatic. More precisely, assuming (a,b,c), then (d) is true iff for any countable pairwise disjoint open sets $U_1, U_2, \dots$ we have $\mu(\bigcup_n U_n) = \sum_n \mu(U_n)$. Prove this fact. (We will not need this fact in the future.)

Remark 23.42. From the monotonicity of μ on $\mathcal{T}_X$, it is clear that $\mu(U) = \mu^*(U)$ if U is open, and $\mu^* : 2^X \to [0, +\infty]$ is monotone increasing. This shows $\mu_*(K) = \mu^*(K)$ if K is compact, and $\mu_* : 2^X \to [0, +\infty]$ is monotone increasing.

Definition 23.43. We say that E is μ**-regular** if $\mu_*(E) = \mu^*(E)$. Note that compact sets and open sets are clearly μ-regular. If E is μ-regular (e.g E is open or compact), we write

$$\mu(E) = \mu^*(E) = \mu_*(E) \qquad \text{if } E \text{ is } \mu\text{-regular}$$

We abbreviate "μ-regular" to "regular" when no confusion arises. (It has nothing to do with regular topological spaces (defined in Def. 9.20).)

Example 23.44. Let $X = \mathbb{R}^N$ and $\mu = m$. Then m satisfies Asmp. 23.40: (a) and (b) are obvious. (c) and (d) were discussed in Pb. 15.7. (A different proof will be given in Thm. 25.21.) (e) was proved in Lem. 23.38.

Our goal is to show that μ^* is countably additive on $\mathfrak{B}_X$. In measure theory, there is a common strategy of proving that the elements inside a σ-algebra satisfy a property P. Let $P : 2^X \to \{\text{true}, \text{false}\}$ be a property. Suppose that we can find $\mathfrak{E} \subset 2^X$ such that every element in $\mathfrak{E}$ satisfies P. Suppose that the set of all $E \subset X$ satisfying P is a σ-algebra. Then P is clearly true for every element of $\sigma(\mathfrak{E})$.

However, countable additivity is not a property about subsets of X, but a property about pairwise disjoint sequences of subsets of X. Nevertheless, one can show that μ is countably additive on a pairwise disjoint sequence of *regular* subsets based on the following simple idea:

$$\boxed{\begin{array}{c} \mu^* \text{ satisfies countable subadditivity} \\ \mu_* \text{ satisfies countable superadditivity} \\ \text{If } \mu^* = \mu_*, \text{ then the countable additivity is true} \end{array}} \qquad (23.10)$$

Therefore, since open sets are regular, if we can show that the class of regular sets is a σ-algebra, then this class contains $\mathfrak{B}_X$. Thus, our goal is accomplished.

As we will see, if $\mu(X) < +\infty$, the class of regular sets is indeed a σ-algebra. However, when $\mu(X) = +\infty$, this statement is not true, so we must find an alternative to regular sets.

23.5.1 Countable subadditivity and countable superadditivity

In this subsection, we prove that μ^* satisfies the countable subadditivity, and μ_* satisfies the countable superadditivity.

Proposition 23.45. $\mu^* : 2^X \to [0, +\infty]$ *satisfies the following conditions:*

(a) $\mu^*(\emptyset) = 0$.

(b) (Monotonicity) If $E \subset F \subset X$*, then* $\mu^*(E) \leqslant \mu^*(F)$.

(c) (Countable subadditivity) For countably many subsets $E_1, E_2, \ldots$ *of* X *we have* $\mu^*\big(\bigcup_n E_n\big) \leqslant \sum_n \mu^*(E_n)$.

Proof. The first two conditions are obvious. Let us check the countable subadditivity. Let $E_1, E_2, \cdots \subset X$. Assume WLOG that each $\mu^*(E_n)$ is finite; otherwise the countable subadditivity is obvious. Let $\varepsilon > 0$. By the definition of μ^*, each E_n is contained in an open set U_n such that $\mu(U_n) \leqslant \mu^*(E_n) + \varepsilon/2^n$. Then $E = \bigcup_n E_n$ is contained in $\bigcup_n U_n$. By the countable subadditivity for open sets, we have $\mu(\bigcup_n U_n) \leqslant \sum_n \mu(U_n) \leqslant \sum_n \mu^*(E_n) + \varepsilon$. By the monotonicity, we have $\mu(E) \leqslant \sum_n \mu^*(E_n) + \varepsilon$. This finishes the proof, since ε can be arbitrary. $\square$

To establish the countable superadditivity for μ_*, we first need to prove the additivity for compact sets:

Lemma 23.46. *Suppose that* K_1, K_2 *are disjoint compact subsets of* X*. Then* $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$.

Proof. By the (countable) subadditivity of μ^*, it remains to prove $\mu^*(K_1 \cup K_2) \geqslant \mu^*(K_1) + \mu^*(K_2)$. By the definition of μ^*, it suffices to prove that for each open U containing $K_1 \cup K_2$ we have $\mu(U) \geqslant \mu^*(K_1) + \mu^*(K_2)$. We claim that there exist disjoint open $U_1, U_2 \subset U$ containing K_1, K_2 respectively. Then by condition (d) of Asmp. 23.40, we get $\mu(U) \geqslant \mu(U_1 \cup U_2) = \mu(U_1) + \mu(U_2) \geqslant \mu^*(K_1) + \mu^*(K_2)$.

The proof of the claim is a routine argument in point-set topology. Choose any $x \in K_1, y \in K_2$. Since U is Hausdorff, there exist disjoint $V_{xy} \in \mathrm{Nbh}_U(x)$ and $W_{xy} \in \mathrm{Nbh}_U(y)$. Since K_2 is compact, there exit $y_1, \ldots, y_n \in K_2$ such that K_2 is contained in $W_x = W_{xy_1} \cup \cdots \cup W_{xy_n}$. Let $V_x = V_{xy_1} \cap \cdots \cap V_{xy_n}$. Then V_x and W_x are disjoint open subsets of U containing respectively x and K_2. By the compactness of K_1, there are $x_1, \dots, x_m \in K_1$ such that K_1 is covered by $U_1 = V_{x_1} \cup \cdots \cup V_{x_m}$. Then $U_2 = W_{x_1} \cap \cdots \cap W_{x_m}$ contains K_2 and is disjoint from U_1. [2] □

Proposition 23.47. *$\mu_* : 2^X \to [0, +\infty]$ satisfies $\mu_*(E) \leqslant \mu^*(E)$ for all $E \subset X$. Moreover, the following are true.*

(a) $\mu_*(\varnothing) = 0$.

(b) (Monotonicity) If $E \subset F \subset X$, then $\mu_*(E) \leqslant \mu_*(F)$.

*(c) (**Countable superadditivity**) For countably many disjoint subsets $E_1, E_2, \dots$ of X we have $\mu_*\big(\bigcup_n E_n\big) \geqslant \sum_n \mu_*(E_n)$.*

Proof. The monotonicity of μ^* implies $\mu_*(E) \leqslant \mu^*(E)$. (a) and (b) are obvious. To prove (c), it suffices to prove $\mu_*\big(\bigcup_k E_k\big) \geqslant \mu_*(E_1) + \cdots + \mu_*(E_n)$ for all n, and hence to prove $\mu_*(E_1 \cup \cdots \cup E_n) \geqslant \mu_*(E_1) + \cdots + \mu_*(E_n)$. By induction on n, it suffices to prove $\mu_*(E_1 \cup E_2) \geqslant \mu_*(E_1) + \mu_*(E_2)$. By the definition of μ_*, it suffices to prove that for every compact $K_1 \subset E_1$ and $K_2 \subset E_2$ we have $\mu_*(E_1 \cup E_2) \geqslant \mu(K_1) + \mu(K_2)$. By Lem. 23.46, it suffices to prove $\mu_*(E_1 \cup E_2) \geqslant \mu(K_1 \cup K_2)$. But this is obvious from the definition of μ_*. □

23.5.2 Criteria for μ-regularity

We now use the idea (23.10) to derive many criteria for μ-regularity.

Corollary 23.48. *Let $E_1, E_2, \dots$ be pairwise disjoint μ-regular subsets of X. Then $\bigcup_n E_n$ is μ-regular, and*

$$\mu\Big(\bigcup_n E_n\Big) = \sum_n \mu(E_n) \tag{23.11}$$

Proof. Let $E = \bigcup_n E_n$. Then $\mu_*(E) \leqslant \mu^*(E)$. Since $\mu_*(E_n) = \mu^*(E_n)$, by Prop. 23.45 and 23.47 we have

$$\mu^*(E) \leqslant \sum_n \mu^*(E_n) = \sum_n \mu_*(E_n) \leqslant \mu_*(E)$$

This proves that $\mu_*(E) = \mu^*(E)$, and that (23.11) is true. □

Corollary 23.49. *Let $E \subset X$ satisfy $\mu^*(E) < +\infty$. Then E is μ-regular iff for every $\varepsilon > 0$ there exist a compact set K and an open set U such that $K \subset E \subset U$ and $\mu(U \backslash K) < \varepsilon$.*

[2] An alternative proof when X is LCH: By Prop. 8.41, the open set $W = U \backslash K_2$ is LCH. Therefore, by Lem. 15.27, there is an open precompact subset $U_1 \Subset W$ containing K_1. Take $U_2 = U \backslash \overline{U}_1$.

Proof. Since $\mu^*(E) < +\infty$, we know that $\mu^*(E) = \mu_*(E)$ iff one can find open $U \subset E$ (satisfying $\mu(U) < +\infty$) and compact $K \subset E$ such that $\mu(U) - \mu(K)$ is small. Since $U, U\backslash K$ are open and K is compact, they are μ-regular. Therefore, by Cor. 23.48, we have $\mu(U) - \mu(K) = \mu(U\backslash K)$. So $\mu(U) - \mu(K)$ being small means $\mu(U\backslash K)$ being small. □

The following lemma is a special case of the main Thm. 23.53. But it can now be proved easily using Cor. 23.49.

Lemma 23.50. *Let E_1, E_2 be μ-regular subsets of X satisfying $\mu(E_1) < +\infty$ and $\mu(E_2) < +\infty$. Then $E_1 \cup E_2, E_1 \cap E_2, E_2\backslash E_1$ are μ-regular.*

Proof. It suffices to prove that $E_2\backslash E_1$ is regular. Then, similarly, $E_1 \cap E_2 = E_2\backslash(E_2\backslash E_1)$ is regular. By Cor. 23.48, $E_1 \cup E_2 = E_1 \sqcup (E_2\backslash E_1)$ is regular. (Note that all the sets involved have finite μ^*-values.)

Choose any $\varepsilon > 0$. By Cor. 23.49, there exist compact $K_i \subset E_i$ and open $U_i \supset E_i$ such that $\mu(U_i\backslash K_i) < \varepsilon/2$. Then $U_2\backslash K_1$ is open and contains $E_2\backslash E_1$, and $K_2\backslash U_1$ is compact and is contained in $E_2\backslash E_1$. Moreover,

$$(U_2\backslash K_1)\backslash(K_2\backslash U_1) \subset (U_2\backslash K_2) \cup (U_1\backslash K_1) \tag{23.12}$$

since $(U_2\backslash K_1)\backslash(K_2\backslash U_1) = U_2 \cap K_1^c \cap (K_2 \cap U_1^c)^c = (U_2 \cap K_1^c \cap K_2^c) \cup (U_2 \cap K_1^c \cap U_1) \subset (U_2\backslash K_2) \cup (U_1\backslash K_1)$. (See also Fig. 23.1.) By the subadditivity of μ^*, we have $\mu^*((U_2\backslash K_1)\backslash(K_2\backslash U_1)) \leqslant \mu(U_2\backslash K_2) + \mu(U_1\backslash K_1) < \varepsilon$. Therefore, by Cor. 23.49, $E_2\backslash E_1$ is μ-regular. □

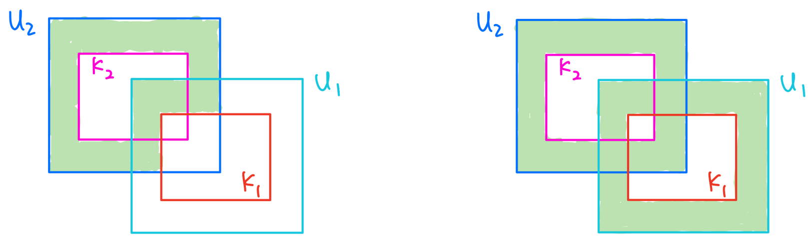


Figure 23.1. The shaded areas are $(U_2\backslash K_1)\backslash(K_2\backslash U_1)$ and $(U_2\backslash K_2) \cup (U_1\backslash K_1)$ respectively.

23.5.3 Locally μ-regular sets

When $\mu(X) < +\infty$, the results in Subsec. 23.5.2 imply easily that the set of μ-regular sets form a σ-algebra $\mathfrak{M}_\mu$ containing all open sets (and hence containing $\mathfrak{B}_X$), and that μ is a measure on $\mathfrak{M}_\mu$. However, when $\mu(X) = +\infty$, μ-regular sets are not good enough. One easily checks that if μ is finite on compact sets (e.g., μ is the Lebesgue measure), Cor. 23.49 fails for regular sets with infinite μ-values. Therefore, one cannot prove that the μ-regular sets form a σ-algebra. In fact, there is an example where $\mu^*(E) = +\infty$ and $\mu_*(E) = 0$ for some $E \in \mathfrak{B}_X$. (See Rem. 26.18.) To overcome this difficulty, we consider the better notion of local μ-regularity:

Definition 23.51. Let $E \subset X$. We say that E is **locally μ-regular** if $\mu^*(E \cap \Omega) = \mu_*(E \cap \Omega)$ for every open set $\Omega \subset X$ satisfying $\mu(\Omega) < +\infty$.

The following proposition shows that when $\mu(X) < +\infty$, the μ-regularity and the local μ-regularity are equivalent.

Proposition 23.52. *Assume that $E \subset X$ satisfies $\mu^*(E) < +\infty$. Then E is μ-regular iff E is locally μ-regular.*

Proof. Assume that E is locally regular. Since $\mu^*(E) < +\infty$, there exists an open $\Omega \supset E$ such that $\mu(\Omega) < +\infty$. So $E \cap \Omega$ is regular. Hence E is regular. Conversely, assume that E is regular. Let $\Omega \subset X$ be an open set satisfying $\mu(\Omega) < \infty$. Since Ω is μ-regular, by Lem. 23.50, $E \cap \Omega$ is regular. So E is locally regular. □

23.5.4 The main theorem

Theorem 23.53. *Let $\mu : \mathcal{T}_X \to [0, +\infty]$ satisfy Asmp. 23.40. Let $\mathfrak{M}_\mu$ be the set of locally μ-regular subsets of X. Then $\mathfrak{M}_\mu$ is a σ-algebra containing $\mathfrak{B}_X$, and μ^* is a complete measure on $\mathfrak{M}_\mu$. We denote the measure $(\mathfrak{M}_\mu, \mu^*)$ by $(\mathfrak{M}_\mu, \mu)$.*

According to the definition of $(\mathfrak{M}_\mu, \mu)$ in Thm. 23.53, if $E \in \mathfrak{M}_\mu$ then $\mu^*(E) = \mu(E)$ (even though $\mu^*(E)$ and $\mu_*(E)$ are not necessarily equal when $\mu^*(E) = +\infty$). The restriction $(\mathfrak{B}_X, \mu)$ will be called a **Radon measure** if X is LCH and μ is finite on compact sets. We will discuss Radon measures in detail in Ch. 25.

Proof. Step 1. We show that $\mathfrak{M}_\mu$ is a σ-algebra. Clearly $\emptyset \in \mathfrak{M}_\mu$. Let $\Omega \in \mathcal{T}_X$ such that $\mu(\Omega) < +\infty$. We want to show that if $E \cap \Omega$ is regular then $E^c \cap \Omega$ is also regular. We want to show that if $E_1, E_2, \dots \subset 2^X$ are such that $E_1 \cap \Omega, E_2 \cap \Omega, \dots$ are regular, then $(\bigcup_n E_n) \cap \Omega$ is regular. By replacing E with $E \cap \Omega$ and E_n with $E_n \cap \Omega$, it suffices to prove:

(a) If $E \subset \Omega$ is regular, then $\Omega \backslash E$ is regular.

(b) If $E_1, E_2, \dots$ are regular subsets of Ω, then $\bigcup_n E_n$ is regular.

By Lem. 23.50, (a) is true (since open sets are regular). Let $E_1, E_2, \dots \subset \Omega$ be regular. Let $F_1 = E_1$ and $F_n = E_n \backslash (E_1 \cup \dots \cup E_{n-1})$ if $n > 1$. Then each F_n is regular by Lem. 23.50. Therefore, by Cor. 23.48, $\bigcup E_n = \bigsqcup_n F_n$ is regular.

Step 2. By Asmp. 23.40-(e), every open set is regular. Thus every open set is also locally regular. Therefore, $\mathfrak{M}_\mu$ contains $\mathcal{T}_X$, and hence contains $\mathfrak{B}_X = \sigma(\mathcal{T}_X)$.

Clearly $\mu^*(\emptyset) = 0$. To show that μ^* is a measure on $\mathfrak{M}_\mu$, we take mutually disjoint $E_1, E_2, \dots \in \mathfrak{M}_\mu$, and we shall show that $\mu^*(E) = \sum_n \mu^*(E_n)$ where $E = \bigcup_n E_n$. If $\mu^*(E) = +\infty$, by the countable subadditivity, we have $+\infty = \mu^*(E) \leqslant \sum_n \mu^*(E_n)$, and hence $\mu^*(E) = \sum_n \mu^*(E_n)$. So it suffices to assume $\mu^*(E) < +\infty$. Therefore $\mu^*(E_n) < +\infty$ for each n. By Prop. 23.52, E_n is regular. Therefore $\mu^*(E) = \sum_n \mu^*(E_n)$ by Cor. 23.48.

Finally, we need to prove that μ^* is complete on $\mathfrak{M}_\mu$. Let $E \in \mathfrak{M}_\mu$ such that $\mu^*(E) = 0$. Let $F \subset E$. Then $\mu^*(F) \leqslant \mu^*(E) = 0$ and hence $\mu^*(F) = 0$. So $\mu_*(F) = \mu^*(F) = 0$. Therefore F is regular and has finite μ^*-value. So $F \in \mathfrak{M}_\mu$ by Prop. 23.52. □

Exercise 23.54. Why do we define μ on $\mathfrak{M}_\mu$ to be the restriction of μ^* but not μ_*?

23.5.5 A relationship between $\mathfrak{M}_\mu$ and the completion of $\mathfrak{B}_X$

By Thm. 23.36, $(\mathfrak{M}_\mu, \mu)$ extends the completion of $(\mathfrak{B}_X, \mu)$. Very often, we only care about the completion of $(\mathfrak{B}_X, \mu)$ but not about the larger set $\mathfrak{M}_\mu$. However, the following proposition shows that in many cases (e.g. when $X = \mathbb{R}^N$ and $\mu = m$), $(\mathfrak{M}_\mu, \mu)$ is equal to the completion.

⋆ **Proposition 23.55.** *Assume that $(\mathfrak{B}_X, \mu)$ is σ-**finite**, i.e., X is a countable union of elements of $\mathfrak{B}_X$ with finite μ-measures. Then $(\mathfrak{M}_\mu, \mu)$ is the completion of $(\mathfrak{B}_X, \mu)$.*

⋆ *Proof.* $(\mathfrak{M}_\mu, \mu)$ extends the completion $(\mathfrak{M}, \mu)$. We want to show that $\mathfrak{M}_\mu \subset \mathfrak{M}$. Let $E \in \mathfrak{M}_\mu$. Since X is σ-finite, we have $X = \bigcup_n A_n$ where $A_n \in \mathfrak{B}_X$ and $\mu(A_n) < +\infty$. It suffices to prove that $E_n := E \cap A_n$ belongs to $\mathfrak{M}$ for each n. Note that $E_n \in \mathfrak{M}_\mu$ and $\mu(E_n) < +\infty$. Thus, by Cor. 23.49, for each $k \in \mathbb{Z}_+$ there exists a compact $K_k \subset E_n$ such that $\mu(E_n \backslash K_k) < 1/k$. By replacing K_k with $K_1 \cup \dots \cup K_k$, we assume (K_k) is increasing. Let $F = \bigcup_k K_k$, which is a Borel set. Then $F \subset E_n$ and $E_n \backslash F$ is μ-null. Since $(\mathfrak{M}, \mu)$ is complete, we see that $E_n \backslash F$ belongs to $\mathfrak{M}$. So $E_n \in \mathfrak{M}$. □

In general, $(\mathfrak{M}_\mu, \mu)$ is larger than the completion $(\mathfrak{M}, \mu)$ of $(\mathfrak{B}_X, \mu)$. $(\mathfrak{M}_\mu, \mu)$ is called the **saturation** of $(\mathfrak{M}, \mu)$. More generally, the measure ν of a measure space $(Y, \mathfrak{N}, \nu)$ is called **saturated** provided that a set $E \subset Y$ belongs to $\mathfrak{N}$ iff $E \cap A \in \mathfrak{N}$ for every $A \in \mathfrak{N}$ such that $\nu(A) < +\infty$. If a measure $(\mathfrak{N}, \nu)$ is not necessarily saturated, the smallest saturated measure extending $(\mathfrak{N}, \nu)$ is called the **saturation** of $(\mathfrak{N}, \nu)$.

23.6 A discussion of measurable sets: from Jordan to Lebesgue to Carathéodory

Let $(X, \mathcal{T}_X)$ be a topological space, and assume that $\mu : \mathcal{T}_X \to [0, +\infty]$ satisfies Asmp. 23.40.

23.6.1 Lebesgue's outer measure and inner measure

Here, I will make some comparisons between our approach in Sec. 23.5 and Lebesgue's original method. We refer the readers to [Haw, Sec. 5.1] and [Jah, Sec. 9.6] for detailed discussions of Lebesgue's approach.

In the 1902 paper where Lebesgue introduced his integral theory, he focused on measurable subsets of $[a, b]$. For the convenience of the following discussion, I will consider the bounded open interval (a, b) instead. For each $E \subset (a, b)$, Lebesgue defined the outer measure $m^*(E)$ by (23.2), i.e., the infinimum of the total sizes of intervals covering E. It is not hard to see that his definition agrees with ours, since any open subset of $\mathbb{R}$ is a countable disjoint union of open intervals (cf. Pb. 8.14).

On the other hand, Lebesgue's definition of inner measure is

$$m_*(E) = b - a - m^*((a, b)\backslash E) \tag{23.13}$$

If we generalize (23.13) to the setting of Sec. 23.5 where E is a subset of an open $\Omega \subset X$ satisfying $\mu(\Omega) < +\infty$, the inner measure of E should be defined by $\mu(\Omega) - \mu^*(\Omega\backslash E)$. We now show that this definition is equal to our definition of $\mu_*(E)$ in (23.9b):

Proposition 23.56. *Let Ω be an open subset of X such that $\mu(\Omega) < +\infty$. Let $E \subset \Omega$. Then*

$$\mu_*(E) = \mu(\Omega) - \mu^*(\Omega\backslash E) \tag{23.14}$$

⋆ *Proof.* Let $F = \Omega\backslash E$. By the definition of μ^*, we have

$$\mu^*(F) = \inf\big\{\mu(V) : V \text{ is open in } \Omega \text{ and contains } F\big\}$$

By Lem. 23.50, $\Gamma := \Omega\backslash V$ is μ-regular, and $\mu(\Omega) = \mu(V) + \mu(\Gamma)$. Therefore, $\mu(\Omega) - \mu^*(\Omega\backslash E)$ equals the supremum of $\mu(\Gamma)$ where Γ is a closed subset of Ω contained in E. Thus, proving (23.14) means proving

$$\sup\big\{\mu(K) : K \subset E \text{ is compact}\big\} = \sup\big\{\mu(\Gamma) : \Gamma \text{ is closed in } \Omega\text{, and } \Gamma \subset E\big\}$$

We clearly have "$\leqslant$". Since $\mu(\Gamma) = \mu_*(\Gamma)$, for each $\varepsilon > 0$ there exists a compact $K \subset \Gamma$ such that $\mu(K) > \mu(\Gamma) - \varepsilon$. So "$\geqslant$" is also true. □

For a further generalization of this proposition, see Pb. 23.11.

23.6.2 Subadditivity and superadditivity

As mentioned in the slogan (23.10), the countable additivity of μ on regular sets follows directly from the fact that the outer measure μ^* is countably subadditive, and the inner measure μ_* is countably superadditive. This simple and intuitive idea is not new. It already appeared in Darboux integrals: Given a bounded real-valued function f defined on $R = [a_1, b_1] \times \cdots [a_N, b_N]$, the upper Darboux integral $\overline{\int} f$ and the lower Darboux integral $\underline{\int} f$ are defined in a similar way as in Thm. 13.41, where the Darboux sums are defined by partitioning the box R into smaller boxes. It is not hard to check that $\overline{\int}$ and $\underline{\int}$ satisfy respectively the subadditivity and the superadditivity:

$$\overline{\int}(f+g) \leqslant \overline{\int} f + \overline{\int} g \qquad \underline{\int}(f+g) \geqslant \underline{\int} f + \underline{\int} g$$

(See also Pb. 15.9.) Thus, since f is Riemann integrable iff $\overline{\int} f = \underline{\int} f$, $\overline{\int}$ and $\underline{\int}$ must be linear on Riemann integrable functions.

In fact, before Lebesgue, Jordan had utilized this idea (subadditivity + superadditivity $\Rightarrow$ additivity) to study measurable sets in 1892 (cf. [Jah, Sec. 9.4] and [Haw, 4.1]): For each bounded $E \subset \mathbb{R}^N$, the **outer content** $c^*(E)$ is defined to be the infimum of the total sizes of *finitely many* boxes covering E, and the **inner content** $c_*(E)$ is the supremum of the total sizes of finitely many disjoint boxes contained in E. Jordan defined E to be measurable if $c^*(E) = c_*(E)$. Using our familiar language,

$$c^*(E) = \overline{\int} \chi_E \qquad c_*(E) = \underline{\int} \chi_E \tag{23.15}$$

and E is Jordan-measurable iff χ_E is Riemann integrable. c^* and c_* satisfy subadditivity and superadditivity respectively, just as $\overline{\int}$ and $\underline{\int}$ do. So c^* satisfies additivity on Jordan-measurable sets.

Example 23.57. Let $E = \mathbb{Q} \cap [0,1]$. We know that $m(E) = 0$. It is not hard to see that $c_*(E) = 0$ and $c^*(E) = 1$. So E is not Jordan-measurable.

Compared to Jordan, an important improvement Lebesgue made is that he extended the finite subadditivity and superadditivity to countable ones. He achieved this goal by giving the better definition of outer measure. Recall that the outer content $c^*(E)$ is defined by the infimum of the sizes of **simple regions** (i.e., finite unions of boxes) covering E. The outer Lebesgue measure $m^*(E)$ is defined in a similar way, except that one allows for *countable* unions of boxes to cover E. From the modern viewpoint (i.e., the viewpoint in Sec. 23.4 and 23.5), $m^*(E)$ is defined by covering E by arbitrary open sets, not just by simple regions. The modern viewpoint is equivalent to the classical one due to the following observation:

Exercise 23.58. Let Ω be an open subset of $\mathbb{R}^N$. Show that Ω is a countable union of boxes $R_1 \cup R_2 \cup \cdots$ where $\mathrm{Int}(R_i) \cap \mathrm{Int}(R_j) = \varnothing$ if $i \neq j$. Here, a box denotes a set $I_1 \times \cdots \times I_N$ where each I_i is a bounded interval in $\mathbb{R}$.

Hint. First treat the case that Ω is inside $R = [0,1]^N$. In step k, partition R equally into 2^{kN} pieces, and take all subboxes inside Ω but not inside the subboxes taken from step 1 to step $k-1$. In the general case, consider $\Omega \cap R$ where $R = [n_1, n_1 + 1] \times \cdots \times [n_N, n_N + 1]$ and $n_1, \dots, n_N \in \mathbb{Z}$. □

23.6.3 The dilemma of μ-regular sets with possibly infinite measures.

We know that when $\mu(X) < +\infty$, the μ-regular sets form a σ-algebra containing $\mathfrak{B}_X$ (Thm. 23.53 and Prop. 23.52). When $\mu(X) = +\infty$, this statement cannot be proved, so we must find alternatives to μ-regular sets.

The alternatives we gave in Sec. 23.5 are locally μ-regular sets, i.e., the sets E such that $E \cap \Omega$ is μ-regular for any open μ-finite set Ω. Our treatment is similar to that in Rudin's book [Rud-R], except that Rudin considered those E such that $E \cap K$ is regular for any compact $K \subset X$. (Rudin assumed that X is LCH, and μ is finite on compact sets.) See the proof of Thm. 2.14 in [Rud-R]. In particular, Step III of the proof of that theorem is similar to Lem. 23.38, Step IV is similar to Cor. 23.48, Step V is similar to Cor. 23.49, Step VI is similar to Lem. 23.50, Step VII and IX are similar to Thm. 23.53, and Step VIII is similar to Prop. 23.52.

Another approach was introduced by Carathéodory in 1914 and is popular among many textbooks (cf. [Fol-R, Sec. 1.4], [RF, Sec. 17.3], [Yu, Sec. 39]). Since you may need to study this approach carefully in the future, I'll explain below how the approach of Carathéodory is related to those of Jordan and Lebesgue based on regular sets. (At first glance, they look very different!)

23.6.4 ⋆ Carathéodory measurable sets

The relationship between our approach (in Sec. 23.5) and Lebesgue's approach was already explained. Now, recall that a subset E of X is locally μ-regular iff $\mu^*(\Omega \cap E) = \mu_*(\Omega \cap E)$ for all open Ω with finite $\mu(\Omega)$. By Prop. 23.56, we have $\mu_*(\Omega \cap E) = \mu(\Omega) - \mu^*(\Omega \backslash E)$. Therefore,

$$\begin{gathered} E \text{ is locally } \mu\text{-regular} \\ \Updownarrow \\ \mu^*(\Omega \cap E) + \mu^*(\Omega \backslash E) = \mu(\Omega) \quad \text{if } \Omega \subset X \text{ is open and } \mu(\Omega) < +\infty \end{gathered} \tag{23.16}$$

Here comes the magic:

Proposition 23.59. *Let $E \subset X$. Then the following are equivalent.*

(1) E is locally μ-regular.

*(2) E is **Carathéodory μ^*-measurable**, which means that for any $A \subset X$ we have*

$$\mu^*(A \cap E) + \mu^*(A \backslash E) = \mu^*(A) \tag{23.17}$$

Proof. By (23.16), clearly (2) implies (1). Conversely, assume (1). To prove (23.17), by the subadditivity of μ^*, it suffices to prove "$\leqslant$". This is obvious when $\mu^*(A) = +\infty$. So we assume WLOG that $\mu^*(A) < +\infty$. Since $\mu^*(A)$ is the infinimum of $\mu(\Omega)$ where $\Omega \supset A$ is open and $\mu(\Omega) < +\infty$, it suffices to prove for such Ω that $\mu^*(A \cap E) + \mu^*(A \backslash E) \leqslant \mu(\Omega)$. By the monotinicity of μ^*, it suffices to prove $\mu^*(\Omega \cap E) + \mu^*(\Omega \backslash E) = \mu(\Omega)$. But this follows from the fact that μ^* is a measure on the σ-algebra $\mathfrak{M}_\mu$ (cf. Thm. 23.53). (Alternatively, it follows directly from Lem. 23.50 and Cor. 23.48.) □

The surprising part of the Carathéodory measurability is that we do not assume A to be either open or compact or Borel or μ-regular. By contrast, $\mu_*(A \cap E) + \mu^*(A \backslash E)$ is not necessarily equal to $\mu^*(A)$ or $\mu_*(A)$ when A is not μ-regular. (It is equal to $\mu(A)$ when A is μ-regular and $\mu(A) < +\infty$. See Prop. 23.71.)

Example 23.60. Let E, F be subsets of X such that there exist disjoint open subsets U, V of X containing E, F respectively. Let $A = E \cup F$. Then $\mu_*(A) = \mu_*(E) + \mu_*(F)$ and $\mu^*(A) = \mu^*(E) + \mu^*(F)$. (Cf. Pb. 23.10.) Assume that E, F have finite outer measures and are not μ-regular, i.e., $\mu_*(E) < \mu^*(E) < +\infty$ and $\mu_*(F) < \mu^*(F) < +\infty$. Then

$$\mu_*(A) < \mu_*(A \cap E) + \mu^*(A \backslash E) < \mu^*(A)$$

since the middle term is equal to $\mu_*(E) + \mu^*(F)$.

We now briefly discuss Carathéodory's theory.

Definition 23.61. Let Y be a set. A function $\nu^* : 2^Y \to [0, +\infty]$ is called an **(abstract) outer measure** if it satisfies the three conditions in Prop. 23.45, namely, it satisfies $\nu^*(\varnothing) = 0$, the monotonicity, and the countable subadditivity (on subsets of Y). A set $E \subset Y$ is called **Carathéodory ν^*-measurable** (or simply **ν^*-measurable**) if for every $A \subset Y$ we have

$$\nu^*(A \cap E) + \nu^*(A \backslash E) = \nu^*(A) \tag{23.18}$$

The construction of Lebesgue measures (and more generally, measures on Hausdorff spaces) using Carathéodory measurable sets is based on the following key theorem.

Theorem 23.62. *Let ν^* be an (abstract) outer measure on a set Y. Then the set $\mathfrak{M}$ of Carathéodory ν^*-measurable sets form a σ-algebra, and the restriction of ν^* to $\mathfrak{M}$ is a complete measure.*

Proof. See [Fol-R, Thm. 1.11]. □

Our main Thm. 23.53 follows easily from 23.62: Since μ satisfies (a,b,c) of Asmp. 23.40, μ^* is an abstract outer measure. By using (d,e) of Asmp. 23.40, one can show that open sets are μ^*-measurable.[3] Therefore, the set $\mathfrak{M}_\mu$ of Carathéodory μ^*-measurable sets contains open subsets of X. Thus, by Thm. 23.62, $\mathfrak{M}_\mu$ is a σ-algebra containing $\mathfrak{B}_X$, and μ^* is a complete measure on $\mathfrak{M}_\mu$.

The reason we use locally regular sets in our course instead of Carathéodory's approach is that the latter is very unintuitive. The intuition "subadditivity + superadditivity ⇒ additivity", which goes back to Lebesgue, Jordan, and even Darboux, is very obscure in the proof of Thm. 23.62. One cannot interpret (23.18) to mean that the outer measure $\nu^*(A \cap E)$ of $A \cap E$ equals the "inner measure" $\nu^*(A) - \nu^*(A\backslash E)$, because Exp. 23.60 tells us that $\mu^*(A) - \mu^*(A\backslash E)$ is often not equal to the actual inner measure $\mu_*(A \cap E)$. On the other hand, the advantage of Carathéodory's approach is that Thm. 23.62 can be applied to more general situations than Thm. 23.53. For example, it can be used to construct measures on Hausdorff spaces not satisfying regularity (such as the Hausdorff measures, cf. [Fol-R, Sec. 11.2]).

I believe that many people will have this confusion when they first see the condition (23.18): Why consider an arbitrary set A instead of just a "good" set, for example, an open set, a compact set, or a Borel set? How can one believe that a definition as strong as Def. 23.61 would have nontrivial examples? The way to understand the motivation behind a definition or a theorem is not to immerse oneself in the technical details of the proof, but to clarify the genealogy of concepts. I hope that the exposition in this section will help the reader to get a general idea of how the concept of measurable sets evolved from its basic and intuitive form to the abstract definition of Carathéodory.

23.7 Problems and supplementary material

23.7.1 Basic properties

Definition 23.63. Let $(X, \mathfrak{M})$ be a measurable space. Let Y be a subset of X. Let

$$\mathfrak{M}|_Y = \iota^{-1}(\mathfrak{M}) \tag{23.19}$$

where $\iota : Y \to X$ is the inclusion map. In other words, $\mathfrak{M}|_Y = \{Y \cap E : E \in \mathfrak{M}\}$. Then $\mathfrak{M}|_Y$ is clearly a σ-algebra on Y, called the **restriction** of $\mathfrak{M}$ to Y.

Exercise 23.64. Let $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ be measurable spaces. Let $f : X \to Y$ be a map. Let $Z \subset Y$. Let $\mathfrak{N}|_Z$ be the restriction of $\mathfrak{N}$ to Z. Prove that $f : (X, \mathfrak{M}) \to (Y, \mathfrak{N})$ is measurable iff $f : (X, \mathfrak{M}) \to (Z, \mathfrak{N}|_Z)$ is measurable.

[3] Sketch of the proof: Let E be open. First prove (23.17) when A is open. Then prove (23.17) for any A by using a similar argument as in the proof of Prop. 23.59.

Problem 23.1. Let Y be a topological space, and let Z be a subspace of Y (equipped with the subspace topology). Prove $\mathfrak{B}_Y|_Z = \mathfrak{B}_Z$.

Hint. Apply Cor. 23.13 to the inclusion map $\iota : Z \to Y$. □

Problem 23.2. Let $(X, \mathfrak{M}, \mu)$ be a measure space with completion $(\overline{\mathfrak{M}}, \overline{\mu})$. Let $\mathcal{V}$ be a separable normed vector space, and let $f : X \to \mathcal{V}$ be $\overline{\mathfrak{M}}$-measurable (i.e., for each $U \in \mathfrak{B}_{\mathcal{V}}$ we have $f^{-1}(U) \in \overline{\mathfrak{M}}$). Prove that there exists $A \in \mathfrak{M}$ with $\mu(X \backslash A) = 0$ such that $f\chi_A$ is $\mathfrak{M}$-measurable.

Problem 23.3. Let X and Y be topological spaces.

1. Let $A \in \mathfrak{B}_X$ and $B \in \mathfrak{B}_Y$. Prove that $A \times B \in \mathfrak{B}_{X \times Y}$.
2. Let $f : X \to [0, +\infty]$ be a Borel function. Prove that R_f is a Borel subset of $X \times \overline{\mathbb{R}}$ where

$$R_f = \{(x, y) \in X \times \overline{\mathbb{R}} : 0 \leqslant y \leqslant f(x)\}$$

Hint. 1. The inverse image of any Borel set under a continuous (and hence Borel) map is Borel. 2. Realize R_f as the inverse image of a closed set under a Borel map. □

23.7.2 Lower and upper semicontinuity

Definition 23.65. Let X be a topological space. A function $f : X \to \overline{\mathbb{R}}$ is called **lower semicontinuous** if $f^{-1}(a, +\infty]$ is open for each $a \in \overline{\mathbb{R}}$. We say that f is **upper semicontinuous** if $f^{-1}[-\infty, a)$ is open for each $a \in \overline{\mathbb{R}}$. By Exp. 23.18, semicontinuous functions are Borel functions.

Problem 23.4. Let X be a topological space.

1. Let $A \subset X$. Prove that χ_A is lower semicontinuous iff A is open.
2. Let $(f_i)_{i \in I}$ be a family of lower semicontinuous functions $X \to \overline{\mathbb{R}}$. Let $f(x) = \sup_{i \in I} f_i(x)$. Prove that $f : X \to \overline{\mathbb{R}}$ is lower semicontinuous.

Recall Pb. 8.2 for the basic properties of $\limsup$ and $\liminf$.

Problem 23.5. Let X be a topological space. Let $f : X \to \overline{\mathbb{R}}$. Prove that the following are equivalent.

(1) f is lower semicontinuous.

(2) For each $x \in X$ and each net (x_α) in X converging to x, we have $\limsup_\alpha f(x_\alpha) \geqslant f(x)$.

(3) For each $x \in X$ and each net (x_α) in X converging to x, we have $\liminf_\alpha f(x_\alpha) \geqslant f(x)$.

Remark 23.66. If X is a metric space, the three conditions in Pb. 23.5 are still equivalent if we replace "each net (x_α)" with "each sequence (x_n)".

Example 23.67. Recall Def. 1.36 for the meaning of summations and multiplications in $\overline{\mathbb{R}}_{\geqslant 0}$. Then the addition map $(a,b) \in \overline{\mathbb{R}}_{\geqslant 0} \times \overline{\mathbb{R}}_{\geqslant 0} \mapsto a+b \in \overline{\mathbb{R}}_{\geqslant 0}$ is continuous. Using Pb. 23.5-(3), one easily checks that the multiplication map $(a,b) \in \overline{\mathbb{R}}_{\geqslant 0} \times \overline{\mathbb{R}}_{\geqslant 0} \mapsto ab \in \overline{\mathbb{R}}_{\geqslant 0}$ is lower semicontinuous. (It is continuous outside $(\{0\} \times \overline{\mathbb{R}}_{\geqslant 0}) \cup (\overline{\mathbb{R}}_{\geqslant 0} \times \{0\})$.)

Remark 23.68. From Pb. 23.5-(3), it is clearly that an $\overline{\mathbb{R}}_{\geqslant 0}$-linear combination of lower semicotinuous functions $X \to \overline{\mathbb{R}}_{\geqslant 0}$ is lower semicontinuous.

Problem 23.6. Let X be a compact topological space, and let $f : X \to \overline{\mathbb{R}}$ be lower (resp. upper) semicontinuous. Show that f attains its minimum (resp. maximum) at some point of X.

23.7.3 ⋆ Weakly measurable functions

Let $(X, \mathfrak{M})$ be a measurable space.

Definition 23.69. Let $\mathcal{V}$ be a normed vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, equipped with the norm topology and the corresponding Borel σ-algebra $\mathfrak{B}_{\mathcal{V}}$. We say that a map $f : X \to \mathcal{V}$ is **weakly measurable** if for every $\varphi \in \mathcal{V}^* = \mathfrak{L}(\mathcal{V}, \mathbb{F})$ the function $\varphi \circ f : X \to \mathbb{F}$ is measurable.

It is clear (from Rem. 23.8) that if f is measurable, then f is weakly measurable.

Exercise 23.70. Let $\mathcal{V}$ be a normed vector space. Let $f : X \to \mathcal{V}$ be a function. Suppose that $\mathcal{W}$ is a linear subspace of $\mathcal{V}$ containing $f(X)$. Prove that $f : X \to \mathcal{V}$ is weakly measurable iff $f : X \to \mathcal{W}$ is weakly measurable.

Hint. Use Hahn-Banach Thm. 16.5. □

Problem 23.7. Let $\mathcal{V}$ be a normed vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $f : X \to \mathcal{V}$. Suppose that $f(X)$ is separable (as a metric subspace of $\mathcal{V}$). Prove that f is measurable iff f is weakly measurable.

Hint. Let $f : X \to \mathcal{V}$ be weakly measurable. Use Exe. 23.70 to show that one can assume WLOG that $\mathcal{V}$ is separable. Then, by Thm. 17.24 and 15.37, there exist countably many elements $\varphi_1, \varphi_2, \dots$ forming a weak-* dense subset of $\overline{B}_{\mathcal{V}^*}(0,1)$. Prove that $\|v\| = \sup_n |\langle \varphi_n, v \rangle|$ for each $v \in \mathcal{V}$. (Hint: Show that v can be viewed as a *continuous* function on $\overline{B}_{\mathcal{V}^*}(0,1)$.) Conclude that for each $v \in \mathcal{V}$, the function $x \in X \mapsto \|f(x) - v\|$ is measurable. Use this fact to show that f is measurable. □

Problem 23.8. Let Y be a metric space. Let (f_n) be a sequence of measurable functions $X \to Y$ converging pointwise to $f : X \to Y$. Assume that $f(X)$ is separable. Prove that f is measurable.

Hint. First method: By Thm. 17.23, Y can be viewed as a metric subspace of a real normed vector space $\mathcal{V}$. Show that $f : X \to \mathcal{V}$ is weakly measurable.

Second method: First prove that for each $y \in Y$, the function $x \in X \mapsto d(f(x), y)$ is measurable. Conclude that the map $f : X \to f(X)$ is measurable. □

In fact, without assuming that $f(X)$ is separable, the statement in Pb. 23.8 is still true, although the proof is more technical. (See the end of Sec. 38 in [Yu].)

Problem 23.9. Let $f : X \to l^2(\mathbb{Z}_+)$ be a map. For each $x \in X$, write $f(x) = (f_1(x), f_2(x), \dots)$. Prove that f is measurable iff $f_n : X \to \mathbb{C}$ is measurable for each $n \in \mathbb{Z}_+$.

23.7.4 ⋆ Outer and inner measures

Let $(X, \mathcal{T}_X)$ be a Hausdorff space. Let $\mu : \mathcal{T}_X \to [0, +\infty]$ and its associated $\mu^*, \mu_* : 2^X \to [0, +\infty]$ be as in Asmp. 23.40. Recall from Thm. 23.53 that μ^* is a measure on the Borel σ-algebra $\mathfrak{B}_X$, and μ^* is denoted by μ when restricted to $\mathfrak{B}_X$.

Problem 23.10. Let E and F be subsets of X. Assume that there are mutually disjoint open subsets $U, V \subset X$ such that $E \subset U$ and $F \subset V$. Prove that $\mu_*(E \cup F) = \mu_*(E) + \mu_*(F)$ and $\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$.

Problem 23.11. Let $E \subset X$. Prove that

$$\mu_*(E) = \sup\big\{\mu(A) : A \in \mathfrak{B}_X \text{ and } A \subset E\big\} \qquad \text{if } \mu^*(E) < +\infty \tag{23.20a}$$

$$\mu^*(E) = \inf\big\{\mu(B) : B \in \mathfrak{B}_X \text{ and } E \subset B\big\} \tag{23.20b}$$

Conclude that if $\Omega \in \mathfrak{B}_X$ satisfies $\mu(\Omega) < +\infty$, then

$$\mu_*(\Omega \cap E) = \mu(\Omega) - \mu^*(\Omega \backslash E) \tag{23.21}$$

This formula generalizes Prop. 23.56.

The following proposition further generalizes (23.21).

Proposition 23.71. *Let $A \subset X$ and $E \subset X$. Assume that $\mu^*(A) < +\infty$. Then*

$$\mu_*(A) \leqslant \mu_*(A \cap E) + \mu^*(A \backslash E) \leqslant \mu^*(A) \tag{23.22}$$

Proof. For each open U containing A and $\mu(U) < +\infty$, we have $\mu_*(A \cap E) + \mu^*(A \backslash E) \leqslant \mu_*(U \cap E) + \mu^*(U \backslash E)$ where the RHS equals $\mu(U)$ by (23.21). Taking $\inf_U$, we get the second "$\leqslant$" of (23.22). For each compact $K \subset A$, by (23.21), we have $\mu(K) = \mu_*(K \cap E) + \mu^*(K \backslash E) \leqslant \mu_*(A \cap E) + \mu^*(A \backslash E)$. Taking $\sup_K$, we get the first "$\leqslant$" of (23.22). □

24 Integrals on measure spaces

The goal of this chapter is to define the integral $\int_X f d\mu$ where (X, μ) is a measure space, and $f : X \to \mathbb{C}$ is measurable. By considering $\mathrm{Re} f$ and $\mathrm{Im} g$ separately, it suffices to define $\int_X f d\mu$ when $f : X \to \mathbb{R}$. We shall first define $\int f$ when $f : X \to [0, +\infty]$. Then we extend the integral to real functions.

24.1 Integrals of extended positive functions

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Recall Def. 1.36 for the addition and multiplication in $\overline{\mathbb{R}}_{\geqslant 0} = [0, +\infty]$.

24.1.1 Integrals of simple functions

Definition 24.1. Let Y be a measurable space. A function $f : X \to Y$ is called a **simple function** if f is measurable and $f(X)$ is a finite set. We let

$$\mathcal{S}(X, Y) = \{\text{simple functions } f \in Y^X\} \tag{24.1}$$

If $Y = [0, +\infty]$, a simple function f is called an **(extended) positive simple function** . This is equivalent to saying that f is of the form

$$f = \sum_{i=1}^{n} a_i \chi_{E_i} \tag{24.2}$$

where $n \in \mathbb{Z}_+$, $a_i \in [0, +\infty]$, and $E_i \in \mathfrak{M}$. Let

$$\mathcal{S}_+(X) = \mathcal{S}(X, \overline{\mathbb{R}}_{\geqslant 0}) = \{\text{extended positive simple functions on } X\} \tag{24.3}$$

which is an $\overline{\mathbb{R}}_{\geqslant 0}$-linear subspace of $[0, +\infty]^X$.

Definition 24.2. For each $f \in \mathcal{S}_+(X)$, define

$$\int_X f \equiv \int_X f d\mu = \sum_{i=1}^{n} a_i \mu(E_i) \tag{24.4}$$

if $f(X) = \{a_1, \ldots, a_n\}$ (where $a_i \neq a_j$ if $i \neq j$) and $E_i = f^{-1}(a_i)$.

To show the linearity of $\int_X$, we need a lemma:

Lemma 24.3. *Suppose that $f = \sum_{i=1}^n a_i \chi_{E_i}$ where $a_i \in \overline{\mathbb{R}}_{\geqslant 0}$, $E_i \in \mathfrak{M}$, and $E_i \cap E_j = \emptyset$ if $i \neq j$. Then $\int_X f = \sum_i a_i \mu(E_i)$.*

Proof. For each $c \in \overline{\mathbb{R}}_{\geqslant 0}$, we let $I_c = \{i \in \mathbb{N} : 1 \leqslant i \leqslant n, a_i = c\}$. Then $f^{-1}(c) = \bigsqcup_{i \in I_c} E_i$. Thus, by the additivity of μ,

$$\int f = \sum_{c \in \overline{\mathbb{R}}_{\geqslant 0}} c \cdot \mu\Big(\bigsqcup_{i \in I_c} E_i\Big) = \sum_c \sum_{i \in I_c} c \cdot \mu(E_i) = \sum_c \sum_{i \in I_c} a_i \cdot \mu(E_i) = \sum_i a_i \mu(E_i)$$

□

Proposition 24.4. *The map $\int_X : \mathcal{S}_+(X) \to \overline{\mathbb{R}}_{\geqslant 0}$ is $\overline{\mathbb{R}}_{\geqslant 0}$-linear.*

Proof. Let $f, g \in \mathcal{S}_+(X)$. Clearly $\int cf = c \int f$ if $c \in \overline{\mathbb{R}}_{\geqslant 0}$. Write $f(X) = \{a_1 < \cdots < a_m\}$ and $g(X) = \{b_1 < \cdots < b_n\}$. Let $E_i = f^{-1}(a_i)$ and $F_j = g^{-1}(b_j)$. Since $X = \bigsqcup_i E_i = \bigsqcup_j F_j$, we have $\chi_{E_i} = \sum_j \chi_{E_i \cap E_j}$ and $\chi_{F_j} = \sum_i \chi_{E_i \cap E_j}$, and hence $f + g = \sum_{i,j}(a_i + b_j)\chi_{E_i \cap F_j}$. Thus, by Lem. 24.3, we get

$$\int (f + g) = \sum_{i,j} (a_i + b_j)\mu(E_i \cap F_j)$$

By the additivity of μ, and by $E_i = \bigsqcup_j E_i \cap F_j$, we have $\mu(E_i) = \sum_j \mu(E_i \cap F_j)$. So

$$\int f = \sum_i a_i \mu(E_i) = \sum_{i,j} a_i \mu(E_i \cap F_j)$$

Similarly, $\int g = \sum_{i,j} b_j \mu(E_i \cap F_j)$. So $\int f + \int g = \sum_{i,j}(a_i + b_j)\mu(E_i \cap F_j)$. □

Corollary 24.5. *If $f, g \in \mathcal{S}_+(X)$ and $f \leqslant g$, then $\int_X f d\mu \leqslant \int_X g d\mu$*

Proof. One easily finds $h \in \mathcal{S}_+(X)$ such that $f + h = g$. So $\int g = \int f + \int h \geqslant \int f$. □

24.1.2 Integrals of measurable functions

Definition 24.6. For each measurable space $(Y, \mathfrak{N})$, we let

$$\mathcal{L}(X, Y) = \{\text{measurable } f \in Y^X\} \tag{24.5}$$

Let $\mathcal{L}_+(X, \mathfrak{M}) \equiv \mathcal{L}_+(X) = \mathcal{L}(X, \overline{\mathbb{R}}_{\geqslant 0})$. In other words,

$$\mathcal{L}_+(X) = \{\text{measurable } f \in [0, +\infty]^X\} \tag{24.6}$$

For each $f \in \mathcal{L}_+(X)$, define the integral

$$\int_X f d\mu = \sup\Big\{\int_X s d\mu : s \in \mathcal{S}_+(X), s \leqslant f\Big\} \tag{24.7}$$

Remark 24.7. It is easy to see that

$$\int_X f d\mu = \sup\left\{\int_X s d\mu : s \in \mathcal{S}(X, \mathbb{R}_{\geqslant 0}), s \leqslant f\right\} \tag{24.8}$$

In other words, to define $\int_X f$, it suffices to consider positive simple functions with finite values.

Remark 24.8. Let $A \in \mathfrak{M}$. Let $\mathfrak{M}|_A$ be the restriction of $\mathfrak{M}$ to A (cf. Def. 23.63). Then $\mathfrak{M}|_A \subset \mathfrak{M}$. So we can restrict μ to $\mathfrak{M}|_A$ so that $(A, \mathfrak{M}|_A, \mu)$ is a measure space. If $f \in \mathcal{L}_+(X)$, noting that $f\chi_A$ is measurable (Cor. 23.22), we clearly have

$$\int_A f|_A d\mu_A = \int_X f\chi_A d\mu \tag{24.9}$$

where $\mu_A : \mathfrak{M}|_A \to \overline{\mathbb{R}}_{\geqslant 0}$ is the restriction of μ. We denote the two sides of (24.9) by $\displaystyle\int_A f$.

We now discuss the linearity of $\int_X$. Clearly $\int_X cf = c\int_X f$ if $c \in [0, +\infty)$. It is not hard to check that $\int_X$ is supperadditive: Suppose that $f, g \in \mathcal{L}_+(X)$. For each simple functions $s \leqslant f$ and $t \leqslant g$ we have $s + t \leqslant f + g$, and hence $\int_X (f + g) \geqslant \int_X (s + t) = \int_X s + \int_X t$. Therefore $\int_X (f + g) \geqslant \int_X f + \int_X g$.

This is not surprising, because the definition clearly suggests that $\int_X$ is actually a lower integral. When f is bounded and $\mu(X) < +\infty$ (which is case that Lebesgue considered in his 1902 paper), we can also define the upper integral

$$\overline{\int_X} f d\mu = \inf\left\{\int_X t d\mu : t \in \mathcal{S}_+(X), f \leqslant t\right\} \tag{24.10}$$

which clearly satisfies the subadditivity $\overline{\int}_X (f + g) \leqslant \overline{\int}_X f + \overline{\int}_X g$. Clearly $\int_X f \leqslant \overline{\int}_X f$. To show the $\overline{\mathbb{R}}_{\geqslant 0}$-linearity of $\int_X$, it suffices to prove that $\int_X f = \overline{\int}_X f$. By scaling f, assume that $f(X) \subset [0, 1]$. For each $k = 1, \dots, n$, let $E_k = f^{-1}(\frac{k-1}{n}, \frac{k}{n}]$. Following Lebesgue's idea of "partitioning the codomain" (cf. Sec. 23.1), define the **Lebesgue sums**

$$s_n = \sum_{k=1}^n \frac{k-1}{n}\chi_{E_k} \qquad t_n = \sum_{k=1}^n \frac{k}{n}\chi_{E_k} \tag{24.11}$$

Then $s_n, t_n \in \mathcal{S}_+(X)$, $s_n \leqslant f \leqslant t_n$, and $\int_X (t_n - s_n) \leqslant \mu(X)/n$. This proves that $\overline{\int}_X f \leqslant \int_X f + \mu(X)/n$ for any n, and hence $\overline{\int}_X f = \int_X f$. The linearity of $\int_X$ on bounded positive measurable functions is thus established. It also shows that $\int_X f$ can be understood as the limit of Lebesgue sums, just as Riemann integrals are the limits of Riemann sums.

It is a subtle task to extend the additivity of $\int_X$ to unbounded functions or to functions f such that $\{x \in X : f(x) > 0\}$ has infinite measures. (Consider for example the case that $f + g = \chi_A$ for some $A \in \mathfrak{M}$ such that $\mu(A) = +\infty$. It is not so obvious why $\mu(A) \leqslant \int f + \int g$ is true.) In the following, we shall study the additivity in a more modern way.

24.1.3 The monotone convergence theorem

Proposition 24.9. *Let $f \in \mathcal{L}_+(X)$. Then there is a sequence $(s_n)_{n\in\mathbb{Z}_+}$ in $\mathcal{S}(X, \mathbb{R}_{\geqslant 0})$ such that $s_1 \leqslant s_2 \leqslant \cdots$ and that $\lim_n s_n$ converges pointwise to f. (In particular, $s_n \leqslant f$.)*

Proof. Choose a strictly increasing homeomorphism $\varphi : [0, +\infty] \xrightarrow{\simeq} [0, 1]$, and let $g = \varphi \circ f$. Similar to (24.11), we let $\sigma_n = \sum_{k=1}^n \frac{k-1}{n}\chi_{E_k}$ where $E_k = g^{-1}(\frac{k-1}{n}, \frac{k}{n}]$. Then $0 \leqslant \sigma_n < 1$, and $\lim_n \sigma_n$ converges uniformly to g (since $\|g - \sigma_n\|_{l^\infty} \leqslant 1/n$). The subsequence $(\sigma_{2^n})_{n\in\mathbb{Z}_+}$ is increasing. Let $s_n = \varphi^{-1} \circ \sigma_{2^n}$. Then (s_n) satisfies the requirement. $\square$

Remark 24.10. The above proof shows that if $\|f\|_{l^\infty} < +\infty$, one can choose an increasing sequence (s_n) in $\mathcal{S}(X, \mathbb{R}_{\geqslant 0})$ converging uniformly to f.

Before proving the linearity of $\int_X$, we first use Prop. 24.9 to give a fun proof of an (almost) special case of Pb. 23.2.

Proposition 24.11. *Let μ be a measure on $\mathfrak{M}$. Let $(\overline{\mathfrak{M}}, \overline{\mu})$ be the completion of $(\mathfrak{M}, \mu)$. Let f be an $\overline{\mathfrak{M}}$-measurable map from X to $[0, +\infty]$ (resp. to $\mathbb{C}$). There there exist $A \in \mathfrak{M}$ with $\mu(X \backslash A) = 0$ such that $f\chi_A$ is $\mathfrak{M}$-measurable.*

Proof. If f is a complex function, by considering $\mathrm{Re}(f)$ and $\mathrm{Im}(g)$ separately, we assume WLOG that f is real. Since $f = f^+ - f^-$ where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ are $\mathfrak{M}$-measurable (by Thm. 23.23), by considering f^+ and f^- separately, it suffices to prove the corollary when $f : X \to [0, +\infty]$.

We first consider the case that $f \in \mathcal{S}_+(X)$. By linearity, it suffices to assume $f = \chi_E$ where $E \in \overline{\mathfrak{M}}$. By Thm. 23.36, we have $B \subset E \subset C$ where $B, C \in \mathfrak{M}$ and $\mu(C \backslash B) = 0$. Set $A = B \cup (X \backslash C)$ and $g = \chi_B$. Then $\mu(X \backslash A) = 0$ and $f\chi_A = \chi_B$ is $\mathfrak{M}$-measurable.

Now consider the general case. By Prop. 24.9, there is an increasing sequence (s_n) of simple functions $X \to \mathbb{R}_{\geqslant 0}$ converging pointwise to f. By the above special case, for each n there exists $A_n \in \mathfrak{M}$ with $\mu(X \backslash A_n) = 0$ such that $s_n\chi_{A_n}$ is $\mathfrak{M}$-measurable. Let $A = \bigcap A_n$. Then $s_n\chi_A = s_n\chi_{A_n} \cdot \chi_A$ is $\mathfrak{M}$-measurable. Since $(s_n\chi_A)$ converges pointwise to $f\chi_A$, by Thm. 23.23, $f\chi_A$ is $\mathfrak{M}$-measurable. $\square$

Theorem 24.12 (Monotone convergence theorem). *Let $(f_n)_{n\in\mathbb{Z}_+}$ be an increasing sequence in $\mathcal{L}_+(X)$. Let f be the pointwise limit $f = \lim_n f_n$, which is in $\mathcal{L}_+(X)$ by Cor. 23.24. Then*

$$\int_X f d\mu = \lim_n \int_X f_n d\mu$$

Proof. We clearly have "$\geqslant$" since $f \geqslant f_n$ implies $\int f \geqslant \int f_n$. To prove the other direction, by Rem. 24.7, it suffices to choose any $s \in \mathcal{S}_+(X)$ satisfying $s \leqslant f$ and $s < +\infty$, and show that $\int_X s \leqslant \lim_n \int_X f_n$. Since for any $0 < \gamma < 1$ we have $\gamma \int_X s = \int_X \gamma s$, by replacing s with γs, we assume that $s(x) < f(x)$ for any $x \in X$ such that $s(x) > 0$.

The idea is to show that for sufficiently large n, we have $f_n > s$ on a sufficiently large region. Write s as a finite sum $s = \sum_i a_i \chi_{E_i}$ where $a_1, a_2, \cdots \in \mathbb{R}_{>0}$, and $E_1, E_2, \cdots \in \mathfrak{M}$ are mutually disjoint. For each i, let

$$E_{i,n} = \{x \in E_i : f_n(x) > s(x)\} = \{x \in E_i : f_n(x) > a_i\}$$

which is in $\mathfrak{M}$. Then $f_n \geqslant \sum_i a_i \chi_{E_{i,n}}$. Thus, by Def. 24.6, we have

$$\int_X f_n \geqslant \sum_i a_i \mu(E_{i,n})$$

Clearly $(E_{i,n})_{n \in \mathbb{Z}_+}$ is increasing. Since $f > s$ on E_i, we have $\bigcup_n E_{i,n} = E_i$. Therefore, by Prop. 23.30-(b) (which is a consequence of the countable additivity of μ), $\lim_n \mu(E_{i,n}) = \mu(E_i)$. So $\lim_n \int_X f_n \geqslant \sum_i a_i \mu(E_i) = \int_X s$ because the map $t \in \mathbb{R}_{\geqslant 0} \mapsto a_i t$ is continuous. $\square$

The countable additivity of μ plays a crucial role in the above proof. Indeed, Prop. 23.30-(b) can be viewed as a special case of the monotone convergence theorem: If (E_n) is an increasing family in $\mathfrak{M}$ and $E = \bigcup_n E_n$, set $f = \chi_E$ and $f_n = \chi_{E_n}$, then $\int f = \lim_n \int f_n$ means precisely $\mu(E) = \lim_n \mu(E_n)$.

We are now ready to prove:

Proposition 24.13. *The map $\int_X : \mathcal{L}_+(X) \to \overline{\mathbb{R}}_{\geqslant 0}$ is $\overline{\mathbb{R}}_{\geqslant 0}$-linear.*

Proof. Let $f, g \in \mathcal{L}_+(X)$. By Prop. 24.9, there are increasing sequences (s_n) and (t_n) in $\mathcal{S}_+(X)$ converging pointwise to f and g respectively. Then $(s_n + t_n)_{n \in \mathbb{Z}_+}$ is increasing and converges pointwise to $f + g$. By the linearity of $\int_X$ on $\mathcal{S}_+(X)$ (Prop. 24.4), we have $\int(s_n + t_s) = \int s_n + \int t_n$. Taking $\lim_n$ and applying the monotone convergence theorem, we have $\int(f + g) = \int f + \int g$. That $\int cf = c \int f$ for all $c \in \overline{\mathbb{R}}_{\geqslant 0}$ can be proved in a similar way using Lem. 24.14. $\square$

Lemma 24.14. *The multiplication map $(a, b) \in \overline{\mathbb{R}}_{\geqslant 0} \times \overline{\mathbb{R}}_{\geqslant 0} \to ab \in \overline{\mathbb{R}}_{\geqslant 0}$ is left continuous, i.e., if $(a_i)_{i \in I}$ and $(b_j)_{j \in J}$ are increasing nets in $\overline{\mathbb{R}}_{\geqslant 0}$ converging to a, b respectively, then $\lim_{i,j} a_i b_j = ab$.*

Proof. This is easy to check. $\square$

Corollary 24.15. *Let $A, B \in \mathfrak{M}$ be disjoint. Let $f \in \mathcal{L}_+(X)$. Then $\int_{A \cup B} f = \int_A f + \int_B f$*

Proof. $\int_{A \cup B} f = \int_X f \chi_{A \cup B} = \int_X (f \chi_A + f \chi_B) = \int_X f \chi_A + \int_X f \chi_B = \int_A f + \int_B f$. $\square$

24.1.4 Criteria for $f = 0$ a.e. and $f < +\infty$ a.e.

Proposition 24.16. *Let $f \in \mathcal{L}_+(X)$. The following are true.*

(a) We have $\int_X f d\mu = 0$ iff $f = 0$ a.e..

(b) Suppose that $\int_X f d\mu < +\infty$. Then $f < +\infty$ a.e.. In other words, there is a null set $\Delta \subset X$ such that $f(x) < +\infty$ for all $x \in X \backslash \Delta$.

Note that part (a) generalizes Exp. 20.14.

Proof. Proof of (a). Assume that $\int_X f > 0$. By the definition of integrals, there is a simple function $\leqslant f$ whose integral is positive. Thus, there exists $E \in \mathfrak{M}$ and $a \in \overline{\mathbb{R}}_{\geqslant 0}$ such that $a\chi_E \leqslant f$, and that $a\mu(E) = \int_X a\chi_E > 0$. So $\mu(E) > 0$ and $a > 0$. Thus f is non-zero on the non-null measurable set E.

Conversely, assume that $\int_X f = 0$. For each $n \in \mathbb{Z}_+$, let $E_n = f^{-1}[1/n, +\infty]$. Then $n^{-1}\chi_{E_n} \leqslant f$ and hence $\int_X f \geqslant \int_X n^{-1}\chi_{E_n} = n^{-1}\mu(E_n)$. So $\mu(E_n) = 0$. So $E = \bigcup_n E_n$ is null, and f is zero outside E.

Proof of (b). Let $I = \int_X f < +\infty$ and assume $I < +\infty$. For each $n \in \mathbb{Z}_+$, let $\Delta_n = f^{-1}[n, +\infty]$. Then $n\chi_{\Delta_n} \leqslant f$, and hence $n\mu(\Delta_n) = \int_X n\chi_{\Delta_n} \leqslant I$. So $\mu(\Delta_n) \leqslant I/n$. Then $f < +\infty$ outside $\Delta = \bigcap_n \Delta_n$ where $\mu(\Delta) \leqslant \mu(\Delta_n) \leqslant I/n$, and hence $\mu(\Delta) = 0$. $\square$

24.2 Integrals of complex functions

Let $(X, \mathfrak{M}, \mu)$ be a measure space. In the last section, we have defined the integral operator $\int_X$ on $\mathcal{L}_+(X)$. Restricting to $\mathcal{L}(X, \mathbb{R}_{\geqslant 0})$, we have an $\mathbb{R}_{\geqslant 0}$-linear map $\int_X \mathcal{L}(X, \mathbb{R}_{\geqslant 0}) \to \overline{\mathbb{R}}_{\geqslant 0}$. In this section, we extend this integral to $\mathcal{L}^1(X, \mu) \to \mathbb{C}$ where

$$\mathcal{L}^1(X, \mu) \equiv \mathcal{L}^1(X, \mu, \mathbb{C}) = \{f \in \mathcal{L}(X, \mathbb{C}) : \|f\|_{L^1} < +\infty\} \tag{24.12}$$

and

$$\|f\|_{L^1} = \int_X |f| d\mu \tag{24.13}$$

is abbreviated to $\|f\|_1$ when no confusion arises. A function $f : X \to \mathbb{C}$ is called **μ-integrable** (or simply **integrable**) if it is in $\mathcal{L}^1(X, \mu)$.

We shall first extend the integral from $\mathcal{L}^1(X, \mu, \mathbb{R}_{\geqslant 0})$ to $\mathcal{L}^1(X, \mu, \mathbb{R})$ using a purely algebraic method, where

$$\mathcal{L}^1(X, \mu, \mathbb{R}) = \{f \in \mathcal{L}(X, \mathbb{R}) : \|f\|_{L^1} < +\infty\}$$
$$\mathcal{L}^1(X, \mu, \mathbb{R}_{\geqslant 0}) = \{f \in \mathcal{L}(X, \mathbb{R}_{\geqslant 0}) : \|f\|_{L^1} < +\infty\}$$

Then we extend it to $\mathcal{L}^1(X, \mu)$ by using Pb. 13.2.

Remark 24.17. $\mathcal{L}^1(X,\mu)$ is a $\mathbb{C}$-linear subspace of $\mathbb{C}^X$.

Proof. Let $f,g\in\mathcal{L}^1(X,\mu)$ and $a\in\mathbb{C}$. Clearly $\int|af|=|a|\int|f|<+\infty$. Since $|f+g|\leqslant|f|+|g|$, we have $\int|f+g|\leqslant\int(|f|+|g|)=\int|f|+\int|g|<+\infty$. □

Definition 24.18. Let V be an $\mathbb{R}$-vector space. A subset $K\subset V$ is called a **convex cone** if K is an $\mathbb{R}_{\geqslant 0}$-linear subspace of V, i.e., for every $u,v\in K$ and $a,b\in\mathbb{R}_{\geqslant 0}$ we have $au+bv\in K$.

Proposition 24.19. *Let K be a convex cone in an $\mathbb{R}$-vector space V. Let $\widetilde{V}$ be a $\mathbb{R}$-linear map. Let $\Gamma:K\to\widetilde{V}$ be an $\mathbb{R}_{\geqslant 0}$-linear map. Suppose that $V=\mathrm{Span}_{\mathbb{R}}K$. Then Γ can be extended uniquely to an $\mathbb{R}$-linear map $\Lambda:V\to\widetilde{V}$.*

Proof. The uniqueness is obvious. To prove the existence, note that any $v\in V$ can be written as

$$v=v^+-v^-$$

where $v^+,v^-\in K$. (Proof: Since $V=\mathrm{Span}_{\mathbb{R}}K$, we have $v=a_1u_1+\cdots+a_mu_m-b_1w_1-\cdots-b_nw_n$ where each u_i,w_j are in K, and each a_i,b_j are in $\mathbb{R}_{\geqslant 0}$. One sets $v^+=\sum_i a_iu_i$ and $v^-=\sum_j b_jw_j$.) We then define $\Lambda(v)=\Gamma(v^+)-\Gamma(v^-)$.

Let us show that this gives a well-defined map $\Lambda:V\to\widetilde{V}$. Assume that $v=w^+-w^-$ where $w^+,w^-\in K$. Then $\Gamma(v^+)-\Gamma(v^-)=\Gamma(w^+)-\Gamma(w^-)$ iff $\Gamma(v^+)+\Gamma(w^-)=\Gamma(v^-)+\Gamma(w^+)$, iff (by the additivity of Γ) $\Gamma(v^++w^-)=\Gamma(v^-+w^+)$. The last statement is true because $v^+-v^-=w^+-w^-$ implies $v^++w^-=v^-+w^+$.

It is easy to see that Λ is additive. If $c\geqslant 0$, then $cv=cv^+-cv^-$ where $cv^+,cv^-\in K$. So $\Lambda(cv)=\Gamma(cv^+)-\Gamma(cv^-)$, which (by the $\mathbb{R}_{\geqslant 0}$-linearity of Γ) equals $c\Gamma(v^+)-c\Gamma(v^-)=c\Lambda(v)$. Since $-v=v^--v^+$, we have $\Lambda(-v)=\Gamma(v^-)-\Gamma(v^+)=-\Lambda(v)$. Hence $\Lambda(-cv)=c\Lambda(-v)=-c\Lambda(v)$. This proves that Λ commutes with the $\mathbb{R}$-multiplication. □

Theorem 24.20. *The integral operator $\int_X:\mathcal{L}^1(X,\mu,\mathbb{R}_{\geqslant 0})\to\mathbb{R}_{\geqslant 0}$ can be extended uniquely to a $\mathbb{C}$-linear map*

$$\int_X:\mathcal{L}^1(X,\mu)\to\mathbb{C}\qquad f\mapsto\int_X fd\mu$$

Proof. $K=\mathcal{L}^1(X,\mu,\mathbb{R}_{\geqslant 0})$ is a convex cone in $V=\mathcal{L}^1(X,\mu,\mathbb{R})$. Note that any $f\in V$ can be written as $f=f^+-f^-$ where

$$f^+(x)=\max\{f(x),0\}\qquad f^-(x)=\max\{-f(x),0\}\tag{24.14}$$

Then $f^\pm$ are measurable (by Thm. 23.23), and $f^\pm\in K$ since $0\leqslant f^\pm\leqslant f$ and (in particular) $\int f^\pm<+\infty$. This proves that $V=\mathrm{Span}_{\mathbb{R}}K$. Therefore, by Prop. 24.19, $\int_X$ can be extended uniquely to an $\mathbb{R}$-linear functional on $\mathcal{L}^1(X,\mu,\mathbb{R})$, i.e.,

$$\int_X f=\int_X f^+-\int_X f^-\tag{24.15}$$

Since $\mathcal{L}^1(X,\mu)$ is $\mathbb{C}$-spanned by $\mathcal{L}^1(X,\mu,\mathbb{R})$ (and hence by K), the extension of $\int_X$ to $\mathcal{L}^1(X,\mu)$ must be unique. We now prove the existence. Let $\Lambda : \mathcal{L}^1(X,\mu) \to \mathbb{R}$ be defined by $\Lambda(f) = \int_X \mathrm{Re}(f)$, noting that $\mathrm{Re}(f) \in \mathcal{L}^1(X,\mu,\mathbb{R})$. Then Λ is $\mathbb{R}$-linear. By Pb. 13.2, we have a $\mathbb{C}$-linear map $\Phi : \mathcal{L}^1(X,\mu) \to \mathbb{C}$ defined by

$$\Phi(f) = \Lambda(f) - \mathbf{i}\Lambda(\mathbf{i}f) = \int_X \mathrm{Re}(f) - \mathbf{i}\int_X \mathrm{Re}(\mathbf{i}f) = \int_X \mathrm{Re}(f) + \mathbf{i}\int_X \mathrm{Im}(f)$$

Then Φ clearly extends $\int_X : \mathcal{L}^1(X,\mu,\mathbb{R}) \to \mathbb{R}$. □

To summarize, the integral $\int_X$ of $f \in \mathcal{L}^1(X,\mu)$ is defined by

$$\int_X f d\mu = \int_X \mathrm{Re}(f)d\mu + \mathbf{i}\int_X \mathrm{Im}(f)d\mu \tag{24.16}$$

Proposition 24.21. *For each $f \in \mathcal{L}^1(X,\mu)$ we have $\Big|\int_X f d\mu\Big| \leqslant \int_X |f|d\mu$.*

Proof. We first assume that f is real. Let $f^\pm$ be defined by (24.14). Then $0 \leqslant f^\pm \leqslant f$. So $0 \leqslant \int f^\pm \leqslant \int f$. Hence, by (24.15), we have $\int f \leqslant \int |f|$. Similarly, $\int(-f) \leqslant \int |f|$. So $|\int f| \leqslant \int |f|$.

Now we consider the general case. We first note that

$$\Big|\mathrm{Re}\int_X f\Big| \leqslant \int_X |f| \tag{24.17}$$

(Namely, $\mathrm{Re}\int_X$ has operator norm $\leqslant 1$ if $\mathcal{L}^1(X,\mu)$ is equipped with the seminorm $\|\cdot\|_{L^1}$.) Indeed, by (24.16), for each $f \in \mathcal{L}^1(X,\mu)$ we have $|\mathrm{Re}\int_X f| = |\int_X \mathrm{Re}(f)|$, which, by the first paragraph, is $\leqslant \int_X |\mathrm{Re}(f)| \leqslant \int_X |f|$.

The following argument is similar to that in the solution of Pb. 13.2-3. Choose $\theta \in \mathbb{R}$ such that $e^{\mathbf{i}\theta}\int f \in \mathbb{R}$. So $\int e^{\mathbf{i}\theta} f \in \mathbb{R}$. By (24.17), we have

$$\Big|\int_X f\Big| = \Big|\int_X e^{\mathbf{i}\theta} f\Big| = \Big|\mathrm{Re}\int_X e^{\mathbf{i}\theta} f\Big| \leqslant \int_X |e^{\mathbf{i}\theta} f| = \int_X |f|$$

□

Corollary 24.22. *Let $f,g \in \mathcal{L}(X,\mathbb{C})$. Assume that there is a measurable $A \subset X$ such that $X\backslash A$ is null, and that $f|_A = g|_A$. Then f is integrable iff g is integrable. If they are integrable, then $\int_X f = \int_X g$.*

Proof. By Prop. 24.16 we have $\int_X |f| = \int_X |f|\chi_A = \int_X |g|\chi_A = \int_X |g|$. Thus $\int |f| < +\infty$ iff $\int |g| < +\infty$. Suppose that $\int |f| < +\infty$. Since $f - g = 0$ a.e., by Prop. 24.16, we have $\int |f-g| = 0$. By Prop. 24.21, we get $\int f - \int g = 0$. □

Example 24.23. Let X be a set, equipped with the counting measure μ on the σ-algebra 2^X (cf. Exp. 23.29). Then for each $f : X \to [0, +\infty]$, we have

$$\sum_{x\in X} f(x) = \sup_{A\in\mathrm{fin}(2^X)} \sum_{x\in A} f(x) = \int_X f d\mu \tag{24.18}$$

Therefore, for each $f \in \mathbb{C}^X$ we have $f \in \mathcal{L}^1(X, \mu)$ iff $\sum_{x\in X} |f(x)| < +\infty$. Namely,

$$\mathcal{L}^1(X, \mu) = l^1(X) \tag{24.19}$$

The linear maps $f \in l^1(X) \mapsto \sum_{x\in X} f(x)$ and $f \in \mathcal{L}^1(X, \mu) \to \int_X f d\mu$ are equal, since they are equal on the subset of all positive l^1-functions (which spans $l^1(X)$).

Proposition 24.24. *Let $f \in \mathscr{R}[a, b] = \mathscr{R}([a, b], \mathbb{C})$ where $-\infty < a < b < +\infty$. Then the Riemann integral of f equals the Lebesgue integral:*

$$\int_a^b f(x)dx = \int_{[a,b]} f dm \tag{24.20}$$

Proof. By considering the real part and the imaginary part separately, it suffices to assume that f is real. Let S be the set of real step functions on $[a, b]$, i.e., $S = \mathrm{Span}_{\mathbb{R}}\{\chi_{[c,d]} : a \leqslant c \leqslant d \leqslant b\}$. Then (24.20) clearly holds when $f \in S$.

Now pick any $f \in \mathscr{R}([a, b], \mathbb{R})$. Since f is strongly Riemann integrable (Def. 13.12), for each $\varepsilon > 0$, there is a partition of $[a, b]$ into compact subintervals $I_1 \cup I_2 \cup \cdots$ such that $\sum_i \mathrm{diam} f(I_i) \cdot |I_i| < \varepsilon$. Let

$$g = \sum_i \inf f(I_i) \cdot \chi_{I_i} \qquad h = \sum_i \sup f(I_i) \cdot \chi_{I_i}$$

Then $g \leqslant f \leqslant h$, and $\int_a^b (h(x) - g(x))dx = \sum_i \mathrm{diam} f(I_i) \cdot |I_i| < \varepsilon$. Since $g, h, h - g \in S$, their Riemann integrals and Lebsgue integrals are equal. (In particular, $\int_{[a,b]}(h - g)dm < \varepsilon$.) Since

$$\int_a^b g(x)dx \leqslant \int_a^b f(x)dx \leqslant \int_a^b h(x)dx \qquad \int_{[a,b]} gdm \leqslant \int_{[a,b]} f dm \leqslant \int_{[a,b]} hdm$$

we have $|\int_a^b f(x)dx - \int_{[a,b]} f dm| < 2\varepsilon$. □

24.3 The convergence theorems

Let $(X, \mathfrak{M}, \mu)$ be a measure space.

Theorem 24.25. *Assume that $\mu(X) < +\infty$. Let (f_α) be a net in $\mathcal{L}^1(X, \mu)$ converging uniformly to $f : X \to \mathbb{C}$. Then $f \in \mathcal{L}^1(X, \mu)$, and $\lim_\alpha \int_X f_\alpha d\mu = \int_X f d\mu$.*

Proof. For each $n \in \mathbb{Z}_+$, there is α_n such that $\|f - f_{\alpha_n}\|_{l^\infty} \leqslant 1/n$. Thus (f_{α_n}) converges uniformly to f. Hence f is measurable (by Cor. 23.24), and $|f| \leqslant |f_{\alpha_1}| + 1$. So $\int |f| \leqslant \int |f_{\alpha_1}| + \mu(X) < +\infty$. This proves $f \in \mathcal{L}^1(X, \mu)$.

For each α, let $\lambda_\alpha = \|f - f_\alpha\|_{l^\infty}$. Then $\int |f - f_\alpha| \leqslant \int \lambda_\alpha = \lambda_\alpha \cdot \mu(X)$. Since $\lim_\alpha \lambda_\alpha = 0$, and since $|\int f - \int f_\alpha| \leqslant \int |f - f_\alpha|$ (by Prop. 24.21), we have $\lim_\alpha |\int f - \int f_\alpha| = 0$. □

24.3.1 The dominated convergence theorem

The following celebrated theorem was proved by Lebesgue in 1904. In fact, in Lebesgue's original theorem, instead of assuming $|f_n| \leqslant g$, the stronger conditions that $\mu(X) < +\infty$ and $\sup_n \|f_n\|_{l^\infty} < +\infty$ were assumed. Moreover, unlike the following proof, Lebesgue's original proof did not use the monotone convergence theorem since the latter was proved by Beppo Levi in 1906. We will explain Lebesgue's original proof in Subsec. 24.4.2.

Theorem 24.26 (Dominated convergence theorem). *Suppose that (f_n) is a sequence in $\mathcal{L}(X, \mathbb{C})$ converging pointwise to $f : X \to \mathbb{C}$. Suppose that there exists $g \in \mathcal{L}^1(X, \mu, \mathbb{R}_{\geqslant 0})$ such that $|f_n| \leqslant g$ for all n. Then $f_n, f \in \mathcal{L}^1(X, \mu)$, and*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

Proof. By Cor. 23.24, f is measurable. Clearly $f_n, f \leqslant g$. So $\int |f_n|, \int |f| \leqslant \int g < +\infty$. Therefore $f_n, f \in \mathcal{L}^1(X, \mu)$. To show that $\lim_n \int f_n = \int f$, since $|\int f_n - \int f| \leqslant \int |f - f_n|$ (by Prop. 24.21), it suffices to prove $\lim_n \int |f_n - f| = 0$. Note that $|f_n - f| \leqslant 2g$. Therefore, by replacing f_n with $|f_n - f|$, it suffices to assume

- (f_n) is a sequence in $\mathcal{L}^1(X, \mu, \mathbb{R}_{\geqslant 0})$ converging pointwise to 0, and there exists $g \in \mathcal{L}^1(X, \mu, \mathbb{R}_{\geqslant 0})$ such that $|f_n| \leqslant g$ for all n.

We shall prove $\lim_n \int_X f_n = 0$.

If (f_n) is decreasing (i.e. $f_1 \geqslant f_2 \geqslant \cdots$), then $g - f_n$ is increasing to g. Since $\int g < +\infty$, and since the monotone convergence theorem implies that $\int g - \int f_n$ converges to $\int g$, we conclude $\lim_n \int f_n = 0$.

In the general case, we let $h_n(x) = \sup_{k \geqslant n}\{f_n(x)\}$. Then h_n is measurable by Thm. 23.23. Since $\lim_n h_n(x) = \limsup_n f_n(x) = 0$, we conclude that (h_n) is a decreasing sequence in $\mathcal{L}^1(X, \mu, \mathbb{R}_{\geqslant 0})$ bounded by g and converging pointwise to 0. Therefore, by the above paragraph, we get $\lim_n \int h_n = 0$. Since $f_n \leqslant h_n$, we have $\lim_n \int f_n = 0$. □

24.3.2 Applications of the dominated convergence theorem

Corollary 24.27. *Let Y be a metric space. Let $f : X \times Y \to \mathbb{C}$. Let $y_0 \in Y$. Assume that the following conditions are satisfied:*

(a) *For each $y \in Y$, the function $f(\cdot, y) : X \to \mathbb{C}$ is in $\mathcal{L}^1(X, \mu)$.*

(b) *For each $x \in X$, the function $f(x, \cdot) : Y \to \mathbb{C}$ is continuous at y_0.*

(c) *There exists $g \in \mathcal{L}^1(X, \mu, \mathbb{R}_{\geqslant 0})$ such that $|f(x, y)| \leqslant g(x)$ for all $x \in X, y \in Y$.*

Then the map $y \in Y \mapsto \int_X f(x, y) d\mu(x)$ is continuous at y_0.

Proof. Choose any sequence (y_n) in Y converging to y_0. Since $|f(\cdot, y_n)| \leqslant g$, and since $\lim_n f(\cdot, y_n)$ converges pointwise to $f(\cdot, y_0)$, by the dominated convergence theorem, we have $\lim_n \int f(\cdot, y_n) d\mu = \int f(\cdot, y_0) d\mu$. □

The following corollary generalizes Thm. 14.34.

Corollary 24.28. *Let $I = [a, b]$ be a compact interval. Assume that $f : X \times I \to \mathbb{C}$ satisfies the following conditions:*

(a) *For each $t \in I$, the function $f(\cdot, y) : X \to \mathbb{C}$ is in $\mathcal{L}^1(X, \mu)$.*

(b) *For each $x \in X$, the function $f(x, \cdot) : I \to \mathbb{C}$ is differentiable.*

(c) *There exists $g \in \mathcal{L}^1(X, \mu, \mathbb{R}_{\geqslant 0})$ such that $|\partial_I f(x, t)| \leqslant g(x)$ for all $x \in X, t \in I$.*

Then $t \in I \to \int_X f(x, t) d\mu(x)$ is differentiable, and its derivative is $\int_X \partial_I f(x, t) d\mu(x)$ where $\partial_I f(\cdot, t) : X \to \mathbb{C}$ is in $\mathcal{L}^1(X, \mu)$.

Proof. Fix $t_0 \in I$. For any sequence in $I \backslash \{t_0\}$ converging to t_0, $\partial_I f(\cdot, t_0)$ is the pointwise limit of the measurable function $h_n = (f(\cdot, t_n) - f(\cdot, t_0))/(t_n - t_0)$ on X, which is measurable by Cor. 23.24. By the finite increment theorem (Cor. 11.29), we have $|h_n| \leqslant g$. Thus $h_n \in \mathcal{L}^1(X, \mu, \mathbb{R}_{\geqslant 0})$. Since h_n converges pointwise to $\partial_I f(\cdot, t_0)$, the dominated convergence theorem shows that $\lim_n \int_X h_n d\mu = \int_X \partial_I f(\cdot, t_0) d\mu$. □

24.3.3 Fatou's lemma

The dominated convergence theorem fails when (f_n) is not bounded by a positive $\mathcal{L}^1$-function.

Example 24.29. Let $f_n : (0, 1) \to \mathbb{C}$ be $f_n = n \cdot \chi_{(0,1/n)}$. Then (f_n) converges pointwise to 0, but $\lim_n \int_{(0,1)} f_n dm = 1$.

Example 24.30. Equip $\mathbb{Z}$ with the counting measure. (Recall Exp. 24.23.) Let $(f_n)_{n \in \mathbb{Z}_+}$ be a sequence in $\overline{B}_{l^2(\mathbb{Z})}(0, 1)$ converging weakly to $f \in \overline{B}_{l^2(\mathbb{Z})}(0, 1)$. In other words, (f_n) converges pointwise to f as a sequence of functions on $\mathbb{Z}$ (cf. Thm. 17.31). By Fatou's lemma for Hilbert spaces (cf. Pb. 21.4), we know that $\liminf_n \|f_n\|^2 \geqslant \|f\|^2$, where the inequality can be strict (e.g. take $f_n = \chi_{\{n\}}$ and $f = 0$). Thus, $g_n = |f_n|^2$ converges pointwise to $g = |f|^2$, but

$$\liminf_n \sum_{\mathbb{Z}} g_n \geqslant \sum_{\mathbb{Z}} g$$

where the inequality could be strict.

Motivated by the above two examples, we prove Fatou's lemma for integrals, which asserts that $\int_X : \mathcal{L}_+(X) \to \overline{\mathbb{R}}_{\geqslant 0}$ is "lower semicontinuous" under *sequential* pointwise convergence. (It is not true for nets of functions. See Rem. 24.33.)

Theorem 24.31 (Fatou's lemma). *Let (f_n) be a sequence in $\mathcal{L}_+(X)$ converging pointwise to $f : X \to \overline{\mathbb{R}}_{\geqslant 0}$. Then $f \in \mathcal{L}_+(X)$, and*

$$\liminf_{n\to\infty} \int_X f_n d\mu \geqslant \int_X f d\mu \tag{24.21}$$

The idea of the proof is to find a sequence (g_n) in $\mathcal{L}_+(X)$ dominated by f and converging pointwise to f such that $g_n \leqslant f_n$. Then, when $\int f < +\infty$, the dominated convergence theorem implies $\lim \int g_n = \int f$, and hence (24.21) follows. When $\int f = +\infty$, this argument needs to be modified.

Proof. By Thm. 23.23 we have $f \in \mathcal{L}_+(X)$ and $g_n \in \mathcal{L}_+(X)$ where $g_n(x) = \min\{f_n(x), f(x)\}$. Note that $0 \leqslant g_n \leqslant f$, and (g_n) converges pointwise to f. So $\liminf_n \int f_n \geqslant \liminf_n \int g_n$. Thus, it suffices to prove

$$\liminf_{n\to\infty} \int_X g_n d\mu \geqslant \int_X f d\mu \tag{24.22}$$

If $\int_X f < +\infty$, the dominated convergence theorem implies $\lim_n \int_X g_n = \int_X f$. So (24.22) is true. Since we do not assume $\int_X f < +\infty$, we need to find another argument.

Let $h_n = \inf_{k\geqslant n} g_k$. Then $h_n \in \mathcal{L}_+(X)$ by Thm. 23.23, and (h_n) converges pointwise to $\liminf_n g_n = f$. Since $h_1 \leqslant h_2 \leqslant \cdots$, by the monotone convergence Thm. 24.12, we get $\lim_n \int_X h_n = \int_X f$. Since $g_n \geqslant h_n$, we get $\liminf_n \int_X g_n \geqslant \lim_n \int_X h_n$. This proves (24.22). [1] $\square$

Exercise 24.32. Although we proved the dominated convergence theorem and Fatou's lemma using the monotone convergence theorem, the latter theorem is an immediate consequence of the first two. Explain the reason.

Remark 24.33. Equip $\mathcal{L}_+(X)$ with the pointwise convergence topology (i.e., the one inherited from $[0, +\infty]^X$). Then the map $\int_X : \mathcal{L}_+(X) \to \overline{\mathbb{R}}_{\geqslant 0}$ is not necessarily lower semicontinuous. For example, take $X = [0, 1]$ and $\mu = m$. Then the net $\{\chi_A\}_{A\in\mathrm{fin}(2^X)}$ converges pointwise to 1. So $\liminf_A \int_X \chi_A = 0 < 1 = \int_X \lim_A \chi_A$. (Recall Pb. 23.5 for the equivalent definition of lower semicontinuity in terms of nets.)

The following version of Fatou's Lemma appears in many textbooks and is a simple corollary of Thm. 24.31. We will not use this version in the future. (The result orginally proved by Fatou is in the form (24.21), not in the form (24.23). See [Haw, Ch. 6, Thm. 6.6].)

[1] This strategy of proving the commutativity of integrals and limits by reducing to monotonic sequences of functions has been used in the proof of the dominated convergence Thm. 24.26.

Corollary 24.34 (Fatou's lemma). *Let (f_n) be a sequence in $\mathcal{L}_+(X)$. Then*

$$\liminf_{n\to\infty}\int_X f_n d\mu \geqslant \int_X \liminf_{n\to\infty} f_n d\mu \tag{24.23}$$

Proof. Let $g_n = \inf_{k\geqslant n} f_n$. Then g_n converges pointwise to $\liminf_n f_n$. By Thm. 24.31, we have $\int_X \lim_n g_n \leqslant \liminf_n \int_X g_n$ where the RHS is $\leqslant \liminf_n \int_X f_n$ since $g_n \leqslant f_n$. □

Thm. 24.31 and Pb. 21.4 (see also Rem. 21.36) suggest that sequential pointwise convergence is related to weak or weak-* convergence. We will discuss this relationship in the future.

24.4 Problems and supplementary material

Let $(X,\mathfrak{M},\mu)$ be a measure space.

Problem 24.1. Let $h \in \mathcal{L}_+(X) = \mathcal{L}_+(X,\mathfrak{M})$, i.e., f is an $\mathfrak{M}$-measurable function $X \to \overline{\mathbb{R}}_{\geqslant 0}$. Prove that there is a unique measure ν on $\mathfrak{M}$ such that

$$\int_X f d\nu = \int_X fh d\mu \tag{24.24}$$

for all $f \in \mathcal{L}_+(X)$. We write

$$d\nu = hd\mu \tag{24.25}$$

Hint. Define $\nu : \mathfrak{M} \to [0,+\infty]$ to be $\nu(E) = \int_X \chi_E h d\mu$. Prove that ν is a measure. First prove (24.24) when f is a simple function. Then prove (24.24) for any $f \in \mathcal{L}_+(X)$. □

Problem 24.2. Let h,ν be as in Pb. 24.1. Let $(\overline{\mathfrak{M}},\overline{\mu})$ be the completion of μ. Let $(\mathfrak{N},\overline{\nu})$ be the completion of ν. Prove that $\overline{\mathfrak{M}} \subset \mathfrak{N}$. Prove that $d\overline{\nu} = hd\overline{\mu}$ on $\overline{\mathfrak{M}}$.

24.4.1 ⋆ Bochner integrals

Let $\mathcal{V}$ be a Banach space over $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$. Recall that $\mathcal{V}^* = \mathfrak{L}(\mathcal{V},\mathbb{F})$ is the dual space. Recall that $\mathcal{L}(X,\mathcal{V})$ is the set of measurable functions $X \to \mathcal{V}$.

Definition 24.35. A function $f : X \to \mathcal{V}$ is called **weakly integrable** if f is weakly measurable (cf. Def. 23.69), and if there exists an element $\int_X f d\mu$ in $\mathcal{V}$ (often abbreviated to $\int_X f$ or $\int f$) such that for every $\varphi \in \mathcal{V}^*$ we have

$$\Big\langle \varphi, \int_X f d\mu \Big\rangle = \int_X \varphi \circ f d\mu \tag{24.26}$$

Note that such vectors $\int_X f d\mu$ are unique because $\mathcal{V}^*$ separates points of $\mathcal{V}$ by the Hahn-Banach Cor. 16.6.

Problem 24.3. Assume that $f : X \to \mathcal{V}$ is weakly integrable, and its absolute value function $|f| : X \to \mathbb{R}_{\geqslant 0}$ is measurable. Prove

$$\Big\| \int_X f d\mu \Big\| \leqslant \int_X |f| d\mu \tag{24.27}$$

Hint. Use the Hahn-Banach Cor. 16.6. □

Pb. 24.3 will be helpful in solving the following problems.

Problem 24.4. Let $\mathcal{W}$ be a Banach space. Let $T \in \mathfrak{L}(\mathcal{V}, \mathcal{W})$. Let $f : X \to \mathcal{V}$. Prove that if f is weakly measurable, then $T \circ f$ is weakly measurable. Prove that if f is weakly integrable, then $T \circ f$ is weakly integrable, and

$$T \int_X f d\mu = \int_X T \circ f d\mu \tag{24.28}$$

Recall that $\mathcal{V}$, as a topological space, is equipped with the Borel σ-algebra $\mathcal{B}_{\mathcal{V}}$. In the following, we let

$$\mathcal{L}_B(X, \mathcal{V}) = \{f \in \mathcal{L}(X, \mathcal{V}) : f(X) \text{ is a separable subset of } \mathcal{V}\} \tag{24.29}$$

(Equivalently, $\mathcal{L}_B(X, \mathcal{V})$ is the set of weakly measurable $f : X \to \mathcal{V}$ with separable $f(X)$. See Pb. 23.7.) A function $f : X \to \mathcal{V}$ is called **Bochner measurable** if $f \in \mathcal{L}_B(X, \mathcal{V})$.[2]

Example 24.36. Suppose that X is a second countable LCH space and $\mathfrak{M} = \mathfrak{B}_X$. Let $f : X \to \mathcal{V}$ be continuous. Then f is measurable. Since X is a union of precompact open subsets, and since X is Lindelöf (by Cor. 8.31), X is a countable union of precompact open subsets $X = \bigcup_n U_n$. Since $f(\overline{U}_n)$ is a compact metric subspace of $\mathcal{V}$, it is separable (Thm. 8.34). Therefore $f(X)$ is a countable union of separable subsets, and hence is separable. This proves $f \in \mathcal{L}_B(X, \mathcal{V})$.

Problem 24.5. Solve the following problems.

1. Let $f \in \mathcal{V}^X$ have separable range $f(X)$. Prove that $f \in \mathcal{L}_B(X, \mathcal{V})$ iff there is a separable closed linear subspace $\mathcal{W} \subset \mathcal{V}$ such that $f(X) \subset \mathcal{W}$ and that the restriction $f : X \to \mathcal{W}$ is measurable.

2. Prove that $\mathcal{L}_B(X, \mathcal{V})$ is an $\mathbb{F}$-linear subspace of $\mathcal{V}^X$.

3. Let $f \in \mathcal{L}_B(X, \mathcal{V})$. Since f is measurable, it is weakly measurable, and $|f| \in \mathcal{L}_+(X)$. Define

$$\|f\|_{L^1} \equiv \|f\|_1 = \int_X |f| d\mu \tag{24.30}$$

[2] In fact, in the usual definition, a function f is called Bochner integrable if $f \in \mathcal{L}(X, \mathcal{V})$, and if $f(X \backslash \Delta)$ is separable for some null set Δ. Here, we do not bother with null sets.

$$\mathcal{L}^1(X,\mu,\mathcal{V}) = \{f \in \mathcal{L}_B(X,\mathcal{V}) : \|f\|_{L^1} < +\infty\} \tag{24.31}$$

Elements in $\mathcal{L}^1(X,\mu,\mathcal{V})$ are called **strongly integrable** (or **Bochner integrable**). Prove that $\mathcal{L}^1(X,\mu,\mathcal{V})$ is a linear subspace of $\mathcal{L}_B(X,\mathcal{V})$. Prove that $\|\cdot\|_{L^1}$ is a seminorm on $\mathcal{L}^1(X,\mu,\mathcal{V})$, i.e., it satisfies

$$\|cf\|_{L^1} = |c| \cdot \|f\|_{L^1} \qquad \|f+g\|_{L^1} \leqslant \|f\|_{L^1} + \|g\|_{L^1} \tag{24.32}$$

for all $f,g \in \mathcal{L}^1(X,\mu,\mathcal{V})$ and $c \in \mathbb{F}$.

4. Let $T \in \mathfrak{L}(\mathcal{V},\mathcal{W})$ where $\mathcal{W}$ is a Banach space. Let $f \in \mathcal{L}_B(X,\mathcal{V})$. Prove that $T \circ f \in \mathcal{L}_B(X,\mathcal{W})$, and

$$\|T \circ f\|_{L^1} \leqslant \|T\| \cdot \|f\|_{L^1} \tag{24.33}$$

Note. When you prove that $\mathcal{L}_B(X,\mathcal{V})$ is closed under addition, pay special attention to the fact that Prop. 23.21 (i.e., f,g measurable $\Rightarrow$ $f \vee g$ measurable) is available only when the codomains of f,g are second countable. □

Theorem 24.37. *Let $f \in \mathcal{L}^1(X,\mu,\mathcal{V})$. Then f is weakly integrable. The integral $\int_X f d\mu$ (which is in $\mathcal{V}$) is called the **Bochner integral** of f.*

Problem 24.6. The goal of this problem is to prove Thm. 24.37. Let

$$\mathcal{S}^1(X,\mu,\mathcal{V}) = \Big\{\sum_{i=1}^n v_i \cdot \chi_{E_i} : n \in \mathbb{Z}_+, v_i \in \mathcal{V}, E_i \in \mathfrak{M}, \mu(E_i) < +\infty\Big\} \tag{24.34}$$

In other words, $\mathcal{S}^1(X,\mu,\mathcal{V})$ is the set of all $f \in \mathcal{S}(X,\mathcal{V})$ such that $\int_X |f| d\mu < +\infty$.

1. Prove that any $f \in \mathcal{S}^1(X,\mu,\mathcal{V})$ is weakly integrable. (What is the explicit expression of $\int_X f$?)
2. Let f be defined by the pointwise limit $f = \sum_{n=1}^\infty v_n \cdot \chi_{E_n}$ where $v_1, v_2, \dots \in \mathcal{V}$, and $E_1, E_2, \dots \in \mathfrak{M}$ are mutually disjoint. Prove that $f \in \mathcal{L}_B(X,\mathcal{V})$, and that

$$\|f\|_{L^1} = \sum_{n=1}^\infty \mu(E_n)\|v_n\| \tag{24.35}$$

3. Prove that $\mathcal{S}^1(X,\mu,\mathcal{V})$ is dense in $\mathcal{L}^1(X,\mu,\mathcal{V})$ under the L^1-seminorm.
4. Use Part 1 and 3 to prove that every $f \in \mathcal{L}^1(X,\mu,\mathcal{V})$ is weakly integrable.

Note: The completeness of $\mathcal{V}$ is only used in part 4.

Hint. Part 2. Use Pb. 23.8 to show that f is measurable.

Part 3. For each $k \in \mathbb{Z}_+$, let $A_k = |f|^{-1}([1/k, k])$ and $f_k = f \cdot \chi_{A_k}$. Show that $\mu(A_k) < +\infty$ and $f_k \in \mathcal{L}^1(X, \mu, \mathcal{V})$. Show that it suffices to approximate each f_k by elements of $\mathcal{S}^1(X, \mu, \mathcal{V})$. To achieve this approximation, for each $\varepsilon > 0$, write $\mathcal{V}$ as a disjoint union of Borel sets whose diameters are $\leqslant \varepsilon$.

Part 4. Pick a sequence (s_n) in $\mathcal{S}^1(X, \mu, \mathcal{V})$ converging to f under the L^1-seminorm. Prove that $(\int_X s_n)_{n\in\mathbb{Z}_+}$ is a Cauchy sequence in $\mathcal{V}$. Let $\int_X f d\mu$ be the limit of this sequence, and show that it satisfies the requirement in Def. 24.35. □

Theorem 24.38 (Dominated convergence theorem). *Let (f_n) be a sequence in $\mathcal{L}^1(X, \mathcal{V})$ converging pointwise to $f : X \to \mathcal{V}$. Suppose that there exists $g \in \mathcal{L}^1(X, \mu, \mathbb{R}_{\geqslant 0})$ such that $|f_n| \leqslant g$ for all n. Then $f \in \mathcal{L}^1(X, \mu, \mathcal{V})$, and*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

Proof. The closure of a countable union of separable subsets of $\mathcal{V}$ is clearly separable. Thus, since $f(X)$ is contained in the closure of $\bigcup_n f_n(X)$, we conclude that $f(X)$ is separable. By Pb. 23.8, f is measurable. Thus $f \in \mathcal{L}_B(X, \mathcal{V})$. Clearly $|f| \leqslant g$. So $f \in \mathcal{L}^1(X, \mu, \mathcal{V})$.

Since $|f - f_n|$ is measurable, and since $|f - f_n| \leqslant 2g$, by the dominated convergence Thm. 24.26, we get $\lim_n \int_X |f - f_n| = 0$. By Pb. 24.3, we have $\| \int_X f - \int_X f_n \| \leqslant \int_X |f - f_n|$. Therefore $\lim_n \| \int_X f - \int_X f_n \| = 0$. □

Exercise 24.39. Extend Cor. 24.27 and 24.28 to Bochner measurable functions.

24.4.2 Lebesgue's proof of dominated convergence theorem

The purpose of this subsection is to present a proof of the dominated convergence theorem that is similar to the original argument of Lebesgue.

Definition 24.40. Let $(f_n)_{n\in\mathbb{Z}_+}$ be a sequence in $\mathcal{L}(X, \mathbb{C})$ (resp. in $\mathcal{L}_+(X)$). Let $f \in \mathcal{L}(X, \mathbb{C})$ (resp. $f = 0$). We say that (f_n) **converges in measure** to f if for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| \geqslant \varepsilon\}) = 0 \tag{24.36}$$

Do not use the monotone convergence theorem in your solutions to Pb. 24.7 and 24.8.

Problem 24.7. Let (f_n) be a sequence in $\mathcal{L}_+(X)$. Consider the following conditions:

(1) $\lim_n \int_X f_n d\mu = 0$.

(2) (f_n) converges in measure to 0.

Prove that (1)⇒(2). Prove that (2)⇒(1) if $\mu(X) < +\infty$ and $\sup_n \|f\|_{l^\infty} < +\infty$.

Problem 24.8. Assume that $\mu(X) < +\infty$. Let (f_n) be a sequence in $\mathcal{L}_+(X)$ converging pointwise to 0.

1. Prove that (f_n) converges in measure to 0 under the assumption that (f_n) is decreasing (i.e. $f_1 \geqslant f_2 \geqslant \cdots$).

2. Without assuming that (f_n) is decreasing, prove that (f_n) converges in measure to 0 by applying part 1 to $h_n(x) = \sup_{k\geqslant n} f_n(x)$.

Note. Can you find the similarity between your solution of Pb. 24.8-1 and the proof of $\lim_n \mu(E_{i,n}) = \mu(E_i)$ in the proof of the monotone convergence Thm. 24.12? □

In Sec. 24.3, we proved the dominated convergence theorem using the monotone convergence theorem. In the following, we give an alternative proof using convergence in measure. First, we prove a special case:

Theorem 24.41 (Bounded convergence theorem). *Let (f_n) be a sequence in $\mathcal{L}(X,\mathbb{C})$ converging pointwise to $f : X \to \mathbb{C}$. Assume that $\mu(X) < +\infty$ and $M := \sup_{n\in\mathbb{Z}_+} \|f_n\|_{l^\infty}$ is finite. Then $f_n, f \in \mathcal{L}^1(X,\mu)$, and $\displaystyle\lim_{n\to\infty}\int_X f_n d\mu = \int_X f d\mu$.*

Proof. Since $0 \leqslant |f_n|, f \leqslant M$, we have $f_n, f \in \mathcal{L}^1(X,\mu)$. Similar to the first paragraph of the proof of Thm. 24.26, by replacing f_n with $|f_n - f|$, it suffices to assume that (f_n) converges pointwise to 0. Then, by Pb. 24.8, (f_n) converges in measure to 0. By Pb. 24.7, we conclude $\lim_n \int_X f_n d\mu = 0$. □

We now use Thm. 24.41 to prove the dominated convergence theorem:

An alternative proof of Thm. 24.26. We are given a sequence (f_n) in $\mathcal{L}(X)$ bounded by $g \in \mathcal{L}^1(X,\mu,\mathbb{R}_{\geqslant 0})$ and converging pointwise to f. Let ν be the measure such that $d\nu = gd\mu$. Then $\nu(X) = \int_X gd\mu < +\infty$. Let $h_n(x) = f_n(x)/g(x)$ and $h(x) = f(x)/g(x)$ if $g(x) \neq 0$, and let $h_n(x) = h(x) = 0$ if $g(x) = 0$. It is easy to see that h_n, h are measurable. Clearly $|h_n| \leqslant 1$, and (h_n) converges pointwise to h. Therefore, by the bounded convergence Thm. 24.41, we have $\lim_n \int_X h_n d\nu = \int_X hd\nu$. Since $h_n g = f_n$ and $hg = f$, we get $\lim_n \int_X f_n d\mu = \int_X f d\mu$. □

Problem 24.9. Assume the setting of the dominated convergence Thm. 24.26, i.e., we are given a sequence (f_n) in $\mathcal{L}(X,\mathbb{C})$ bounded by $g \in \mathcal{L}^1(X,\mu,\mathbb{R}_{\geqslant 0})$ and converging pointwise to f. For each $k \in \mathbb{Z}_+$, let

$$E_k = g^{-1}[1/k, k] \tag{24.37}$$

Prove that $\mu(E_k) < +\infty$ and $\lim_k \int_X |g - g\chi_{E_k}| = 0$. Give another proof of Thm. 24.26 by applying the bounded convergence Thm. 24.41 to the sequence $(f_n|_{E_k})_{n\in\mathbb{Z}_+}$ in $\mathcal{L}(E_k,\mathbb{C})$, where k is "large enough".

Note. The idea of passing from the finite-measure set E_k to the whole set X would be similar to that of solving Pb. 5.9. $\square$

Remark 24.42. The original theorem of dominated convergence theorem proved by Lebesgue (in 1904) is the bounded convergence theorem. More precisely, Lebesgue proved Thm. 24.41 where X is a compact interval $[a, b]$ and $\mu = m$.

The idea of using convergence in measure to prove Thm. 24.41, presented in this subsection, is almost identical to Lebesgue's original idea: In the setting of Thm. 24.41, assume WLOG that $f_n \geqslant 0$ and $f = 0$. Then Lebesgue proved $\lim_n \int f_n = 0$ by proving that for each $\varepsilon > 0$, the μ-measure of

$$E_{n,\varepsilon} = \Big\{x \in X : \sup_{k \geqslant n} f_k(x) \geqslant \varepsilon\Big\} \tag{24.38}$$

converges to 0. (In the language of this subsection, what Lebesgue proved is that $h_n = \sup_{k \geqslant n} f_k$ converges in measure to 0 as $n \to \infty$.) See Sec. 5.1 (especially p. 128) of [Haw].

In fact, Lebesgue's contribution to the convergence theorem is not primarily in the realization that $\lim_n \mu(E_{n,\varepsilon}) = 0$ implies $\lim_n \int f_n = 0$. In 1878, Kronecker already noticed that if (f_n) is a uniformly bounded sequence of continuous functions $[a, b] \to \mathbb{R}_{\geqslant 0}$, then $\lim_n \int f_n = 0$ iff (f_n) "**converges in content**", i.e., $\lim_n c^*(K_{n,\varepsilon}) = 0$ where c^* is the outer content (cf. (23.15)) of

$$K_{n,\varepsilon} = \{x \in X : f_n(x) \geqslant \varepsilon\} \tag{24.39}$$

In 1897, Osgood showed that if (f_n) converges pointwise to 0, then $\lim_n c^*(K_{n,\varepsilon}) = 0$. (Thus, combined with Kronecker's result, one concludes $\lim_n \int f_n = 0$.) The consideration of (24.38) is implicit in Osgood's argument. See [Haw, Sec. 4.4] for details.

Since f_n is continuous, $K_{n,\varepsilon}$ is a closed subset of $[a, b]$ and hence is compact. Therefore, the finite covering property implies $c^*(K_{n,\varepsilon}) = m(K_{n,\varepsilon})$. Therefore, if we look back at history from the perspective of measure theory, it is easy to understand why $\lim_n c^*(K_{n,\varepsilon}) = 0$ if (f_n) converges pointwise to 0.

The real novelty of Lebesgue's theory is that his convergence theorem applies to a broader class of functions whose associated $K_{n,\varepsilon}$ are not necessarily Jordan-measurable, and therefore do not necessarily satisfy $\lim_n c^*(K_{n,\varepsilon}) = 0$ when (f_n) converges pointwise to 0. (Consider for example $f_n = |g_n - g|$ where (g_n) is a uniformly bounded sequence of continuous functions on $[a, b]$ converging pointwise to g. Then g is not necessarily continuous or Riemann-integrable.) Lebesgue developed a theory of measure m satisfying (most importantly) the countable additivity. Therefore, $E_{1,\varepsilon} \supset E_{2,\varepsilon} \supset \cdots$ and $\bigcap_n E_{n,\varepsilon} = \varnothing$ imply $\lim_n m(E_{n,\varepsilon}) = 0$. (One takes $E_{n,\varepsilon}$ to be (24.38).) Thus, since $K_{n,\varepsilon} \subset E_{n,\varepsilon}$, one obtains $\lim_n m(K_{n,\varepsilon}) = 0$, generalizing the arguments of Kronecker and Osgood.

From this observation, it seems fair to say that Lebesgue was not the first to realize that the problem of proving $\lim_n \int f_n = \int \lim_n f_n$ can be transform into proving that the measure/content of $K_{n,\varepsilon}$ converges to 0 as $n \to \infty$ (i.e., proving the convergence in measure). However, Lebesgue was the first person to define the correct measure that allowed this idea to be realized under very loose requirements. □

Remark 24.43. Our proof of the dominated convergence Thm. 24.26 relies on the linearity of $\int_X$, which in turn relies on the monotone convergence theorem proved by Beppo Levi in 1906. (See [Haw, p. 161].) However, when $\mu(X) < +\infty$, the linearity of $\int_X$ on bounded measurable functions can be established without using the monotone convergence theorem. (See Subsec. 24.1.2.) Therefore, Lebesgue's proof of the bounded convergence Thm. 24.41 in 1904 clearly does not rely on the theorem of Beppo Levi (in 1906).

We close this subsection by giving an application of convergence in measure.

Theorem 24.44 (Egorov's theorem). *Assume that $\mu(X) < +\infty$. Let $\mathcal{V}$ be a normed vector space. Assume that (f_n) is a sequence in $\mathcal{L}(X,\mathcal{V})$ converging pointwise to some $f \in \mathcal{L}(X,\mathcal{V})$. Then (f_n) converges* ***almost uniformly*** *to f, which means that for every $\delta > 0$ there exists $A \in \mathfrak{M}$ such that $\mu(X\backslash A) < \delta$, and that (f_n) converges uniformly on A to f.*

Problem 24.10. Prove Egorov's theorem. (Hint: Let $g_n(x) = \sup_{k\geqslant n} \|f(x) - f_k(x)\|$. Then (g_n) converges in measure to 0. For each $k \in \mathbb{Z}_+$, choose $n_k \in \mathbb{Z}_+$ such that $A_k = g_{n_k}^{-1}([0,1/k])$ is large enough. Let $A = \bigcap_n A_n$.)

25 Positive linear functionals and Radon measures

The goal of this chapter is to study a class of Borel measures on LCH spaces generalizing the Lebesgue measure on $\mathbb{R}^N$. Our starting point is an LCH space $(X, \mathcal{T}_X)$ and a linear functional $\Lambda : C_c(X, \mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$ where

$$C_c(X, \mathbb{R}_{\geqslant 0}) = \{f \in C_c(X, \mathbb{R}) : f \geqslant 0\} \tag{25.1}$$

We use Λ to define a function $\mu : \mathcal{T}_X \to \overline{\mathbb{R}}_{\geqslant 0}$ in the same way that we define the Lebesgue measures of open subsets of $\mathbb{R}^N$. Namely, for each $U \in \mathcal{T}_X$ we let

$$\mu(U) = \sup\{\Lambda(f) : f \in C_c(U, [0, 1])\}$$

(Recall that a compactly supported continuous function on U is equivalent to a continuous function on X with compact support in U, cf. Rem. 15.19.) Then we use Thm. 23.53 to extend μ to a measure on the Borel σ-algebra $\mathfrak{B}_X$. Such measure will be called a **Radon measure**.

In fact, we shall first define Radon measure to be a Borel measure satisfying certain regular conditions, and then show that they correspond bijectively to linear functionals on $C_c(X, \mathbb{R}_{\geqslant 0})$.

25.1 Radon measures

Let $(X, \mathcal{T}_X)$ be an LCH space with topology $\mathcal{T}_X$. Recall Rem. 15.20 for the two equivalent descriptions of precompact subsets (and their closures) of an open $U \subset X$. Recall that every open subset of an LCH space is LCH (cf. Prop. 8.41).

Definition 25.1. Let $\mathfrak{M} \subset 2^X$ be a σ-algebra containing $\mathfrak{B}_X$. Let $\mu : \mathfrak{M} \to \overline{\mathbb{R}}_{\geqslant 0}$ be a measure. Let $E \in \mathfrak{M}$. We say that μ is **outer regular** on E (or that E is **outer μ-regular**) if

$$\mu(E) = \inf\{\mu(U) : U \supset E, U \text{ is open}\}$$

We say that μ is **inner regular** on E (or that E is **inner μ-regular**) if

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ is compact}\}$$

We say that μ is **regular** on E (or that E is **μ-regular**) if μ is outer regular and inner regular on E.

Example 25.2. Assume that $\mu : \mathcal{T}_X \to [0, +\infty]$ satisfies conditions (a)-(e) in Asmp. 23.40. Define $\mu^* : 2^X \to [0, +\infty]$ by (23.9). Then by Thm. 23.53, μ^* restricts to a complete measure on $\mathfrak{M}_\mu$ containing $\mathfrak{B}_X$, and $(\mathfrak{M}_\mu, \mu)$ is *defined to be* $(\mathfrak{M}_\mu, \mu^*)$. Then μ is clearly outer regular on any $E \in \mathfrak{M}_\mu$, and is inner regular on open sets by condition (e) of Asmp. 23.40.

For each $E \in \mathfrak{M}_\mu$, the meaning of μ-regularity in Def. 23.43 clearly agrees with the meaning in Def. 25.1: The former says $\mu^*(E) = \mu_*(E)$, and the latter says $\mu^*(E) = \mu(E)$ (which is a tautology) and $\mu_*(E) = \mu(E)$. [1] However, Def. 25.1 cannot be applied to non-measurable sets, but Def. 23.43 can be applied to any subset of X. $\square$

The inner regularity on open sets can also be described in the following way:

Lemma 25.3. *Let μ be a measure on $\mathfrak{B}_X$. Let $U \in \mathcal{T}_X$. Then*

$$\sup\left\{\mu(K) : K \subset U, K \text{ is compact}\right\} = \sup\left\{\int_X f d\mu : f \in C_c(U,[0,1])\right\} \tag{25.2}$$

Proof. Let A and B denote the LHS and the RHS. If $f \in C_c(U,[0,1])$, then $K = \mathrm{Supp}(f)$ is compact in U. So $0 \leqslant f \leqslant \chi_K$, and hence $\mu(K) = \int_X \chi_K \geqslant \int_X f d\mu$. This proves $A \geqslant B$.

Conversely, let $K \subset U$ be compact. By Urysohn's lemma (Thm. 15.25), there exists $f \in C_c(U,[0,1])$ such that $f|_K = 1$. So $\chi_K \leqslant f$, and hence $\mu(K) \leqslant \int_X f d\mu$. This proves $A \leqslant B$. $\square$

In the definition of Radon measures, compact sets are assumed to have finite measures. This property has an equivalent description:

Lemma 25.4. *Let μ be a measure on $\mathfrak{B}_X$. Then $\mu(K) < +\infty$ for all compact $K \subset X$ iff $\displaystyle\int_X f d\mu < +\infty$ for all $f \in C_c(X,\mathbb{R}_{\geqslant 0})$.*

Proof. Suppose that $\mu(K) < +\infty$ for each compact $K \subset X$. Then for each $f \in C_c(X,\mathbb{R}_{\geqslant 0})$, letting $K = \mathrm{Supp}(f)$ and $M = \|f\|_{l^\infty}$, we have $f \leqslant M\chi_K$, and hence $\int f \leqslant \int M\chi_K = M\mu(K) < +\infty$. Conversely, assume that $\int f < +\infty$ for every $f \in C_c(X,\mathbb{R}_{\geqslant 0})$. Let $K \subset X$ be compact. By Urysohn's lemma, there exists $f \in C_c(X,[0,1])$ such that $f|_K = 1$. So $\chi_K \leqslant f$, and hence $\mu(K) = \int \chi_K \leqslant \int f < +\infty$. $\square$

Definition 25.5. A Borel measure $\mu : \mathfrak{B}_X \to \overline{\mathbb{R}}_{\geqslant 0}$ is called a **Radon measure** if the following conditions are satisfied:

(a) μ is outer regular on Borel sets.

(b) μ is inner regular on open sets. Equivalently (by Lem. 25.3), for each open $U \subset X$, we have

$$\mu(U) = \sup\left\{\int_X f d\mu : f \in C_c(U,[0,1])\right\} \tag{25.3}$$

[1] Note that μ_* is defined in terms of the outer measures $\mu^*(K)$ of compact sets K. But since $K \in \mathfrak{B}_X \subset \mathfrak{M}_\mu$, we know $\mu^*(K) = \mu(K)$.

(c) $\mu(K) < +\infty$ if K is a compact subset of X. Equivalently (by Lem. 25.4), for each $f \in C_c(X, \mathbb{R}_{\geqslant 0})$ we have

$$\int_X f d\mu < +\infty \tag{25.4}$$

Example 25.6. A (finite) $\mathbb{R}_{\geqslant 0}$-linear combination of Radon measures on X is a Radon measure.

Example 25.7. Suppose that $\mu : \mathfrak{B}_X \to [0, +\infty]$ is a Radon measure. Let U be an open subset of X (which is LCH by Prop. 8.41). Then the restriction of μ to $\mathfrak{B}_U$ is clearly a Radon measure on U.

Example 25.8. Let $x_0 \in X$. The Dirac measure δ_{x_0} is Radon when restricted to $\mathfrak{B}_X$.

Example 25.9. Assume that $\mathcal{T}_X$ is the discrete topology, i.e., $\mathcal{T}_X = 2^X$. (So compact sets are exactly finite sets.) The counting measure is Radon.

Example 25.10. The Lebesgue measure m on $\mathbb{R}^N$ is Radon when restricted to $\mathfrak{B}_{\mathbb{R}^N}$. In fact, the outer and inner regularities were explained in Exp. 25.2. Since m is finite on bounded measurable subsets (cf. Thm. 23.39), it is finite on compact sets.

In application, it is often more convenient to consider the completion of a Radon measure:

Exercise 25.11. Assume that $(\mathfrak{M}, \mu)$ is the completion of a Radon measure on X. Show that μ is outer regular on any $E \in \mathfrak{M}$.

One of the most important features of Radon measures is that they are determined by the integral of functions in $C_c(X, \mathbb{R}_{\geqslant 0})$.

Proposition 25.12. *Let μ_1, μ_2 be Radon measures on $\mathfrak{B}_X$. Suppose that for each $f \in C_c(X, \mathbb{R}_{\geqslant 0})$ we have $\int_X f d\mu_1 = \int_X f d\mu_2$. Then $\mu_1 = \mu_2$.*

Proof. By (25.3), we have $\mu_1(U) = \mu_2(U)$ when U is open. By the outer regularity, for each $E \in \mathfrak{B}_X$ we must have $\mu_1(E) = \mu_2(E)$. $\square$

25.2 Extending positive linear functionals from $C_c(X, \mathbb{R}_{\geqslant 0})$ to $\mathrm{LSC}_+(X)$

Fix an LCH space $(X, \mathcal{T}_X)$.

Definition 25.13. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. A linear map $\Lambda : C_c(X, \mathbb{F}) \to \mathbb{F}$ is called a **positive linear functional** if $\Lambda(C_c(X, \mathbb{R}_{\geqslant 0})) \subset \mathbb{R}_{\geqslant 0}$.

Remark 25.14. There exist canonical bijections among:

- $\mathbb{R}_{\geqslant 0}$-linear maps $C_c(X, \mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$
- Positive linear functionals on $C_c(X, \mathbb{R})$.
- Positive linear functionals on $C_c(X) = C_c(X, \mathbb{C})$.

Proof. An $\mathbb{R}_{\geqslant 0}$-linear map $\Lambda : C_c(X, \mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$ can be extended uniquely to a linear map $\Lambda : C_c(X, \mathbb{R}) \to \mathbb{R}$ due to Prop. 24.19. The latter can be extended to a linear functional on $C_c(X)$ by setting $\Lambda(f) = \Lambda(\mathrm{Re} f) + \mathbf{i}\Lambda(\mathrm{Im} f)$ for all $C_c(X)$. (This is similar to the proof of Thm. 24.20. It is also the complexification of the $\mathbb{R}$-linear map $f \in C_c(X) \mapsto \Lambda(\mathrm{Re} f) \in \mathbb{R}$, cf. Pb. 13.2.) □

Remark 25.15. Let $\Lambda : C_c(X, \mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$ be linear. Then Λ is (monotonically) increasing, i.e., if $f, g \in C_c(X, \mathbb{R}_{\geqslant 0})$ and $f \leqslant g$, then $\Lambda(f) \leqslant \Lambda(g)$. This is because $g - f \in C_c(X, \mathbb{R}_{\geqslant 0})$ and $\Lambda(g) = \Lambda(f) + \Lambda(g - f) \geqslant \Lambda(f)$.

25.2.1 Toward the proof of the Riesz-Markov representation theorem

As mentioned at the beginning of this chapter, our goal is to construct a Radon measure μ associated to a linear $\Lambda : C_c(X, \mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$ in the same way that we constructed the Lebesgue measure from the Riemann integrals of continuous compactly supported functions on $\mathbb{R}^N$. Thus, Λ can be viewed as an "abstract Riemann integral" on X. This correspondence between Λ and μ is called the **Riesz-Markov representation theorem**.

There are two main difficulties in the proof of Riesz-Markov. The first one is the construction of μ. But we have already studied this part in detail in Sec. 23.5. The second one is to show that the Radon measure μ constructed from Λ satisfies

$$\Lambda(f) = \int_X f d\mu \tag{25.5}$$

for all $f \in C_c(X, \mathbb{R}_{\geqslant 0})$. In my opinion, a direct proof of (25.5) is usually very technical, and the idea of the proof is very isolated, making it difficult to connect with the main ideas in measure theory. (The readers can make their own judgment by reading Step X of the proof of Thm. 2.14 in [Rud-R, Ch. 2], or the last part of the proof of Thm. 7.2 in [Fol-R, Ch. 7].)

The goal of this section is to prepare for a more conceptual proof of (25.5) for $f \in C_c(X, \mathbb{R}_{\geqslant 0})$. To motivate our proof, recall that in Prop. 24.24, we proved that the Riemann integral of f equals the Lebesgue integral by sandwiching f between step functions g and h, and show that the Riemann integrals of g and h agree with their Lebesgue integrals. But step functions are usually not continuous functions. Therefore, to generalize the proof of Prop. 24.24, we must first extend Λ to a suitable larger class of positive functions.

Through laborious work, one can extend Λ to a large class of Borel measurable functions without resorting to measures. The value $\Lambda(f)$ of f in this class is called the **Daniell integral** of f. Then one constructs the Radon measure satisfying (25.5). This approach can be found e.g. in [HS, Sec. 9], [HR-1, Sec. 11], and [Ped, Sec. 6.1]. (A systematic treatment of Daniell integrals, not necessarily in the context of LCH spaces, can be found in [Roy, Ch. 16].)

We will partially adopt the idea of Daniell integrals, but will exclude many irrelevant results so that the proofs are as concise and clear as possible. In particular, we will only extend Λ to positive lower semicontinuous functions. This will be sufficient for the purpose of proving (25.5), because functions of the form $\sum_i a_i\chi_{U_i}$ (where $a_i \in \overline{\mathbb{R}}_{\geqslant 0}$ and U_i is open) will play the same role as that of step functions in the proof of Prop. 24.24.

The method of extending Λ to semicontinuous functions was already used by Riesz in [Rie14] to simplify his original proof of the "Riesz representation theorem for $C([a,b],\mathbb{R})$", see Subsec. 25.7.3 for details. In [Rie13], a similar method was used by Riesz to prove the spectral theorem for bounded self-adjoint operators, cf. Sec. 27.7. Therefore, it is very valuable for us to study this method.

25.2.2 Extending positive linear functionals to $\mathrm{LSC}_+(X)$

In this subsection, we fix an $\mathbb{R}_{\geqslant 0}$-linear map $\Lambda : C_c(X,\mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$.

Recall Subsec. 23.7.2 for the basic facts about lower semicontinuous functions. By Rem. 23.68,

$$\mathrm{LSC}_+(X) = \{\text{lower semicontinuous } f : X \to \overline{\mathbb{R}}_{\geqslant 0}\} \tag{25.6}$$

is an $\overline{\mathbb{R}}_{\geqslant 0}$-linear subspace of $[0,+\infty]^X$. It clearly contains $C_c(X,\mathbb{R}_{\geqslant 0})$ as an $\mathbb{R}_{\geqslant 0}$-linear subspace.

Definition 25.16. For each $f \in \mathrm{LSC}_+(X)$, define

$$\Lambda(f) = \sup\{\Lambda(h) : h \in C_c(X,\mathbb{R}_{\geqslant 0}), h \leqslant f\} \tag{25.7a}$$

Equivalently, noting that $\Omega_f = f^{-1}(0,+\infty]$ is open (by the lower semicontinuity of f), define

$$\Lambda(f) = \sup\{\Lambda(h) : h \in C_c(\Omega_f,\mathbb{R}_{\geqslant 0}), h \leqslant f\} \tag{25.7b}$$

This defines a map $\Lambda : \mathrm{LSC}_+(X) \to \overline{\mathbb{R}}_{\geqslant 0}$ which (by Rem. 25.15) extends the original map $\Lambda : C_c(X,\mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$. We call this new Λ the **canonical extension** of the original Λ. The extension Λ is clearly monotonically increasing, i.e., if $f,g \in \mathrm{LSC}_+(X)$ and $f \leqslant g$, then

$$\Lambda(f) \leqslant \Lambda(g)$$

Proof of the equivalence. Let A and B denote respectively the RHS of (25.7a) and (25.7b). We need to prove $A = B$. Since $C_c(\Omega_f,\mathbb{R}_{\geqslant 0})$ is naturally a subspace of

$C_c(X,\mathbb{R}_{\geqslant 0})$ (cf. Rem. 15.19), we have $A \geqslant B$. To prove $A \leqslant B$, we pick any $h \in C_c(X,\mathbb{R}_{\geqslant 0})$ satisfying $h \leqslant f$, and we shall prove that $\Lambda(h) \leqslant B$.

For each $\varepsilon > 0$, let $h_\varepsilon = (h-\varepsilon)^+ = \max\{h-\varepsilon, 0\}$. Then $\mathrm{Supp}(h_\varepsilon)$ is the closure of $h^{-1}(\varepsilon,+\infty]$, which is contained in the closed set $h^{-1}[\varepsilon/2,+\infty]$ and hence in Ω_f. Therefore $h_\varepsilon \in C_c(\Omega_f,\mathbb{R}_{\geqslant 0})$. To prove $\Lambda(h) \leqslant B$, it suffices to prove

$$\lim_{\varepsilon\to 0} \Lambda(h-h_\varepsilon) = 0$$

We shall prove this by using the fact that $0 \leqslant h-h_\varepsilon \leqslant \varepsilon\chi_K$ where $K = \mathrm{Supp}(h)$. If Λ were defined on χ_K and $0 \leqslant \Lambda(\chi_K) < +\infty$, then one could argue that $\Lambda(h-h_\varepsilon) \leqslant \varepsilon\Lambda(\chi_K)$ where the RHS converges to 0 as $\varepsilon \to 0$. Unfortunately, we do not know whether χ_K is in the domain of Λ. To fix this issue, note that by Urysohn's lemma, there exists $\varphi \in C_c(X,[0,1])$ such that $\varphi|_K = 1$. So $h - h_\varepsilon \leqslant \varepsilon\varphi$, and hence

$$0 \leqslant \Lambda(h-h_\varepsilon) \leqslant \Lambda(\varepsilon\varphi) = \varepsilon\Lambda(\varphi)$$

where the RHS converges to 0 as $\varepsilon \to 0$. □

Example 25.17. Let $U \subset X$ be open. By (25.7b) we have

$$\Lambda(\chi_U) = \sup\big\{\Lambda(f) : f \in C_c(U,[0,1])\big\} \tag{25.8}$$

where the RHS will be the definition of the measure $\mu(U)$.

We shall prove that the canonical extension Λ is $\overline{\mathbb{R}}_{\geqslant 0}$-linear using the same strategy in Subsec. 24.1.3 where we proved that $\int_X : \mathcal{L}_+(X) \to \overline{\mathbb{R}}_{\geqslant 0}$ is $\overline{\mathbb{R}}_{\geqslant 0}$-linear. Therefore, we first need to prove:

Theorem 25.18 (Monotone convergence theorem). *Let $(f_\alpha)_{\alpha\in I}$ be an increasing net of elements in $\mathrm{LSC}_+(X)$. (So $f_\alpha \leqslant f_\beta$ if $\alpha \leqslant \beta$.) Let f be the pointwise limit $\lim_\alpha f_\alpha$. Then $f \in \mathrm{LSC}_+(X)$, and*

$$\Lambda(f) = \lim_{\alpha\in I} \Lambda(f_\alpha)$$

The following proof is close in spirit to the proof of the monotone convergence Thm. 24.12.

Proof. Since f is the pointwise supremum $\sup_\alpha f_\alpha$, by Pb. 23.5, we have $f \in \mathrm{LSC}_+(X)$. Since $f \geqslant f_\alpha$, we clearly have $\Lambda(f) \geqslant \lim_\alpha \Lambda(f_\alpha)$. To prove "$\leqslant$", by (25.7b), it suffices to choose any $g \in C_c(\Omega_f,\mathbb{R}_{\geqslant 0})$ (where $\Omega_f = f^{-1}(0,+\infty]$) satisfying $g \leqslant f$, and prove that $\Lambda(g) \leqslant \sup_\alpha \Lambda(f_\alpha)$.

Since $g \leqslant f$ and $f|_K > 0$ where $K = \mathrm{Supp}(g)$, we have $\gamma g|_K < f|_K$ where $0 < \gamma < 1$. By the linearity of Λ on $C_c(X,\mathbb{R}_{\geqslant 0})$, we have $\gamma\Lambda(g) = \Lambda(\gamma g)$. Thus, it suffices to prove that $\Lambda(\gamma g) \leqslant \sup_\alpha \Lambda(f_\alpha)$ for each γ. Thus, by replacing g with γg, it suffices to assume that $g|_K < f|_K$.

The proof will be finished by finding some $\alpha \in I$ such that $g|_K < f_\alpha|_K$ (and hence $g \leqslant f_\alpha$). This follows from a standard compactness argument: For each $x \in K$, since $g(x) < f(x)$, there exists $\alpha_x \in I$ such that $g(x) < f_{\alpha_x}(x)$. Since g is continuous and f_{α_x} is lower semicontinuous, the function $f_{\alpha_x} - g$ is lower semicontinuous (e.g. by Pb. 23.5-(3)). Therefore $U_x = \{p \in X : f_{\alpha_x}(p) - g(p) > 0\}$ is an open subset of X containing x. Since K is compact, there exist $x_1, \dots, x_n \in K$ such that $K \subset U_{x_1} \cup \cdots \cup U_{x_n}$. Since I is directed, there exists $\alpha \in I$ that is $\geqslant \alpha_{x_1}, \dots, \alpha_{x_n}$. Then $f_\alpha \geqslant f_{\alpha_{x_i}} > g$ on U_{x_i}. Therefore $f_\alpha|_K > g|_K$. $\square$

The following lemma is similar to Prop. 24.9.

Lemma 25.19. *Let $f \in \mathrm{LSC}_+(X)$ and $\Omega_f = f^{-1}(0, +\infty]$. Then there is an increasing net (f_α) in $C_c(\Omega_f, \mathbb{R}_{\geqslant 0})$ converging pointwise to f.*

Proof. Let $\mathscr{I}$ be the set of all $g \in C_c(\Omega_f, \mathbb{R}_{\geqslant 0})$ such that $g \leqslant f$. Then $(\mathscr{I}, \leqslant)$ is a directed set, because if $g_1, g_2 \in \mathscr{I}$ then $\max\{g_1, g_2\} \in \mathscr{I}$. Let us prove that the (clearly increasing) net $(g)_{g \in \mathscr{I}}$ converges pointwise to f. Equivalently, we shall prove for each $x \in X$ that $f(x) = \sup_{g \in \mathscr{I}} g(x)$.

It suffices to prove that for every $\varepsilon > 0$ there exists $g \in \mathscr{I}$ such that $g(x) \geqslant f(x) - \varepsilon$. Assume WLOG that $A := f(x) - \varepsilon$ is > 0. (Otherwise, one can simply take $g = 0$.) Since f is lower semicontinuous, $U = f^{-1}(A, +\infty]$ is a neighborhood of x in X. By Urysohn's lemma, there exists $g \in C_c(U, \mathbb{R}_{\geqslant 0})$ such that $0 \leqslant g \leqslant A$ and $g(x) = A$. Then $g \in \mathscr{I}$. $\square$

Proposition 25.20. *The canonical extension $\Lambda : \mathrm{LSC}_+(X) \to \overline{\mathbb{R}}_{\geqslant 0}$ is $\overline{\mathbb{R}}_{\geqslant 0}$-linear.*

Proof. Choose any $f, g \in \mathrm{LSC}_+(X)$. By Lem. 25.19, there exist increasing nets $(f_\alpha)_{\alpha \in I}$ and $(g_\beta)_{\beta \in J}$ in $C_c(X, \mathbb{R}_{\geqslant 0})$ converging pointwise to f and g respectively. Then $(f_\alpha + g_\beta)_{(\alpha,\beta) \in I \times J}$ is increasing and converges pointwise to $f + g$. By the monotone convergence Thm. 25.18, we have

$$\Lambda(f) + \Lambda(g) = \lim_{\alpha,\beta} \Lambda(f_\alpha) + \lim_{\alpha,\beta} \Lambda(g_\beta) = \lim_{\alpha,\beta} \Lambda(f_\alpha + g_\beta) = \Lambda(f + g)$$

Choose an increasing sequence (c_n) in $\mathbb{R}_{\geqslant 0}$ converging to $c \in \overline{\mathbb{R}}_{\geqslant 0}$. Then by Lem. 24.14 and Thm. 25.18, $c\Lambda(f) = \lim_{\alpha,n} c_n \Lambda(f_\alpha) = \lim_{\alpha,n} \Lambda(c_n f_\alpha) = \Lambda(cf)$. $\square$

25.3 The Riesz-Markov representation theorem

Fix an LCH space $(X, \mathcal{T}_X)$.

Theorem 25.21 (Riesz-Markov representation theorem). *For every $\mathbb{R}_{\geqslant 0}$-linear $\Lambda : C_c(X, \mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$ there exists a unique Radon measure $\mu : \mathfrak{B}_X \to \overline{\mathbb{R}}_{\geqslant 0}$ such that*

$$\Lambda(f) = \int_X f d\mu \tag{25.9}$$

for all $f \in C_c(X, \mathbb{R}_{\geqslant 0})$. Moreover, every Radon measure on X arises from some Λ in this way.

We call μ the **Radon measure associated to** Λ. Note that if (25.9) holds for all $f \in C_c(X, \mathbb{R}_{\geqslant 0})$, then for each $f \in C_c(X)$, since $\int_X |f| d\mu < +\infty$, the RHS of (25.9) can be defined, and (25.9) holds true by the $\mathbb{C}$-linearity.

Proof. The uniqueness follows from Prop. 25.12. Every Radon measure μ arises from the Λ defined by $\Lambda(f) = \int_X f d\mu$ for all $f \in C_c(X, \mathbb{R}_{\geqslant 0})$. Note that $\Lambda(f) < +\infty$ by Lem. 25.4.

We now fix an $\mathbb{R}_{\geqslant 0}$-linear $\Lambda : C_c(X, \mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$, and construct the Radon measure μ satisfying (25.9).

Step 1. Extend Λ canonically to $\Lambda : \mathrm{LSC}_+(X) \to \overline{\mathbb{R}}_{\geqslant 0}$ which is increasing (cf. Def. 25.16) and $\overline{\mathbb{R}}_{\geqslant 0}$-linear (by Prop. 25.20). For each $U \in \mathcal{T}_X$, define

$$\mu(U) = \Lambda(\chi_U)$$

So $\mu(U) = \sup\{\Lambda(f) : f \in C_c(U, [0,1])\}$ by Exp. 25.17. We need to check that $\mu : \mathcal{T}_X \to [0, +\infty]$ satisfies conditions (a)-(e) in Asmp. 23.40. Clearly $\mu(\emptyset) = \Lambda(\chi_\emptyset) = \Lambda(0) = 0$. The monotonicity of μ follows from that of Λ.

Conditions (c) and (d) can be proved in the same way as in Pb. 15.7. But here we provide a different proof without using the partition of unity. Choose countably many open sets $U_1, U_2, \dots$, and let $U = \bigcup_n U_n$. So $\chi_U \leqslant \sum_n \chi_{U_n}$. By the monotone convergence Thm. 25.18 and the linearity of Λ, we have $\Lambda(\sum_n \chi_{U_n}) = \sum_n \Lambda(\chi_{U_n})$. Therefore

$$\mu(U) = \Lambda(\chi_U) \leqslant \Lambda\Big(\sum_n \chi_{U_n}\Big) = \sum_n \Lambda(\chi_{U_n}) = \sum_n \mu(U_n)$$

This proves the countable subadditivity. If $U_1, U_2 \in \mathcal{T}_X$ are disjoint, then $\chi_U = \chi_{U_1} + \chi_{U_2}$ where $U = U_1 \cup U_2$. So

$$\mu(U) = \Lambda(\chi_U) = \Lambda(\chi_{U_1} + \chi_{U_2}) = \Lambda(\chi_{U_1}) + \Lambda(\chi_{U_2}) = \mu(U_1) + \mu(U_2)$$

This proves the additivity. We have finished proving (c) and (d).

Finally, the μ-regularity on any open subset $U \subset X$ (i.e., condition (e)) can be proved in the same way as Lem. 23.38: Clearly $\mu_*(U) \leqslant \mu(U)$. To prove $\mu_*(U) \geqslant \mu(U)$, it suffices to prove $\mu_*(U) \geqslant \Lambda(f)$ for each $f \in C_c(U, [0,1])$. Let $K = \mathrm{Supp}(f)$, which is in U. So $\mu_*(U) \geqslant \mu^*(K)$. Clearly $\mu^*(K) \geqslant \Lambda(f)$ (since $\mu(V) = \Lambda(\chi_V) \geqslant \Lambda(f)$ for any open V containing K). Therefore $\mu_*(U) \geqslant \Lambda(f)$.

We have finished proving that μ satisfies Asmp. 23.40. Therefore, by Thm. 23.53, the outer measure $\mu^* : 2^X \to \overline{\mathbb{R}}_{\geqslant 0}$ (defined by $\mu^*(E) = \inf\{\mu(U) : U \in \mathcal{T}_X, U \supset E\}$) restricts to a measure on $\mathfrak{B}_X$ and is denoted by $(\mathfrak{B}_X, \mu)$.

Since $(\mathfrak{B}_X, \mu)$ is *defined to be* $(\mathfrak{B}_X, \mu^*)$, it is clear that μ is outer regular on Borel sets. For each $U \in \mathcal{T}_X$, we have proved that $\mu(U) = \mu_*(U)$, i.e., that $\mu(U)$ can be approximated by $\mu^*(K)$ where $K \subset U$ is compact. Since $\mu^*(K) = \mu(K)$ (because μ is outer regular on Borel sets), μ is inner regular on any open set U. To prove that μ is Radon, it remains to prove that $\int_X f d\mu < +\infty$ for each $f \in C_c(X, \mathbb{R}_{\geqslant 0})$. This follows from $\int_X f d\mu = \Lambda(f)$, to be proved in the next step.

Step 2. Let us prove (25.9) for every $f \in C_c(X, \mathbb{R}_{\geqslant 0})$. In fact, we shall prove the more general fact that (25.9) is true for all $f \in \mathrm{LSC}_+(X)$.

First, note that if U is open, then $\Lambda(\chi_U) = \mu(U) = \int_X \chi_U d\mu$. Therefore (25.9) holds whenever $f = \chi_U$. By linearity, and by the two monotone convergence theorems (i.e., Thm. 24.12 and 25.18), Eq. (25.9) holds if f is the pointwise limit $f = \sum_{n=1}^{\infty} a_n \chi_{U_n}$ where $a_n \in \overline{\mathbb{R}}_{\geqslant 0}$ and $U_n \in \mathcal{T}_X$. We let $\mathscr{S}$ be the set of all such f.

Now we choose any $f \in \mathrm{LSC}_+(X)$. To prove (25.9), by the two monotone convergence theorems, it suffices to find an increasing sequence $(f_n)_{n\in\mathbb{Z}_+}$ in $\mathscr{S}$ converging pointwise to f.

Choose any $\varepsilon > 0$. For each $k \in \mathbb{N}$, take $E_k = f^{-1}(k\varepsilon, (k+1)\varepsilon]$ and $E_\infty = f^{-1}(+\infty)$. Motivated by the proof of Prop. 24.9, we define $g_\varepsilon : X \to \overline{\mathbb{R}}_{\geqslant 0}$ to be the pointwise limit

$$g_\varepsilon = \sum_{k\in\mathbb{N}} k\varepsilon \cdot \chi_{E_k} + \infty \cdot \chi_{E_\infty} \tag{25.10a}$$

so that $\Lambda(g_\varepsilon)$ can be viewed as an infinite Lebesgue sum. Then $\lim_{\varepsilon\to 0} g_\varepsilon$ converges pointwise to f. Let $f_n = g_{1/2^n}$. Then (f_n) is increasing and converges pointwise to f. To finish the proof, it remains to show that $g_\varepsilon \in \mathscr{S}$ (and hence $f_n \in \mathscr{S}$).

For each $k \in \mathbb{N}$, let $U_k = f^{-1}(k\varepsilon, +\infty]$. Then U_k is open because f is lower semicontinuous. One checks easily that

$$g_\varepsilon = \sum_{k\in\mathbb{Z}_+} \varepsilon \cdot \chi_{U_k} \tag{25.10b}$$

(See also Fig. 25.1.) This proves $g_\varepsilon \in \mathscr{S}$. $\square$

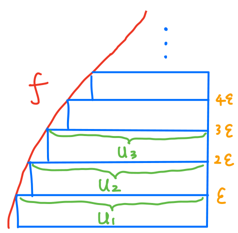


Figure 25.1. g_ε is the sum of all horizontal bars

Theorem 25.22. *Choose an $\overline{\mathbb{R}}_{\geqslant 0}$-linear $\Lambda : C_c(X, \mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$, and let μ be the Radon measure associated to Λ. Extend Λ canonically to $\Lambda : \mathrm{LSC}_+(X) \to \overline{\mathbb{R}}_{\geqslant 0}$. Then for each $f \in \mathrm{LSC}_+(X)$ we have $\Lambda(f) = \displaystyle\int_X f d\mu$.*

Proof. This was proved in Step 2 of the proof of Thm. 25.21. □

Corollary 25.23 (Monotone convergence theorem). *Let μ be a Radon measure on $\mathfrak{B}_X$. Let $(f_\alpha)_{\alpha\in I}$ be an increasing net in $\mathrm{LSC}_+(X)$. Let f be the pointwise limit $\lim_\alpha f_\alpha$. Then $f \in \mathrm{LSC}_+(X)$, and*

$$\int_X f d\mu = \lim_{\alpha\in I} \int_X f_\alpha d\mu$$

Proof. This follows immediately from Thm. 25.22 and the monotone convergence Thm. 25.18. □

For more traditional proofs of the Riesz-Markov representation theorem without resorting to the canonical extension of Λ, see [Rud-R, Thm. 2.14] and [Fol-R, Thm. 7.2]. See also [Fol-R, Prop. 7.12] for a direct proof of Cor. 25.23 without extending Λ.

25.4 Regularity and Lusin's theorem

In this section, we fix an LCH space $(X, \mathcal{T}_X)$, and let $(\mathfrak{M}, \mu)$ be the completion of a Radon measure on X.

Example 25.24. Let $p \in X$, and let $(\mathfrak{M}, \mu)$ be the completion of the Dirac Radon measure $(\mathfrak{B}_X, \delta_p)$. Choose any $E \subset X$. If $p \notin E$, then $E \subset X\backslash\{p\}$ and $X\backslash\{p\}$ is null. So $E \in \mathfrak{M}$ and E is null. If $p \in E$, then $E = \{p\} \sqcup (E\backslash\{p\})$ where $\{p\}$ is Borel and $E\backslash\{p\} \in \mathfrak{M}$. So $E \in \mathfrak{M}$ and $\mu(E) = \mu(\{p\}) = 1$. We conclude that $(\mathfrak{M}, \mu) = (2^X, \delta_p)$.

25.4.1 Regularity=measurability for sets with finite (outer) measures

Theorem 25.25. *Let $E \in \mathfrak{M}$. Then μ is outer regular on E. Moreover, if $\mu(E) < +\infty$, then μ is inner regular on E.*

Proof. Let μ^* and μ_* be as in (23.9). From the Riesz-Markov representation Thm. 25.21, we can assume that $(\mathfrak{B}_X, \mu)$ arises from some linear $\Lambda : C_c(X, \mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$. Now, recall that when we constructed the Radon measure μ from Λ (cf. Step 1 of the proof of Thm. 25.21), we showed that $\mu|_{\mathcal{T}_X}$ satisfies conditions (a)-(e) in Asmp. 23.40 so that we can apply Thm. 23.53 to show that $(\mathfrak{B}_X, \mu^*)$ is a measure, and we *defined* $(\mathfrak{B}_X, \mu)$ to be $(\mathfrak{B}_X, \mu^*)$.

Note that Thm. 23.53 says that $(\mathfrak{M}_\mu, \mu^*)$ is a complete measure where $\mathfrak{M}_\mu$ contains $\mathfrak{B}_X$. Therefore $(\mathfrak{M}_\mu, \mu^*)$ extends the completion $(\mathfrak{M}, \mu)$ of $(\mathcal{B}_X, \mu)$ (cf. Thm. 23.36). Thus, for each $E \in \mathfrak{M}$, we have $\mu^*(E) = \mu(E)$. This proves that E is outer μ-regular. [2]

[2] An alternative proof: Write $E = A \cup B$ where $A \in \mathfrak{B}_X$ and B is a subset of a Borel null set C. Then use the fact that A and C are outer μ-regular (according to the definition of Radon measures).

Assume that $E \in \mathfrak{M}$ satisfies $\mu(E) < +\infty$. So $\mu^*(E) < +\infty$. Since $E \in \mathfrak{M}_\mu$, by Prop. 23.52, we have $\mu^*(E) = \mu_*(E)$ and hence $\mu(E) = \mu_*(E)$. This means that $\mu(E)$ can be approximated from below by $\mu^*(K)$ where $K \subset E$ is compact. Note that $\mu^*(K) = \mu(K)$ by the last paragraph. So μ is inner regular on E. □

Example 25.26. From the proof of Thm. 25.25, if $(\mathfrak{B}_X, \mu)$ is a Radon measure, and if we use Thm. 23.53 to construct a new Borel measure, then this new Borel measure (which is the restriction of μ^* to $\mathfrak{B}_X$) is equal to the original one μ. We now show that if μ is not Radon, the new measure might be different from μ.

We know that the counting measure is a Radon measure if X is equipped with the discrete topology. Now, consider $X = \mathbb{R}$ equipped with the *Euclidean* topology. Let $(\mathfrak{B}_X, \mu)$ be the counting measure which is not Radon. Then the new Borel measure constructed from Thm. 23.53 is $(\mathfrak{B}_X, \mu^*)$ and satisfies $\mu^*(E) = +\infty$ iff $E \neq \emptyset$. So $\mu^*(E) \neq \mu(E)$ if E is a finite set. □

As an application of Thm. 25.25, we give a corollary similar to Cor. 23.49. (In fact, it shows that if $\mathfrak{M}_\mu$ is as in the proof of Thm. 25.25, then for any $E \in \mathrm{fin}(2^X)$ satisfying $\mu^*(E) < +\infty$, we have $E \in \mathfrak{M}_\mu$ iff $E \in \mathfrak{M}$.) We first need a definition.

Definition 25.27. Let Y be a topological space. A subset $E \subset Y$ is called a $\boldsymbol{G_\delta}$ **set** (of Y) if E is a countable intersection of open subsets of Y. E is called an $\boldsymbol{F_\sigma}$ **set** (of Y) if E^c is a G_δ set, equivalently, if E is a countable union of closed subsets of Y. E is called a $\boldsymbol{\sigma}$**-compact** set if E is a countable union of compact sets.[3] If Y is Hausdorff, a σ-compact set is clearly F_σ in Y.

Note that the meanings of G_δ and F_σ depend on the ambient space Y, but the meaning of σ-compactness does not.

⋆ **Exercise 25.28.** Assume that the LCH space X is σ-compact (e.g. when X is second countable, cf. Exp. 25.36). Prove that a subset E of X is σ-compact iff E is an F_σ subset of X.

Corollary 25.29. *Let $E \subset X$ such that E is contained in an open set with finite μ-measure. Then the following are equivalent.*

(1) $E \in \mathfrak{M}$.

(2) *For every $\varepsilon > 0$ there exist a compact set $K \subset X$ and an open set $U \subset X$ such that $K \subset E \subset U$ and $\mu(U \backslash K) < \varepsilon$.*

(2′) *There exist a σ-compact set $A \subset X$ and a G_δ set $B \subset X$ such that $A \subset E \subset B$ and $\mu(B \backslash A) = 0$.*

[3] I wonder why σ-compact sets are not called K_σ sets, or why G_δ sets and F_σ sets are not called δ-open sets and σ-closed sets.

Note that the assumption that E is contained in a finite-measure open set simply means that $\mu^*(E) < +\infty$.

Proof. (1)⇒(2): This is clear from Thm. 25.25.

(2)⇒(2'): Assume (2). For each $n \in \mathbb{Z}_+$, one can choose an open $U_n \supset E$ and a compact $K_n \subset E$ such that $\mu(U_n \backslash K_n) < 1/n$. Take $A = \bigcup_n K_n$ and $B = \bigcap_n U_n$. Since $B \backslash A \subset \bigcap_n (U_n \backslash K_n)$, we have $\mu(B \backslash A) \leqslant \mu(U_n \backslash K_n) < 1/n$ for all n. Therefore $\mu(B \backslash A) = 0$.

(2')⇒(1): This follows immediately from the fact that $A, B \in \mathfrak{B}_X$ and $(\mathfrak{M}, \mu)$ is the completion of $(\mathfrak{B}_X, \mu)$. □

One can also prove (2')⇒(2) directly (by Prop. 23.30-(c)) without first proving (2')⇒(1). Therefore, the readers should regard (2) and (2') as two ways of expressing the same fact. (But note that we do not have (2')⇒(2) when $\mu^*(E) = +\infty$.)

Remark 25.30. Recall that from the definition of measure completion (cf. Thm. 23.36), we know that a set $E \subset X$ belongs to $\mathfrak{M}$ iff one can find Borel sets A, B such that $A \subset E \subset B$ and $\mu(B \backslash A) = 0$. Now, Cor. 25.29 tells us that if $E \in \mathfrak{M}$ and $\mu(E) < +\infty$, then A and B can be chosen to be G_δ and σ-compact, which are much more explicit than general Borel sets.

Corollary 25.31. *Let $E \subset X$. Then the following are equivalent.*

(1) $E \in \mathfrak{M}$, and $\mu(E) = 0$.

(2) There is a G_δ set B containing E such that $\mu(B) = 0$.

Proof. "(2)⇒(1)" is clear due to the completeness of μ. Assume (1). By Thm. 25.25, for each $n \in \mathbb{Z}_+$ there is an open $U_n \supset E$ such that $\mu(U_n) < 1/n$. So E is contained in the G_δ-set $U = \bigcap_n U_n$. Since $\mu(U) \leqslant \mu(U_n) < 1/n$ for all n, we have $\mu(U) = 0$. □

25.4.2 Approximation by continuous functions

As an important application of Thm. 25.25, we prove Lusin's theorem, which says that any Radon-measurable function is "almost continuous" on any measurable subset with finite measure. This result will be used in the future to show that $C_c(X)$ is dense in $L^p(X, \mu)$ if $1 \leqslant p < +\infty$ (cf. Thm. 27.17).

Recall that $(\mathfrak{M}, \mu)$ is the completion of a Radon measure on X.

Theorem 25.32 (Lusin's theorem). *Let $f : X \to \mathbb{C}$ be measurable. Let $A \in \mathfrak{M}$ such that $\mu(A) < +\infty$. Then for every $\varepsilon > 0$ there is a compact $K \subset A$ such that $\mu(A \backslash K) < \varepsilon$, and that $f|_K : K \to \mathbb{C}$ is continuous.*

It follows from the Tietze extension Thm. 15.22 that there exists $g \in C_c(X)$ such that $g|_K = f|_K$, and that $\|g\|_{l^\infty(X)} \leqslant \|f\|_{l^\infty(X)}$.

Proof. By replacing f with $f\chi_A$, we assume that f vanishes outside A.

Case 1: $f = \chi_E$ where $E \in \mathfrak{M}$ and $E \subset A$. Let $F = A\backslash E$. By Thm. 25.25, there exist compact $K_1 \subset E$ and $K_2 \subset F$ such that $\mu(E\backslash K_1) < \varepsilon/2$ and $\mu(F\backslash K_2) < \varepsilon/2$. Let $K = K_1 \cup K_2$. Since $K_1 \cap K_2 = \emptyset$, we have $K_1 = K\backslash K_2$ and $K_2 = K\backslash K_1$. So K_1, K_2 are open subsets of K. Therefore, since $f|_{K_1}$ and $f|_{K_2}$ are constant functions, $f|_K$ is continuous by the local to global principle (Exe. 7.119). Clearly $\mu(A\backslash K) < \varepsilon$.

Case 2: f is a simple function $X \to \mathbb{C}$ vanishing outside A. Then $f = a_1 f_1 + \cdots + a_n f_n$ where $a_i \in \mathbb{C}$ and f_i is as in Case 1. Thus, by Case 1, there exists a compact $K_i \subset A$ with $\mu(A\backslash K_i) < \varepsilon/n$ such that $f_i|_{K_i}$ is continuous. So f is continuous on $K = K_1 \cap \cdots \cap K_n$, and $\mu(A\backslash K) < \varepsilon$.

Case 3: $f \in \mathcal{L}(X, \mathbb{C})$ vanishes outside A, and $\|f\|_{l^\infty} < +\infty$. By considering $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ separately, we assume that f is real. By considering $f^+ = \max\{f, 0\}$ and $f^- = \min\{-f, 0\}$ separately, we assume $f \geqslant 0$. By Rem. 24.10, we can find an increasing sequence of simple functions $s_n : X \to \mathbb{R}_{\geqslant 0}$ converging uniformly to f. Since $s_n \leqslant f$, s_n vanishes outside A. Therefore, by case 2, there is a compact $K_n \subset A$ such that $\mu(A\backslash K_n) < \varepsilon/2^n$, and that $s_n|_{K_n}$ is continuous. Let $K = \bigcap_n K_n$, which is a compact subset of A satisfying $\mu(A\backslash K) < \varepsilon$. Since each s_n is continuous on K, and since (s_n) converges uniformly on K to f, we conclude that f is continuous on K.

Case 4: the general case. For each $n \in \mathbb{Z}_+$, let $E_n = f^{-1}(n, +\infty)$. Since $\bigcap_n E_n = \emptyset$ and $\mu(E_1) \leqslant \mu(A) < +\infty$, we have $\lim_n \mu(E_n) = 0$. Choose n such that $\mu(E_n) < \varepsilon/2$. Let $F_n = A\backslash E_n$. By Case 3, there exists a compact $K \subset F_n$ such that $\mu(F_n\backslash K) < \varepsilon/2$, and that $f|_K$ is continuous. We have $\mu(A\backslash K) < \varepsilon$. $\square$

The converse of Lusin's theorem is given in Pb. 25.2 and Rem. 25.53. The readers should compare Lusin's theorem and its converse with Lebesgue's criterion for Riemann integrable functions (Thm. 14.10).

25.5 Regularity beyond finite measures

Fix an LCH space $(X, \mathcal{T}_X)$, and let $(\mathfrak{M}, \mu)$ be the completion of a Radon measure on X.

We have said that regularity for sets of infinite measures is not as useful as regularity for sets of finite measures. (That's why we introduced local regularity in Subsec. 23.5.3.) But why do we need outer regularity for any measurable set with possibly infinite measure (cf. Thm. 25.25)? Because it shows that the measure μ is determined by its values on open sets. Similarly, why do we need inner regularity for open sets? Because it implies that μ on open sets is determined by the integrals of functions in $C_c(X, \mathbb{R}_{\geqslant 0})$, cf. Lem. 25.3. The regularity on sets with infinite measures is used to ensure certain uniqueness properties of the measures, rather than approximating these sets with open sets or compact sets.

If $E \in \mathfrak{M}$ and $\mu(E) = +\infty$, we cannot find open $U \subset E$ and $K \subset E$ such that $\mu(U \backslash K)$ is small. This is because $\mu(K)$ is finite. Nevertheless, we shall show that in many cases, if we replace K by closed sets, then such an approximation of E is possible. The crucial condition is σ-finiteness:

Definition 25.33. Let $(Y, \mathfrak{N}, \nu)$ be a measure space. We say that ν is ***σ*-finite** on E if E is a countable union of measurable subsets with finite ν-measures. We say that ν is a ***σ*-finite measure** if ν is σ-finite on Y.

Note that any measurable subset of a σ-finite set is σ-finite.

Remark 25.34. The following conditions are equivalent.

(1) μ is a σ-finite measure on $\mathfrak{M}$.

(2) There exists an increasing sequence of open sets $U_1 \subset U_2 \subset \cdots$ such that $X = \bigcup_n U_n$ and that $\mu(U_n) < +\infty$ for each n.

Proof. Clearly (2) implies (1). Assume (1). Then $X = E_1 \cup E_2 \cup \cdots$ where $E_n \in \mathfrak{M}$ and $\mu(E_n) < +\infty$. By Thm. 25.25, μ is outer regular on E_n. So we can find an open $V_n \supset E_n$ such that $\mu(V_n) < +\infty$. Take $U_n = V_1 \cup \cdots \cup V_n$. □

Example 25.35. If X is compact, then μ is finite, and hence is σ-finite.

Example 25.36. Assume that X is second countable. Then X is a countable union of precompact open subsets. In particular, μ is σ-finite on X.

Proof. Since X is LCH, X is a union of precompact open sets. Since X is second countable, X is Lindelöf (by Cor. 8.31). Therefore, X is a countable union of precompact open sets (which have finite measures because μ is Radon). □

We are ready to prove a variant of Cor. 25.29. In this course, the following Thm. 25.37 will only be used in the proof of Thm. 25.38.

Theorem 25.37. *Assume that μ is σ-finite (on X). Let $E \subset X$. Then the following are equivalent.*

(1) $E \in \mathfrak{M}$.

(2) For every $\varepsilon > 0$ there exist a closed set $F \subset X$ and an open set $U \subset X$ such that $F \subset E \subset U$ and $\mu(U \backslash F) < \varepsilon$.

Proof. (2)⇒(1): Similar to the proof of Cor. 25.29, (2) implies that there is an F_σ set A and a G_δ set B such that $A \subset E \subset B$ and $\mu(B \backslash A) = 0$. [4] Since A, B are Borel and $(\mathfrak{M}, \mu)$ is complete, we conclude $E \in \mathfrak{M}$.

[4] We didn't single out this property and call it (2') because we will not use this property in the future of this course, and also because without assuming $\mu^*(E) < +\infty$ as in Cor. 25.29, one cannot prove (2')⇒(2) directly without first proving (2')⇒(1).

(1)⇒(2): Let $E \in \mathfrak{M}$. To prove (2), it suffices to find an open U containing E such that $\mu(U\backslash E) < \varepsilon/2$. Then, a similar argument gives an open set V containing E^c such that $\mu(V\backslash E^c) < \varepsilon/2$. Write $V = X\backslash F$ where F is closed. Then $F \subset E \subset U$ and $\mu(E\backslash F) < \varepsilon/2$. So $\mu(U\backslash F) < \varepsilon$.

Since every σ-finite measure is σ-finite on every measurable subset, we can write E as a countable union $E = \bigcup_n E_n$ where $E_n \in \mathfrak{M}$ and $\mu(E_n) < +\infty$. By Thm. 25.25, there exists an open U_n containing E_n such that $\mu(U_n) < \mu(E_n) + \varepsilon/2^{n+1}$, and hence $\mu(U_n\backslash E_n) < \varepsilon/2^{n+1}$. Let $U = \bigcup_n U_n$. Then $U\backslash E \subset \bigcup_n (U_n\backslash E_n)$, and hence $\mu(U\backslash E) < \varepsilon/2$. □

We will only use Thm. 25.37 in the special case that X is second countable. (Then the σ-finiteness of μ follows automatically.) So the readers may as well assume this slightly stronger condition which was assumed in many previous theorems.

25.6 A criterion for Radon measures

Theorem 25.38. *Let X be a second countable LCH space. Let $\mu : \mathfrak{B}_X \rightarrow [0, +\infty]$ be a measure such that $\mu(K) < +\infty$ for every compact subset $K \subset X$. Then μ is a Radon measure.*

Proof. By Lem. 25.4, we have $\int_X f d\mu < +\infty$ for every $f \in C_c(X, \mathbb{R}_{\geqslant 0})$. Therefore, we have an $\mathbb{R}_{\geqslant 0}$-linear map

$$\Lambda : C_c(X, \mathbb{R}_{\geqslant 0}) \rightarrow \mathbb{R}_{\geqslant 0} \qquad \Lambda(f) = \int_X f d\mu$$

By Riesz-Markov, there is a unique Radon measure λ on X satisfying

$$\Lambda(f) = \int_X f d\lambda$$

for all $f \in C_c(X, \mathbb{R}_{\geqslant 0})$. We shall prove $\mu = \lambda$.

We first prove that $\mu(U) = \lambda(U)$ for any nonempty open $U \subset X$. Since λ is inner regular on open sets, we know that $\lambda(U)$ is the supremum of $\int f d\lambda = \Lambda(f) = \int f d\mu$ for all $f \in C_c(U, [0, 1])$. Suppose that we can prove that μ is inner regular on open sets, then by Lem. 25.3, $\mu(U)$ is also the supremum of $\int f d\mu$ where $f \in C_c(U, [0, 1])$. This proves $\mu(U) = \lambda(U)$.

To prove that μ is inner regular on U, note that since U is LCH (cf. Prop. 8.41) and second countable, by Exp. 25.36, U is a countable union of precompact open subsets. Therefore, we have a sequence of compact subsets (K_n) of U whose union is U. By replacing each K_n with $K_1 \cup \cdots \cup K_n$ we assume that (K_n) is increasing. Therefore $\mu(U) = \lim_n \mu(K_n)$. This proves that $\mu(U)$ can be approximated by the measures of compact subsets. So μ is inner regular on U.

Now we choose any $E \in \mathfrak{B}_X$. Since X is second countable and hence μ is σ-finite (Exp. 25.36), by Thm. 25.37, for each $\varepsilon > 0$ there exist an open $U \subset X$ and a closed $F \subset X$ such that $F \subset E \subset U$ and $\lambda(U \backslash F) < \varepsilon$. So $\lambda(U \backslash E) < \varepsilon$, and hence

$$\lambda(U) - \varepsilon \leqslant \lambda(E) \leqslant \lambda(U) \tag{25.11}$$

Since both U and $U \backslash F$ are open, we have $\mu(U) = \lambda(U)$ and $\mu(U \backslash F) = \lambda(U \backslash F) < \varepsilon$. Therefore $\mu(U \backslash E) < \varepsilon$. Thus $\mu(U) - \varepsilon \leqslant \mu(E) \leqslant \mu(U)$, and hence

$$\lambda(U) - \varepsilon \leqslant \mu(E) \leqslant \lambda(U) \tag{25.12}$$

Since (25.11) and (25.12) hold for every $\varepsilon > 0$, we conclude $\lambda(E) = \mu(E)$. □

Remark 25.39. The readers may wonder if there is a direct proof of Thm. 25.38 without appealing to linear functionals and Riesz-Markov. Note that part of the above proof of Thm. 25.38 shows that every open set U is inner μ-regular. Therefore, it remains to prove that every Borel set is outer μ-regular. A natural idea is to prove that the set of μ-regular Borel sets form a σ-algebra. This idea actually works when μ is a finite measure, and also works when $\mu(X) = +\infty$ if one considers local μ-regular sets instead.

We will present such a proof in Subsec. 25.8.3, and we encourage the readers to read through that subsection. This will help the readers understand why Thm. 25.38 can also be proved by the Riesz-Markov representation Thm. 25.21: It is because the proof of Riesz-Markov relies on Thm. 23.53, and the proof of the latter theorem is similar to the approach in Subsec. 25.8.3. □

Example 25.40. Let X be a second countable LCH space. Let μ be a Radon measure on X. Let $\varphi : X \to \overline{\mathbb{R}}_{\geqslant 0}$ be a Borel function such that $\int_K \varphi d\mu < +\infty$ for every compact $K \subset X$. Then the Borel measure ν defined by $d\nu = \varphi d\mu$ (cf. Pb. 24.1) is finite on compact sets. Therefore, by Thm. 25.38, ν is Radon.

Example 25.41. Let X be a compact metric space. Then, by Thm. 8.34, X is second countable. Therefore, by Thm. 25.38, a Radon measure on X is equivalently a finite Borel measure.

25.7 Stieltjes integrals and Radon measures on $[a, b]$

Let $-\infty < a < b < +\infty$. The original version of the Riesz-Markov representation was proved for $X = [a, b]$ by Riesz in 1909 without appeal to measure theory. This **Riesz representation theorem** says that the (positive) linear functionals on $C[a, b]$ can be realized by Stieltjes integrals. In this section, we will explain how this result follows from the Riesz-Markov representation theorem.

25.7.1 Stieltjes integrals

Recall Subsec. 13.2.1 for the meaning of partitions and their refinements. The set of partitions of $[a,b]$ is denoted by $\mathcal{P}[a,b]$. For simplicity, we define Stieltjes integrals using Darboux sums instead of Riemann sums.

Let $\rho : [a,b] \to \mathbb{R}_{\geqslant 0}$ be increasing (i.e. $\rho(x) \leqslant \rho(y)$ if $x \leqslant y$). Let $f : [a,b] \to \mathbb{R}$ be bounded. For each partition

$$\sigma = \big(\{a_0 = a < a_1 < \cdots < a_n = b\}\big)$$

of $[a,b]$, define the **lower Darboux sum** and the **upper Darbox sum** to be

$$\underline{S}(f,\sigma,\rho) = \sum_{i=1}^{n} (\inf f(a_{i-1},a_i]) \cdot (\rho(a_i) - \rho(a_{i-1})) \tag{25.13a}$$

$$\overline{S}(f,\sigma,\rho) = \sum_{i=1}^{n} (\sup f(a_{i-1},a_i]) \cdot (\rho(a_i) - \rho(a_{i-1})) \tag{25.13b}$$

Clearly $\underline{S}(f,\sigma,\rho) \leqslant \overline{S}(f,\sigma,\rho)$, and $\underline{S}(f,\sigma,\rho)$ increases as σ is refined, and $\overline{S}(f,\sigma,\rho)$ decreases as σ is refined. Let

$$A(f,\sigma) = \sup_i \operatorname{diam} f(a_{i-1},a_i] \tag{25.14}$$

Then

$$\overline{S}(f,\sigma,\rho) - \underline{S}(f,\sigma,\rho) \leqslant A(f,\sigma) \cdot (\rho(b) - \rho(a)) \tag{25.15}$$

Define the **lower Darboux integral** and the **upper Darboux integral** to be

$$\underline{\int_a^b} f d\rho = \sup_{\sigma \in \mathcal{P}[a,b]} \underline{S}(f,\sigma,\rho) \qquad \overline{\int_a^b} f d\rho = \inf_{\sigma \in \mathcal{P}[a,b]} \overline{S}(f,\sigma,\rho) \tag{25.16}$$

Clearly $\underline{S}(f,\sigma,\rho) \leqslant \underline{\int_a^b} f d\rho \leqslant \overline{\int_a^b} f d\rho \leqslant \overline{S}(f,\sigma,\rho)$.

Definition 25.42. We say that a bounded function $f : [a,b] \to \mathbb{R}$ is **Stieltjes integrable** with respect to ρ if the lower integral $\underline{\int_a^b} f d\rho$ is equal to the upper integral $\overline{\int_a^b} f d\rho$. When these two values are equal, we denote them by $\displaystyle\int_a^b f d\rho$ and call it the **Stieltjes integral** of f with respect to ρ.

Theorem 25.43. *Let $\rho : [a,b] \to \mathbb{R}_{\geqslant 0}$ be increasing. Then each $f \in C([a,b],\mathbb{R})$ is Stieltjes integrable with respect to ρ, and*

$$\mathcal{I}_\rho : C([a,b],\mathbb{R}) \to \mathbb{R} \qquad \mathcal{I}_\rho(f) = f(a)\rho(a) + \int_a^b f d\rho \tag{25.17}$$

is a positive linear functional.

The Stieltjes integral $\int_a^b f d\rho$ should be viewed as an integral on $(a, b]$ rather than on $[a, b]$. (This is compatible with the fact that $\int_a^b f d\rho + \int_b^c f d\rho = \int_a^c f d\rho$, because $(a, c] = (a, b] \sqcup (b, c]$ if $c > b$.) This is why we need the extra term $f(a)\rho(a)$ in the definition of $\mathcal{I}_\rho(f)$.

Proof. Define $\underline{\mathcal{I}}_\rho, \overline{\mathcal{I}}_\rho : C([a, b], \mathbb{R}) \to \mathbb{R}$ by $\underline{\mathcal{I}}_\rho(f) = f(a)\rho(a) + \underline{\int_a^b} f d\rho$ and $\overline{\mathcal{I}}_\rho(f) = f(a)\rho(a) + \overline{\int_a^b} f d\rho$. It is easy to check that $\underline{\mathcal{I}}_\rho(cf) = c\underline{\mathcal{I}}_\rho(f)$, $\overline{\mathcal{I}}_\rho(cf) = c\overline{\mathcal{I}}_\rho(f)$, $\underline{\mathcal{I}}_\rho(f + g) \geqslant \underline{\mathcal{I}}_\rho(f) + \underline{\mathcal{I}}_\rho(g)$, $\overline{\mathcal{I}}_\rho(f + g) \leqslant \overline{\mathcal{I}}_\rho(f) + \overline{\mathcal{I}}_\rho(g)$ if $f, g \in C([a, b], \mathbb{R})$ and $c \in \mathbb{R}_{\geqslant 0}$. Also $\underline{\mathcal{I}}_\rho(-f) = -\overline{\mathcal{I}}_\rho(f)$. The linearity of $\mathcal{I}_\rho$ follows immediately if we can prove $\underline{\mathcal{I}}_\rho = \overline{\mathcal{I}}_\rho$, and the positivity is obvious.

Since $f \in C([a, b], \mathbb{R})$ is continuous and $[a, b]$ is compact, f is uniformly continuous. Therefore, for each $\varepsilon > 0$ there exists a partition σ such that $A(f, \rho) \leqslant \varepsilon$, and hence $\overline{\mathcal{I}}_\rho(f) - \underline{\mathcal{I}}_\rho(f) \leqslant \varepsilon(\rho(b) - \rho(a))$ by (25.15). This finishes the proof since ε can be arbitrary. □

Example 25.44. Let $\rho(x) = x$. For each $f \in C([a, b], \mathbb{R})$, $\int_a^b f d\rho$ is the Riemann integral of f.

Example 25.45. Let $c \in [a, b]$ and $\rho = \chi_{[c,b]}$. Let $f : [a, b] \to \mathbb{R}$ be bounded and left continuous at c. Then f is Stieltjes integrable with respect to ρ, and $\mathcal{I}_\rho(f) = f(c)$.

Example 25.46. If f is Stieltjes integrable with respect to ρ_1, ρ_2, then f is Stieltjes integrable with respect to $\rho = k_1\rho_1 + k_2\rho_2$ (where $k_1, k_2 \in \mathbb{R}_{\geqslant 0}$), and

$$\mathcal{I}_\rho(f) = k_1\mathcal{I}_{\rho_1}(f) + k_2\mathcal{I}_{\rho_2}(f)$$

Example 25.47. Let $\{c_1 < \cdots < c_n\} \subset [a, b]$. Let $\rho = \mathrm{id} + \sum_{i=1}^n \chi_{[c_i,b]}$ where $\mathrm{id} : x \in [a, b] \mapsto x \in \mathbb{R}$. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then by the above examples, we have

$$\mathcal{I}_\rho(f) = \int_a^b f dx + \sum_{i=1}^n f(c_i)$$

25.7.2 The Riesz representation theorem

A function $\rho : [a, b] \to \mathbb{R}$ is called **right continuous** if for each $p \in [a, b]$ we have $\lim_{x \to p^+} f(x) = f(p)$. This is equivalent to saying that for each $p \in [a, b]$ and each sequence (x_n) in $[p, b]$ converging to p we have $\lim_n f(x_n) = f(p)$.

The following lemma shows that ρ can be recovered from $\mathcal{I}_\rho$ if ρ is right continuous.

Lemma 25.48. *Let $\rho : [a, b] \to \mathbb{R}_{\geqslant 0}$ be increasing. Assume $a \leqslant c < d \leqslant b$. Let $f \in C([a, b], [0, 1])$ such that $f|_{[a,c]} = 1$ and $f|_{[d,b]} = 0$. Then*

$$\rho(c) \leqslant \mathcal{I}_\rho(f) \leqslant \rho(d) \tag{25.18}$$

Therefore, if ρ is right continuous, and if (f_n) is a sequence in $C([a,b],[0,1])$ such that $f|_{[a,c]} = 1$ and $f|_{[c+1/n,b]} = 0$, then

$$\lim_{n\to\infty} \mathcal{I}_\rho(f_n) = \rho(c) \tag{25.19}$$

Proof. Let $\sigma_1 = \{a,c,b\}$ and $\sigma_2 = \{a,d,b\}$. Then $\underline{S}(f,\sigma_1,\rho) = \rho(c) - \rho(a)$ and $\overline{S}(f,\sigma_2,\rho) = \rho(d) - \rho(a)$. This proves $\rho(c) - \rho(a) \leqslant \int_a^b f d\rho \leqslant \rho(d) - \rho(a)$, and hence proves (25.18).

We now have $\rho(c) \leqslant \mathcal{I}_\rho(f_n) \leqslant \rho(c+1/n)$. Let $n \to \infty$. Then the right continuity of ρ implies (25.19). □

Theorem 25.49 (Riesz representation theorem). *We have a bijection from the set of increasing right continuous functions $[a,b] \to \mathbb{R}_{\geqslant 0}$ to the set of positive linear functionals on $C([a,b],\mathbb{R})$ defined by*

$$\rho \mapsto \mathcal{I}_\rho \tag{25.20}$$

Proof. By Lem. 25.48, the map (25.20) is injective. Let us prove that (25.20) is surjective. Choose any positive linear functional $\Lambda : C([a,b],\mathbb{R}) \to \mathbb{R}$. By the Riesz-Markov representation Thm. 25.21, there is a Radon measure μ on $[a,b]$ such that $\int_{[a,b]} f d\mu = \Lambda(f)$ for all $f \in C([a,b],\mathbb{R})$. Since $[a,b]$ is compact, μ is a finite measure. Define

$$\rho_\mu : [a,b] \to \mathbb{R}_{\geqslant 0} \qquad \rho_\mu(x) = \mu([a,x]) \tag{25.21}$$

Clearly ρ_μ is increasing. If (x_n) is a decreasing sequence in $[x,b]$ converging to x, then $\bigcap_{n\in\mathbb{Z}_+}[a,x_n] = [a,x]$. Therefore $\mu([a,x]) = \lim_n \mu([a,x_n])$. This proves that ρ_μ is right continuous. [5] Let us prove for each $f \in C([a,b],\mathbb{R})$ that $\mathcal{I}_{\rho_\mu}(f) = \Lambda(f)$, i.e., that

$$\mathcal{I}_{\rho_\mu}(f) = \int_{[a,b]} f d\mu \tag{25.22}$$

In the following, we write ρ_μ as ρ for simplicity.

Since f is uniformly continuous, for each $\varepsilon > 0$ there is a partition $\sigma = \{a_0 < \cdots < a_n\}$ of $[a,b]$ such that $A(f,\sigma)$ (defined by (25.14)) satisfies $A(f,\sigma) \leqslant \varepsilon$. In other words, if

$$m_i = \inf f(a_{i-1},a_i] \qquad M_i = \sup f(a_{i-1},a_i]$$

then $M_i - m_i \leqslant \varepsilon$ for all i. Thus, we have (cf. (25.15))

$$\big(f(a)\rho(a) + \overline{S}(f,\sigma,\rho)\big) - \big(f(a)\rho(a) - \underline{S}(f,\sigma,\rho)\big) \leqslant \varepsilon\big(\rho(b) - \rho(a)\big) \tag{25.23}$$

[5] If (x_n) is an increasing sequence in $[a,x)$ converging to x, then $\bigcup_n [a,x_n] = [a,x)$. Thus, ρ might not be left continuous because $\mu([a,x))$ and $\mu([a,x])$ are possibly different.

Our goal is to find bounded Borel functions $g, h : [a,b] \to \mathbb{R}$ such that $g \leqslant f \leqslant h$ and

$$f(a)\rho(a) + \underline{S}(f,\sigma,\rho) = \int_{[a,b]} g d\mu \qquad f(a)\rho(a) + \overline{S}(f,\sigma,\rho) = \int_{[a,b]} h d\mu \tag{25.24}$$

Then both $\int_{[a,b]} f d\mu$ and $\mathcal{I}_\rho(f)$ are between $f(a)\rho(a) + \underline{S}(f,\sigma,\rho)$ and $f(a)\rho(a) + \overline{S}(f,\sigma,\rho)$. Therefore, by (25.23), the difference of $\int_{[a,b]} f d\mu$ and $\mathcal{I}_\rho(f)$ is bounded by $2\varepsilon(\rho(b) - \rho(a))$. This finishes the proof since ε can be arbitrary.

Define

$$g = f(a)\chi_{\{a\}} + \sum_{i=1}^n m_i \cdot \chi_{(a_{i-1},a_i]} \qquad h = f(a)\chi_{\{a\}} + \sum_{i=1}^n M_i \cdot \chi_{(a_{i-1},a_i]} \tag{25.25}$$

Then clearly $g \leqslant f \leqslant h$. Since

$$\mu(\{a\}) = \rho(a) \qquad \mu((a_{i-1},a_i]) = \rho(a_i) - \rho(a_{i-1})$$

(25.24) is clearly satisfied. □

25.7.3 The second proof of the Riesz representation theorem

Since the statement of the Riesz representation Thm. 25.49 does not involve measures, one naturally wonders whether it is possible to prove this theorem without resorting to measures. The answer is yes:

Second proof of Thm. 25.49. By Lem. 25.48, $\rho \mapsto \mathcal{I}_\rho$ is injective. By Prop. 25.20, Λ can be extended (canonically) to an $\mathbb{R}_{\geqslant 0}$-linear map $\Lambda : \mathrm{LSC}_+([a,b]) \to \overline{\mathbb{R}}_{\geqslant 0}$ which is monotonically increasing (i.e. $f \leqslant g$ implies $\Lambda(f) \leqslant \Lambda(g)$). Let

$$\mathscr{C}_1 = \{\text{bounded lower semicontinuous } f : [a,b] \to \mathbb{R}_{\geqslant 0}\}$$

Then for each $f \in \mathscr{C}_1$ we have $f \leqslant \|f\|_{l^\infty}$, and hence $\Lambda(f) \leqslant \|f\|_{l^\infty} \cdot \Lambda(1) < +\infty$. Therefore, we have an increasing $\mathbb{R}_{\geqslant 0}$-linear $\Lambda : \mathscr{C}_1 \to \mathbb{R}_{\geqslant 0}$. Let

$$\mathscr{C}_2 = \mathrm{Span}_{\mathbb{R}} \mathscr{C}_1 = \{f^+ - f^- : f^\pm \in \mathscr{C}_1\}$$

Then by Prop. 24.19, Λ can be extended uniquely to an $\mathbb{R}$-linear functional $\Lambda : \mathscr{C}_2 \to \mathbb{R}$.

This extension is increasing, i.e., if $f, g \in \mathscr{C}_2$ and $f \leqslant g$ then $\Lambda(f) \leqslant \Lambda(g)$. To prove this, it suffices to prove $\Lambda(h) \geqslant 0$ where $h = g - f$. Write $h = h^+ - h^-$ where $h^\pm \in \mathscr{C}_1$. Then $h^+ \geqslant h^-$. Therefore, by the monotonicity of $\Lambda : \mathscr{C}_1 \to \mathbb{R}_{\geqslant 0}$ we have $\Lambda(h^+) \geqslant \Lambda(h^-)$, and hence $\Lambda(h) \geqslant 0$.

For each $x \in [a,b]$, the upper semicontinuous function $\chi_{[a,x]}$ belongs to $\mathscr{C}_2$. Therefore, we can define $\rho(x) = \Lambda(\chi_{[a,x]})$. Then $\rho : [a,b] \to \mathbb{R}_{\geqslant 0}$ is increasing. For each decreasing sequence (x_n) in $[a,b]$ converging to x, the increasing sequence $(1-\chi_{[a,x_n]})_{n\in\mathbb{Z}_+}$ converges pointwise to $1-\chi_{[a,x]}$. Therefore, by the monotone convergence Thm. 25.18, we have $\lim_n(\Lambda(1)-\rho(x_n)) = \Lambda(1)-\rho(x)$. This proves that ρ is right continuous.

Finally, we show that $\Lambda = \mathcal{I}_\rho$ on $C([a,b],\mathbb{R})$. As in the first proof, for each $\varepsilon>0$, choose a partition $\sigma=\{a_0<\cdots<a_n\}$ of $[a,b]$ such that $\mathrm{diam} f(a_{i-1},a_i]\leqslant\varepsilon$ for all i. Let g,h be defined by (25.25). Then $g\leqslant f\leqslant h$ and $g,h\in\mathscr{C}_2$. Therefore, by the monotonicity of $\Lambda:\mathscr{C}_2\to\mathbb{R}_{\geqslant 0}$ proved above, we have $\Lambda(g)\leqslant\Lambda(f)\leqslant\Lambda(h)$. By the definition of ρ, it is easy to see

$$f(a)\rho(a)+\underline{S}(f,\sigma,\rho)=\Lambda(g)\qquad f(a)\rho(a)+\overline{S}(f,\sigma,\rho)=\Lambda(h)$$

Therefore, since $\overline{S}(f,\sigma,\rho)-\underline{S}(f,\sigma,\rho)\leqslant\varepsilon(\rho(b)-\rho(a))$, we conclude that $|\mathcal{I}_\rho(f)-\Lambda(f)|\leqslant 2\varepsilon(\rho(b)-\rho(a))$. Since ε is arbitrary, we get $\mathcal{I}_\rho(f)=\Lambda(f)$. □

Remark 25.50. The Riesz representation Thm. 25.49 was proved by Riesz in 1909 (cf. [Rie09, Rie11]) and also by Helly in 1912. Riesz's originally proof is quite complicated. In 1914, Riesz gave a simplified proof in [Rie14]. The second proof we presented above is similar to the one in [Rie14]. Before that, in 1913, Riesz used the same idea to prove the spectral theorem for bounded self-adjoint operators on Hilbert spaces, cf. [Rie13]. We will discuss this topic in Sec. 27.7.

Riesz and Helly's interest in this theorem is closely related to their interest in the moment problem in $C([a,b],\mathbb{R})$. As mentioned in Sec. 17.5, the early solution of moment problems used a compactness argument which was later abstracted into the Banach-Alaoglu theorem. Now, since positive linear functionals on $C([a,b],\mathbb{R})$ take the explicit form of Stieltjes integrals with respect to increasing functions, the readers may ask whether the Banach-Alaoglu theorem for $C([a,b],\mathbb{R})^*$ also takes an explicit form. The answer is yes: the explicit formulation of the Banach-Alaoglu theorem for $C([a,b],\mathbb{R})^*$ in terms of increasing functions is called **Helly's selection theorem**. We will explain this in detail in Subsec. 25.8.4. □

25.7.4 Classification of Radon measures on $[a,b]$

In view of the bijection between positive linear functionals and Radon measures, the following corollary is more or less an equivalent formulation of the Riesz representation Thm. 25.49.

Corollary 25.51. *There is a bijection $\mu\mapsto\rho_\mu$ from the set of Radon measures on $[a,b]$ to the set of increasing right continuous functions $[a,b]\to\mathbb{R}_{\geqslant 0}$ such that*

$$\rho_\mu:[a,b]\to\mathbb{R}_{\geqslant 0}\qquad \rho_\mu(x)=\mu([a,x]) \tag{25.26}$$

The Radon measure μ is determined by ρ_μ by

$$\int_{[a,b]} f d\mu = f(a)\rho_\mu(a) + \int_a^b f d\rho_\mu \tag{25.27}$$

for each $f \in C([a,b],\mathbb{R})$.

Proof. Let $\Phi : \rho \mapsto \mathcal{I}_\rho$ be the bijection in the Riesz representation Thm. 25.49. By Riesz-Markov, $\mathcal{I}_\rho$ can be identified with its associated Radon measure μ_ρ. This gives a bijection $\rho \mapsto \mu_\rho$ determined by

$$\mathcal{I}_\rho(f) = \int_{[a,b]} f d\mu_\rho \tag{25.28}$$

for all $f \in C([a,b],\mathbb{R})$.

Moreover, in the proof of Thm. 25.49, we have shown (cf. (25.22)) that $\mathcal{I}_{\rho_{\mu_\rho}}(f) = \int_{[a,b]} f d\mu_\rho$. This proves $\mathcal{I}_\rho = \mathcal{I}_{\rho_{\mu_\rho}}$ (by (25.28)), and hence $\rho = \rho_{\mu_\rho}$. Thus, $\mu \mapsto \rho_\mu$ is the inverse of the bijection $\rho \mapsto \mu_\rho$. So $\mu \mapsto \rho_\mu$ is bijective. Eq. (25.27) follows from (25.28). $\square$

Definition 25.52. Let $\rho : [a,b] \to \mathbb{R}_{\geqslant 0}$ be increasing and right continuous. Let $(\mathfrak{M},\mu)$ be the completion of the Radon measure on $[a,b]$ associated to ρ due to Cor. 25.51. (So $\mu|_{\mathfrak{B}_{[a,b]}}$ is the unique Radon measure satisfying $\mu([a,x]) = \rho(x)$ for all $x \in [a,b]$.) We call μ the **Lebesgue-Stieltjes measure** associated to ρ. If $f : [a,b] \to \overline{\mathbb{R}}_{\geqslant 0}$ (resp. $f : [a,b] \to \mathbb{C}$) is $\mathfrak{M}$-measurable (resp. $(\mathfrak{M},\mu)$-integrable), we say that f is **Lebesgue-Stieltjes measurable** (resp. **integrable**), and define the **Lebesgue-Stieltjes integral** of f with respect to ρ to be

$$\int_{[a,b]} f d\rho := \int_{[a,b]} f d\mu$$

If $f : [a,b] \to \mathbb{R}$ is (bounded and) Stieltjes integrable, then f is Lebesgue-Stieltjes integrable, and the Stieltjes integral $\int_a^b f d\rho$ satisfies

$$\int_{[a,b]} f d\rho = f(a)\rho(a) + \int_a^b f d\rho \tag{25.29}$$

where the LHS is the Lebesgue-Stieltjes integral. (Thus, $\int_a^b f d\rho$ should be understood as $\int_{(a,b]} f d\rho$). See Pb. 25.15.

25.8 Problems and supplementary material

Problem 25.1. Let $(X,\mathcal{T}_X)$ be a topological space, and let $(\mathfrak{M},\mu)$ be a measure on X such that $\mathfrak{B}_X \subset \mathfrak{M}$. Define the **support** $\mathrm{Supp}(\mu)$ of μ such that

$$X\backslash\mathrm{Supp}(\mu) = \bigcup_{U\in\mathcal{T}_X,\mu(U)=0} U$$

In other words, $x \in X$ belongs to $\mathrm{Supp}(\mu)$ iff every neighborhood V of x satisfies $\mu(V_x) > 0$. Prove that $\mu(X \backslash \mathrm{Supp}(\mu)) = 0$ if one of the following is true:

(a) μ is inner regular on open sets.

(b) X is second countable.

Problem 25.2. Prove the converse of Lusin's theorem: Let X be LCH, and let $(\mathfrak{M}, \mu)$ be the completion of a σ-finite Radon measure on X. Let $f : X \to \mathbb{C}$ such that for every $A \in \mathfrak{B}_X$ satisfying $\mu(A) < +\infty$, and for every $\varepsilon > 0$, there exists a compact $K \subset A$ such that $\mu(A \backslash K) < \varepsilon$ and $f|_K$ is continuous. Then f is $\mathfrak{M}$-measurable.

⋆ **Remark 25.53.** Without assuming σ-finiteness, we have the following version of Lusin's theorem and its converse: A function $f : X \to \mathbb{C}$ satisfies the description in Pb. 25.2 iff f is $\mathfrak{M}_\mu$-measurable, where $\mathfrak{M}_\mu$ is defined by Thm. 23.53, i.e., the set of all $E \subset X$ such that μ is regular on $E \cap \Omega$ for each open Ω with finite μ-measure. (Equivalently, $\mathfrak{M}_\mu$ is the saturation of $\mathfrak{M}$, cf. Subsec. 23.5.5. Note that if $\mu|_{\mathfrak{B}_X}$ is σ-finite then $\mathfrak{M}_\mu = \mathfrak{M}$ by Prop. 23.55.) We leave the proof to the readers.

25.8.1 ⋆ Approximation by upper and lower semicontinuous functions

Let X be an LCH space, and let $(\mathfrak{M}, \mu)$ be the completion of a Radon measure on X.

Definition 25.54. For each topological space X (not necessarily LCH), we let

$$\mathrm{USC}(X, \mathbb{R}_{\geqslant 0}) = \{\text{upper semicontinuous } f \in [0, +\infty)^X\} \tag{25.30}$$

$$\mathrm{USC}_c(X, \mathbb{R}_{\geqslant 0}) = \{f \in \mathrm{USC}(X, \mathbb{R}_{\geqslant 0}) : \mathrm{Supp}(f) \text{ is compact}\} \tag{25.31}$$

(We do not let $\mathrm{USC}_+(X)$ denote the RHS of (25.30) since, according to our usual conventions, $\mathrm{USC}_+(X)$ refers to the set of all upper semicontinuous $f : X \to [0, +\infty]$.)

Problem 25.3. Let $f : X \to [0, +\infty]$ be measurable. Prove that

$$\int_X f d\mu = \inf\left\{\int_X h d\mu : h \in \mathrm{LSC}_+(X), h \geqslant f\right\} \tag{25.32a}$$

Prove that if $\int_X f d\mu < +\infty$, then

$$\int_X f d\mu = \sup\left\{\int_X g d\mu : g \in \mathrm{USC}_c(X, \mathbb{R}_{\geqslant 0}), g \leqslant f\right\} \tag{25.32b}$$

Hint. Assume WLOG that $\int f < +\infty$. (Why?) If $f = \chi_E$ for some finite measure $E \in \mathfrak{M}$, use the μ-regularity. In general, write $f = \sum_{n \in \mathbb{Z}_+} s_n$ where each $s_n : X \to \mathbb{R}_{\geqslant 0}$ is simple. Approximate each s_n from above and from below by some $h_n \in \mathrm{LSC}_+(X)$ and $g_n \in \mathrm{USC}_c(X, \mathbb{R}_{\geqslant 0})$ respectively. Take $h = \sum_{n=1}^\infty h_n$ and $g = \sum_{n=1}^N g_n$ for some large enough N. □

Remark 25.55. Lower semicontinuous functions are the function version of open sets. Upper semicontinuous functions with compact supports are the function version of compact sets. Therefore, Pb. 25.3 can be viewed as the function version Thm. 25.25.

Problem 25.4. Let $(f_\alpha)_{\alpha\in I}$ be a decreasing net in $\mathrm{USC}(X,\mathbb{R}_{\geqslant 0})$ converging pointwise to $f : X \to \mathbb{R}_{\geqslant 0}$. Assume that there exists a measurable $g : X \to \overline{\mathbb{R}}_{\geqslant 0}$ such that $\int_X g d\mu < +\infty$, and that $f_\alpha \leqslant g$ for all α. Prove $\displaystyle\int_X f d\mu = \lim_\alpha \int_X f_\alpha d\mu$.

Hint. By Pb. 25.3, assume WLOG that $g \in \mathrm{LSC}_+(X)$. □

Exercise 25.56. Let K be a compact subset of X. Use Pb. 25.4 to show that

$$\mu(K) = \inf\Big\{\int_X f d\mu : f \in C_c(X,[0,1]), f|_K = 1\Big\} \tag{25.33}$$

by constructing a decreasing net in $C_c(X,[0,1])$ (bounded by some χ_U where $U \supset K$ is open and $\mu(U) < +\infty$) converging pointwise to χ_K.

Note. It is not necessary to prove (25.33) using Pb. 25.4. You can try to find a direct proof. However, the purpose of Pb. 25.4 is to let you know how (25.33) can fit into a broader picture. □

25.8.2 The dual space $C_c(X)^*$

Let X be an LCH space.

Problem 25.5. Let $\Lambda : C_c(X,\mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$ be $\mathbb{R}_{\geqslant 0}$-linear. By Rem. 25.14, Λ can be extended uniquely to a $\mathbb{C}$-linear map $\Lambda : C_c(X) \to \mathbb{C}$. Let μ be the Radon measure associated to Λ. Prove that the following four numbers are equal:

$$\begin{aligned}&\sup\{\Lambda(f) : f \in C_c(X,\mathbb{R}_{\geqslant 0}), f \leqslant 1\} = \sup\{|\Lambda(f)| : f \in C_c(X,\mathbb{R}), |f| \leqslant 1\}\\ =&\sup\{|\Lambda(f)| : f \in C_c(X), |f| \leqslant 1\} = \mu(X)\end{aligned}$$

These four identical numbers are called (unambiguously) the **operator norm** of Λ and denoted by $\|\Lambda\|$.

Problem 25.6. Equip $C_c(X,\mathbb{R})$ with the l^∞-norm. Let $\Lambda : C_c(X,\mathbb{R}) \to \mathbb{R}$ be a bounded linear map with operator norm M. Define $\Lambda^\pm : C_c(X,\mathbb{R}_{\geqslant 0}) \to \overline{\mathbb{R}}$ sending each $f \in C_c(X,\mathbb{R}_{\geqslant 0})$ to

$$\Lambda^+(f) = \sup\big\{\Lambda(h) : h \in C_c(X,\mathbb{R}_{\geqslant 0}), h \leqslant f\big\} \tag{25.34a}$$
$$\Lambda^-(f) = \sup\big\{-\Lambda(h) : h \in C_c(X,\mathbb{R}_{\geqslant 0}), h \leqslant f\big\} \tag{25.34b}$$

Clearly $(-\Lambda)^\pm = \Lambda^\mp$.

1. Prove that Λ^+ has range in $\mathbb{R}_{\geqslant 0}$, that $\Lambda^+ : C_c(X, \mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$ is $\mathbb{R}_{\geqslant 0}$-linear, and that $\|\Lambda^+\| \leqslant M$. (Replacing Λ with $-\Lambda$, we see that Λ^- satisfies the same property.)

2. Prove that

$$\Lambda(f) = \Lambda^+(f) - \Lambda^-(f) \tag{25.35}$$

for all $f \in C_c(X, \mathbb{R}_{\geqslant 0})$. This is called the **Jordan decomposition** of Λ.

Hint. 1. For $f, g \in C_c(X, \mathbb{R}_{\geqslant 0})$, to prove $\Lambda^+(f + g) \leqslant \Lambda^+(f) + \Lambda^+(g)$, let $h \in C_c(X, \mathbb{R}_{\geqslant 0})$ such that $h \leqslant f + g$. Let $h_1 = \min\{h, f\}$ and $h_2 = \max\{h - f, 0\}$. Show that $0 \leqslant h_1 \leqslant f$, $0 \leqslant h_2 \leqslant g$, and $h = h_1 + h_2$.

2. Prove $\Lambda + \Lambda^- \leqslant \Lambda^+$, and replace Λ with $-\Lambda$. □

⋆ **Problem 25.7.** Let μ be a finite Radon measure on X. Let $A \in \mathfrak{B}_X$ and $B = X \backslash A$. Define a linear map $\Lambda : C_c(X, \mathbb{R}) \to \mathbb{R}$ by

$$\Lambda(f) = \int_A f d\mu - \int_B f d\mu$$

which is clearly bounded (with operator norm $\leqslant \mu(X)$). Prove that for every $f \in C_c(X, \mathbb{R}_{\geqslant 0})$ we have

$$\Lambda^+(f) = \int_A f d\mu \qquad \Lambda^-(f) = \int_B f d\mu \tag{25.36}$$

Hint. To prove $\Lambda^+(f) \geqslant \int_A f d\mu$, find compact $K \subset A$ and $L \subset B$ such that $\mu(A \backslash K)$ and $\mu(B \backslash L)$ are small. Multiply f by an Urysohn function associated to K and $X \backslash L$. □

Theorem 25.57 (Riesz-Markov representation theorem). *Any linear functional on $C_c(X)$ defined by $\Lambda_\mu : f \mapsto \int_X f d\mu$ for some finite Radon measure μ is in $C_c(X)^*$. Moreover, such linear functionals span $C_c(X)^*$.*

A similar classification of $C_c(X, \mathbb{R})^*$ is left to the readers.

Proof. If μ is Radon and $\mu(X) < +\infty$, by Pb. 25.5, we know $\|\Lambda_\mu\| < +\infty$. So $\Lambda_\mu \in C_c(X)^*$.

We now show that $\Lambda \in C_c(X)^*$ can be written as a finite sum $\sum_i a_i \Lambda_{\mu_i}$ where $a_i \in \mathbb{C}$ and μ_i is a finite Radon measure. For each $f \in C_c(X, \mathbb{R})$, let $\Lambda_1(f) = \mathrm{Re}\Lambda(f)$ and $\Lambda_2(f) = \mathrm{Im}\Lambda(f)$. So $\Lambda_i \in C_c(X, \mathbb{R})^*$, and $\Lambda = \Lambda_1 + \mathbf{i}\Lambda_2$ on $C_c(X, \mathbb{R})$. By Pb. 25.6, there exist $\mathbb{R}_{\geqslant 0}$-linear $\Lambda_i^\pm : C_c(X, \mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$ with finite operator norms such that $\Lambda_i = \Lambda_i^+ - \Lambda_i^-$ on $C_c(X, \mathbb{R}_{\geqslant 0})$. So $\Lambda = \Lambda_1^+ - \Lambda_1^- + \mathbf{i}\Lambda_2^+ - \mathbf{i}\Lambda_2^-$ on $C_c(X, \mathbb{R}_{\geqslant 0})$ and hence on $C_c(X)$ by the $\mathbb{C}$-linearity. By the Riesz-Markov representation Thm. 25.21, each of the four positive linear functionals is represented by the integral of a Radon measure. The finiteness of these Radon measures is due to Pb. 25.5. □

Remark 25.58. Recall that if $\mathcal{V}$ is a normed vector space with dense subspace $\mathcal{U}$, there is a canonical isomorphism of Banach spaces $\mathcal{V}^* \simeq \mathcal{U}^*$ (cf. Rem. 21.32). Therefore, since $C_0(X)$ is the completion of $C_c(X)$ (cf. Pb. 15.5), the Riesz-Markov Thm. 25.57 also characterizes the dual space of $C_0(X)$.

25.8.3 ⋆ An alternative proof of Thm. 25.38

Let X be a (not necessarily LCH) Hausdorff space. Let $\mu : \mathfrak{M} \to [0, +\infty]$ be a measure where $\mathfrak{M}$ is a σ-algebra containing $\mathfrak{B}_X$. Define $\mu^*, \mu_* : \mathfrak{M} \to [0, +\infty]$ by

$$\mu^*(E) = \inf \{\mu(U) : U \text{ is an open subset of } X \text{ containing } E\} \tag{25.37a}$$

$$\mu_*(E) = \sup \{\mu(K) : K \text{ is a compact subset of } E\} \tag{25.37b}$$

Clearly $\mu_*(E) \leqslant \mu(E) \leqslant \mu^*(E)$. Clearly $\mu^*(U) = \mu(U)$ if U is open, and $\mu_*(K) = \mu(K)$ if K is compact.

Note that the definition of $\mu_*(E)$ (using $\mu(K)$) is slightly different from that in (23.9b) (using $\mu^*(K)$), since we do not assume $\mu^*(K) = \mu(K)$. Therefore, you cannot directly use the results proved in Sec. 23.5. (But you can use the methods in that section.)

Recall from Def. 25.1 that a set $E \in \mathfrak{M}$ is **μ-regular** iff $\mu^*(E) = \mu_*(E)$.

Exercise 25.59. Let $E \in \mathfrak{M}$ such that $\mu^*(E) < +\infty$. Prove that μ is regular on E iff for each $\varepsilon > 0$ there exist an open $U \supset E$ and a compact $K \subset E$ such that $\mu(U \backslash K) < \varepsilon$.

Problem 25.8. Let $E_1, E_2, \dots \in \mathfrak{M}$ be mutually disjoint. Suppose that μ is regular on each E_n. Prove that μ is regular on $E = \bigcup_n E_n$, and $\mu(E) = \sum\limits_n \mu(E_n)$.

Problem 25.9. Let $E_1, E_2 \in \mathfrak{M}$ be μ-regular with finite μ-measures. Prove that $E_2 \backslash E_1$ is μ-regular.

We say that $E \in \mathfrak{M}$ is **locally μ-regular** if for each *μ-regular* open $\Omega \subset X$ satisfying $\mu(\Omega) < +\infty$, the set $E \cap \Omega$ is μ-regular. Let

$$\mathfrak{M}^\mu = \{E \in \mathfrak{M} : E \text{ is locally } \mu\text{-regular}\} \tag{25.38}$$

Problem 25.10. Prove that $\mathfrak{M}^\mu$ is a σ-algebra.

We are ready to give an alternative proof of Thm. 25.38.

Theorem 25.60 (=Thm. 25.38)**.** *Let X be a second countable LCH space. Let μ be a Borel measure on X which is finite on compact subsets. Then μ is a Radon measure.*

Proof. As in the proof of Thm. 25.38, we can find a countable increasing chain of compact subsets $K_1 \subset K_2 \subset \cdots$ of U such that $U = \bigcup_n K_n$. So $\mu(U) = \lim_n \mu(K_n)$. This proves that U is inner regular, and hence is regular. In particular, $\mathfrak{M}^\mu$ contains all open sets.

It remains to prove that every Borel set is outer regular. By Pb. 25.10, $\mathfrak{M}^\mu$ is a σ-algebra. So $\mathfrak{M}^\mu$ contains any Borel set E. Let us prove that E is outer regular.

Since X is a countable union of open subsets, and since μ is finite on compact sets, X is a countable union $X = \bigcup \Omega_n$ where each Ω_n is open and has finite μ-measure. Since $E \in \mathfrak{M}^\mu$, we know that $E_n := E \cap \Omega_n$ is μ-regular. Therefore, for each $\varepsilon > 0$ there is an open $U_n \subset X$ such that $\mu(U_n \backslash E_n) < \varepsilon/2^n$. Let $U = \bigcup_n U_n$. Since $E = \bigcup_n E_n$, we have $U \backslash E \subset \bigcup_n U_n \backslash E_n$, and hence $\mu(U \backslash E) < \varepsilon$. Therefore $\mu(U) \leqslant \mu(E) + \varepsilon$. □

25.8.4 Stieltjes integrals and the Banach-Alaoglu theorem for $C([a,b],\mathbb{R})^*$

Let $-\infty < a < b < +\infty$. If $\rho : [a,b] \to \mathbb{R}_{\geqslant 0}$ is increasing, for each $f : [a,b] \to \mathbb{C}$, the Stieltjes integral $\int_a^b f d\rho$ is understood as $\int_a^b \mathrm{Re}(f) d\rho + \mathbf{i}\int_a^b \mathrm{Im}(f) d\rho$ whenever it can be defined. Recall that

$$\mathcal{I}_\rho(f) = f(a)\rho(a) + \int_a^b f d\rho$$

Problem 25.11. Let $\rho : [a,b] \to \mathbb{R}_{\geqslant 0}$ be increasing. Prove that $\mathcal{I}_\rho : C[a,b] \to \mathbb{C}$ has operator norm

$$\|\mathcal{I}_\rho\| = \rho(b)$$

Lemma 25.61. *Let $I \subset \mathbb{R}$ be an interval. Suppose that $\rho : I \to \mathbb{R}$ is increasing. Then ρ is continuous outside countably many points. In particular, if $I = [a,b]$, then ρ is Riemann integrable on I.*

Proof. Since I is a countable union of compact subintervals, by restricting ρ to each compact subinterval, it suffices to assume $I = [a,b]$.

For each $x \in I$, let $\rho_-(x), \rho_+(x)$ be the left resp. right limit of ρ at x, i.e., $\rho_\pm(x) = \lim_{t \to x^\pm} \rho(t)$. Then $\rho_-(x_1) \leqslant \rho_+(x_1) \leqslant \rho_-(x_2) \leqslant \rho_+(x_2)$ if $x_1 < x_2$. Let Δ be the set of all $x \in I$ at which ρ is not continuous. Then $x \in \Delta$ iff $\rho_-(x) < \rho_+(x)$. If $x_1 < \cdots < x_n$ are in Δ, then $\sum_{i=1}^n (\rho_+(x_i) - \rho_-(x_i)) \leqslant \rho(b) - \rho(a)$. It follows that

$$\sum_{x \in \Delta} (\rho_+(x) - \rho_-(x)) \leqslant \rho(b) - \rho(a) < +\infty$$

Therefore, by Pb. 5.3, Δ is countable. □

Corollary 25.62. *Let I be an interval, and let $\rho : I \to \mathbb{R}$ be increasing. There there is an increasing right continuous $\widetilde{\rho} : I \to \mathbb{R}$ such that ρ equals $\widetilde{\rho}$ outside a countable subset of I.*

Proof. Let $\widetilde{\rho}(x) = \lim_{t\to x^+} \rho(t)$. Then $\widetilde{\rho}$ is right continuous, and $\widetilde{\rho}(x) = \rho(x)$ if ρ is continuous at x. □

The following problem shows that the Stieltjes integral of a C^1 function can be calculated by a Riemann integral.

Problem 25.12. Let $\rho : [a,b] \to \mathbb{R}_{\geqslant 0}$ be increasing. Let $g \in C^1[a,b]$. (Namely, $g : [a,b] \to \mathbb{C}$ is continuous and has continuous differentials.) Prove the **integration by parts**

$$\int_a^b g d\rho = g(b)\rho(b) - g(a)\rho(a) - \int_a^b g'\rho dx \tag{25.39}$$

Hint. Assume WLOG that $g \in C^1([a,b],\mathbb{R})$. Let $\sigma = \{a_0 = a < a_1 < \cdots < a_n = b\}$ be a partition of $[a,b]$. By the summation by parts (cf. Pb. 4.2, and set $g_k = g(a_k)$, $f_0 = \rho(a)$, $f_i = \rho(a_i) - \rho(a_{i-1})$ when $i > 0$), we have

$$\sum_{k=1}^n g(a_k)\big(\rho(a_k) - \rho(a_{k-1})\big) = \rho(b)g(b) - \rho(a)g(a) - \sum_{k=0}^{n-1} \rho(a_k)\big(g(a_{k+1}) - g(a_k)\big)$$

Apply the mean value theorem to $g(a_{k+1}) - g(a_k)$. □

Recall that m is the Lebesgue measure.

Problem 25.13. Let $(\rho_n)_{n\in\mathbb{Z}_+}$ be a net of increasing functions $[a,b] \to \mathbb{R}_{\geqslant 0}$. Let $\rho : [a,b] \to \mathbb{R}_{\geqslant 0}$. Assume that the following are true:

(1) $\sup_n \rho_n(b) < +\infty$.

(2) (ρ_n) converges m-a.e. to ρ, and $\lim_n \rho_n(b) = \rho(b)$.

Prove that $(\mathcal{I}_{\rho_n})$ converges weak-* (in $(C[a,b])^*$) to $\mathcal{I}_\rho$. In other words, prove for each $f \in C[a,b]$ that

$$\lim_{n\to\infty} \mathcal{I}_{\rho_n}(f) = \mathcal{I}_\rho(f)$$

Hint. Use a density argument to reduce to the case that f is a polynomial. Then use integration by parts and the dominated convergence theorem. □

In Pb. 27.2, we will describe a similar relationship between pointwise (or a.e.) convergence and the weak-* convergence in L^p spaces.

The following problem is Problem 13 from Chapter 7 of Rudin's *Principles of Mathematical Analysis* [Rud-P]. We shall see the background of this problem, which was not given in Rudin's book.

Theorem 25.63 (Helly's selection theorem). *Let (ρ_n) be a sequence of increasing functions $[a,b] \to [0,1]$. Then (ρ_n) has a subsequence converging pointwise to an increasing $\rho : [a,b] \to [0,1]$.*

⋆ **Problem 25.14.** Prove Helly's selection Thm. 25.63.

Hint. Choose a subsequence (ρ_{n_k}) converging pointwise on $[a,b]\cap\mathbb{Q}$ to an increasing $\tau:[a,b]\cap\mathbb{Q}\rightarrow[0,1]$. Extend τ to an increasing right continuous function $\tau:[a,b]\rightarrow[0,1]$ by $\tau(x)=\lim\limits_{t\rightarrow x^+,t\in\mathbb{Q}\cap[a,b]}\tau(t)$. Let Δ be the (countable) set of all $x\in[a,b]$ at which τ is not continuous. Prove that (ρ_{n_k}) converges pointwise on $[a,b]\backslash\Delta$ to τ. Conclude that (ρ_{n_k}) has a subsequence converging everywhere on $[a,b]$. □

Helly's selection theorem is a prototype of the Banach-Alaoglu Thm. 17.21. In a 1912 paper [Hel12], Helly proved this theorem and used it to study the moment problem in $C([a,b],\mathbb{R})$: Given $c_1,c_2,\dots\in\mathbb{R}$ and $f_1,f_2,\dots\in C([a,b],\mathbb{R})$ such that there exists $M\in\mathbb{R}_{\geqslant0}$ satisfying

$$\Big|\sum_{i=1}^n\lambda_ic_i\Big|\leqslant M\Big\|\sum_{i=1}^n\lambda_if_n\Big\|_{l^\infty}\qquad(\forall n\in\mathbb{Z}_+,\lambda_1,\dots,\lambda_n\in\mathbb{R})\tag{25.40}$$

find a function of bounded variation $\rho:[a,b]\rightarrow\mathbb{R}$ such that, for all n,

$$\int_a^b f_nd\rho=c_n\tag{25.41}$$

A **function of bounded variation** (simply called a **BV function**) $\rho:[a,b]\rightarrow\mathbb{R}$ is a function that can be written as $\rho^+-\rho^-$ where $\rho^+,\rho^-:[a,b]\rightarrow\mathbb{R}_{\geqslant0}$ are increasing. See Sec. 17.5 for the relationship between the moment problems and the Banach-Alaoglu theorem. (F. Riesz has also studied this problem in 1911. His interest in the study of dual space of $C[a,b]$ is clearly related to moment problems. See [Die-H, Sec. 6.3] for a detailed history.)

Corollary 25.64 (Banach-Alaoglu theorem for $C([a,b],\mathbb{R})^*$). *The closed unit ball of $C([a,b],\mathbb{R})^*$ is weak-* sequentially compact.*

Proof. Let (Λ_n) be a sequence in $C([a,b],\mathbb{R})^*$ such that $\sup_n\|\Lambda_n\|\leqslant1$. By Pb. 25.6, each Λ_n has Jordan decomposition $\Lambda_n=\Lambda_n^+-\Lambda_n^-$ where $\Lambda_n^\pm$ is positive and $\|\Lambda_n^\pm\|\leqslant1$. By considering $\Lambda_n^\pm$ separately, it suffices to assume that Λ_n is positive.

By the Riesz representation Thm. 25.49, for each n there is an increasing $\rho_n:[a,b]\rightarrow\mathbb{R}_{\geqslant0}$ such that $\Lambda_n(f)=\mathcal{I}_{\rho_n}(f)$ for all $f\in C([a,b],\mathbb{R})$. By Helly's selection Thm. 25.63, (ρ_n) has a subsequence (ρ_{n_k}) converging pointwise to an increasing $\rho:[a,b]\rightarrow\mathbb{R}_{\geqslant0}$. By Pb. 25.13, (Λ_{n_k}) converges weak-* to $\mathcal{I}_\rho$. □

Another prototype of the Banach-Alaoglu theorem is the compactness of the closed unit ball of $l^2(\mathbb{Z}_+)$ under the pointwise convergence topology, cf. Thm. 17.31 and Pb. 17.5. We have seen in Sec. 22.5 that it plays a crucial role in the Hilbert-Schmidt theorem.

Understanding how abstract theorems like the Banach-Alaoglu theorem evolved from early explicit versions is important because it helps us understand the nature of mathematical development. We have emphasized this point throughout this course.

25.8.5 ⋆ Lebesgue-Stieltjes integrals

Let $\rho : [a, b] \to \mathbb{R}_{\geqslant 0}$ be increasing and right continuous. Let μ_ρ be the Lebesgue-Stieltjes measure associated to ρ (cf. Def. 25.52), i.e., the completion of the unique Radon measure whose value at each $[a, x]$ equals $\rho(x)$.

Problem 25.15. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Assume that f is Stieltjes integrable with respect to ρ. Prove that f is μ_ρ-measurable, and

$$\int_{[a,b]} f d\mu_\rho = f(a)\rho(a) + \int_a^b f d\rho$$

where $\int_a^b f d\rho$ is the Stieltjes integral.

Hint. To prove that f is μ_ρ-measurable, use the lower and upper Darboux sums to find bounded Borel functions $g, h : [a, b] \to \mathbb{R}$ satisfying $g \leqslant f \leqslant h$ and $\int_{[a,b]}(h - g)d\mu_\rho = 0$. Show that $f - g = 0$ μ_ρ-a.e.. (Where is the completeness of μ_ρ used?) □

Problem 25.16. Assume that ρ has continuous derivative $\rho' \in C([a, b], \mathbb{R}_{\geqslant 0})$. Let $(\mathfrak{M}, m)$ be the Lebesgue measure on $[a, b]$. Prove that every Lebesgue measurable set is μ_ρ-measurable. Prove that on $\mathfrak{M}$ we have

$$d\mu_\rho = \rho(a)d\delta_a + \rho' dm \tag{25.42}$$

(cf. Pb. 24.1 for the notation), where δ_a is the Dirac measure associated to a.

Hint. By Pb. 24.2, it suffices to prove (25.42) on $\mathfrak{B}_{[a,b]}$. Let ν be the Borel measure on $[a, b]$ such that $d\nu = \rho(a)d\delta_a + \rho' dm$. Show that ν is Radon. Show that it suffices to prove $\int_{[a,b]} f d\mu_\rho = \int_{[a,b]} f d\nu$ for any polynomial f. Prove this relation by using integration by parts (Pb. 25.12). □

26 Theorems of Fubini and Tonelli for Radon measures

26.1 Products of Radon measures

Fix LCH spaces $X_1, \dots, X_N$. For each $1 \leqslant i \leqslant N$, let $\Lambda_i : C_c(X_i, \mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$ be an $\mathbb{R}_{\geqslant 0}$-linear map, equivalently (cf. Rem. 25.14), a positive linear functional $\Lambda : C_c(X_i) \to \mathbb{C}$. Let $(\mathfrak{M}_i, \mu_i)$ be *the completion of* the Radon measure associated to Λ_i.

Our goal of this section and the next one is to prove the Fubini theorem for integrals of Radon measures. In the special case that $N = 2$, $X_i = \mathbb{R}^{k_i}$, and $(\mathfrak{M}_i, \mu_i)$ is the Lebesgue measure m^{k_i}, the theorem implies that

$$\int_{\mathbb{R}^{k_1} \times \mathbb{R}^{k_2}} f(x_1, x_2) dm^{k_1+k_2} = \int_{\mathbb{R}^{k_1}} \int_{\mathbb{R}^{k_2}} f(x_1, x_2) dm^{k_2} dm^{k_1} \tag{26.1}$$

for any $f \in \mathcal{L}^1(\mathbb{R}^{k_1} \times \mathbb{R}^{k_2}, m^{k_1+k_2})$. However, to prove such a theorem for Radon measures, the first task is to define the "product Radon measure" $\mu_1 \times \cdots \times \mu_N$ on $X_1 \times \cdots \times X_N$ generalizing $m^{k_1+k_2}$. This will be achieved by defining the corresponding positive linear functional $\Lambda_1 \otimes \cdots \otimes \Lambda_N$.

Lemma 26.1. *Let X be an LCH space, and let $\Lambda : C_c(X) \to \mathbb{C}$ be a positive linear functional. Then for each precompact open $U \subset X$, the restriction of Λ to $C_c(U)$ is bounded (and hence continuous) with respect to the l^∞-norm.*

Proof. Let μ be the associated Radon measure of Λ. Then for each $f \in C_c(U)$ we have $|\Lambda(f)| = |\int f d\mu| \leqslant \|f\|_{l^\infty} \cdot \mu(U)$. So $\Lambda|_{C_c(U)}$ has operator norm $\leqslant \mu(U)$ which is finite because μ is finite on the compact set $\overline{U}$. □

Exercise 26.2. Use Urysohn's lemma to give a direct proof of Lem. 26.1 without using the associated Radon measure.

Hint. First show that the $\mathbb{R}$-linear map $\Lambda|_{C_c(U,\mathbb{R})}$ has finite operator norm $\leqslant \Lambda(\varphi)$ where $\varphi \in C_c(X, [0,1])$ and $\varphi|_{\overline{U}} = 1$. □

Theorem 26.3. *There exists a unique positive linear functional*

$$\Lambda : C_c(X_1 \times \cdots \times X_N) \to \mathbb{C}$$

satisfying that for each $f_i \in C_c(X_i)$, by viewing $f_1 \cdots f_N$ as a function $X_1 \times \cdots \times X_N \to \mathbb{C}$ in the obvious way (i.e. sending $(x_1, \dots, x_N)$ to $f_1(x_1) \cdots f_N(x_N)$), we have

$$\Lambda(f_1 \cdots f_N) = \Lambda_1(f_1) \cdots \Lambda_N(f_N) \tag{26.2}$$

Proof. Uniqueness: Let Γ satisfy the same properties as Λ. Let $X_\bullet = X_1 \times \cdots \times X_N$. By Lem. 15.27, it is easy to see that $C_c(X_\bullet)$ is the union of all $C_c(U_\bullet)$ where $U_\bullet = U_1 \times \cdots \times U_N$ for some precompact open $U_1 \subset X_1, \dots, U_N \subset X_N$. Therefore, it suffices to restrict Λ and Γ to each $C_c(U_\bullet)$ and show that they are equal.

Let $\mathcal{E} = \{f_1 \cdots f_N : f_i \in C_c(U_i)\}$. Then $\Lambda|_{\mathcal{E}} = \Gamma|_{\mathcal{E}}$. Note that $\mathrm{Span}_{\mathbb{C}}\mathcal{E}$ is clearly a $*$-subalgebra of $C_0(U_\bullet)$ vanishing nowhere and separating points of $U_\bullet$. (This is because $C_c(U_i)$ vanishes nowhere and separates points of U_i, cf. Cor. 15.24.) Applying the Stone-Weierstrass Thm. 15.49 to the LCH space $U_\bullet$, we conclude that $\mathrm{Span}_{\mathbb{C}}\mathcal{E}$ is dense in $C_c(U_\bullet)$ (under the l^∞-norm). [1] Therefore, by Lem. 26.1, we see that Λ equals Γ on $C_c(U_\bullet)$.

Existence: By induction on N, it suffices to assume $N = 2$. Choose any $f \in C_c(X_\bullet)$, and let K_i be the projection of $\mathrm{Supp}(f)$ to X_i. (So K_1, K_2 are compact, and $\mathrm{Supp}(f) \subset K_1 \times K_2$.) For each net $(p_\alpha)_{\alpha \in I}$ converging in X_1 to p, the net of functions $(f(p_\alpha, \cdot)|_{K_2})_{\alpha \in I}$ in $C(K_2)$ converges uniformly to $f(p, \cdot)|_{K_2}$ due to Thm. 9.3. Therefore, by Thm. 24.25, we have

$$\lim_\alpha \int_{K_2} f(p_\alpha, \cdot) d\mu_2 = \int_{K_2} f(p, \cdot) d\mu_2$$

And we can clearly replace the K_2 under the integral with X_2. This proves that $x_1 \in X_1 \mapsto \int_{K_2} f(x_1, \cdot) d\mu_2$ is a continuous function on X_1 which clearly has compact support in K_1. Thus, we can define

$$\Lambda(f) = \int_{X_1} \int_{X_2} f(x, y) d\mu_2(y) d\mu_1(x) \tag{26.3}$$

This defines a map $\Lambda : C_c(X) \to \mathbb{C}$ which is clearly $\mathbb{C}$-linear, positive, and satisfying (26.2). $\square$

Definition 26.4. The positive linear functional Λ in Thm. 26.3 is denoted by $\Lambda_1 \otimes \cdots \otimes \Lambda_N$ and called the **tensor product** of $\Lambda_1, \dots, \Lambda_N$. The completion of the associated Radon measure of Λ is denoted by $\mu_1 \times \cdots \times \mu_N$ and called the **Radon product** of $\mu_1, \dots, \mu_N$.

Remark 26.5. There is a definition of product measure $\mu_1 \times \cdots \times \mu_N$ for general measure spaces $(X_1, \mu_1), \dots, (X_N, \mu_N)$. (See [Rud-R, Ch. 8] or [Fol-R, Sec. 2.5].) Unfortunately, this product measure is in general not complete. However, when each X_i is LCH *and second countable* and each μ_i is Radon, this product measure is defined on a σ-algebra containing $\mathfrak{B}_{X_1 \times \cdots \times X_N}$, and its completion is equal to the Radon product in Def. 26.4. Without assuming second countability, this statement is not true.

[1] In fact, one can use the Stone-Weierstrass theorem for compact Hausdorff spaces since all functions involved have compact supports.

Exercise 26.6. Prove that

$$(\Lambda_1 \otimes \Lambda_2) \otimes \Lambda_3 = \Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3 = \Lambda_1 \otimes (\Lambda_2 \otimes \Lambda_3)$$
$$(\mu_1 \times \mu_2) \times \mu_3 = \mu_1 \times \mu_2 \times \mu_3 = \mu_1 \times (\mu_2 \times \mu_3)$$

Generalize these relations to tensor products of more than three positive linear functionals, and Radon products of more than three Radon measures.

Example 26.7. Let m^k be the Lebesgue measure of $\mathbb{R}^k$. Then $m^{k_1+k_2}$, the Lebesgue measure of $\mathbb{R}^{k_1+k_2} = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$, equals the Radon product of m^{k_1} and m^{k_2}.

26.2 Theorems of Fubini and Tonelli

In this section, we fix LCH spaces X and Y. Let $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ be the completions of Radon measures on X and Y respectively. Let $\mu \times \nu$ be the Radon product of μ and ν (which is the completion of a Radon measure on $X \times Y$).

Whenever the integrals can be defined, we adopt the abbreviations

$$\int_X \int_Y f d\nu d\mu = \int_X \bigg(\int_Y f(x,y) d\nu(y)\bigg) d\mu(x)$$
$$\int_Y \int_X f d\mu d\nu = \int_Y \bigg(\int_X f(x,y) d\mu(x)\bigg) d\nu(y)$$

We let

$$\int_Y f d\nu : x \mapsto \int_Y f(x,y) d\nu(y) \tag{26.4a}$$
$$\int_X f d\mu : y \mapsto \int_Y f(x,y) d\mu(x) \tag{26.4b}$$

whenever $x \in X$ and $y \in Y$ are such that the integrals on the RHS can be defined.

26.2.1 Tonelli's theorem without σ-finiteness

Theorem 26.8 (Tonelli's theorem). *Let $f \in \mathrm{LSC}_+(X \times Y)$. Then the functions $\int_Y f d\nu : X \to \overline{\mathbb{R}}_{\geqslant 0}$ and $\int_X f d\mu : Y \to \overline{\mathbb{R}}_{\geqslant 0}$ are lower semicontinuous, and*

$$\int_{X\times Y} f d(\mu \times \nu) = \int_X \int_Y f d\nu d\mu = \int_Y \int_X f d\mu d\nu \tag{26.5}$$

Proof. We first consider the special case that $f \in C_c(X \times Y)$. By the definition of Radon product, we know that $\int_{X\times Y} f d(\mu \times \nu)$ equals $(\Lambda_1 \otimes \Lambda_2)(f)$ where Λ_1, Λ_2 are the positive linear functionals inducing μ, ν respectively. From the proof of Thm. 26.3, we know that $\int_Y f d\nu \in C_c(X)$. Note that the second term of (26.5) defines a positive linear functional $C_c(X\times Y)\to\mathbb{C}$ sending f_1f_2 to $\Lambda_1(f_1)\Lambda_2(f_2)=\int_X f_1d\mu\cdot\int_Y f_2d\nu$ where $f_1\in C_c(X)$ and $f_2\in C_c(Y)$. Therefore, by the uniqueness in Thm. 26.3, this positive linear functional equals Λ. This proves the first equality in (26.5) in the special case that $f\in C_c(X\times Y,\mathbb{R}_{\geqslant 0})$. The second half of (26.5) can be proved in the same way.

Now we consider the general case that $f\in \mathrm{LSC}_+(X\times Y)$. By Lem. 25.19, there is an increasing net $(f_\alpha)_{\alpha\in I}$ in $\mathrm{LSC}_+(X)$ converging pointwise to f. We know that $(\int_Y f_\alpha d\nu)_{\alpha\in I}$ is an increasing net in $C_c(X,\mathbb{R}_{\geqslant 0})$. Therefore, its pointwise limit must be in $\mathrm{LSC}_+(X)$ (by Pb. 23.4), and must be equal to $\int_Y fd\nu$ by the monotone convergence Cor. 25.23. Therefore, $\int_Y fd\nu\in\mathrm{LSC}_+(X)$. By Cor. 25.23 again, and by the special case already proved, we have

$$\int_{X\times Y}fd(\mu\times\nu)=\lim_\alpha\int_{X\times Y}f_\alpha d(\mu\times\nu)=\lim_\alpha\int_X\int_Y f_\alpha d\nu d\mu=\int_X\int_Y fd\nu d\mu$$

This proves a half of the theorem. The other half can be proved in the same way. □

Remark 26.9. Thm. 26.8 can be formulated (and proved) without the language of measure theory: For each $f\in\mathrm{LSC}_+(X\times Y)$, define $\Lambda_2(f):X\to\overline{\mathbb{R}}_{\geqslant 0}$ sending $x\mapsto\Lambda_2(f(x,\cdot))$. By using Lem. 26.1, it is easy to check that $\Lambda_2(f)\in C_c(X,\mathbb{R}_{\geqslant 0})$ if $f\in C_c(X\times Y,\mathbb{R}_{\geqslant 0})$, and hence $\Lambda_2(f)\in\mathrm{LSC}_+(X)$ in general (by Pb. 23.4). The element $\Lambda_1(f)\in\mathrm{LSC}_+(Y)$ can be defined in the same way, and

$$(\Lambda_1\otimes\Lambda_2)(f)=\Lambda_1(\Lambda_2(f))=\Lambda_1(\Lambda_2(f))\tag{26.6}$$

The key to the proof is (again) the monotone convergence theorem for nets, i.e., Thm. 25.18.

For example, take $X=Y=\mathbb{R}$ and let Λ_1,Λ_2 be both the Riemann integrals of continuous compactly supported functions, extended canonically to positive lower semicontinuous functions. Then, for each continuous function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}_{\geqslant 0}$, the functions $x\in\mathbb{R}\mapsto\int_{\mathbb{R}}f(x,y)dy$ and $y\in\mathbb{R}\mapsto\int_{\mathbb{R}}f(x,y)dx$ from $\mathbb{R}$ to $\overline{\mathbb{R}}_{\geqslant 0}$ (defined by improper Riemann integrals) are lower semicontinuous, and we have Tonelli's theorem for improper integrals

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x,y)dydx=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x,y)dxdy\tag{26.7}$$

Recall that in the last semester, we were only able to prove Fubini's theorems for the exchangeability of an improper integral and a definite integral. (See Thm. 14.32.) Now, we see that the exchangeability of two improper integrals for continuous positive functions can be easily proved without using the heavy machinery of measure theory and Lebesgue integrals. All we need is the method of extending positive linear functionals developed in Sec. 25.2. □

26.2.2 Fubini-Tonelli with σ-finiteness

Theorem 26.10 (Tonelli's theorem). *Assume that μ and ν are σ-finite. Let $f \in \mathcal{L}_+(X \times Y)$. Then the following are true.*

(a) $f(x,\cdot) \in \mathcal{L}_+(Y)$ for almost every $x \in X$.

(b) Extend the function $\int_Y f d\nu$ (originally defined a.e. on X) to $X \rightarrow \overline{\mathbb{R}}_{\geqslant 0}$ in an arbitrary way. Then $\int_Y f d\nu \in \mathcal{L}_+(X)$.

(c) We have

$$\int_{X\times Y} f d(\mu\times\nu) = \int_X \int_Y f d\nu d\mu \tag{26.8}$$

It follows that $y \in Y \mapsto f(\cdot, y)$ satisfies similar conditions, and hence $\displaystyle\int_X \int_Y f d\nu d\mu = \int_Y \int_X f d\mu d\nu$.

Proof. Step 1. We let $\mathscr{F}$ be the set of all $f \in \mathcal{L}_+(X\times Y)$ satisfying conditions (a,b,c). We make the following observations:

(i) $\mathscr{F}$ is an $\overline{\mathbb{R}}_{\geqslant 0}$-linear subspace of $\mathcal{L}_+(X \times Y)$.

(ii) By Thm. 23.23 and the monotone convergence Thm. 24.12, if (f_n) is an increasing sequence in $\mathscr{F}$, then its pointwise limit is also in $\mathscr{F}$.

Since μ, ν are σ-finite, by Rem. 25.34, there exist μ-finite open sets $\Omega_1 \subset \Omega_2 \subset \cdots \subset X$ whose union is X, and there exist ν-finite open sets $O_1 \subset O_2 \subset \cdots \subset Y$ whose union is Y. To show that each $f \in \mathcal{L}_+(X \times Y)$ belongs to $\mathscr{F}$, by (ii), it suffices to prove that $f\chi_{\Omega_n \times O_n} \in \mathscr{F}$ for each n.

Therefore, it suffices to choose any open $\Omega \subset X$ and $O \subset Y$ satisfying $\mu(\Omega) < +\infty$ and $\nu(O) < +\infty$, choose any $f \in \mathcal{L}_+(X \times Y)$ vanishing outside $\Omega \times O$, and prove that $f \in \mathscr{F}$.

Step 2. By Prop. 24.9, f is the pointwise limit of an increasing sequence of simple functions in $\mathcal{L}_+(X \times Y, \mathbb{R}_{\geqslant 0})$. Therefore, by (ii), it suffices to prove that each simple $f : X \times Y \rightarrow \mathbb{R}_{\geqslant 0}$ vanishing outside $\Omega \times O$ belongs to $\mathscr{F}$. By (i), it suffices to assume that $f = \chi_E$ where $E \subset \Omega \times O$ and E is $(\mu \times \nu)$-measurable.

By Tonelli's Thm. 26.8, if $E \subset X$ is open then $\chi_E \in \mathscr{F}$. Note that Thm. 26.8 also implies

$$(\mu\times\nu)(\Omega\times O) = \int_{X\times Y} \chi_{\Omega\times O} d(\mu\times\nu) = \int_X\int_Y \chi_{\Omega\times O} d\nu d\mu = \mu(\Omega)\nu(O) \tag{26.9}$$

Since $\mu(\Omega) < +\infty$ and $\nu(O) < +\infty$, we have $(\mu \times \nu)(\Omega \times O) < +\infty$. Therefore, by Thm. 23.23 and the dominated convergence Thm. 24.26, we have $\chi_E \in \mathscr{F}$

if $E \subset \Omega \times O$ is G_δ (since E is the intersection of a *decreasing* sequence of open subsets of $\Omega \times O$).

Now we consider the general case that $E \subset \Omega \times O$ and E is $\mu \times \nu$-measurable. By the regularity Cor. 25.29, we can find a G_δ set $B \subset \Omega \times O$ such that $E \subset B$ and $(\mu \times \nu)(B \backslash E) = 0$. Since $\chi_B \in \mathscr{F}$, and since all the integrals involved for χ_B are finite (since this is true for $\chi_{\Omega \times O}$), if we can show that $\chi_{B\backslash E} \in \mathscr{F}$, then $\chi_E = \chi_B - \chi_{B\backslash E}$ clearly belongs to $\mathscr{F}$.

Step 3. Therefore, it suffices choose any $E \subset \Omega \times O$ such that E is $(\mu \times \nu)$-measurable and $(\mu \times \nu)$-null, and prove $\chi_E \in \mathscr{F}$. By Cor. 25.31, there is a G_δ set $A \subset \Omega \times O$ such that $(\mu \times \nu)(A) = 0$ and $E \subset A$. We have proved in Step 2 that $\chi_A \in \mathscr{F}$. In fact, the proof in Step 2 actually shows (a',b',c) where

(a') $\chi_A(x, \cdot) \in \mathcal{L}_+(Y)$ for all $x \in X$.

(b') $\int_Y \chi_A d\nu \in \mathcal{L}_+(X)$.

(They are true when A is open, due to Tonelli's Thm. 26.8. By the dominated convergence theorem, they are also true when A is G_δ in $\Omega \times O$.)

Since (c) holds for χ_A, and since A is null, we have

$$0 = \int_{X \times Y} \chi_A d(\mu \times \nu) = \int_X \int_Y \chi_A d\nu d\mu \tag{26.10}$$

By Prop. 24.16, the function $x \in X \mapsto \int_Y \chi_A(x, \cdot) d\nu$ is 0 outside a μ-null set $\Delta \subset X$. By Prop. 24.16 again, for each $x \in X \backslash \Delta$, the function $\chi_A(x, \cdot)$ on Y is 0 a.e.. Since $\chi_E \leqslant \chi_A$, we conclude that for each $x \in X \backslash \Delta$, $\chi_E(x, \cdot)$ is 0 a.e. on Y. (However, when $x \in \Delta$, it is not known whether $\chi_E(x, \cdot)$ is measurable.) This proves that χ_E satisfies (a,b). Since $\chi_E \leqslant \chi_A$, (26.10) clearly holds if A is replaced by E. This proves that χ_E satisfies (c). Hence $\chi_E \in \mathscr{F}$. □

Corollary 26.11. *Assume that μ and ν are σ-finite, and let $f : X \times Y \to \mathbb{C}$ be measurable. Then $f \in \mathcal{L}^1(X \times Y, \mu \times \nu)$ iff $\int_X \int_Y |f| d\nu d\mu < +\infty$ iff $\int_Y \int_X |f| d\mu d\nu < +\infty$.*

Proof. Apply Tonelli's Thm. 26.10 to $|f|$. □

Example 26.12. Let $X = Y = [0, 1]$ and $\mu = \nu = m$. Let $A \subset [0, 1]$ be non-measurable. Let $E = \{0\} \times A$. Then E is an m^2-null subset of $[0, 1]^2$. So $\chi_E : [0, 1]^2 \to \mathbb{R}_{\geqslant 0}$ is Lebesgue measurable. However, $\chi_E(x, \cdot) : [0, 1] \to \mathbb{R}_{\geqslant 0}$ is not measurable when $x = 0$. This shows that the phrase "for almost every $x \in X$" in statement (a) of Tonelli's Thm. 26.10 cannot be replaced by "for every $x \in X$".

Theorem 26.13 (Fubini's theorem). *Assume that μ and ν are σ-finite. Let $f \in \mathcal{L}^1(X \times Y, \mu \times \nu)$. Then the following are true.*

(a) $f(x, \cdot) \in \mathcal{L}^1(Y, \nu)$ for almost every $x \in X$.

(b) *Extend the function $\int_Y f d\nu$ (originally defined a.e. on X) to $X \to \mathbb{C}$ in an arbitrary way. Then $\int_Y f d\nu \in \mathcal{L}^1(X,\mu)$.*

(c) *We have*

$$\int_{X\times Y} f d(\mu\times\nu) = \int_X \int_Y f d\nu d\mu \tag{26.11}$$

Proof. By considering $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ separately, it suffices to assume that f is real. Apply Tonelli's Thm. 26.10 to $f^\pm = \max\{\pm f, 0\}$, and notice Prop. 24.16-(b) (which is needed to prove that $\int_Y f^\pm(x,\cdot)d\nu < +\infty$ for almost every $x \in X$). We leave the details to the readers. □

Thm. 26.10 and Thm. 26.13 are often jointly referred to as the **Fubini-Tonelli theorem**.

26.3 Discussion on Fubini-Tonelli

26.3.1 Some easy consequences

Let $(\mathfrak{M},\mu)$ and $(\mathfrak{N},\nu)$ be completions of Radon measures on LCH spaces X and Y respectively. Let $\mu\times\nu$ be the Radon product.

Proposition 26.14. *Assume that μ and ν are σ-finite. Let $A \subset X$ and $B \subset Y$ be measurable. Then $A \times B$ is $(\mu\times\nu)$-measurable, and*

$$(\mu\times\nu)(A\times B) = \mu(A)\cdot\nu(B) \tag{26.12}$$

Compare this proposition with the fact that if A, B are Borel, then $A \times B$ is Borel and hence measurable, cf. Pb. 23.3.

Proof. Eq. (26.12) follows immediately from Tonelli's Thm. 26.10 once one can prove that $A\times B$ is measurable. Since X and Y can be written as countable unions $X = \bigcup_n E_n$ and $Y = \bigcup_k F_k$ where each E_n and F_k are measurable and have finite measures, it suffices to prove that $(A\cap E_n)\times(B\cap F_k)$ is measurable for each n, k.

In other words, it suffices to prove that $A \times B$ is measurable under the extra assumption that $\mu(A) < +\infty$ and $\nu(B) < +\infty$. By Cor. 25.29, for each $\varepsilon > 0$ there exist open sets $U \supset A$ and $V \supset B$, and compact sets $K \subset A$ and $L \subset B$, such that $\mu(U\backslash K) < \varepsilon$ and $\nu(V\backslash L) < \varepsilon$. Thus $U \times V$ is open and contains $A\times B$, $K \times L$ is compact and is contained in $A \times B$. Moreover, since

$$(U\times V)\backslash(K\times L) = ((U\backslash K)\times V)\cup(U\times(V\backslash L))$$

and since $(U\backslash K)\times V$ and $(U\times(V\backslash L))$ are open and hence measurable, we have

$$(\mu\times\nu)((U\times V)\backslash(K\times L)) \leqslant \mu(U\backslash K)\nu(V) + \mu(U)\nu(V\backslash L) \leqslant \varepsilon(\mu(U)+\nu(V))$$

Moreover, when applying Cor. 25.29, we can choose U, V such that $\mu(U) \leqslant \mu(A) + 1$ and $\nu(V) \leqslant \nu(B) + 1$. Thus, the open set $(U \times V)\backslash(K \times L)$ has measure $\leqslant \varepsilon(\mu(A) + \nu(B) + 2)$. Since ε is arbitrary, by Cor. 25.29, we conclude that $A \times B$ is measurable. □

Example 26.15. The projection $\pi : X \times Y \to X$ is continuous, and hence is Borel. This does not imply that π is $(\mu \times \nu)$-measurable (where X is equipped with the σ algebra $\mathfrak{M}$). However, if μ and ν are σ-finite, then π is $(\mu \times \nu)$-measurable because for each $A \in \mathfrak{M}$, the set $\pi^{-1}(A) = A \times Y$ is measurable by Prop. 26.14.

Example 26.16. Assue that μ is σ-finite. Let $f : X \to \mathbb{R}_{\geqslant 0}$ be μ-measurable. Let R_f be the region between the graph of f and the X-axis, i.e.,

$$R_f = \big\{(x, y) \in X \times \mathbb{R} : 0 \leqslant y \leqslant f(x)\big\} \tag{26.13}$$

Then R_f is an $(\mu \times m)$-measurable subset of $X \times \mathbb{R}$ (where m is the Lebesgue measure), and

$$(\mu \times m)(R_f) = \int_X f d\mu \tag{26.14}$$

Compare this example with Pb. 23.3.

Proof. It suffices to prove that R_f is measurable. Then (26.14) follows from Tonelli's Thm. 26.10. By Exp. 26.15, the projections $\pi_1 : X \times \mathbb{R} \to X$ and $\pi_2 : X \times \mathbb{R} \to \mathbb{R}$ are measurable. Therefore, $\Phi = (f \circ \pi_1) \vee \pi_2 : X \times \mathbb{R} \to \mathbb{R}^2$ is measurable by Prop. 23.21. Since $E = \{(s, t) \in \mathbb{R}^2 : 0 \leqslant t \leqslant s\}$ is closed, $R_f = \Phi^{-1}(E)$ is measurable. □

26.3.2 Other approaches to Fubini-Tonelli

We have mentioned in Rem. 26.5 that there is a general notion of product measure space which (after completion) agrees with the Radon product when the LCH spaces are second countable. The general definition is as follows. Suppose that $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ are measure spaces. Let $\mathfrak{M} \otimes \mathfrak{N}$ be the σ-algebra generated by all $E \times F$ where $E \in \mathfrak{M}$ and $F \in \mathfrak{N}$. Using Carathéodory's Thm. 23.62, one can naturally construct a measure $\mu \widehat{\times} \nu : \mathfrak{M} \otimes \mathfrak{N} \to [0, +\infty]$ satisfying $(\mu \widehat{\times} \nu)(E \times F) = \mu(E)\nu(F)$.

Assuming that μ and ν are σ-finite, Tonelli's Thm. 26.10 and Fubini's Thm. 26.13 can be proved for $(\mathfrak{M} \otimes \mathfrak{N})$-measurable functions on $X \times Y$ with one improvement: the statement "for almost every $x \in X$" in Thm. 26.13-(a) can be replaced by "for every $x \in X$". However, $(\mathfrak{M} \otimes \mathfrak{N}, \mu \widehat{\times} \nu)$ is in general not complete. If we consider its completion instead, then we still need "for almost every $x \in X$" in Thm. 26.13-(a). The readers are referred to [Fol-R, Sec. 2.5] for details.

Alternatively, one can first prove $\int_X \int_Y f = \int_Y \int_X f$ for σ-finite measures, and then define the measure on $\mathfrak{M} \otimes \mathfrak{N}$ by using $(\mu \widehat{\times} \nu)(A) = \int_X \int_Y \chi_A$. This approach was adopted by Rudin. See [Rud-R, Ch. 8]. Of course, this approach does not define $\mu \widehat{\times} \nu$ when μ or ν is not σ-finite. But you won't lose anything if you only care about σ-finite measures.

When X, Y are LCH spaces and $(\mathfrak{M}, \mu), (\mathfrak{N}, \nu)$ are completions of Radon measures, the Radon product $\mu \times \nu$ extends the completion of $\mu \widehat{\times} \nu$ when μ, ν are σ-finite, and agrees with the completion of $\mu \widehat{\times} \nu$ when X, Y are second countable. See [Fol-R, Sec. 7.4]. [2]

In many approaches to the Fubini-Tonelli theorem, one uses either Carathéodory's theory (cf. [SS-R, Ch. 6, Sec. 3.1]), or monotone classes (cf. [Rud-R]), or both (cf. [Fol-R], [Tao, Sec. 1.7], or [Yu, Sec. 44]). We used neither of these, but used regularity instead. It is clear that regularity runs through our treatment of measure theory from beginning to end.

26.4 Failure of Tonelli's theorem without LSC and σ-finiteness; a Borel set not inner regular

We have proved two versions of Tonelli's theorem, i.e., Thm. 26.8 and 26.10. The first one assumes that the functions are lower semicontinuous, and the second one assumes that X, Y are σ-finite. The readers should compare them with the two monotone convergence theorems, i.e., Cor. 25.23 and Thm. 24.12. The first one assumes lower semicontinuity but can be applied to nets. The latter can be applied only to sequences. The relationship between a general Radon measure and a σ-finite one is similar to the relationship between a net and a sequence.

In the following, let us see a class of (counter)examples that can be interpreted both by the failure of Fubini-Tonelli without σ-finiteness and lower semicontinuity, and by the failure of the monotone convergence theorem for nets of functions without lower semicontinuity.

Let $(\mathfrak{M}, \mu)$ be the completion of a Radon measure on an LCH space X. Let Y be a set, equipped with the discrete topology 2^Y, and let $\nu : 2^Y \to \overline{\mathbb{R}}_{\geqslant 0}$ be the counting measure. Then ν is a complete Radon measure on Y which is σ-finite iff Y is countable.

For each $E \subset X \times Y$ and $y \in Y$, let $E_y \subset X$ such that

$$E_y \times \{y\} = E \cap (X \times \{y\}) \tag{26.15}$$

It is clear that E is open iff each E_y is open. Tonelli's Thm. 26.8 implies that if E is

[2] Folland used the word "Radon product" in a different way. Since Folland focused on Borel measurable sets and functions, his Radon product is the Radon measure associated to $\Lambda_1 \otimes \Lambda_2$ (defined on $\mathfrak{B}_{X \times Y}$). Therefore, our Radon product is the completion of Folland's Radon product.

open then

$$(\mu \times \nu)(E) = \sum_y \mu(E) \tag{26.16}$$

Let $f : X \times Y \to \overline{\mathbb{R}}_{\geqslant 0}$. For each $y \in Y$, define $f_y : X \to \overline{\mathbb{R}}_{\geqslant 0}$ be $f_y(x) = f(x, y)$. It follows from (26.15) that f is lower semicontinuous iff f_y is lower semicontinuous for each $y \in Y$.

We now assume that μ is σ-finite and f is Borel. Consider the relation

$$\sum_{y \in Y} \int_X f_y d\mu = \int_X \Big(\sum_{y \in Y} f_y\Big) d\mu \tag{26.17}$$

which holds by Tonelli's Theorems if Y is countable or if f is lower semicontinuous. Let $I = \mathrm{fin}(2^Y)$. For each $\alpha \in I$, let $g_\alpha : X \to \overline{\mathbb{R}}_{\geqslant 0}$ be defined by $g_\alpha = \sum_{y \in \alpha} f_\alpha$. Then (26.17) is equivalent to

$$\lim_{\alpha \in I} \int_X g_\alpha d\mu = \int_X \Big(\lim_{\alpha \in I} g_\alpha\Big) d\mu \tag{26.18}$$

which holds when f is lower semicontinuous by the monotone convergence Cor. 25.23. When $Y = \mathbb{Z}_+$, (g_α) has a subnet $(g_{\alpha_n})_{n \in \mathbb{Z}_+}$ where $\alpha_n = \{1, \dots, n\}$. Since any increasing net in $\overline{\mathbb{R}}_{\geqslant 0}$ converges in $\overline{\mathbb{R}}_{\geqslant 0}$, and since any of its subnet converges to the same value, we see that (26.18) is equivalent to

$$\lim_{n \to \infty} \int_X g_{\alpha_n} d\mu = \int_X \Big(\lim_{n \to \infty} g_{\alpha_n}\Big) d\mu$$

which holds by the monotone convergence Thm. 24.12.

We now give an example where Y is uncountable, f is not lower semicontinuous, and (26.17) fails (equivalently, (26.18) fails).

Example 26.17. Let $X = [0, 1]$ and μ is the Lebesgue measure m. Let $Y = [0, 1]$, equipped with the discrete topology and the counting measure (denoted by ν). Define $f : X \times Y \to \mathbb{R}_{\geqslant 0}$ by $f(x, y) = \delta_{x,y}$. Namely, f is the characteristic function χ_Δ where Δ is the diagonal line $\{(x, x) : x \in [0, 1]\}$. Δ clear has open complement. So Δ is closed. Therefore f is upper semicontinuous (and hence is Borel). We have

$$\sum_{y \in Y} \int_X f_y dm = \sum_{y \in Y} 0 = 0 \neq 1 = \int_X 1 dm = \int_X \Big(\sum_{y \in Y} f_y\Big) dm$$

Moreover, the above two iterated integrals are not equal to $\int_{X \times Y} f d(m \times \nu) = (m \times \nu)(\Delta)$. Let us prove that

$$\int_{X \times Y} f d(m \times \nu) = +\infty$$

By the outer regularity (cf. Thm. 25.25), it suffices to prove that $(m\times\nu)^*(\Delta) = +\infty$, i.e., any an open set E containing Δ has infinite measure. Let E_y be as in (26.15), which is nonempty for each y. So $m(E_y) > 0$. Since Y is uncountable, we have $\sum_y m(E_y) = +\infty$ by Pb. 5.3. So $(m \times \nu)(E) = +\infty$ by (26.16). □

Remark 26.18. In Exp. 26.17, we proved that the closed set Δ satisfies $(m \times \nu)^*(\Delta) = (m \times \nu)(\Delta) = +\infty$. Note that every compact subset K of Δ is a finite set (since its image under the projection $X \times Y \to Y$ is compact and hence finite). Hence K is $(m\times\nu)$-null. Therefore $(m\times\nu)_*(\Delta) = 0$. This gives an example of Borel subset with infinite Radon measure which is not inner regular.

Exercise 26.19. The condition $\int_{X\times Y} f d(\mu \times \nu) < +\infty$ in Fubini's Thm. 26.13 is similar to the condition that $|f_n| \leqslant g$ and $\int g < +\infty$ in the dominated convergence Thm. 24.26. Find a counterexample that can be explained by both perspectives.

27 The marriage of Hilbert spaces and integral theory

Definition 27.1. Let V be a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. A **seminorm** on V is a function $\|\cdot\| : V \to \mathbb{R}_{\geqslant 0}$ satisfying:

- (Subadditivity) If $u, v \in V$, then $\|u + v\| \leqslant \|u\| + \|v\|$.
- (Absolute homogeneity) If $v \in V$ and $\lambda \in \mathbb{F}$ then $\|\lambda v\| = |\lambda| \cdot \|v\|$.

The relationship between seminorms and norms is similar to that between positive sesquilinear forms and inner products. In particular, we have the following generalization of Prop. 20.13.

Proposition 27.2. *Let $\|\cdot\|$ be a seminorm on V. Then $\mathscr{N} = \{v \in V : \|v\| = 0\}$ is a linear subspace of V. There is a canonical norm $\|\cdot\|_U$ on $U = V/\mathscr{N}$ such that*

$$\|v + \mathscr{N}\|_U = \|v\| \qquad (\forall v \in V)$$

Proof. It is easy to check that $\mathscr{N}$ is a linear subspace. If $u + \mathscr{N} = v + \mathscr{N}$, then $v - u \in \mathscr{N}$. So $\|v\| \leqslant \|u\| + \|v - u\| = \|u\|$, and similarly $\|u\| \leqslant \|v\|$. This proves that $\|\cdot\|_U : U \to \mathbb{R}_{\geqslant 0}$ is well-defined. We leave it to the readers to check that $\|\cdot\|_U$ is a norm. □

27.1 The definition of L^p spaces

Let $(X, \mathfrak{M}, \mu)$ be a measure space.

27.1.1 The space $L^p(X, \mu)$ where $1 \leqslant p < +\infty$

We fix a number p satisfying $1 \leqslant p < +\infty$.

Definition 27.3. For each $f \in \mathcal{L}(X, \mathbb{C})$, define

$$\|f\|_{L^p} \equiv \|f\|_p = \Big(\int_X |f|^p d\mu\Big)^{\frac{1}{p}} \tag{27.1}$$

In particular, we will not let $\|f\|_p$ denote $\|f\|_{l^p}$ unless otherwise stated.

Theorem 27.4. *Assume that $f, g \in \mathcal{L}(X, \mathbb{C})$ or $f, g \in \mathcal{L}_+(X)$. We have* ***Minkowski's inequality***

$$\|f + g\|_p \leqslant \|f\|_p + \|g\|_p \tag{27.2}$$

and, if $1 < p, q < +\infty$ and $p^{-1} + q^{-1} = 1$, ***Hölder's inequality***

$$\Big|\int_X fg d\mu\Big| \leqslant \|f\|_p \cdot \|g\|_q \tag{27.3}$$

Proof. Since $\|f+g\|_p \leqslant \|(|f|+|g|)\|_p$ and $|\int fg| \leqslant \int |fg|$, by replacing f, g with $|f|, |g|$, it suffices to assume $f, g \in \mathcal{L}_+(X)$. By Prop. 24.9, there exist increasing sequences in $\mathcal{S}(X, \mathbb{R}_{\geqslant 0})$ converging to f and g respectively. By the monotone convergence theorem, it suffices to prove the two inequalities for elements in $\mathcal{S}(X, \mathbb{R}_{\geqslant 0})$.

Let's prove Hölder's inequality for $f, g \in \mathcal{S}_+(X)$ assuming that $1 < p, q < +\infty$. The proof of Minkowski's inequality is similar and is left to the readers.

We assume WLOG that $\|f\|_p \cdot \|g\|_q < +\infty$, otherwise, the inequality is obvious. If $\|f\|_p = +\infty$, then $\|g\|_q = 0$. So $\int |g|^p = 0$. By Prop. 24.16, we have $g = 0$ a.e., and hence $fg = 0$ a.e.. So $\int fg = 0$ by Prop. 24.16. Similarly, if $\|g\|_q = +\infty$, then $\int fg = 0$. The inequality holds.

So we can assume that $\|f\|_p$ and $\|g\|_q$ are both finite. By Prop. 24.16, $f < +\infty$ and $g < +\infty$ outside a null set Δ. Replacing f, g with $f\chi_{\Delta^c}, g\chi_{\Delta^c}$, it suffices to assume $f, g \in \mathcal{S}(X, \mathbb{R}_{\geqslant 0})$. Write f and g as finite sums $f = \sum_i a_i \chi_{E_i}$ and $g = \sum_j b_j \chi_{F_j}$ where $a_i, b_j \in \mathbb{R}_{>0}$, $E_1, E_2, \dots \in \mathfrak{M}$ are pairwise disjoint, and $F_1, F_2, \dots \in \mathfrak{M}$ are pairwise disjoint. Since $a_i \chi_{E_i} \leqslant f$, we have $a_i^p \mu(E_i) \leqslant \int |f|^p < +\infty$ and hence $\mu(E_i) < +\infty$. Similarly, we have $\mu(F_j) < +\infty$. Let $G_{i,j} = E_i \cap F_j$. So $f = \sum_{i,j} a_i \chi_{G_{i,j}}$ and $g = \sum_{i,j} b_j \chi_{G_{i,j}}$ and $fg = \sum_{i,j} a_i b_j \chi_{G_{i,j}}$. By Hölder's inequality for finite sums (Thm. 12.31) we have

$$\int_X fg = \sum_{i,j} a_i b_j \mu(G_{i,j}) = \sum_{i,j} a_i \mu(G_{i,j})^{\frac{1}{p}} \cdot b_j \mu(G_{i,j})^{\frac{1}{q}}$$

$$\leqslant \Big(\sum_{i,j} a_i^p \mu(G_{i,j}) \Big)^{\frac{1}{p}} \cdot \Big(\sum_{i,j} b_j^q \mu(G_{i,j}) \Big)^{\frac{1}{q}} = \|f\|_p \cdot \|g\|_q$$

□

Definition 27.5. We let

$$\mathcal{L}^p(X, \mu) = \{ f \in \mathcal{L}(X, \mathbb{C}) : \|f\|_p < +\infty \} \tag{27.4}$$

Then, by Minkowski's inequality, $\mathcal{L}^p(X, \mu)$ is a linear subspace of $\mathbb{C}^X$ with seminorm $\|\cdot\|_p$. Thus, by Prop. 27.2, $L^p(X, \mu)$ is a normed vector space with norm $\|\cdot\|_{L^p} = \|\cdot\|_p$ if we define

$$L^p(X, \mu) = \mathcal{L}^p(X, \mu) / \{ f \in \mathcal{L}^p(X, \mu) : \|f\|_p = 0 \} \tag{27.5}$$

By Prop. 24.16, we have $\|f\|_p = 0$ iff $f = 0$ a.e.. So

$$L^p(X, \mu) = \mathcal{L}^p(X, \mu) / \{ f \in \mathcal{L}(X, \mathbb{C}) : f = 0 \ \mu\text{-a.e.} \}$$

In other words, elements in $L^p(X, \mu)$ are measurable functions $f : X \to \mathbb{C}$, and two elements f, g are viewed as the same iff $f = g$ a.e..

Remark 27.6. The L^2 norm on $L^2(X, \mu)$ is clearly induced by the inner product

$$\langle f | g \rangle = \int_X f g^* d\mu \qquad (\forall f, g \in L^2(X, \mu)) \tag{27.6}$$

where fg^* is integrable by Hölder's inequality. We shall always understand $L^2(X, \mu)$ as an inner product space whose inner product is defined by (27.6).

27.1.2 The space $L^\infty(X,\mu)$

Definition 27.7. For each $f \in \mathcal{L}(X,\mathbb{C})$, define

$$\|f\|_{L^\infty} \equiv \|f\|_\infty = \inf\big\{a \in \overline{\mathbb{R}}_{\geqslant 0} : \mu\{x \in X : |f(x)| > a\} = 0\big\} \tag{27.7}$$

(Note that the set inside the $\inf$ is nonempty since it contains $+\infty$.) Clearly

$$\|f\|_{L^\infty} \leqslant \|f\|_{l^\infty}$$

Unless otherwise stated, we will not let $\|f\|_\infty$ denote $\|f\|_{l^\infty}$.

We give some elementary facts about L^∞.

Proposition 27.8. *Let $f \in \mathcal{L}(X,\mathbb{C})$ and $\lambda = \|f\|_{L^\infty}$. Then*

$$\big\{a \in \overline{\mathbb{R}}_{\geqslant 0} : \mu\{|f| > a\} = 0\big\} = [\lambda, +\infty] \tag{27.8}$$

In particular, $\lambda = \|f\|_{L^\infty}$ is the smallest number in $\overline{\mathbb{R}}_{\geqslant 0}$ such that $\{x \in X : |f(x)| > \lambda\}$ is null. Moreover, if we let

$$A = \{|f| \leqslant \lambda\}$$

then $X\backslash A$ is null, and for any measurable $B \subset A$ satisfying $\mu(X\backslash B) = 0$, we have

$$\|f\chi_B\|_{l^\infty} = \|f\|_{L^\infty} \tag{27.9}$$

Proof. Let $E = \big\{a \in \overline{\mathbb{R}}_{\geqslant 0} : \mu\{|f| > a\} = 0\big\}$. Clearly, if $a \in E$ and $b \geqslant a$ then $b \in E$. Therefore, E equals $(\lambda, +\infty]$ or $[\lambda, +\infty]$. Pick a decreasing sequence (a_n) in E converging to λ. Then $\{|f| > \lambda\}$ is the union of $\{|f| > a_n\}$, which is null. This proves $\lambda \in E$, and hence $E = [\lambda, +\infty]$.

That $\lambda \in E$ means that $X\backslash A$ is null. Let $\kappa = \|f\chi_B\|_{l^\infty}$. Since $|f|_B| \leqslant \lambda$, we have $\kappa \leqslant \lambda$. The set $\{|f| > \kappa\}$ is a measurable subset of $X\backslash B$, which is null. Therefore, κ belongs to $E = [\lambda, +\infty]$, and hence $\kappa \geqslant \lambda$. □

Corollary 27.9. *Choose countably many $f_1, f_2, \dots \in \mathcal{L}(X,\mathbb{C})$. There there exists $A \in \mathfrak{M}$ such that $\mu(X\backslash A) = 0$ and $\|f_n\chi_A\|_{l^\infty} = \|f_n\|_{L^\infty}$ for each n.*

Proof. By Prop. 27.8, $A = \bigcap_n A_n$ satisfies the desired property if we let $A_n = \{|f_n| \leqslant \lambda_n\}$ and $\lambda_n = \|f_n\|_{L^\infty}$. □

Proposition 27.10. *Let $f, g \in \mathcal{L}(X,\mathbb{C})$. The following are true.*

(a) If $f = g$ a.e., then $\|f\|_{L^\infty} = \|g\|_{L^\infty}$.

(b) We have $f = 0$ a.e. iff $\|f\|_{L^\infty} = 0$.

Proof. Suppose that $f = g$ a.e.. Then $\{|f| > a\}$ is null iff $\{|g| > a\}$ is null. So $\|f\|_{L^\infty} = \|g\|_{L^\infty}$.

In particular, if $f = 0$ a.e., then $\|f\|_{L^\infty} = \|0\|_{L^\infty} = 0$. Conversely, if $\|f\|_{L^\infty} = 0$, then by Prop. 27.8, $\{x \in X : |f(x)| > 0\}$ is null. So $f = 0$ a.e.. □

Thanks to Cor. 27.9, we can prove many properties of L^∞ with the help of l^∞. Let us see some examples.

Proposition 27.11. *Let (f_n) be a sequence in $\mathcal{L}(X, \mathbb{C})$. Then the following are equivalent.*

(1) $\lim_{n\to\infty} \|f_n\|_{L^\infty} = 0$.

(2) There exists $A \in \mathfrak{M}$ such that $\mu(X\backslash A) = 0$ and $\lim_{n\to\infty} \|f_n\chi_A\|_{l^\infty} = 0$.

Proof. Assume (1). By Cor. 27.9, there exists $A \in \mathfrak{M}$ such that $X\backslash A$ is null and $\|f_n\|_{L^\infty} = \|f_n\chi_A\|_{l^\infty}$. Thus (2) holds. Assume (2). By Prop. 27.10 we have $\|f_n\|_{L^\infty} = \|f_n\chi_A\|_{L^\infty} \leqslant \|f_n\chi_A\|_{l^\infty}$. So (1) is true. □

Proposition 27.12. *For every $a \in \mathbb{C}$ and $f, g \in \mathcal{L}(X, \mathbb{C})$ we have*

$$\|f + g\|_{L^\infty} \leqslant \|f\|_{L^\infty} + \|g\|_{L^\infty} \qquad \|af\|_{L^\infty} = |a| \cdot \|f\|_{L^\infty}$$

Proof. By Cor. 27.9, there exists $A \in \mathfrak{M}$ with null complement such that $\|f\|_{L^\infty} = \|f\chi_A\|_{l^\infty}$, $\|g\|_{L^\infty} = \|g\chi_A\|_{l^\infty}$, and $\|f + g\|_{L^\infty} = \|(f + g)\chi_A\|_{l^\infty}$. Therefore

$$\|f + g\|_{L^\infty} = \|f\chi_A + g\chi_A\|_{l^\infty} \leqslant \|f\chi_A\|_{l^\infty} + \|g\chi_A\|_{l^\infty} = \|f\|_{L^\infty} + \|g\|_{L^\infty}$$

Similarly, let $B \in \mathfrak{M}$ with null complement such that $\|af\|_{L^\infty} = \|af\chi_B\|_{l^\infty}$ and $\|f\|_{L^\infty} = \|f\chi_B\|_{l^\infty}$. Then

$$\|af\|_{L^\infty} = \|af\chi_B\|_{l^\infty} = a\|f\chi_B\|_{l^\infty} = a\|f\|_{L^\infty}$$

□

Remark 27.13. Hölder's inequality clearly holds when $p = 1, q = +\infty$. Namely, if $f, g \in \mathcal{L}(X, \mathbb{C})$, since $|fg| \leqslant |f| \cdot \|g\|_\infty$ a.e., we have $\int |fg| \leqslant \int |f| \cdot \|g\|_\infty$, i.e.

$$\|fg\|_{L^1} \leqslant \|f\|_{L^1} \cdot \|g\|_{L^\infty}$$

Definition 27.14. We let

$$\mathcal{L}^\infty(X) = \{f \in \mathcal{L}(X, \mathbb{C}) : \|f\|_{l^\infty} < +\infty\}$$

Then, by Prop. 27.10 and 27.12, we can define the normed vector space

$$L^\infty(X, \mu) = \mathcal{L}^\infty(X)/\{f \in \mathcal{L}^\infty(X) : f = 0 \text{ a.e.}\} \tag{27.10}$$

with the (well-defined) norm $\|\cdot\|_{L^\infty}$.

Theorem 27.15. *$L^\infty(X,\mu)$ is a Banach space.*

Proof. Let (f_n) be a Cauchy sequence in $L^\infty(X,\mu)$. Choose $f_n \in \mathcal{L}^\infty(X)$ representing the corresponding element in $L^\infty(X,\mu)$. Then $\lim_{m,n\to\infty} \|f_m - f_n\|_{L^\infty} = 0$. By Cor. 27.9, there is $A \in \mathfrak{M}$ with null complement such that $\|(f_m - f_n)\chi_A\|_{l^\infty}$ equals $\|f_m - f_n\|_{L^\infty}$, and hence converges to 0. By the completeness of $l^\infty(X)$, $(f_n\chi_A)_{n\in\mathbb{Z}_+}$ converges uniformly to some $f \in l^\infty(X)$. By Cor. 23.24, $f \in \mathcal{L}^\infty(X)$. Thus

$$\|f - f_n\|_{L^\infty} = \|(f - f_n)\chi_A\|_{L^\infty} \leqslant \|(f - f_n)\chi_A\|_{l^\infty(X)} = \|f - f_n\|_{l^\infty(A)} \to 0$$

where Prop. 27.10 is used in the first equality. □

Exercise 27.16. Let $1 \leqslant p \leqslant +\infty$. Let $\overline{\mu}$ be the completion of μ. Use Pb. 23.2 or Prop. 24.11 to prove that the map

$$L^p(X,\mu) \to L^p(X,\overline{\mu}) \qquad f \mapsto f$$

is an isomorphism of normed vector spaces.

27.2 Approximation in L^p spaces

In this section, we provide two useful dense subspaces of an L^p space.

27.2.1 Approximation by continuous functions

Theorem 27.17. *Let X be LCH. Let μ be the completion of a Radon measure on X. Let $1 \leqslant p < +\infty$. Then $C_c(X)$ is dense in $L^p(X,\mu)$. More precisely, the (non-necessarily injective) map $f \in C_c(X) \mapsto f \in L^p(X,\mu)$ has dense range.*

Note that by Exe. 27.16, the theorem will be no different if we deal with the original Radon measure rather than its completion.

Proof. Let $f \in L^p(X,\mu)$. We shall show that f can be approximated by elements of $C_c(X)$.

We first consider the special case that $M = \|f\|_{l^\infty} < +\infty$ and that f is zero outside some $A \in \mathfrak{M}$ such that $\mu(A) < +\infty$. By the regularity Thm. 25.25, A is contained in an open set with finite measure. By replacing A with this larger open set, we may assume that A is open.

By Lusin's Thm. 25.32, for every $\varepsilon > 0$, there is a compact $K \subset A$ such that $\mu(A\backslash K) < \varepsilon$ and $f|_K$ is continuous. Since A is open and hence is LCH (cf. Prop. 8.41), by the Tietze extension Thm. 15.22, there exists $g \in C_c(X)$ compactly supported in A such that $g|_K = f|_K$ and $\|g\|_{l^\infty} \leqslant M$. Thus $|f - g| \leqslant 2M$, and hence

$$\int_X |f-g|^p = \int_{A\backslash K} |f-g|^p \leqslant (2M)^p \cdot \mu(A\backslash K) \leqslant (2M)^p \cdot \varepsilon$$

Therefore $\|f-g\|_p \leqslant 2M \cdot \varepsilon^{\frac{1}{p}}$. Since ε can be arbitrary, we conclude that f can be approximated by elements of $C_c(X)$.

Now we treat the general case. Let $E_n = \{x \in X : 1/n \leqslant |f(x)| \leqslant n\}$. Then (E_n) is increasing and $\bigcup_n E_n = X$. Moreover, since $n^{-1}\chi_{E_n} \leqslant |f|$, we have $n^{-p}\mu(E_n) \leqslant \|f\|_p^p$ and hence $\mu(E_n) < +\infty$. Since $|f - f\chi_{E_n}| \leqslant |f|$ and $\int |f|^p < +\infty$, we have $\lim_n \int |f - f\chi_{E_n}|^p = 0$ by the dominated convergence theorem. By the above special case, $f\chi_{E_n}$ can be approximated by elements of $C_c(X)$. This finishes the proof. $\square$

Note that $C_c(X)$ is in general not dense in $L^\infty(X)$ becasue the uniform limit of a sequence of continuous functions is continuous.

27.2.2 Applications of continuous function approximation

Definition 27.18. Fix any $\theta \in \mathbb{R}$. A subset $E \subset \mathbb{S}^1$ is called **Lebesgue measurable** if it is of the form $\exp(\mathbf{i}F)$ where F is a Lebesgue measuarble subset of $[\theta-\pi, \theta+\pi)$. In that case, we defined the **Lebesgue measure** $m(E)$ to be $m(F)$. It is easy see that this definition is independent of the choice of θ. Since $(\mathbb{S}^1, m)$ is equivalent to $([-\pi,\pi), m)$, the Lebesgue measure on $\mathbb{S}^1$ is the completion of a Radon measure.

Corollary 27.19. *Let $e_n \in C[-\pi,\pi]$ be defined by $e_n(x) = e^{\mathbf{i}nx}$. Let $1 \leqslant p < +\infty$. Then $(e_n)_{n\in\mathbb{Z}}$ spans a dense linear subspace of $L^p([-\pi,\pi], \frac{m}{2\pi})$. In particular, $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2([-\pi,\pi], \frac{m}{2\pi})$.*

Proof. Clearly $L^p([-\pi,\pi], \frac{m}{2\pi})$ can be identified naturally with $L^p(\mathbb{S}^1, \frac{m}{2\pi}) \simeq L^p([-\pi,\pi), \frac{m}{2\pi})$. By Thm. 27.17, $C(\mathbb{S}^1)$ is dense in $L^p(\mathbb{S}^1, \frac{m}{2\pi})$. By Stone-Weierstrass, $V = \mathrm{Span}\{e_n : n \in \mathbb{Z}\}$ is l^∞-dense in $C(\mathbb{S}^1)$. Therefore V is L^p-dense in $C(\mathbb{S}^1)$ since

$$\|f\|_p^p = \frac{1}{2\pi}\int_{[-\pi,\pi]} |f|^p dm \leqslant \|f\|_{l^\infty}^p$$

shows that l^∞-convergence implies L^p-convergence. Therefore, V is dense in $L^p([-\pi,\pi], \frac{m}{2\pi})$.

It is obvious that (e_n) is an orthonormal sequence in $L^2([-\pi,\pi], \frac{m}{2\pi})$. Therefore, the density proved above shows that (e_n) is an orthonormal basis of $L^2([-\pi,\pi], \frac{m}{2\pi})$. $\square$

Corollary 27.20 (Riemann-Lebesgue). *Let $f \in L^1(\mathbb{R}, m)$. Then*

$$\lim_{t\to+\infty}\int_{\mathbb{R}} f(x)e^{\mathbf{i}tx}dm = \lim_{t\to-\infty}\int_{\mathbb{R}} f(x)e^{\mathbf{i}tx}dm = 0 \tag{27.11}$$

Proof. By Thm. 27.17, for each $\varepsilon > 0$ there exists $g \in C_c(\mathbb{R})$ such that $\|f-g\|_{L^1} \leqslant \varepsilon$. Then $|\int_{\mathbb{R}}(f-g)e^{\mathbf{i}tx}dm| \leqslant \varepsilon$. Therefore, if we can show that $\lim_{t\to\pm\infty}\int_{\mathbb{R}} ge^{\mathbf{i}tx}dm = 0$, then $\limsup_{t\to\pm\infty}\int_{\mathbb{R}} fe^{\mathbf{i}tx}dm \leqslant \varepsilon$ for all $\varepsilon > 0$, finishing the proof.

That $\lim_{t\to\pm\infty}\int_{\mathbb{R}} ge^{\mathbf{i}tx}dm = 0$ follows from the Riemann-Lebesgue lemma for Riemann integrals, Thm. 14.52. But we can also prove it without using the method of Riemann integrals. Since $\int g(x)e^{\mathbf{i}tx}dx = \int \lambda g(\lambda x)e^{\mathbf{i}t\lambda x}dx$ (where $\lambda \in \mathbb{R}_{>0}$), by replacing g with $g(\lambda x)$, it suffices to assume that $\mathrm{Supp} g \subset [-\pi, \pi]$ and prove

$$\lim_{t\to\pm\infty} \int_{-\pi}^{\pi} g(x)e^{\mathbf{i}tx}dx = 0 \tag{27.12}$$

In fact, we can prove this for every $g \in C(\mathbb{S}^1)$. By Stone-Weierstrass, it suffices to prove this relation if $g : x \in [-\pi, \pi] \mapsto e^{\mathbf{i}nx}$ for some n. Then $\int_{-\pi}^{\pi} g(x)e^{\mathbf{i}tx}dx$ can be calculated using the fundamental theorem of calculus, and one easily checks that it converges to 0 as $t \to \pm\infty$. □

27.2.3 The separability of L^p spaces

The following theorem is a further application of Thm. 27.17.

Theorem 27.21. *Let X be a second countable LCH space. Let $(\mathfrak{M}, \mu)$ be a Radon measure (or its completion) on X. Let $1 \leqslant p < +\infty$. Then $L^p(X, \mu)$ is separable.*

In the next section, we will see that $L^p(X, \mu)$ is complete. Therefore, if X is second countable LCH, then the Hilbert space $L^2(X, \mu)$ has a countable orthonormal basis (Cor. 21.6) and hence is isomorphic to $l^2(\mathbb{Z})$ or $l^2(\{1, \dots, n\})$ (Thm. 21.7).

Proof. Step 1. We consider the special case that X is compact (and second countable). Then $\mu(X) < +\infty$. By Thm. 27.17, $C(X)$ is dense in $L^p(X, \mu)$. Therefore, it suffices to prove that $C(X)$ is separable under the L^p-seminorm, i.e., there is a countable subset $\mathcal{E} \subset C(X)$ such that for each $f \in C(X)$ there is a sequence (f_n) in $\mathcal{E}$ satisfying $\lim_n \|f - f_n\|_p = 0$. Since

$$\|f - f_n\|_p^p = \int_X |f - f_n|^p d\mu \leqslant \|f - f_n\|_{l^\infty}^p \cdot \mu(X)$$

it suffices to find a countable $\mathcal{E} \subset C(X)$ which is l^∞-dense in $C(X)$. But we have already proved this before: Thm. 15.37 says that a compact Hausdorff space is second countable iff its space of continuous functions is (l^∞-)separable.

Step 2. We consider the general case. Since X is second countable, X is Lindelöf. Therefore, X is a countable union of precompact open subsets. So X is σ-compact, i.e., we have compact $K_1, K_2, \dots \subset X$ such that $X = \bigcup_n K_n$. By replacing K_n with $K_1 \cup \cdots \cup K_n$ we assume $K_1 \subset K_2 \subset \cdots \subset X$.

The restriction $\mu_n := \mu|_{\mathfrak{B}_{K_n}}$ is a finite Borel measure on the second countable compact Hausdorff space K_n. Therefore, μ_n is a Radon measure on K_n by Thm. 25.38. Thus, by Step 1, $L^p(K_n, \mu_n)$ has a countable dense subset $\mathcal{E}_n$. Note that we can view $L^p(K_n, \mu_n)$ as a subset of $L^p(X, \mu)$. So $\mathcal{E} = \bigcup_n \mathcal{E}_n$ is a countable subset of $L^p(X, \mu)$.

Let us prove that $\mathcal{E}$ is dense in $L^p(X, \mu)$. Choose any $f \in L^p(X, \mu)$. By Prop. 24.11, we may assume that f is Borel. So each $f\chi_{K_n}$ is Borel. The dominated convergence theorem implies that $\lim_n \|f - f\chi_{K_n}\|_p^p = 0$. Since $\mathcal{E}_n$ is dense in $L^p(K_n, \mu_n)$, the Borel function $f|_{K_n} : K_n \to \mathbb{C}$ can be L^p-approximated by elements of $\mathcal{E}_n$. So $f\chi_{K_n} : X \to \mathbb{C}$ can be approximated by elements of $\mathcal{E}_n$. Thus f can be approximated by elements of $\mathcal{E}$. □

Remark 27.22. In the above proof, we have used the criterion Thm. 25.38 to show that μ restricts to a Radon measure on the second countable compact set K_n. Since Thm. 25.38 is a deep result, one may wonder if Thm. 27.21 can be proved without using Thm. 25.38. The answer is yes. We sketch such a proof below.

First, consider the special case that $\mu(X) < +\infty$. By Thm. 27.17, $C_c(X)$ is dense in $L^p(X, \mu)$. By Pb. 15.14, $C_c(X)$ is l^∞-separable, and hence is L^p-separable (by the finiteness of $\mu(X)$). Therefore $L^p(X, \mu)$ is separable. Second, consider the general case. Since X is second countable, we can write X as a countable union $X = \bigcup_n U_n$ where $U_1 \subset U_2 \subset \cdots$ are open and precompact subsets of X. So $\mu(U_n) < +\infty$. The restriction $\mu_n = \mu|_{\mathfrak{B}_{U_n}}$ is Radon. (This is easy to check by using the openness of U_n, and does not rely on the second countability. So Thm. 25.38 is not needed here.) Therefore, $L^p(U_n, \mu_n)$ is separable. Similar to Step 2 of the proof of Thm. 27.21, one concludes that $L^p(X, \mu)$ is separable. □

Remark 27.23. The importance of Thm. 27.21 has been discussed in Rem. 17.27: Let (X, μ) be a measure space. Assume that $L^p(X, \mu)$ is separable (where $1 \leqslant p < +\infty$). Then, equivalently, the closed unit ball of the dual space $L^p(X, \mu)^*$ is weak-* metrizable by Thm. 17.24. A deep theorem originally due to Riesz (cf. Thm. 27.34) says that $L^p(X, \mu)^*$ is naturally isomorphic to $L^q(X, \mu)$ where $p^{-1} + q^{-1} = 1$. (In the case that $p = 1$, one should assume that μ is σ-finite.) Thus, when $L^p(X, \mu)$ is separable, we can use sequences (rather than nets) to study the weak-* topology and the weak-* compactness of norm-bounded subsets of $L^q(X, \mu)$.

27.2.4 Approximation by simple functions

Thm. 27.17 can only be applied to (completions of) Radon measures. For a general measure space, we have the following approximation:

Theorem 27.24. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Let $1 \leqslant p \leqslant +\infty$. Then*

$$L^p(X, \mu) \cap \mathcal{S}(X, \mathbb{C}) \tag{27.13}$$

is a dense subset of $L^p(X, \mu)$.

Remark 27.25. The set (27.13) has an explicit description: If $1 \leqslant p < +\infty$, a function $f : X \to \mathbb{C}$ belongs to (27.13) iff f is equivalent to a finite sum $g = \sum_i a_i \chi_{E_i}$ where $a_i \in \mathbb{C}$ and each $E_i \in \mathfrak{M}$ has finite measure. The word "equivalent" means that $f = g$ a.e.. If $p = +\infty$, then f belongs to (27.13) iff f is equivalent to an element of $\mathcal{S}(X, \mathbb{C})$.

Proof of Thm. 27.24. By approximating $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ separately, we can assume that f is real. By considering f^+ and f^- separately, we assume $f \geqslant 0$. By Prop. 24.9, there is an increasing sequence (s_n) in $\mathcal{S}(X, \mathbb{R}_{\geqslant 0})$ converging pointwise to f. Therefore, when $p < +\infty$, we have $\lim_n \|f - s_n\|_p^p = 0$ by the dominated convergence theorem (since $|f - s_n|^p \leqslant |f|^p$). If $p = +\infty$, by Prop. 27.8, we can replace f with an equivalent function whose l^∞-norm is finite. Then, by Rem. 24.10, we can assume that (s_n) converges uniformly to f. This proves $\lim_n \|f - s_n\|_{L^\infty} = 0$. □

In Pb. 27.2 and 27.4, we will give applications of Thm. 27.24 to the study of weak-* convergence in L^p spaces.

27.3 The Riesz-Fischer theorem

Let $(X, \mathfrak{M}, \mu)$ be a measure space.

In this section, we shall show that $L^p(X, \mu)$ is complete. Recall how to prove that $l^p(X)$ is complete (cf. Thm. 12.32): Let (f_n) be a Cauchy sequence in $l^p(X)$. We first show that for each $x \in X$, $(f_n(x))$ is a Cauchy sequence, and hence converges to some $f(x)$. Then we show that $\sum_X |f - f_n|^p$ is small for large enough n. In particular, $\|f\|_{l^p} \leqslant \|f - f_n\|_{l^p} + \|f_n\|_{l^p} < +\infty$. Thus $f \in l^p(X)$ and $\lim_n \|f - f_n\|_{l^p} = 0$.

Our proof of the completeness of $L^p(X, \mu)$ will follow a similar strategy, except for the following difference: If (f_n) is a Cauchy sequence in $L^p(X, \mu)$, we cannot show that (f_n) converges a.e.. But as a fallback, we can prove that (f_n) has an a.e. convergent subsequence.

Theorem 27.26 (Riesz-Fischer theorem). *Let $1 \leqslant p < +\infty$. Then $L^p(X, \mu)$ is a Banach space. Moreover, if (f_n) is a sequence in $L^p(X, \mu)$, f is in $L^p(X, \mu)$, and $\lim_n \|f - f_n\|_p = 0$, then (f_n) has a subsequence converging a.e. to f.*

It follows that $L^2(X, \mu)$ is a Hilbert space under the inner product (27.6).

Proof. Step 1. Let (f_n) be a Cauchy sequence in $L^p(X, \mu)$. We claim that (f_n) has a subsequence $(g_k) = (f_{n_k})$ converging pointwise outside a null set Δ.

Suppose the claim is true. Let $g : X \to \mathbb{C}$ be defined by $g(x) = \lim_k g_k(x)$ if $x \in X \backslash \Delta$, and $g(x) = 0$ if $x \in \Delta$. Then g is measurable. Since (g_k) is L^p-Cauchy, for each $\varepsilon > 0$ there exists $K \in \mathbb{Z}_+$ such that for each $k, l \geqslant K$ we have $\|g_k - g_l\|_p \leqslant \varepsilon$. By Fatou's lemma (Thm. 24.31), we get

$$\int_X |g_k - g|^p \leqslant \liminf_l \int_X |g_k - g_l|^p \leqslant \varepsilon^p$$

for all $k \geqslant K$. In particular, $\|g\|_p \leqslant \|g - g_k\|_p + \|g_k\|_p < +\infty$. This proves that $g \in L^p(X, \mu)$ and that (g_k) converges to g in L^p. Therefore, (f_n) converges in L^p to g since (f_n) is L^p-Cauchy (cf. Thm. 3.23).

If (f_n) converges in L^p to $f \in L^p(X, \mu)$, then f and g are the same elements in $L^p(X, \mu)$. So $\|f - g\|_p = 0$, and hence $f = g$ a.e.. This proves that the subsequence

(g_k) converges to f a.e..

Step 2. Let us prove the claim. Since (f_n) is Cauchy, for each $k \in \mathbb{Z}_+$ there is $n_k \in \mathbb{Z}_+$ such that $\|f_n - f_m\|_p \leqslant 2^{-k}$ for all $m, n \geqslant n_k$. By increasing the value of each n_k we assume $n_1 < n_2 < \cdots$ so that $(g_k) = (f_{n_k})$ is a subsequence of (f_n), and

$$\|g_{k+1} - g_k\|_p \leqslant 2^{-k}$$

Let $g_0 = 0$. We shall show that $\sum_{k\in\mathbb{N}}(g_{k+1} - g_k)$ converges absolutely a.e.. Then this series converges a.e., and hence (g_k) converges a.e..

Let $h_k = |g_{k+1} - g_k|$. Let $H = \sum_{k\in\mathbb{N}} h_k$ which is a measurable function $X \to \overline{\mathbb{R}}_{\geqslant 0}$. We shall show that $H < +\infty$ a.e.. By Prop. 24.16, it suffices to prove that $\|H\|_p^p \equiv \int_X H^p$ is finite. Let $k \in \mathbb{N}$. By Minkowski's inequality, we have $\|h_0 + \cdots + h_k\|_p \leqslant \|h_0\|_p + \cdots + \|h_k\|_p \leqslant 1 + 2^{-1} + \cdots + 2^{-k} \leqslant 2$. Thus, by Fatou's lemma (or by the monotone convergence theorem), we have

$$\|H\|_p^p \leqslant \liminf_k \|h_1 + \cdots + h_k\|_p^p \leqslant 2^p$$

□

Remark 27.27. The use of Fatou's lemma in the proof of Thm. 27.26 is very typical: If (f_n) is a sequence of measurable functions converging pointwise to f, one can use the L^p-norms of (f_n) to give an upper bound for the L^p-norm of f.

Remark 27.28. Thm. 27.26 clearly also holds when $p = +\infty$. Indeed, we have proved in Thm. 27.15 that $L^\infty(X, \mu)$ is complete. If (f_n) is a sequence converging in $L^\infty(X, \mu)$ to $f \in L^\infty(X, \mu)$, then by Prop. 27.11, (f_n) converges uniformly to f outside a null set. In particular, (f_n) converges a.e. to f. There is no need to choose a subsequence.

Corollary 27.29 (Riesz-Fischer). *Let $e_n \in C(\mathbb{S}^1)$ be defined by $e_n(x) = e^{\mathbf{i}nx}$. Then we have a unitary map*

$$L^2\big([-\pi, \pi], \frac{m}{2\pi}\big) \xrightarrow{\simeq} l^2(\mathbb{Z}) \qquad f \mapsto \widehat{f} \tag{27.14}$$

where $\widehat{f} : \mathbb{Z} \to \mathbb{C}$ is the Fourier series of f, i.e.,

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{[-\pi,\pi]} f e_{-n} dm \tag{27.15}$$

Proof. $L^2([-\pi, \pi], \frac{m}{2\pi})$ is a Hilbert space by Thm. 27.26 and has an orthonormal basis $(e_n)_{n\in\mathbb{Z}}$ by Cor. 27.19. Therefore, by Thm. 21.7, (27.14) defines a unitary map. □

Remark 27.30. Cor. 27.29 is in fact the original theorem proved by Riesz and by Fischer in 1907. At that time, it was already known that (e_n) is an orthonormal basis of $L^2[-\pi,\pi]$. (This was proved by Fatou in 1906, and Fatou's lemma was proved in the same paper as an auxiliary result. See [Haw, Ch. 6]. [1]) Therefore, to show that the map (27.14) is unitary, it remains to prove one the following (clearly) equivalent conditions:

(1) If $\varphi \in l^2(\mathbb{Z})$, then there exists $f \in L^2[-\pi,\pi]$ such that $\frac{1}{2\pi}\int fe_{-n} = \varphi(n)$ for all n.

(2) If (f_n) is a sequence in $L^2[-\pi,\pi]$ such that $\lim_{m,n\to\infty}\int |f_m - f_n|^2 = 0$, then there exists $f \in L^2[-\pi,\pi]$ such that $\lim_n |f - f_n|^2 = 0$.

Note that (1) simply says that the map (27.14) is surjective, and (2) simply says that $L^2[-\pi,\pi]$ is complete. Riesz proved (1). (Indeed, he formulated his theorem in a slightly more general fashion, although he reduced the problem to proving (1).) Fischer proved (2). (It is noteworthy that the vague idea of completeness in L^2 spaces already appeared in Fischer's treatment.) Both of them are different from the proof we have given for Thm. 27.26. See [Ber65, Sec. IV.3] and [Haw, Ch. 6] for a detailed account of the relevant history.

27.4 Introduction to dualities in L^p spaces

Let $(X,\mathfrak{M},\mu)$ be a measure space.

Proposition 27.31. *Let $1 \leqslant p,q \leqslant +\infty$ and $p^{-1}+q^{-1}=1$. Assume that μ is σ-finite if $p=+\infty, q=1$. Then there is a linear isometry*

$$\Psi : L^p(X,\mu) \to L^q(X,\mu)^* \qquad f \mapsto \Psi(f) \tag{27.16a}$$

such that for each $g \in L^q(X,\mu)$, we have

$$\langle \Psi(f), g\rangle = \int_X fg\,d\mu \tag{27.16b}$$

where fg is integrable by Hölder's inequality.

Proof. Let $f \in L^p(X,\mu)$. Hölder's inequality shows that $|\langle \Psi(f), g\rangle| \leqslant \|f\|_p \cdot \|g\|_q$. Therefore $\Psi(f)$ is bounded and has operator norm $\|\Psi(f)\| \leqslant \|f\|_p$. In particular, if $f=0$ in $L^p(X,\mu)$, then $\Psi(f)=0$. Thus, we may assume that $\|f\|_p > 0$, and we shall prove that $\|\Psi(f)\| = \|f\|_p$.

[1] Fatou's idea was to first show that if $f \in L^2[-\pi,\pi]$, then $\lim_{r\to 1^-}\sum_n r^n \widehat{f}(n)e_n$ converges a.e. to f. (A proof of this result can be found in [SS-R, Ch. 4 Sec. 3].) Then Fatou's lemma can be applied to show that $\|f\|_2^2 \leqslant \sum_n |\widehat{f}(n)|^2$. This, together with Bessel's inequality, implies Parseval's identity. The latter is equivalent to that $\{e_n\}$ is an orthonormal basis.

Case 1: $1 < p < +\infty$. Similar to the proof of Thm. 12.33, we let $g : X \to \mathbb{C}$ be $g = \frac{\overline{f}}{|f|} \cdot |f|^{p-1}$ on

$$\Omega_f = \{x \in X : f(x) \neq 0\}$$

and let $g = 0$ on Ω_f^c. Then g is measurable, and one checks that $\|g\|_q^q = \int |f|^{pq-q} = \int |f|^p = \|f\|_p^p$. So $g \in L^q(X, \mu)$. Moreover,

$$\langle \Psi(f), g \rangle = \int |f|^p = \|f\|_p^p = \|f\|_p \cdot \|f\|_p^{p-1} = \|f\|_p \cdot \|g\|_q$$

Since $\|g\|_q > 0$, we must have $\|\Psi(f)\| \geqslant \|f\|_p$.

Case 2: $p = 1, q = +\infty$. Let $g : X \to \mathbb{C}$ be defined by $g = \frac{\overline{f}}{|f|}$ on Ω_f (which is not null since $\int |f| > 0$), and $g = 0$ on Ω_f^c. So $\|g\|_\infty = 1$. And $\langle \Psi(f), g \rangle = \int |f| = \|f\|_1 = \|f\|_1 \cdot \|g\|_\infty$. So again $\|\Psi(f)\| \geqslant \|f\|_p$.

Case 3: $p = +\infty, q = 1$, and μ is σ-finite. We know $0 < \|f\|_\infty < +\infty$, and we want to prove $\|\Psi(f)\| \geqslant \|f\|_\infty$. Let us prove that if $0 < a < \|f\|_\infty$ then $\|\Psi(f)\| \geqslant a$. The reason for considering such a is that by the definition of $\|f\|_\infty$, the set

$$A = \{x \in X : |f(x)| > a\}$$

is not null. Let us first consider the special case that $\mu(A) < +\infty$. [2] Let g be $\overline{f}/|f|$ on A, and $g = 0$ on A^c. Then $\|g\|_1 = \mu(A) < +\infty$ and hence g is a nonzero element of $L^1(X, \mu)$. Moreover, $\langle \Psi(f), g \rangle = \int_A |f| \geqslant a\mu(A) = a\|g\|_1$. This proves $\|\Psi(f)\| \geqslant a$.

Now we consider the case $\mu(A) = +\infty$. Since μ is σ-finite, A is a countable union of finite-measure subsets. So there exists a measurable $B \subset A$ such that $0 < \mu(B) < +\infty$. Let $g = \overline{f}/|f|$ on B and $g = 0$ on B^c. Then the same argument as above proves $\|\Psi(f)\| \geqslant a$. $\square$

Example 27.32. Let $\mathfrak{M} = 2^X$ and $\mu : \mathfrak{M} \to \overline{\mathbb{R}}_{\geqslant 0}$ be $\mu(\varnothing) = 0$ and $\mu(E) = +\infty$ if $E \subset X$ is nonempty. Then $L^\infty(X, \mu)$ is nontrivial, but $L^1(X, \mu) = 0$. So the canonical map $\Psi : L^\infty(X, \mu) \to L^1(X, \mu)^*$ is not an isometry.

Corollary 27.33. *Let $1 \leqslant p, q \leqslant +\infty$ and $p^{-1} + q^{-1} = 1$. Assume that μ is σ-finite if $p = +\infty, q = 1$. Let $f \in L^p(X, \mu)$. Then $f = 0$ a.e. iff $\int_X fgd\mu = 0$ for all $g \in L^q(X, \mu)$.*

Proof. Let Ψ be the linear isometry in Prop. 27.31. Then $f = 0$ a.e. iff $\|f\|_p = 0$ iff $\Psi(f) = 0$. $\square$

Theorem 27.34 (Riesz representation theorem for L^p spaces). *Assume that $1 < p \leqslant +\infty$ and $p^{-1} + q^{-1} = 1$. (So $1 \leqslant q < +\infty$.) Moreover, assume that if $p = +\infty$ then μ is σ-finite. Then the canonical linear isometry $\Psi : L^p(X, \mu) \to L^q(X, \mu)^*$ in Prop. 27.31 is surjective, and hence is an isomorphism of Banach spaces.*

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[2]When proving a result about σ-finite measures, it is always a good idea to first prove it for finite measures.

Proof. When $p = q = 2$, by Riesz-Fischer, $L^2(X, \mu)$ is a Hilbert space. Therefore, the theorem follows from the Riesz-Fréchet representation Thm. 21.22. If $p \neq 2$, the proof is more difficult and will not be given in the notes. See [Fol-R, Sec. 6.2] or [Rud-R, Ch. 6] for a proof. [3] □

Although we will not prove Thm. 27.34 for $p \neq 2$, we can make the following definition:

Definition 27.35. Let $1 < p \leqslant +\infty$ and $1 \leqslant q < +\infty$ satisfy $p^{-1} + q^{-1} = 1$. Assume that if $p = +\infty$ then μ is σ-finite. If (f_α) is a net in $L^p(X, \mu)$ and $f \in L^p(X, \mu)$, we say that (f_α) **converges weak-*** to f if $\lim_\alpha \int_X f_\alpha g d\mu = \int_X f g d\mu$ for all $g \in L^q(X, \mu)$. If (g_α) is a net in $L^q(X, \mu)$ and $g \in L^q(X, \mu)$, we say that (g_α) **converges weakly** to g if $\lim_\alpha \int_X f g_\alpha d\mu = \int_X f g d\mu$ for all $f \in L^p(X, \mu)$.

Remark 27.36. Let $1 \leqslant p \leqslant +\infty$, and assume that μ is σ-finite if $p = +\infty$. Then the weak convergence in $L^p(X, \mu)$ was defined when $p \in [1, +\infty)$, the weak-* convergence in $L^p(X, \mu)$ was defined when $p \in (1, +\infty]$, and the two notions agree when $p \in (1, +\infty)$. Due to Thm. 27.34, if $p \in [1, +\infty)$, the weak convergence in $L^p(X, \mu)$ defined by Def. 27.35 is induced by the weak topology (cf. Def. 21.25). If $p \in (1, +\infty]$, and if we fix the predual (cf. Def. 21.20) $\Psi : L^p(X, \mu) \xrightarrow{\simeq} L^q(X, \mu)^*$ so that $L^p(X, \mu)$ is viewed as the dual space of $L^q(X, \mu)$ (where $p^{-1} + q^{-1} = 1$), then the weak-* convergence in Def. 27.35 is induced by the weak-* topology.

Of course, one can talk about weak convergence in $L^\infty(X, \mu)$. According to Def. 21.25, a net (f_α) in $L^\infty(X, \mu)$ converges weakly to $f \in L^\infty(X, \mu)$ if $\lim_\alpha \langle f_\alpha, \varphi \rangle = \langle f, \varphi \rangle$ for all $\varphi \in L^\infty(X, \mu)^*$. Then (f_α) must converge weak-* to f (if μ is σ-finite). However, weak-* convergence does not imply weak convergence in $L^\infty(X, \mu)$, since $L^\infty(X, \mu)^*$ may contain more elements than $L^1(X, \mu)$, cf. Pb. 17.8.

27.5 The spectral theorem for bounded self-adjoint operators

Fix a Hilbert space $\mathcal{H}$.

In 1906, Hilbert established the spectral theorem for bounded Hermitian forms ($\simeq$ bounded self-adjoint operators) on Hilbert spaces (cf. the fourth part of [Hil12]). Hilbert's method is no longer used today. In this section, we will study the spectral theorem based on Riesz's method of functional calculus introduced in his seminal paper [Rie13, Ch. V]. Besides the crucial roles played by spectral theorem in the mathematical theory of quantum mechanics, there are several important reasons for studying the spectral theorem and Riesz's method:

[3] In [Rud-R], Rudin assumed that μ is σ-finite for all p.

- [Rie13] is one of the most important papers that mark the shift in the study of function spaces from the perspective of sesquilinear/quadratic forms (due to Hilbert) to the viewpoint of linear operators.[4] It shows us how some important modern mathematical ideas were born and evolved.

- Riesz's treatment of spectral theorem is an important application of the Riesz representation Thm. 25.49. Moreover, it provides a deep application (probably the first in history) of the operator norm on $\mathfrak{L}(\mathcal{H})$ by using it as an equicontinuity condition, not just a continuity condition. Its significance should be compared with Riesz's application (in [Rie10]) of the operator norm on $L^p[a,b]^*$ (as an equicontinuity condition) to the weak(-*) compactness and the moment problems, cf. Ch. 17.

- Riesz's treatment incorporates ideas with very different backgrounds and origins: (1) Classification of dual spaces (originating in moment problems). (2) Integral theory, especially Stieltjes integrals. (3) Hilbert spaces and their bounded linear operators (originating in integral equations and differential equations). Therefore, Riesz's spectral theorem is one of the culminations of his work.

However, we will first present the spectral theorem in its modern and *representation theoretic* form, which was due to Segal [Seg51] and further promoted by Halmos [Hal63]. After that, we will explain how this modern viewpoint is related to Riesz's *integral theoretic* version of spectral theorem. The proof of the representation theoretic version of spectral theorem integrates the Riesz-Markov representation theorem, the Riesz-Fischer theorem, the density of $C(X)$ in $L^2(X)$, while retaining Riesz's crucial idea of functional calculus.

27.5.1 The spectral theorem

The Hilbert-Schmidt theorem implies that if $T \in \mathfrak{L}(\mathcal{H})$ is self-adjoint *and completely continuous*, then $\mathcal{H}$ is spanned by the eigenvectors of T. This statement is clearly not true without assuming complete continuity, as we now see below.

Definition 27.37. Let (X,μ) be a measure space. Let $f \in L^\infty(X,\mu)$. Define the **multiplication operator**

$$M_f : L^2(X,\mu) \to L^2(X,\mu) \qquad g \mapsto fg \tag{27.17}$$

Then $M_f \in \mathfrak{L}(L^2(X,\mu))$, and clearly $\|M_f\| \leqslant \|f\|_{L^\infty}$.

[4] Another paper is Riesz's 1918 paper [Rie18] studying compact operators on $C[a,b]$ (where $-\infty < a < b < +\infty$) entirely from the viewpoint of linear operators. Since $V = C[a,b]$ is not reflexive, a bounded sesquilinear form $V \times V^* \to \mathbb{C}$ only gives a bounded linear map $V \to V^{**}$ but not necessarily $V \to V$. Therefore, the method of sesquilinear forms completely fails.

Example 27.38. Let μ be a Radon measure on a compact interval $[a,b]$. Let $x \in L^\infty([a,b],\mu)$ denote the identity map $t \in [a,b] \mapsto t \in \mathbb{C}$. Let $\lambda \in \mathbb{C}$. Then λ is an eigenvalue of M_x iff $\lambda \in [a,b]$ and $\mu\{\lambda\} > 0$. Consequently, if μ is the Lebesgue measure m, then M_x has no eigenvalues.

Proof. Suppose that $\lambda \in [a,b]$ and $\mu\{\lambda\} > 0$. Then $\chi_{\{\lambda\}}$ is an λ-eigenvalue of M_x. Conversely, suppose that λ is an eigenvalue. Choose a nonzero $g \in L^2([a,b],\mu)$ such that $M_x g - \lambda g = 0$. Then $\int_{[a,b]} |(x-\lambda)g|^2 d\mu = 0$. By Prop. 24.16, we get $(x-\lambda)g = 0$ a.e..

If $\lambda \notin [a,b]$, then $x-\lambda$ is nonzero everywhere on $[a,b]$, and hence $g = 0$ a.e. on $[a,b]$. This is impossible, since we assume that g is nonzero in $L^2([a,b],\mu)$.

Therefore, we have $\lambda \in [a,b]$. Then $x-\lambda$ is nonzero on $[a,b]\backslash\{\lambda\}$, and hence $g = 0$ a.e. on $[a,b]\backslash\{\lambda\}$. If $\mu\{\lambda\} = 0$, then $g = 0$ a.e. on $[a,b]$, impossible. So $\mu\{\lambda\} > 0$. □

The above example shows that the traditional meaning of diagonalization (i.e., that $\mathcal{H}$ has an orthonormal basis whose elements are eigenvectors of T) should be extended. We should view the traditional diagonalization as discrete, and regard the multiplication operator M_f as "continuously diagonal". That the traditional diagonal operator can be viewed as a special case of M_f is clear from the following example:

Example 27.39. Let $(e_i)_{i\in I}$ be an orthonormal basis of $\mathcal{H}$. Assume that $T \in \mathfrak{L}(\mathcal{H})$ satisfies $Te_i = \lambda_i e_i$ where $\lambda_i \in \mathbb{R}$ and $\sup_i |\lambda_i| < +\infty$. Let μ be the counting measure on I. (So $L^2(X,\mu) = l^2(X)$.) Let $f \in L^\infty(I,\mu) = l^\infty(I)$ be defined by $f(i) = \lambda_i$. Let $\Phi : \mathcal{H} \xrightarrow{\simeq} l^2(I)$ be the canonical unitary map (cf. Thm. 21.7). Then T is unitarily equivalent to M_f via Φ, i.e., $\Phi T \Phi^{-1} = M_f$.

We now state the main spectral theorem for self-adjoint operators. Recall Subsec. 21.5.3 for the basic facts about direct sums and invariant subspaces. Recall that if $T \in \mathfrak{L}(\mathcal{H})$ is self-adjoint, then $-\|T\| \leqslant T \leqslant \|T\|$ (Exp. 22.22).

Theorem 27.40. *Let $T \in \mathfrak{L}(\mathcal{H})$ be self-adjoint. Let $x : \mathbb{R} \to \mathbb{R}$ be the identity map. Choose $-\infty < a \leqslant b < +\infty$ such that*

$$a \leqslant T \leqslant b \tag{27.18}$$

Then there exist a family $(\mu_i)_{i\in I}$ of Radon measures on $[a,b]$ and a unitary map $U : \mathcal{H} \xrightarrow{\simeq} \bigoplus_{i\in I} L^2([a,b],\mu_i)$ such that

$$UTU^{-1} = \bigoplus_{i\in I} M_x \tag{27.19}$$

If $\mathcal{H}$ is separable, then I can be chosen to be countable.

In other words, for each $\oplus_i\xi_i \in \bigoplus_i L^2([a,b],\mu_i)$, we have

$$UTU^{-1}(\oplus_i\xi_i) = \oplus_i x\xi_i \tag{27.20}$$

$\star$ **Remark 27.41.** If $(X_i)_{i\in I}$ is a family of LCH spaces and each X_i is equipped with a Radon measure μ_i, then the disjoint union $X = \bigsqcup_i X_i$ (equipped with the disjoint union topology, cf. Exp. 7.68) can be equipped with the unique Radon measure μ whose restriction to X is μ_i. (Proof: First define its associated positive linear functional.) Then we have a unitary map

$$\Phi : \bigoplus_{i\in I} L^2(X_i,\mu_i) \xrightarrow{\simeq} L^2(X,\mu) \tag{27.21}$$

(Proof: First define Φ on the dense subspace $\mathcal{V}$ of finite direct sums of elements in $C_c(X_\bullet,\mu_\bullet)$, which is a linear isometry. By Prop. 10.28, Φ can be extended to a linear isometry $\bigoplus_i L^2(X_i,\mu_i) \to L^2(X,\mu)$. The range of this map contains $C_c(X)$, and hence is dense in $L^2(X,\mu)$ by Thm. 27.17. So Φ is surjective because $\bigoplus_i L^2(X_i,\mu_i)$ is complete.)

Now let $X_i = [a,b]$ and let μ_i be as in Thm. 27.40. Let $\Psi = \Phi U$, which is a unitary map $\mathcal{H} \xrightarrow{\simeq} L^2(X,\mu)$. Then we have

$$\Psi T\Psi^{-1} = M_f \tag{27.22}$$

where $f : X \to \mathbb{R}$ is defined by sending each $x \in [a,b]$ (where $[a,b]$ is any of the $\mathrm{card}(I)$ components) to $x \in \mathbb{R}$. Then $f \in C(X,\mathbb{R}) \cap l^\infty(X,\mathbb{R})$. □

27.5.2 Proof of the spectral Thm. 27.40

Let $\mathscr{A}$ be a (complex) unital $*$-algebra (cf. Def. 15.4).

Definition 27.42. A **unitary representation** of $\mathscr{A}$ on $\mathcal{H}$ is defined to be a linear map $\pi : \mathscr{A} \to \mathfrak{L}(\mathcal{H})$ satisfying

$$\pi(ab) = \pi(a)\pi(b) \qquad \pi(1) = \mathbf{1}_{\mathcal{H}} \qquad \pi(a^*) = \pi(a)^* \tag{27.23}$$

for all $a,b \in \mathscr{A}$. In other words, a representation is a **unital $*$-homomorphism** from $\mathscr{A}$ to $\mathfrak{L}(\mathcal{H})$.

If $\Omega \in \mathcal{H}$ is such that the linear subspace

$$\pi(\mathscr{A})\Omega = \{\pi(a)\Omega : a \in \mathscr{A}\}$$

is dense in $\mathcal{H}$, we say that π is a **cyclic representation**, and that Ω is a **cyclic vector**.

If $\mathcal{K}$ is a *closed* linear subspace of $\mathcal{H}$, and if $\mathcal{K}$ is $\mathscr{A}$-invariant, then π restricts to a unital $*$-homomorphism

$$\pi|_{\mathcal{K}} : \mathscr{A} \to \mathfrak{L}(\mathcal{K}) \qquad a \mapsto \pi(a)|_{\mathcal{K}} \tag{27.24}$$

We call $\pi|_{\mathcal{K}}$ a **(unitary) subrepresentation** of π. □

Note that $\pi(a)^*$ is the adjoint of the bounded linear operator $\pi(a)$, and $\pi(a)\pi(b)$ is the composition of the operators $\pi(a)$ and $\pi(b)$.

Definition 27.43. Let $(\mathcal{H}, \pi)$ and $(\mathcal{H}', \pi')$ be two unitary representations of $\mathscr{A}$. If $U : \mathcal{H} \to \mathcal{H}'$ is a unitary map satisfying

$$U\pi(a) = \pi'(a)U \tag{27.25}$$

for all $a \in \mathscr{A}$, then U is called a **unitary equivalence** of representations, and $(\mathcal{H}, \pi)$ and $(\mathcal{H}', \pi')$ are called **unitarily equivalent**.

Example 27.44. Let μ be a Radon measure on a compact Hausdorff space X. Then $(M, L^2(X, \mu))$ is a cyclic representation of $C(X)$, where

$$M : C(X) \to \mathfrak{L}(L^2(X, \mu)) \qquad f \mapsto M_f \tag{27.26}$$

and M_f is the multiplication operator of f. The constant function $1 \in L^2(X, \mu)$ is a cyclic vector since, by Thm. 27.17, $\{M_f 1 : f \in C(X)\}$ is dense in $L^2(X, \mu)$. M is called the **multiplication representation**.

Example 27.45. Let $T \in \mathfrak{L}(\mathcal{H})$ be self-adjoint. Let $\mathbb{C}[x]$ be the unital $*$-algebra whose linear structure and multiplication are the usual ones for polynomials, and whose $*$-structure is defined by

$$(a_0 + a_1 x + \cdots + a_n x^n)^* = \overline{a_0} + \overline{a_1} x + \cdots + \overline{a_n} x^n \tag{27.27}$$

where $a_0, \dots, a_n \in \mathbb{C}$. For each $p \in \mathbb{C}[x]$, if $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, let

$$p(T) = a_0 + a_1 T + \cdots + a_n T^n$$

Then we have a unitary representation

$$\pi_T : \mathbb{C}[x] \to \mathfrak{L}(\mathcal{H}) \qquad f \mapsto f(T) \tag{27.28}$$

We call this representation the **polynomial functional calculus** with respect to T.

By the above example, we can transform the study of spectral theory to the study of unitary representations of $\mathbb{C}[x]$. Moreover, the study of arbitrary unitary representations can be reduced to the study of cyclic representations:

Proposition 27.46. *Let $\pi : \mathscr{A} \to \mathfrak{L}(\mathcal{H})$ be a unitary representation of a unital $*$-algebra $\mathscr{A}$. Then there is a family $(\mathcal{H}_i)_{i \in I}$ of unitary subrepresentations of π satisfying the following conditions:*

(a) Each $\mathcal{H}_i$ is a cyclic representation of $\mathscr{A}$.

(b) This family is pairwise orthogonal, i.e., $\mathcal{H}_i \perp \mathcal{H}_j$ if $i \neq j$.

(c) $\mathrm{Span}_{i\in I}\mathcal{H}_i$ *is dense in* $\mathcal{H}$

Moreover, if $\mathcal{H}$ *is separable, then* I *can be chosen to be countable.*

Proof. By Zorn's lemma, we can choose a maximal family $(\mathcal{H}_i)_{i\in I}$ of *nonzero* subrepresentations satisfying (a) and (b). Let $\mathcal{V} = \mathrm{Span}_{i\in I}\mathcal{H}_i$. To show that $\mathcal{V}$ is dense in $\mathcal{H}$, by Cor. 21.14, it suffices to prove that $\mathcal{V}^\perp = 0$.

By assumption, for each $a \in \mathscr{A}$, $\mathcal{V}$ is invariant under $\pi(a)$ and $\pi(a)^* = \pi(a^*)$. Therefore, by Prop. 22.25, $\mathcal{V}^\perp$ is $\pi(a)$-invariant. Thus, $\mathcal{V}^\perp$ is a subrepresentation. If $\mathcal{V}^\perp$ is nonzero, we can choose a nonzero $\Omega \in \mathcal{V}^\perp$. Then the family $(\mathcal{H}_i)_{i\in I}$, together with cyclic representation $\overline{\pi(\mathscr{A})\Omega}$, form a larger family contradicting the maximality of $(\mathcal{H}_i)_{i\in I}$. This proves that $(\mathcal{H}_i)_{i\in I}$ satisfies (c).

Since each $\mathcal{H}_i$ is nonzero and hence contains a nonempty set of orthonormal basis, if I is uncountable, then $\mathcal{H}$ will have an uncountable orthonormal basis, and hence $\mathcal{H} \simeq l^2(X)$ for some uncountable set X. We leave it to the readers to check that $l^2(X)$ is not separable. □

Remark 27.47. Let $(\mathcal{H}_i)_{i\in I}$ be a pairwise orthogonal family of subrepresentations of $\pi : \mathscr{A} \to \mathfrak{L}(\mathcal{H})$ spanning a dense subspace of $\mathcal{H}$. By Pb. 21.12, we have a unitary map $\Phi : \mathcal{H} \to \bigoplus_{i\in I}\mathcal{H}_i$, and for each $a \in \mathscr{A}$ we have

$$\Phi\pi(a)\Phi^{-1} = \oplus_i \pi|_{\mathcal{H}_i}(a) \tag{27.29}$$

We denote the representation on the RHS by $\bigoplus_{i\in I}\pi|_{\mathcal{H}_i}$ and call it the **direct sum** of the family of unitary representations $(\mathcal{H}_i)_{i\in I}$. Then Prop. 27.46 says that every unitary representation is unitarily equivalent a direct sum of cyclic representations.

Compared to $\mathbb{C}[x]$, it is easier to study the cyclic representations of $C[a,b]$:

Theorem 27.48. *Let* X *be a compact Hausdorff space. Let* $\pi : C(X) \to \mathfrak{L}(\mathcal{H})$ *be a cyclic representation with cyclic vector* Ω. *Then there is a Radon measure* μ *on* X *and a unitary equivalence* U *of representations satisfying*

$$U : (\mathcal{H}, \pi) \to (L^2(X,\mu), M) \qquad U\Omega = 1 \tag{27.30}$$

where $M : f \in C(X) \mapsto M_f \in \mathfrak{L}(L^2(X,\mu))$ *is the multiplication representation.*

Proof. Define linear functional

$$\Lambda : C(X) \to \mathbb{C} \qquad f \mapsto \langle \pi(f)\Omega|\Omega\rangle \tag{27.31}$$

If $f \geqslant 0$, let $g = \sqrt{f}$. Then $g^* = g$ and $f = g^*g$. So $\Lambda(f) = \langle \pi(g)^*\pi(g)\Omega|\Omega\rangle = \|\pi(g)\Omega\| \geqslant 0$. So Λ is positive. Therefore, by the Riesz-Markov representation Thm. 25.21, there is a Radon measure μ on X such that

$$\langle \pi(f)\Omega|\Omega\rangle = \int_X f d\mu \tag{27.32}$$

for all $f \in C(X)$. Therefore, for each $f, g \in C(X)$ we have

$$\langle \pi(f)\Omega | \pi(g)\Omega \rangle_{\mathcal{H}^2} = \langle f | g \rangle_{L^2(X,\mu)} \tag{27.33}$$

since LHS$= \langle \pi(g^* f)\Omega | \Omega \rangle = \int_X g^* f d\mu = \langle f | g \rangle$.

Define a map

$$U : \pi(C(X))\Omega \to L^2(X, \mu) \qquad \pi(f)\Omega \mapsto f$$

We first check that U is well-defined: Suppose that $\pi(f)\Omega = \pi(f')\Omega$. Then by (27.33), we have $\langle f | g \rangle = \langle f' | g \rangle$. Since $C(X)$ is dense in $L^2(X, \mu)$ (Thm. 27.17), we get $f = f'$ as elements of $L^2(X, \mu)$, finishing checking the well-definedness.

Clearly U is linear, and (27.33) shows that U is an isometry. By the Riesz-Fischer Thm. 27.26, $L^2(X, \mu)$ is complete. Therefore, by Prop. 10.28, U can be uniquely extended to a bounded linear map $U : \mathcal{H} \to L^2(X, \mu)$. Since $\|U\xi\| = \|\xi\|$ for each ξ in the dense subspace $\pi(C(X))\Omega$, by the continuity of U, we have $\|U\xi\| = \|\xi\|$ for each $\xi \in \mathcal{H}$. So U is a linear isometry. Therefore, $U(\mathcal{H})$ is a complete linear subspace of $L^2(X, \mu)$ because $\mathcal{H}$ is complete. In particular, $U(\mathcal{H})$ is a closed subspace of $L^2(X, \mu)$. Since $U(\mathcal{H})$ contains all continuous functions (forming a dense subspace of $L^2(X, \mu)$), we must have $U(\mathcal{H}) = L^2(X, \mu)$. This proves that U is unitary.

Finally, we check that $U\pi(f) = M_f U$ for all $f \in C(X)$. By the density of $\pi(C(X))\Omega$ in $\mathcal{H}$, it suffices to check that $U\pi(f)\xi = M_f U\xi$ if $\xi = \pi(g)\Omega$ and $g \in C(X)$. We compute that

$$U\pi(f)\xi = U\pi(f)\pi(g)\Omega = U\pi(fg)\Omega = fg = M_f g = M_f U\pi(g)\Omega = M_f U\xi$$

This proves that U is a unitary equivalence of representations. $\square$

Remark 27.49. We have mentioned in Subsec. 21.1.2 that the operator viewpoint (rather than the sesquilinear form viewpoint) is closely related to the completeness (rather than compactness) in Hilbert spaces. The readers should notice the important role played by the completeness of $L^2(X, \mu)$ (due to the Riesz-Fischer Thm. 27.26) in the above proof. Without knowing this completeness, we cannot extend U from $\pi(C(X))\Omega$ to $\mathcal{H}$. (Choose a sequence $\pi(f_n)\Omega$ in $\pi(C(X))\Omega$ converging to a given $\xi \in \mathcal{H}$. In order to define $U\xi$ as the limit of $U\pi(f_n)\Omega$, i.e., the limit of the Cauchy sequence (f_n) in $L^2(X, \mu)$, we need the completeness of $L^2(X, \mu)$.)

It remains to extend the polynomial functional calculus $p \mapsto p(T)$ to a unitary representation of $C[a, b]$. This is not an easy task. It relies on the following deep theorem to be proved in the next section. This theorem is a variant of the (so called) **spectral mapping theorem**.

Theorem 27.50. *Let $T \in \mathfrak{L}(\mathcal{H})$ be self-adjoint. Choose $-\infty < a \leqslant b < +\infty$ such that $a \leqslant T \leqslant b$. Then, for each $p \in \mathbb{C}[x]$, we have*

$$\|p(T)\| \leqslant \|p\|_{l^\infty[a,b]} \tag{27.34}$$

In other words, if $\mathbb{C}[x]$ is equipped with the $l^\infty[a,b]$-seminorm [5]*, the polynomial functional calculus $\pi_T : \mathbb{C}[x] \to \mathfrak{L}(\mathcal{H})$ with respect to T has operator norm $\leqslant 1$.*

Corollary 27.51. *Let $T \in \mathfrak{L}(\mathcal{H})$ be self-adjoint. Choose $-\infty < a \leqslant b < +\infty$ such that $a \leqslant T \leqslant b$. Then the polynomial functional calculus $\pi_T : \mathbb{C}[x] \to \mathfrak{L}(\mathcal{H})$ with respect to T can be extended to a unitary representation*

$$\pi_T : C[a,b] \to \mathfrak{L}(\mathcal{H}) \qquad f \mapsto f(T) \tag{27.35}$$

with operator norm $\leqslant 1$. We call π_T the ***continuous functional calculus*** *with respect to T.*

The completeness of $\mathfrak{L}(\mathcal{H})$ (under the operator norm) plays an important role in the following proof.

Proof. Note that $\mathbb{C}[x]$ is dense in $C[a,b]$ by the Weierstrass approximation Thm. 14.45. Since $\pi_T : \mathbb{C}[x] \to \mathfrak{L}(\mathcal{H})$ is a bounded linear map, and since $\mathfrak{L}(\mathcal{H})$ is complete (Thm. 17.35), by Prop. 10.28, this linear map can be extended (uniquely) to a bounded linear map $\pi_T : C[a,b] \to \mathfrak{L}(\mathcal{H})$ with operator norm $\leqslant 1$. Since $\pi_T(fg) = \pi_T(f)\pi_T(g)$ and $\pi_T(f^*) = \pi_T(f)^*$ hold for any $f,g \in \mathbb{C}[x]$, by a density argument, they hold for all $f,g \in C[a,b]$. This proves that π_T is a unitary representation. □

Proof of Thm. 27.40. By Cor. 27.51, the polynomial functional calculus π_T can be extended to the continuous functional calculus $\pi_T : C[a,b] \to \mathfrak{L}(\mathcal{H})$. By Prop. 27.46 (and Rem. 27.47), $(\mathcal{H}, \pi_T)$ is unitarily equivalent to a direct sum of cyclic representations $(\mathcal{H}_i, \pi_i)$ of $C[a,b]$. By Thm. 27.48, each $(\mathcal{H}_i, \pi_i)$ is unitarily equivalent to the multiplication representation $(M, L^2([a,b],\mu_i))$ for some Radon measure μ_i on $[a,b]$. Therefore, π_T is unitarily equivalent to a direct sum of multiplication representations. In particular, $T = x(T)$ is unitarily equivalent to a direct sum of M_x. □

27.6 Riesz's proof of Thm. 27.50

Let $\mathcal{H}$ be a Hilbert space, let $T \in \mathfrak{L}(\mathcal{H})$ be self-adjoint, and let $-\infty < a \leqslant b < +\infty$ such that $a \leqslant T \leqslant b$. Let $\pi_T : \mathbb{C}[x] \to \mathfrak{L}(\mathcal{H})$ be the polynomial functional calculus $p \mapsto p(T)$. Our goal of this section is to prove Thm. 27.50 saying that π_T has operator norm $\leqslant 1$.

There are several different proofs of this theorem. In some modern textbooks, this theorem is proved by first extending π_T to a **holomorphic functional calculus**: when f is a holomorphic function on a neighborhood of $[a,b]$ in $\mathbb{C}$, then $f(T)$ is defined to be $f(T) = \oint_\gamma f(z)(z-T)^{-1}\frac{dz}{2i\pi}$ where γ is an anticlockwise curve in the domain of f surrounding $[a,b]$. Other textbooks prove this by first proving the

[5] It is a norm when $a < b$.

spectral mapping theorem and the spectral radius formula. Both methods involve many deep ideas, the motivations and historical origins of which would require many pages to explain.

The proof we shall give in this section is very similar to Riesz's original method in [Rie13, Sec. 86-88]. Unfortunately, it is difficult to find this brilliant and elegant approach in modern textbooks. (Even Riesz himself did not use this method in his textbook [RN].) However, I believe it is worth learning this method because the idea behind it is extremely simple and straightforward, much more so than any modern approach I've seen.

Riesz's idea is as follows. Assume for simplicity that $\mathcal{H}$ is separable. For each $f \in \mathbb{C}[x]$, let $C_f = \|f\|_{l^\infty[a,b]}$.

1. First, assume that $\mathcal{H}$ is finite dimensional. Then T can be diagonalized, say $T \simeq \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then $a \leqslant \lambda_i \leqslant b$, and hence $|f(\lambda_i)| \leqslant C_f$. For each $f \in \mathbb{C}[x]$ we have $f(T) \simeq \mathrm{diag}(f(\lambda_1), \ldots, f(\lambda_n))$. Then clearly $\|f(T)\| \leqslant \sup_i |f(\lambda_i)| \leqslant C$, finishing the proof.

2. Now we don't assume that $\dim \mathcal{H} < +\infty$. However, if T has finite rank, then step 1 implies $\|f(T)\| \leqslant C_f$.

3. Finally, we don't assume that T has finite rank. However, we can assume that $\mathcal{H} = l^2(\mathbb{Z}_+)$. Let $T_n = \chi_{\{1,\ldots,n\}} T \chi_{\{1,\ldots,n\}}$. Then $T_n \in \mathfrak{L}(\mathcal{H})$ is self-adjoint and has finite rank. Therefore, by Step 2, we have $\|f(T_n)\| \leqslant C_f$. Finally, using the fact that $\sup_n \|T_n\| < +\infty$ and (T_n) converges pointwise to T, one shows that $f(T_n)$ converges pointwise to $f(T)$. This proves $\|f(T)\| \leqslant C_f$.

Let us present the detailed proof of Thm. 27.50. We do not assume that $\mathcal{H}$ is separable.

Lemma 27.52. *Let (A_α) be a net in $\mathfrak{L}(\mathcal{H})$ satisfying $\sup_\alpha \|A_\alpha\| < +\infty$. Assume that (A_α) converges pointwise to $A \in \mathfrak{L}(\mathcal{H})$. Let (ξ_β) be a net in $\mathcal{H}$ converging pointwise to $\xi \in \mathcal{H}$. Then $\lim\limits_{\alpha,\beta} A_\alpha \xi_\beta = A\xi$.*

Consequently, if (A_α) and (B_β) are nets in $\mathfrak{L}(\mathcal{H})$ converges pointwise to $A, B \in \mathfrak{L}(\mathcal{H})$ respectively, and if $\sup_\alpha \|A_\alpha\| < +\infty$, then $\lim_{\alpha,\beta} A_\alpha B_\beta$ converges pointwise to AB.

Proof. The fact that $C = \sup_\alpha \|A_\alpha\|$ is finite implies that (A_α) is an equicontinuous family of functions $\mathcal{H} \to \mathcal{H}$. Therefore, the lemma follows easily from (3)⇒(1) of Thm. 9.12 (together with Prop. 9.16). But we can also check it directly:

$$\|A\xi - A_\alpha \xi_\beta\| \leqslant \|A\xi - A_\alpha \xi\| + \|A_\alpha \xi - A_\alpha \xi_\beta\| \leqslant \|A\xi - A_\alpha \xi\| + C\|\xi - \xi_\beta\|$$

where the RHS converges to 0. □

Corollary 27.53. *Let (A_α) be a net in $\mathfrak{L}(\mathcal{H})$ satisfying $\sup_\alpha \|A_\alpha\| < +\infty$. Assume that (A_α) converges pointwise to $A \in \mathfrak{L}(\mathcal{H})$. Let $f \in \mathbb{C}[x]$. Then $\lim_\alpha f(T_\alpha)$ converges pointwise to $f(T)$. In particular, $\|f(T)\| \leqslant \sup_\alpha \|f(T_\alpha)\|$.*

Proof. If $f(T_\alpha)$ converges pointwise to $f(T)$, then for each $\xi \in \mathcal{H}$, we have by Lem. 27.52 that

$$\lim_\alpha (xf)(T_\alpha)\xi = \lim_\alpha T_\alpha f(T_\alpha)\xi = Tf(T)\xi$$

Therefore, a proof by induction on the degree of the polynomial $f \in \mathbb{C}[x]$ implies that $f(T_\alpha)$ converges pointwise to $f(T)$ for all $f \in \mathbb{C}[x]$. Let $C = \sup_\alpha \|f(T_\alpha)\|$. Then $\|f(T)\xi\| = \lim_\alpha \|f(T_\alpha)\xi\| \leqslant C\|\xi\|$. This proves $\|f(T)\| \leqslant C$. □

Remark 27.54. The readers should notice the crucial role played by the condition $\sup_\alpha \|A_\alpha\| < +\infty$ in the proof of Cor. 27.53. As the proof of Lem. 27.52 suggests, this is a condition of equicontinuity which ensures the convergence of double limits.

Proof of Thm. 27.50. Fix $f \in \mathbb{C}[x]$ and let $C_f = \|f\|_{l^\infty[a,b]}$. We need to show that $\|f(T)\| \leqslant C_f$. Recall that T is self-adjoint.

Let $(e_i)_{i\in I}$ be an orthonormal basis of $\mathcal{H}$. For each $J \in \mathrm{fin}(2^I)$, let P_J be the projection of $\mathcal{H}$ onto $V_J = \mathrm{Span}\{e_j : j \in J\}$, namely, for each $\xi \in \mathcal{H}$ we have

$$P_J\xi = \sum_{j\in J}\langle\xi|e_j\rangle e_j$$

Then $\lim_{J\in\mathrm{fin}(2^X)} P_J\xi = \xi$ by Thm. 20.35. Therefore, $\lim_J P_JTP_J$ converges pointwise to T by Lem. 27.52. Since P_J is self-adjoint (Pb. 21.2), $T_J := P_JTP_J$ is also self-adjoint since $(P_JTP_J)^* = P_J^*T^*P_J^* = P_JTP_J$. Clearly $\|T_J\| \leqslant \|T\|$ and hence $\sup_J \|T_J\| < +\infty$. Therefore, to prove $\|f(T)\| \leqslant C_f$, by Cor. 27.53, it suffices to prove $\|f(T_J)\| \leqslant C_f$.

Note that $\mathcal{H} \simeq V_J \bigoplus V_J^\perp$ and $T_J|_{V_J^\perp} = 0$. Thus, T_J is unitarily equivalent to $(T_J|_{V_J})\oplus 0$ on $V_J \bigoplus V_J^\perp$. Let $S_J = T_J|_{V_J}$ which is an element of $\mathfrak{L}(V_J)$. Then $f(T_J)$ is unitarily equivalent to $f(S_J)\oplus 0$. Therefore, it suffices to prove that $\|f(S_J)\| \leqslant C_f$.

Since $\dim V_J < +\infty$, by (e.g.) Thm. 22.29, S_J is unitarily equivalent to the diagonal matrix $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$ where $\lambda_k \in \mathbb{R}$ for each $1 \leqslant k \leqslant n$. Note that $a \leqslant T \leqslant b$ implies that

$$a\|\xi\|^2 \leqslant \langle T\xi|\xi\rangle \leqslant b\|\xi\|^2$$

for all $\xi \in \mathcal{H}$, and hence for all $\xi \in V_J$. So $a \leqslant S_J \leqslant b$. Therefore $\lambda_k \in [a, b]$. Combining this fact with $f(S_J) \simeq \mathrm{diag}(f(\lambda_1), \dots, f(\lambda_n))$, we get $\|f(S_T)\| \leqslant C_f$. □

27.7 The birth of functional calculus: Riesz 1913

Let $\mathcal{H}$ be a Hilbert space. Let $T \in \mathfrak{L}(\mathcal{H})$ be self-adjoint and $a \leqslant T \leqslant b$ where $-\infty < a \leqslant b < +\infty$.

As mentioned in Sec. 21.1, historically, the research on Hilbert spaces has shifted from the perspective of sesquilinear/quadratic forms to the perspective of linear operators. Accompanying this shift is a transition from an emphasis on compactness to a focus on completeness. Our spectral Thm. 27.40 was presented entirely in the spirit of linear operators. The readers have also seen that completeness has played an important role in the proof of Thm. 27.40:

(1) In the proof of Thm. 27.48, to extend the linear isometry $U : \pi(C(X))\Omega \rightarrow L^2(X, \mu)$ to the closure $\mathcal{H}$, the completeness of $L^2(X, \mu)$ is needed.

(2) To extend the polynomial functional calculus $\mathbb{C}[x] \rightarrow \mathfrak{L}(\mathcal{H})$ to the continuous functional calculus $C[a, b] \rightarrow \mathfrak{L}(\mathcal{H})$, the completeness of $\mathfrak{L}(\mathcal{H})$ is needed. (Cf. Cor. 27.51.)

In contrast, the weak compactness of $\overline{B}_{\mathcal{H}}(0, 1)$ was never used.

F. Riesz proved his spectral theorem for bounded self-adjoint operators in Ch. V of [Rie13]. This great article made an extremely important contribution (probably the first important contribution) to the shift in perspective mentioned above by introducing the powerful functional calculus. However, any article that presents brand new ideas must still retain a considerable amount of traditional views. Therefore, we can see in [Rie13] that Riesz still often adopted the viewpoint of quadratic forms rather than that of linear operators. The transitional nature of this great text means that completeness plays a very ambiguous role in the text. (In particular, the Riesz-Fischer theorem was not used.)

In the following, I will make some comments on [Rie13], especially on Ch. V where the spectral theorem was proved. As we shall see, Riesz's original treatment of the spectral theorem is very close to his proof in [Rie14] (and also our second proof in Subsec. 25.7.3) of the Riesz representation Thm. 25.49: The extension of functional calculus from polynomials to a larger class of functions is similar to the extension of integrals from a smaller class of positive functions to a larger class.

27.7.1 The roles of the linear operator perspective and the operator norm on $\mathfrak{L}(\mathcal{H})$

In [Rie13, Sec. 75,88], Riesz proved the following parallel to Thm. 27.50.

Theorem 27.55. *Let $f \in \mathbb{R}[x]$. Let $\alpha = \inf f([a, b])$ and $\beta = \sup f([a, b])$. Then $\alpha \leqslant f(T) \leqslant \beta$.*

Riesz's proof is almost identical to the proof we gave for Thm. 27.50, i.e., by finding a net/sequence (T_α) of finite rank self-adjoint operators satisfying $\sup_\alpha \|T_\alpha\| < +\infty$ and converging pointwise to T, then showing $f(T_\alpha)$ converges pointwise to $f(T)$. The linear operator perspective is essential to this theorem and its proof. Moreover, as pointed out in Rem. 27.54, the operator norm on $\mathfrak{L}(\mathcal{H})$ plays a crucial role in the proof of this theorem because $\sup_\alpha \|T_\alpha\| < +\infty$, as a condition of equicontinuity, ensures the convergence of double limits.

Oddly enough, Riesz's operator viewpoint almost ended here. In the rest of the proof of spectral theorem, Riesz mostly adopted the viewpoint of quadratic forms because it was consistent with his familiar approach to moment problems. This attachment to old ideas resulted in Riesz hardly making any evident use of the completeness of $\mathcal{H}$ or $\mathfrak{L}(\mathcal{H})$ in his treatment of spectral theorem, even though he proved in Ch. IV the completeness of $\mathfrak{L}(\mathcal{H})$ in order to show that $1 - T$ is invertible if $\|T\| < 1$.

27.7.2 Riesz's continuous functional calculus

Although Riesz's Thm. 27.55 is similar to our Thm. 27.50 (saying that $\pi_T : \mathbb{C}[x] \to \mathfrak{L}(\mathcal{H})$ is a bounded linear map), Thm. 27.55 does not use the language of operator norms, even though operator norms were introduced by Riesz. (This is another example that truly innovative work inevitably retains a great deal of old ideas.)

Riesz's key viewpoint behind Thm. 27.55 can be revealed by looking at its special case (explicitly pointed out in [Rie13, Sec. 88]):

Corollary 27.56. *Let $f \in \mathbb{R}[x]$. Suppose that $f|_{[a,b]} \geqslant 0$. Then $f(T) \geqslant 0$. Consequently, if $f, g \in \mathbb{R}[x]$ and $f(x) \leqslant g(x)$ for all $x \in [a,b]$, then $f(T) \leqslant g(T)$.*

Note that one can choose g to be the constant $C_f = \|f\|_{l^\infty[a,b]}$ to conclude $0 \leqslant f(T) \leqslant C_f$ (if $f|_{[a,b]} \geqslant 0$).

The readers should compare this corollary with the notion of *positive linear functionals*. Here, the crucial fact conveyed in Riesz's paper is

$$\pi_T : f \mapsto f(T) \text{ is a "positive" linear map} \tag{27.36}$$

Consequently, Riesz extended the functional calculus from polynomials to continuous functions by using the **positivity** of π_T, not by using the boundedness of π_T and the completeness of $\mathfrak{L}(\mathcal{H})$. [6] This process is similar to the (modern) idea of extending integrals from a smaller class of functions to a larger class that we learned before:

[6] In [Die-H, Sec. 7.2], Dieudonné interpreted Riesz's method of extension as using the completeness of $\mathfrak{L}(\mathcal{H})$. This is misleading.

(a) Given a measure space (X, μ), we first define the integral on $\mathcal{S}_+(X)$, and then extend it to $\mathcal{L}_+(X)$ by approximation. This approximation can be monotonic: If $f \in \mathcal{L}_+(X)$, then there is an increasing sequence (s_n) in $\mathcal{S}_+(X)$ converging pointwise to f. Then $\int f$ equals $\lim_n \int s_n$. See Sec. 24.1.

(b) Given an LCH X and a linear $\Lambda : C_c(X, \mathbb{R}_{\geqslant 0}) \rightarrow \mathbb{R}_{\geqslant 0}$, we extend Λ to $\mathrm{LSC}_+(X) \rightarrow \overline{\mathbb{R}}_{\geqslant 0}$ by approximation. This approximation can be monotonic: If $f \in \mathrm{LSC}_+(X)$, there is an increasing net (f_α) converging pointwise to f. Then $\Lambda(f)$ equals $\lim_\alpha \Lambda(f_\alpha)$. See Sec. 25.2.

Note that positivity and monotonicity are crucial to both of the above extensions. After all, Lebesgue's theory is distinguished from Riemann's theory by its heavy reliance on the posivity of the codomains. This explains why the Riemann integral can be easily defined for vector-valued functions, whereas the generalization of Lebesgue's approach to the Bochner integral is more painful (cf. Subsec. 24.4.1): A general normed vector space does not have a positivity structure. Therefore, the straightforward generalization of Lebesgue's theory to infinite dimensional codomains is not the Bochner integral but the spectral theorem for self-adjoint operators, since the latter has a natural positivity structure, i.e., the one defined by the positive operators. [7]

In [Rie13, Sec. 90-92], Riesz used an idea similar to (a) and (b) to prove the following analogue of Cor. 27.51.

Theorem 27.57. *The polynomial functional calculus $\pi_T : \mathbb{R}[x] \rightarrow \mathfrak{L}(\mathcal{H})$ can be extended to a unital homomorphism $\pi_T : C([a,b], \mathbb{R}) \rightarrow \mathfrak{L}(\mathcal{H})$ such that any element of the range is self-adjoint. Moreover, π_T is positive in the sense that $f(T) \geqslant 0$ if $f \in C([a,b], \mathbb{R}_{\geqslant 0})$.*

Riesz's main idea of the proof is as follows. For each $f \in C([a,b], \mathbb{R}_{\geqslant 0})$, pick a sequence of polynomial (p_n) in $\mathbb{R}[x]$ converging uniformly to f on $[a,b]$ such that $p_1(x) \leqslant p_2(x) \leqslant \cdots$ for all $x \in [a,b]$. [8] By Cor. 27.56, for each $\xi \in \mathcal{H}$, the sequence $(\langle p_n(T)\xi|\xi\rangle)_{n \in \mathbb{Z}_+}$ is increasing in $[0, C_f]$ (where $C_f = \|f\|_{l^\infty[a,b]}$), and hence converges to a number $\leqslant C_f$. This defines *the quadratic form*

$$\omega_{f(T)}(\xi|\xi) := \lim_n \langle p_n(T)\xi|\xi\rangle \tag{27.37}$$

By the polarization identity, $\omega_{f(T)}(\xi|\eta)$ can be defined. This in turn gives a self-adjoint $f(T) \in \mathfrak{L}(\mathcal{H})$.

[7] Riesz later proposed to develop the Lebesgue integral theory by extending the integral from the class of step functions to larger classes by using monotonic approximation, cf. [RN]. I guess this idea was motivated by (or at least closely related to) his treatment of spectral theorem in [Rie13].

[8] Proof: By Weierstrass approximation, for each n, there is $p_n \in \mathbb{R}[x]$ such that the $l^\infty[a,b]$-distance between $f - 1/n$ and p_n is $\leqslant 10^{-n}$. Then one checks easily that $(p_n|_{[a,b]})$ is increasing.

27.7.3 Riesz's semicontinuous functional calculus

In fact, Riesz established the functional calculus on a larger class of functions ([Rie13, Sec.90,91]): Since any $f \in \mathrm{LSC}([a,b],\mathbb{R}_{\geqslant 0})$ is the limit of an increasing sequence in $C([a,b],\mathbb{R})$ (compare this with Lem. 25.19), f is also the limit of a sequence of polynomials (p_n) positive and increasing (with respect to n) on $[a,b]$. Therefore, the above method gives $f(T)$ for each $f \in \mathscr{C}_1$ where

$$\mathscr{C}_1 = \{\text{bounded lower semicontinuous } f : [a,b] \to \mathbb{R}_{\geqslant 0}\} \tag{27.38}$$

The map $f \mapsto f(T)$ is again linear, and is also **multiplicative** (i.e. $(fg)(T) = f(T)g(T)$).

We modern people can understand Riesz's method in the following way, which is similar to the method used in Sec. 24.1 and 25.2: For each $f \in \mathscr{C}_1$, define $f(T) \in \mathfrak{L}(\mathcal{H})$ to be the positive operator determined by

$$\langle f(T)\xi|\xi\rangle = \sup\left\{\langle p(T)\xi|\xi\rangle : p \in \mathbb{R}[x], p|_{[a,b]} \leqslant f\right\} \tag{27.39}$$

Then, similar to Thm. 25.18, we can prove a monotone convergence theorem, namely, if (f_n) is an increasing sequence in $\mathscr{C}_1$ converging pointwise to $f \in \mathscr{C}_1$, then

$$\langle f(T)\xi|\xi\rangle = \lim_n \langle f_n(T)\xi|\xi\rangle \qquad \in \mathbb{R}_{\geqslant 0} \tag{27.40}$$

Therefore, similar to Prop. 25.20, since the polynomial functional calculus is linear and multiplicative, with the help of the monotone convergence theorem, one shows that $f \in \mathscr{C}_1 \mapsto f(T) \in \mathfrak{L}(\mathcal{H})$ is linear and multiplicative; the operator perspective is used here. [9]

What Riesz did next amounts to defining

$$\mathscr{C}_2 = \mathrm{Span}_{\mathbb{R}}\mathscr{C}_1 = \{f^+ - f^- : f^\pm \in \mathscr{C}_1\} \tag{27.41}$$

so that $\mathscr{C}_1$ is a spanning convex cone in $\mathscr{C}_2$. It is clear that $\mathscr{C}_2$ is a unital $\mathbb{R}$-algebra since $\mathscr{C}_1$ is a unital $\mathbb{R}_{\geqslant 0}$-algebra. Therefore, by Prop. 24.19, the lower semicontinuous functional calculus can be extended to a linear $f \in \mathscr{C}_2 \mapsto f(T) \in \mathfrak{L}(\mathcal{H})$. Since $f \in \mathscr{C}_1 \mapsto f(T)$ is increasing (i.e., if $f_1 \leqslant f_2$ then $f_1(T) \leqslant f_2(T)$), we see that $f(T)$ is positive if $f \in \mathscr{C}_2$ and $f \geqslant 0$. (This is very similar to our second proof of the Riesz representation Thm. 25.49.) Finally, using the fact that $f \in \mathscr{C}_1 \mapsto f(T)$ is multiplicative, an easy algebraic argument shows that $f \in \mathscr{C}_2 \mapsto f(T)$ is also multiplicative. Therefore, Riesz essentially obtained the following theorem on $\mathscr{C}_2$-functional calculus.

[9] Let $(f_n),(g_n)$ be increasing sequences of real polynomials converging pointwise to f,g. Then $\langle (f_m g_n)(T)\xi|\eta\rangle = \langle f_m(T)g_n(T)\xi|\eta\rangle = \langle g_n(T)\xi|f_m(T)\eta\rangle$. Taking $\lim_m$, we get $\langle (fg_n)(T)\xi|\eta\rangle = \langle g_n(T)\xi|f(T)\eta\rangle$. Taking $\lim_n$, we get $\langle (fg)(T)\xi|\eta\rangle = \langle g(T)\xi|f(T)\eta\rangle = \langle f(T)g(T)\xi|\eta\rangle$.

Theorem 27.58. *The polynomial functional calculus $\pi_T : \mathbb{R}[x] \to \mathfrak{L}(\mathcal{H})$ can be extended to a (linear) unital homomorphism $\pi_T : \mathscr{C}_2 \to \mathfrak{L}(\mathcal{H})$ such that any element of $\pi_T(\mathscr{C}_2)$ is self-adjoint. Moreover, π_T is positive in the sense that $f(T) \geqslant 0$ if $f \in \mathscr{C}_2$ and $f \geqslant 0$.*

We remark that in the setting of our spectral Thm. 27.40, we have

$$f(T) = U^{-1}\big(\oplus_i M_f\big)U \tag{27.42}$$

Note that the RHS above can also be defined for every bounded Borel $f : [a,b] \to \mathbb{R}$. Therefore, our Thm. 27.40 actually implies a stronger theorem on **Borel functional calculus**, namely, Thm. 27.58 holds if $\mathscr{C}_2$ is replaced by the unital $\mathbb{R}$-algebra of bounded Borel functions $[a,b] \to \mathbb{R}$.

27.7.4 Riesz's spectral theorem

For each $\lambda \in [a,b]$, the function $\chi_{[a,\lambda]}$ is positive upper semicontinuous and hence is in $\mathscr{C}_2$. Therefore, Riesz obtained a positive operator

$$E(\lambda) = \chi_{[a,\lambda]}(T) \tag{27.43}$$

Then $\chi_{[a,\lambda]} \cdot \chi_{[a,\lambda]} = \chi_{[a,\lambda]}$ implies $E(\lambda)^2 = E(\lambda)$, and hence $E(\lambda)$ is a projection operator (cf. Pb. 21.3). These projections play the role of eigenvectors in the traditional diagonalization: In the finite dimensional case, $E(\lambda)$ is the projection onto the subspace spanned by eigenvectors whose corresponding eigenvalues are $\leqslant \lambda$; thus, the eigenvalues can be recovered from these projections.

With these preparations, Riesz proved in [Rie13, Sec. 94] his spectral theorem in a similar way to our second proof of the Riesz representation theorem in Subsec. 25.7.3:

Theorem 27.59. *There is an increasing right continuous map $E : \lambda \in [a,b] \mapsto E(\lambda)$ such that each $E(\lambda)$ is a projection, that E is right continuous (i.e. $\lim_{\lambda\to\lambda_0^+} E(\lambda)$ converges pointwise to $E(\lambda_0)$), and that for each $f \in C([a,b],\mathbb{R})$ we have*

$$f(T) = f(a)E(a) + \int_a^b f(\lambda)dE(\lambda) \tag{27.44}$$

The word "increasing" means that if $\lambda \leqslant \lambda'$, then the range of $E(\lambda)$ is contained in that of $E(\lambda')$. (In that case, $E(\lambda') - E(\lambda)$ is the projection onto the relative orthogonal complement.) The integral $\int_a^b f(T)dE(\lambda)$ in (27.44) should be understood as an **operator-valued Stieltjes integral**. Since f is continuous, $\int_a^b f(T)dE(\lambda)$ can be defined to be the limit of Riemann-Stieltjes sums, i.e.,

$$\sum_{i=1}^n f(\xi_i)(E(a_i) - E(a_{i-1}))$$

where $\{a_0 < \cdots < a_n\}$ is a partition of $[a,b]$ and $\xi_i \in (a_{i-1}, a_i]$.

27.7.5 Integral theory vs. unitary representation theory

I believe that the best name for Riesz's spectral Thm. 27.59 is the **Riesz representation theorem for the continuous functional calculus**, since (27.44) is close in form and spirit to the formula

$$\Lambda(f) = f(a)\rho(a) + \int_a^b f(\lambda)d\rho(\lambda) \tag{27.45}$$

in the Riesz representation Thm. 25.49. This is also clear from the proofs: Recall that in our second proof of the Riesz representation theorem in Subsec. 25.7.3, we first extended Λ to the set $\mathscr{C}_1$ of bounded lower semicontinuous functions. (This part was done in Sec. 25.2.) Then, we extended Λ to $\mathscr{C}_2$, and related Λ to Stieltjes integrals. Similarly, to prove the spectral Thm. 27.59, Riesz first extended $\pi_T : f \mapsto f(T)$ to $\mathscr{C}_1$, [10] then to $\mathscr{C}_2$, and finally related π_T to quadratic-form valued Stieltjes integrals.

To summarize, unlike our proof of the modern spectral Thm. 27.40 which uses the Riesz-Markov theorem as a black box, Riesz's proof of his spectral theorem uses ideas and perspectives from Thm. 25.49 rather than just the conclusion.

In contrast to Riesz's **integral theoretic** approach, our spectral Thm. 27.40 should be viewed as a **unitary representation theory** for T or for $C[a,b]$. Riesz's spectral theorem is closer to the viewpoint of quadratic/sesequilinear forms than ours. On the other hand, our Thm. 27.40 fully exhibits the spirit of linear operators and makes full use of the completeness of $\mathcal{H}$ and $\mathfrak{L}(\mathcal{H})$. The disadvantage of the viewpoint of quadratic/sesquilinear forms is clear: it can only deal with the linear structure of linear operators but not the *ring* structure.

Nevertheless, although some of Riesz's ideas in [Rie13] are not so modern, by studying this work, we can understand how the ideas we take for granted today were incubated and grew in the work of a great mathematician. We can also understand how the various very different theories developed by a great mathematician have a common core. The growth and vitality of mathematical ideas cannot be grasped merely by reading texts written entirely in modern language.

27.8 Problems and supplementary material

We assume Riesz's representation Thm. 27.34 for L^p spaces, although its full proof is not given in the notes.

⋆ **Problem 27.1.** Let $1 \leqslant p < +\infty$ and $(X, \mathfrak{M}, \mu)$ is a σ-finite measure space. Let $p^{-1}+q^{-1} = 1$. Let $f : X \to \mathbb{C}$ be measurable. Prove that $f \in \mathcal{L}^p(X,\mu)$ iff $\int_X |fg|d\mu < +\infty$ for all $g \in \mathcal{L}^q(X,\mu)$.

[10] In fact, Riesz first extended π_T to the set of *upper* semicontinuous functions.

Hint. To prove "$\Leftarrow$", choose an increasing sequence (E_n) of measurable sets whose union is X such that $\mu(E_n) < +\infty$ and $\|f\chi_{E_n}\|_{l^\infty} < +\infty$ for all n. Prove $\sup_n \|f\chi_{E_n}\|_p < +\infty$. □

27.8.1 The weak-* topology of L^p spaces

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Let $1 \leqslant p, q \leqslant +\infty$ and $p^{-1} + q^{-1} = 1$.

Exercise 27.60. Let (f_n) be a sequence in $L^p(X, \mu)$. Assume that (f_n) converges a.e. to $f : X \to \mathbb{C}$. Assume that there exists $g \in L^p(X, \mu)$ such that $\lim_n \|f_n - g\|_p = 0$. Prove that $f = g$ a.e. using the Riesz-Fischer Thm. 27.26.

Assume that $p > 1$. Assume that μ is σ-finite if $p = +\infty, q = 1$.Then we have a canonical isomorphism of Banach spaces $L^p(X, \mu) \simeq L^q(X, \mu)^*$ (cf. Thm. 27.34). By Rem. 21.36, the norm function $\|\cdot\|_p$ is lower weak-* semicontinuous on $L^p(X, \mu)$. Namely, if (f_α) is a net in $L^p(X, \mu)$ converging weak-* to $f \in L^p(X, \mu)$, then $\lim_\alpha \int_X |f_\alpha|^p d\mu \geqslant \int_X |f|^p d\mu$. A comparison of this inequality with Fatou's lemma (Thm. 24.31) suggests that there is a close relationship between pointwise convergence and weak-* convergence. We now study this relationship. The following problem is close in spirit to Pb. 25.13.

Problem 27.2. Assume that $p > 1$, and μ is σ-finite if $p = +\infty, q = 1$. Let (f_n) be a bounded sequence in $L^p(X, \mu)$. ("Bounded" means $\sup_n \|f_n\|_p < +\infty$.) Let $f : X \to \mathbb{C}$ be measurable.

1. Assume $f \in L^p(X, \mu)$. Prove that (f_n) converges weak-* to f iff for each $A \in \mathfrak{M}$ satisfying $\mu(A) < +\infty$ we have
$$\lim_{n\to\infty} \int_A f_n d\mu = \int_A f d\mu \tag{27.46}$$

2. Assume that (f_n) converges a.e. to f. Prove that $f \in L^p(X, \mu)$. Use part 1 to prove that (f_n) converges weak-* to f.

3. Let m be the Lebesgue measure. Construct a sequence (f_n) in $L^1([0,1], m)$ satisfying that $\sup_n \|f_n\| \leqslant 1$, that (f_n) converges pointwise to 0, and that (f_n) does not converge weakly to 0.

4. Construct a sequence (g_n) in $L^2([0,1], m)$ such that $\sup_n \|g_n\| < +\infty$, that (g_n) converges weakly to 0, and that (g_n) does not converge a.e. to 0.

Hint. 1. Use the density of simple functions in $L^q(X, \mu)$ (Thm. 27.24).

2. Case $1 < p < +\infty$: Use Fatou's lemma to prove $f \in L^p(X, \mu)$. Use convergence in measure (cf. Pb. 24.8, or use Egorov's Thm. 24.44) and Hölder's inequality to prove (27.46). □

Now we assume that μ is a Radon measure (or its completion) on an LCH space X. By Thm. 27.17, we know that if $1 \leqslant p < +\infty$, then $C_c(X)$ is L^p-dense in $L^\infty(X, \mu)$. This is not true when $p = +\infty$. However, we shall show that $C_c(X)$ is weak-* dense in $L^\infty(X, \mu)$. To prove this result, we need some preparation.

Definition 27.61. Let V be a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $\mathfrak{S}$ be a set of linear functional $V \to \mathbb{F}$ separating points of V (i.e., if $v \in V$ satisfies $\langle v, \varphi \rangle = 0$ for all $\varphi \in \mathfrak{S}$ then $v = 0$). Then V can be viewed as a subset of $\mathbb{F}^{\mathfrak{S}}$. The $\boldsymbol{\sigma(V, \mathfrak{S})}$**-topology** on V is defined to be the subspace topology inherited from the product topology of $\mathbb{F}^{\mathfrak{S}}$. Therefore, if (v_α) is a net in V and $v \in V$, then (v_α) converges under $\sigma(V, \mathfrak{S})$ to v iff

$$\lim_\alpha \langle v_\alpha, \varphi \rangle = \langle v, \varphi \rangle \qquad (\forall \varphi \in \mathfrak{S}) \tag{27.47}$$

Moreover, by the definition of product topology in terms of basis (cf. Def. 7.71), we know that the sets

$$V_{v_0, A, \varepsilon} = \{v \in V : |\langle v - v_0, \varphi \rangle| < \varepsilon \text{ for all } \varphi \in A\} \tag{27.48}$$

(where $v_0 \in V$, $A \in \operatorname{fin}(2^{\mathfrak{S}})$, and $\varepsilon \in \mathbb{R}_{>0}$) form a basis for the $\sigma(V, \mathfrak{S})$-topology.

Example 27.62. Let V be a normed vector space. Then the $\sigma(V, V^*)$-topology is the weak topology on V. The $\sigma(V^*, V)$-topology is the weak-* topology on V^*.

Problem 27.3. Let V be a normed vector space. Let $E \subset V$ and $\mathfrak{S} \subset V^*$. Suppose that $\mathfrak{S}$ is bounded (i.e. $\sup_{\varphi \in \mathfrak{S}} \|\varphi\| < +\infty$), and $\operatorname{Span} E$ is norm-dense in V. Prove that E separates points of V^*. Prove that the $\sigma(V^*, V)$-topology and the $\sigma(V^*, E)$-topology are equal when restricted to $\mathfrak{S}$.

Note. The content of this problem is not very new; compare it with Prop. 17.19 (setting $W = \mathbb{F}$). Note that Thm. 17.31 is a special case of this problem. $\square$

$\star$ **Problem 27.4.** Let μ be a σ-finite Radon measure (or its completion) on an LCH space X.

1. Let $\mathfrak{S}_0 \subset \mathfrak{S} \subset L^\infty(X, \mu)$. Assume that $\mathfrak{S}$ is L^∞-bounded. Prove that $\mathfrak{S}_0$ is weak-* dense in $\mathfrak{S}$ iff for any $f \in \mathfrak{S}$, any finitely many Borel subsets $A_1, \dots, A_n \subset X$ with finite measures, and every $\varepsilon > 0$, there exists $g \in \mathfrak{S}_0$ such that $|\int_{A_i} f - \int_{A_i} g| < \varepsilon$ for all $1 \leqslant i \leqslant n$.

2. Prove that $\overline{B}_{C_c(X)}(0, 1)$ is weak-* dense in $\overline{B}_{L^\infty(X,\mu)}(0, 1)$. Conclude that, in particular, $C_c(X)$ is weak-* dense in $L^\infty(X, \mu)$.

3. Assume that X is second countable. Prove that for each $f \in L^\infty(X, \mu)$ there exists a sequence (f_n) in $C_c(X)$ such that $\sup_n \|f_n\|_{l^\infty(X)} \leqslant \|f\|_{L^\infty}$ and that (f_n) converges weak-* in $L^\infty(X, \mu)$ to f.

Hint. 1. Use the basis (27.48) and the density of simple functions (Thm. 27.24).

2. Use part 1, Lusin's theorem, and the Tietze extension Thm. 15.22.

3. $L^1(X,\mu)$ is separable (Thm. 27.21). Therefore, the weak-* topology on $\overline{B}_{L^\infty(X,\mu)}(0,1)$ is (compact and) metrizable, cf. Thm. 17.24. $\square$

⋆ **Exercise 27.63.** In part 3 of Pb. 27.4, give a more explicit construction of (f_n) without citing the big Thm. 17.24, and without first proving part 2.

More precisely: Choose a sequence of Borel sets $A_1, A_2, \dots \subset X$ with finite measures such that $\chi_{A_1}, \chi_{A_2}, \dots$ span a dense subspace of $L^1(X,\mu)$. (Why can we do so?) For each n, find f_n such that $|\int_{A_i} f - \int_{A_i} f_n|$ is small for all $1 \leqslant i \leqslant n$. Show that (f_n) converges weak-* to f.

If X is $[a,b]$, $\mathbb{R}$, or $\mathbb{R}^N$, can you give a more explicit choice of $A_1, A_2, \dots$? $\square$

27.8.2 ⋆ L^p spaces and Fubini-Tonelli

Let $1 \leqslant p < +\infty$. Let X, Y be LCH spaces. Let μ, ν be the completions of σ-finite Radon measures on X, Y respectively. Let $\mu \times \nu$ be the Radon product (cf. Def. 26.4).

Problem 27.5. Let

$$\mathscr{A} = \mathrm{Span}_{\mathbb{C}}\{fg : f \in \mathscr{L}^p(X,\mu), g \in \mathscr{L}^p(Y,\mu)\}$$

Prove that $\mathscr{A} \subset \mathscr{L}^p(X \times Y, \mu \times \nu)$, and that $\mathscr{A}$ is a dense linear subspace of $L^p(X \times Y, \mu \times \nu)$.

Note. It is a non-trivial fact that a measurable function on X can be viewed as a measurable function on $X \times Y$. This fact relies on the fact that the projection $X \times Y \to X$ is measurable, cf. Exp. 26.15. $\square$

Hint. Use (e.g.) Stone-Weierstrass to show that $\mathrm{Span}\{fg : f \in C_c(X), g \in C_c(Y)\}$ is l^∞-dense in $C_c(X \times Y)$. $\square$

The following problem aims to interpret Minkowski's integral inequality from the perspective of vector-valued integrals/Bochner integrals. To begin with, note that for each measurable $f : X \times Y \to \mathbb{C}$, by Tonelli's Thm. 26.8, the function

$$x \in X \qquad \mapsto \qquad \|f(x,\cdot)\|_{L^p(Y)} = \Big(\int_Y |f(x,y)|^p d\nu(y)\Big)^{\frac{1}{p}} \in \overline{\mathbb{R}}_{\geqslant 0} \tag{27.49}$$

can be defined for almost every x. Extend this function to the whole domain X, which is measurable by Tonelli's Thm. 26.8.

Problem 27.6. Let q satisfy $p^{-1} + q^{-1} = 1$. Assume that $f : X \times Y \to \mathbb{C}$ is measurable, and

$$\int_X \|f(x,\cdot)\|_{L^p(Y)} d\mu(x) < +\infty \tag{27.50}$$

In particular, by Prop. 24.16, $\|f(x,\cdot)\|_{L^p(Y)} < +\infty$ for almost every x. Replacing f by $\chi_{A\times Y}$ where $A \subset X$ is a measurable set with null complement, we assume that $\|f(x,\cdot)\|_{L^p(Y)} < +\infty$ for every x.

1. Prove that for a.e. $y \in Y$, the function $f(\cdot,y) : x \in X \mapsto f(x,y)$ is μ-integrable. Prove that the function $\int_X f d\mu$ (sending $y \in Y$ to $\int_X f(\cdot,y)d\mu$) is ν-measurable, and is in $L^p(Y,\nu)$.

2. Let $\varphi : X \to L^p(Y,\nu)$ such that for each $x \in X$, $\varphi(x)$ and $f(x,\cdot)$ are the same element in $L^p(Y)$. Prove that φ is weakly integrable and its integral $\int_X \varphi d\mu$ equals $\int_X f d\mu$, cf. Def. 24.35. In other words, prove for each $g \in L^q(Y,\nu)$ that

$$\int_X \langle \varphi(x), g\rangle = \Big\langle \int_X f d\mu, g \Big\rangle \tag{27.51}$$

3. By Pb. 24.3, we have $\|\int_X \varphi d\mu\| \leqslant \int_X \|\varphi(x)\| d\mu(x)$. Use this fact to conclude **Minkowski's integral inequality**

$$\bigg(\int_Y \Big|\int_X f(x,y)d\mu(x)\Big|^p d\nu(y)\bigg)^{\frac{1}{p}} \leqslant \int_X \bigg(\int_Y |f(x,y)|^p d\nu(y)\bigg)^{\frac{1}{p}} d\mu(x) \tag{27.52}$$

4. Assume that Y is second countable. Prove that φ is Bochner integrable (cf. Pb. 24.5).

Hint. Part 1. For each $g \in L^q(Y,\nu)$, apply Fubini's theorem to $f(x,y)g(y)$ to show that $f(\cdot,y)g(y)$ is μ-integrable for a.e. $y \in Y$, and $y \in Y \mapsto \int_X f(\cdot,y)g(y)d\mu$ is ν-integrable. By choosing g to be suitable characteristic functions, show that $f(\cdot,y)$ is μ-integrable for a.e. $y \in Y$, and $\int_X f d\mu$ is ν-measurable. The same conclusions hold for $|f|$. For each $g \in L^q(Y,\nu)$, prove $\int_Y |\int_X f(x,y)d\mu(x)| \cdot |g(y)| d\nu(y) < +\infty$. Conclude from Pb. 27.1 that $\int_X f d\mu$ belongs to $L^p(Y,\nu)$.

Part 2. Use Fubini's theorem.

Part 4. By Thm. 27.21, $L^p(Y,\nu)$ is separable. So φ is measurable iff φ is weakly measurable (cf. Pb. 23.7). □

27.8.3 Essential ranges and spectra

Let $(X,\mathfrak{M},\mu)$ be a measure space.

Definition 27.64. Let Y be a measurable space. Let $\varphi : X \to Y$ be measurable. The **pushforward measure** $\varphi_*\mu : \mathfrak{N} \to [0,+\infty]$ is defined by

$$(\varphi_*\mu)(E) = \mu(\varphi^{-1}(E)) \qquad (\forall E \in \mathfrak{N}) \tag{27.53}$$

Then we have

$$\int_Y f d\varphi_*\mu = \int_X (f\circ\varphi) d\mu$$

for all $f = \chi_E$ where $E \in \mathfrak{N}$, and hence for all $f \in \mathcal{L}_+(Y)$ by the monotone convergence theorem.

Remark 27.65. If the σ-algebra of a set Y is not assigned and $\varphi : X \to Y$ is an arbitrary map, we let $\varphi_*\mu$ be defined on the **pushforward σ-algebra**

$$\varphi_*\mathfrak{M} = \{E \in 2^Y : \varphi^{-1}(E) \in \mathfrak{M}\} \tag{27.54}$$

Example 27.66. The Lebesgue measure on $\mathbb{S}^1$ (together with its σ-algebra) is the pushforward of the Lebesgue measure on $[\theta - \pi, \theta + \pi)$.

Recall Pb. 25.1 for the meaning of support of a measure.

Definition 27.67. Let Y be a topological space. Let $\varphi : X \to Y$ be measurable. The **essential range** $\mathrm{Rng}^{\mathrm{ess}}(\varphi)$ of φ (with respect to the measure μ) is defined to be $\mathrm{Supp}(\varphi_*\mu)$, which is clearly a subset of the closure $\overline{\varphi(X)}$.

Problem 27.7. Let $f \in \mathcal{L}(X, \mathbb{C})$. Prove that $\|f\|_{L^\infty}$ is the supremum of the essential range of $|f| : X \to \mathbb{R}_{\geqslant 0}$.

⋆ **Problem 27.8.** Let $\mathcal{H}$ be a Hilbert space. Let $T \in \mathfrak{L}(\mathcal{H})$. Prove that $\mathrm{Ker}(T^*) = \mathrm{Im}(T)^\perp$. Conclude that T^* is injective iff T has dense range.

Hint. A subspace of $\mathcal{H}$ is dense iff any vector orthogonal to this subspace must be zero (Cor. 21.14). □

Recall Def. 27.37 for the meaning of multiplication operators.

Problem 27.9. Solve the following problems.

1. Assume that μ is σ-finite. Prove that there exists a measurable $h : X \to (0, +\infty)$ such that the measure $\nu : \mathfrak{M} \to [0, +\infty]$ defined by $d\nu = hd\mu$ is finite.

2. Let $h \in \mathcal{L}(X, \mathbb{R}_{\geqslant 0})$ such that $\{x \in X : h(x) = 0\}$ is μ-null. Define a measure $\nu : \mathfrak{M} \to [0, +\infty]$ by $d\nu = hd\mu$. Prove that $E \in \mathfrak{M}$ is μ-null iff E is ν-null. Use this to show:

   (a) The essential range of φ in Def. 27.67 defined by μ is equal to the one defined by ν, and hence that

   $$L^\infty(X, \mu) = L^\infty(X, \nu) \tag{27.55}$$

   (b) If $(\overline{\mathfrak{M}}, \mu)$ and $(\widehat{\mathfrak{M}}, \mu)$ are the completions of $(\mathfrak{M}, \mu)$ and $(\mathfrak{M}, \nu)$ respectively, then $\overline{\mathfrak{M}} = \widehat{\mathfrak{M}}$.

⋆ 3. Let h, ν be as in Part 2. Prove that there is a unitary map

$$U : L^2(X, \nu) \to L^2(X, \mu) \qquad \xi \mapsto \sqrt{h}\xi \tag{27.56}$$

For each f in $L^\infty(X, \mu)$ (equivalently, in $L^\infty(X, \nu)$), let M_f^μ and M_f^ν denote the multiplication operators on $L^2(X, \mu)$ and $L^2(X, \nu)$ respectively. Prove that

$$UM_f^\nu U^{-1} = M_f^\mu \qquad (\text{on } L^2(X, \mu)) \tag{27.57}$$

⋆ **Problem 27.10.** Assume that μ is σ-finite. Let $\mathcal{H} = L^2(X, \mu)$. Let $f : X \to \mathbb{R}$ be bounded and measurable. Let $T = M_f \in \mathfrak{L}(\mathcal{H})$ be the multiplication operator, i.e., $Tg = fg$ for all $g \in \mathcal{H}$. Let $\mathrm{Rng}^{\mathrm{ess}}(f)$ be the essential range of f, which is a bounded subset of $\mathbb{R}$. Let $\lambda \in \mathbb{C}$.

1. Suppose that $\lambda \notin \mathrm{Rng}^{\mathrm{ess}}(f)$. Prove that $\lambda - T$ is **invertible** in $\mathfrak{L}(\mathcal{H})$, i.e., there is $S \in \mathfrak{L}(\mathcal{H})$ such that $(\lambda - T)S = S(\lambda - T) = 1$.

2. Prove that $\mu(f^{-1}(\lambda)) > 0$ iff λ is an eigenvalue of T. (Note that if $\mu(f^{-1}(\lambda)) > 0$, then $\lambda \in \mathrm{Rng}^{\mathrm{ess}}(f) \subset \mathbb{R}$. So $\lambda - T$ is self-adjoint, and hence $\lambda - T$ does not have dense range by Pb. 27.8.)

3. Suppose that $\lambda \in \mathrm{Rng}^{\mathrm{ess}}(f)$ and $\mu(f^{-1}(\lambda)) = 0$. Prove that $\lambda - T$ is not surjective. (But note that $\lambda - T$ has dense range by Part 2 and Pb. 27.8.)

Hint. Part 1: Prove that $M_{(\lambda - f)^{-1}}$ is a bounded linear map. Part 2: Generalize the proof of Exp. 27.38. Part 3: By Pb. 27.9, it suffices to assume that μ is finite. Let

$$E_n = \{x \in X : (n+1)^{-1} \leqslant |f(x) - \lambda| < n^{-1}\}$$

Let $\alpha_n = \dfrac{1}{n^2\mu(E_n)}$ if $\mu(E_n) > 0$, and $\alpha_n = 0$ if $\mu(E_n) = 0$. Let $\xi = \sum_n \sqrt{\alpha_n}\chi_{E_n}$. Prove that $\xi \in \mathcal{H}$. If $g \in \mathcal{H}$ and $(\lambda - T)g = \xi$, then $(\lambda - f)g = \xi$. Then $\chi_{E_n} g = (\lambda - f)^{-1}\sqrt{\alpha_n}\chi_{E_n}$. Prove that $\|\chi_{E_n} g\|_{L^2} \geqslant 1$ if $\mu(E_n) > 0$. Prove that there are infinitely many n such that $\mu(E_n) > 0$. Conclude that g cannot be in $L^2(X, \mu)$. □

⋆ **Remark 27.68.** Let $\mathcal{H}$ be a separable Hilbert space, let $T \in \mathfrak{L}(\mathcal{H})$ be self-adjoint, and let $\lambda \in \mathbb{C}$. Continuing the story in Ch. 22, we ask if for every $\eta \in \mathcal{H}$, the "integral equation"

$$\lambda\xi - T\xi = \eta \tag{27.58}$$

has a solution $\xi \in \mathcal{H}$, i.e., if $\lambda - T$ is surjective. Moreover, we ask how the surjectivity is related to the injectivity of $\lambda - T$. (In concrete integral equation problems, it is often easier to check that λ is not an eigenvalue of T than to show that $\lambda - T$ is surjective; see Subsec. 22.6.2 and the reference therein.)

By Rem. 27.41, T is unitarily equivalent to the multiplication operator M_f on $L^2(X, \mu)$ where μ is a Radon measure on a second countable LCH space X and $f \in C(X, \mathbb{R}) \cap l^\infty(X, \mathbb{R})$. Define the **spectrum** of T to be

$$\mathrm{Sp}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not invertible in } \mathfrak{L}(\mathcal{H})\} \tag{27.59}$$

Then, by Pb. 27.10, we have

$$\mathrm{Sp}(T) = \mathrm{Rng}^{\mathrm{ess}}(f) \tag{27.60}$$

Therefore, if $\lambda \in \mathbb{C}\backslash\mathbb{R}$, then $\lambda \notin \mathrm{Sp}(T)$. Moreover, if $\lambda \in \mathbb{R}$, then one and only one of the following there cases happens:

(1) $\lambda \notin \mathrm{Sp}(T)$. In particular, λ is not an eigenvalue, and $\lambda - T$ is surjective.

(2) λ is an eigenvalue, and $\lambda - T$ does not have dense range.

(3) λ is not an eigenvalue, $\lambda - T$ has dense range but is not surjective.

These three cases correspond respectively to the three cases in Pb. 27.10, i.e.,

(1) $\lambda \notin \mathrm{Rng}^{\mathrm{ess}}(f)$.

(2) $\mu(f^{-1}(\lambda)) > 0$. In particular, $\lambda \in \mathrm{Rng}^{\mathrm{ess}}(f)$.

(3) $\lambda \in \mathrm{Rng}^{\mathrm{ess}}(f)$ and $\mu(f^{-1}(\lambda)) = 0$.

The set of λ satisfying (2) resp. (3) is called the **point spectrum** resp. **continuous spectrum** of T. Thus, if T has nonempty continuous spectrum, the Fredholm alternative (Cor. 22.33) does not hold for T.

From the above three cases, it is clear that for each $\lambda \in \mathbb{C}$, the map $\lambda - T$ is surjective iff $\lambda \notin \mathrm{Sp}(T)$. This suggests that $\mathrm{Sp}(T)$ is useful for the study of the "integral equation" (27.58). □

Index

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