Dual fusion products of modules of vertex operator algebras associated to compact Riemann surfaces

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arXiv:2305.10180
June 2023

Conformal blocks

- We fix an $\mathbb{N}$-graded VOA $\mathbb{V} = \bigoplus_{n\in\mathbb{N}} \mathbb{V}(n)$. In this talk, we assume that $\mathbb{V}$ is C_2-cofinite (i.e. $C_2(\mathbb{V}) = \{Y(u)_{-2}v : u, v \in \mathbb{V}\}$ has finite codimention) for simplicity, although some discussions also apply to more general VOAs. By a $\mathbb{V}$-module, we always mean a grading restricted generalized $\mathbb{V}$-module (equivalently, a finitely generated admissible $\mathbb{V}$-module).
- Goal: develop a (geometric) theory for conformal blocks associated to modules of C_2-cofinite VOAs and compact Riemann surfaces, aiming at proving a sewing-factorization theorem.
- The above goal has more or less been achieved when the compact Riemann surfaces have genera 0 or 1, or when $\mathbb{V}$ is also rational (i.e. all $\mathbb{V}$-modules are completely reducible).

Conformal blocks

- An N-pointed compact Riemann surface with local coordinates is the data
$$\mathfrak{X} = (C; x_1, \dots, x_N; \eta_1, \dots, \eta_N)$$
where C is a (possibly disconnected) compact Riemann surface, $x_1, \dots, x_N$ are distinct points of C. Each η_i is a local coordinate at x_i, i.e. a biholomorphism from a neighborhood of x_i to an open subset of $\mathbb{C}$ such that $\eta_i(x_i) = 0$. We assume that each component of C contains one of the marked points $x_1, \dots, x_N$.
- Let $\mathbb{W}$ be a $\mathbb{V}^{\otimes N}$-module. A conformal block (Zhu, 94; E.Frenkel&Ben-Zvi, 03) associated to $\mathfrak{X}$ and $\mathbb{W}$ is a linear functional $\phi : \mathbb{W} \to \mathbb{C}$ "invariant under the action of $\mathbb{V}$" (more precisely: invariant under a holomorphic bundle $\mathscr{V}_{\mathfrak{X}}$ on C associated to $\mathfrak{X}$ and $\mathbb{V}$). The space of all such conformal blocks is denoted by $\mathscr{T}^*_{\mathfrak{X}}(\mathbb{W})$ and is finite-dimensional.

Examples of conformal blocks: genus 0

- Let $z \in \mathbb{C}\backslash\{0\}$. If $\mathbb{W} = \mathbb{W}_2 \otimes \mathbb{W}_1 \otimes \mathbb{W}_3'$ where each $\mathbb{W}_i$ is a $\mathbb{V}$-module, $\mathbb{W}_3'$ is the contragredient (dual) of $\mathbb{W}_3$, $\mathfrak{X} = (\mathbb{P}^1; 0, z, \infty; \zeta, \zeta - z, 1/\zeta)$ where ζ is the standard coordinate of $\mathbb{C}$. Then an element of $\mathscr{T}_{\mathfrak{X}}^*(\mathbb{W})$ are precisely intertwining operators of type $\binom{\mathbb{W}_3}{\mathbb{W}_1\mathbb{W}_2}$ at z, and is often viewed as a linear map
$$\mathbb{W}_1 \otimes \mathbb{W}_2 \to \overline{\mathbb{W}_3} = (\mathbb{W}_3')^* \qquad w_1 \otimes w_3 \mapsto \mathcal{Y}(w_1, z)w_2$$
- If $\mathcal{Y}_1$ is of type $\binom{\mathbb{W}_4}{\mathbb{W}_1\mathbb{W}_5}$ at z_1, $\mathcal{Y}_2$ is of type $\binom{\mathbb{W}_5}{\mathbb{W}_2\mathbb{W}_3}$ at z_2, where $0 < |z_2| < |z_1|$, then
$$\mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \mathbb{W}_3 \to \overline{\mathbb{W}_4} \qquad w_1 \otimes w_2 \otimes w_3 \mapsto \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3$$
defines an element of $\mathscr{T}_{\mathfrak{Y}}^*(\mathbb{W}_3 \otimes \mathbb{W}_2 \otimes \mathbb{W}_1 \otimes \mathbb{W}_4)$ where $\mathfrak{Y} = (\mathbb{P}^1; 0, z_2, z_1, \infty; \zeta, \zeta - z_2, \zeta - z_1, 1/\zeta)$. All conformal blocks on $\mathfrak{Y}$ can be obtained in this way. (Huang-Lepowsky 95, Huang 05, Huang-Lepowsky-Zhang 07.)

Examples of conformal blocks: genus 1

- If $\mathcal{Y}$ is of type $\binom{\mathbb{M}}{\mathbb{W}\mathbb{M}}$ at $z \in \mathbb{C}$, $0 < |q| < |z| < 1$, then
$$\mathbb{W} \mapsto \mathbb{C} \qquad w \mapsto \mathrm{Tr}\mathcal{Y}(w,z)q^{L(0)}$$
is an element of $\mathscr{T}_{\mathfrak{T}}^*(\mathbb{W})$. Here, $\mathfrak{T} = (\mathbb{T}^1; z; \zeta - z)$ where $\mathbb{T}^1$ is the torus obtained by gluing the boundaries of the annulus $\{\gamma \in \mathbb{C} : |q| \leqslant |\gamma| \leqslant 1\}$ such that $e^{\mathbf{i}\theta}$ is glued with $qe^{\mathbf{i}\theta}$.
- If $\mathbb{V}$ is (C_2-cofinite and) rational, all conformal blocks of $\mathfrak{T}$ arise in this way. This property is closely related to the so-called modular invariance property. (Zhu 96, Huang 05)
- If $\mathbb{V}$ is not rational, the above trace construction is not enough to produce all conformal blocks on $\mathfrak{T}$. Instead, one needs to consider pseudo-traces. (Miyamoto 04, Arike-Nagatomo 13, Fiordalisi 15, Huang 23)

Sewing-factorization theorem

- Now let $\mathfrak{X} = (C; x_1, \dots, x_N; y', y''; \eta_1, \dots, \eta_N; \xi, \varpi)$ be $(N+2)$-pointed with distinct marked points $x_1, \dots, x_N, y', y''$. Assume that each component of C contains some x_i. Associate $\mathbb{W}_i$ to x_i, $\mathbb{M}$ to y', and the contragredient module $\mathbb{M}'$ to y''. (So the $\mathbb{V}^{\otimes(N+2)}$-module $\mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N \otimes \mathbb{M} \otimes \mathbb{M}'$ is associated to $\mathfrak{X}$.)
- Let $\upphi \in \mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}_\bullet \otimes \mathbb{M} \otimes \mathbb{M}')$ where $\mathbb{W}_\bullet = \bigotimes_{i=1}^N \mathbb{W}_i$. Let $q \in \mathbb{C}\backslash\{0\}$. If q is reasonably large, we can define the sewing of $\upphi$ to be a linear functional $\mathcal{S}_q\upphi : \mathbb{W}_\bullet \to \mathbb{C}$ by taking contraction

$$\mathcal{S}_q\upphi(w_\bullet) = \upphi(w_\bullet \otimes q^{L(0)} - \otimes -)$$

The sewing theorem says that $\mathcal{S}_q\upphi$ converges absolutely and is an element of $\mathscr{T}^*_{\mathcal{S}_q\mathfrak{X}}(\mathbb{W}_\bullet)$ where $\mathcal{S}_q\mathfrak{X}$ (N-pointed) is obtained by removing small discs centered at y', y'' and gluing the remaining annuli via the rule

$$p' \text{ is identified with } p'' \qquad \Leftrightarrow \qquad \xi(y')\varpi(y'') = q$$

Sewing-factorization theorem

- Assume that $\mathbb{V}$ is C_2-cofinite and rational. A version of factorization theorem (as the converse of sewing theorem) says that every element of $\mathscr{T}^*_{\mathcal{S}_q\mathfrak{X}}(\mathbb{W}_\bullet)$ arises from $\mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}_\bullet \otimes \mathbb{M} \otimes \mathbb{M}')$ via sewing.

Theorem (Sewing-factorization theorem, rational version)

Let $\mathbb{V}$ be (C_2-cofinite and) rational. Let $\mathcal{E}$ be a (necessarily finite) set of representatives of equivalence classes of simple $\mathbb{V}$-modules. Then

$$\mathfrak{S}_q : \bigoplus_{\mathbb{M}\in\mathcal{E}} \mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}_\bullet \otimes \mathbb{M} \otimes \mathbb{M}') \to \mathscr{T}^*_{\mathcal{S}_q\mathfrak{X}}(\mathbb{W}_\bullet)$$
$$\bigoplus_{\mathbb{M}} \phi_{\mathbb{M}} \mapsto \sum_{\mathbb{M}} \mathcal{S}_q\phi_{\mathbb{M}}$$

is well defined (i.e. the RHS converges absolutely to an element of $\mathscr{T}^*_{\mathcal{S}_q\mathfrak{X}}(\mathbb{W}_\bullet)$) and is a linear isomorphism.

Sewing-factorization theorem

This final version of rational sewing-factorization theorem was proved by Damiolini-Gibney-Tarasca in 2022 (for the part that the domain and the codomain of $\mathfrak{S}_q$ have the same dimension) and G. in 2023 (for the part that $\mathfrak{S}_q$ is well-defined and injective). But the proofs of its preliminary versions took more than 30 years and have been one of the central problems in the history of VOA theory:

- $\mathbb{V}$ is affine VOA of positive integer level: Tsuchiya-Ueno-Yamada (89), for sufficiently small q.
- Arbitrary (C_2-cofinite and) rational $\mathbb{V}$, self-sewing a 3-pointed sphere: Zhu (96). This is very close to the modular invariace property.
- Arbitrary rational $\mathbb{V}$, self-sewing a $\geqslant 4$-pointed sphere: Huang (05). This result is crucial to the solution of Verlinde conjecture and the proof of the rigidity and modularity of the category $\mathrm{Mod}(\mathbb{V})$ of $\mathbb{V}$-modules.

Sewing-factorization theorem

- Arbitrary rational $\mathbb{V}$, sewing two (or more) spheres to get a sphere: Huang-Lepowsky (95), Huang (05). This is crucial to the construction of the braided tensor category $\mathrm{Mod}(\mathbb{V})$, especially in the proof of associativity and the pentagon and hexagon axioms. Also, in this case, Nagatomo-Tsuchiya (05) proved independently that the domain and the codomain of $\mathfrak{S}_q$ have the same dimension.
- Some relevant results: $\dim \mathscr{T}_{\mathfrak{X}}^*(\mathbb{W}_\bullet)$ is finite (Abe-Nagatomo 03) and depends only on the topology (but not the complex structure) of $\mathfrak{X}$ (Damiolini-Gibney-Tarasca 22). Here, $\mathbb{V}$ is not assumed rational.

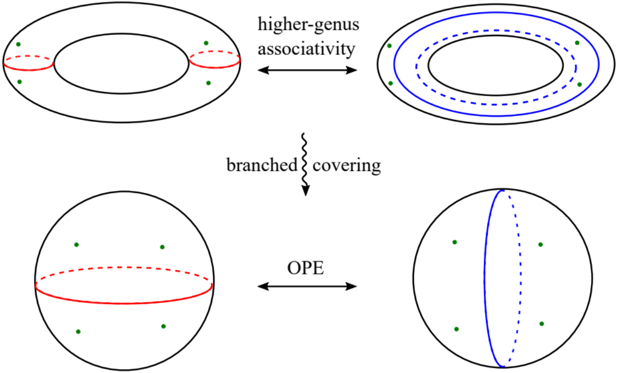
Irrational sewing-factorization

- From now on, we do not assume that our C_2-cofinite VOA $\mathbb{V}$ is rational. Then only a half of the previous sewing-factorization theorem holds: $\mathfrak{S}_q$ is well-defined and injective. It is not surjective, above all because it is not enough to consider sewing along *simple* $\mathbb{V}$-modules $\mathbb{M}$ and $\mathbb{M}'$.
- Now we drop the assumption that $\mathbb{M}$ is simple, and ask whether the factorization property holds, namely, whether every element of $\mathscr{T}^*_{\mathcal{S}_q}(\mathbb{W}_\bullet)$ equals $\mathcal{S}_q\phi$ for some $\phi \in \mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}_\bullet \otimes \mathbb{M} \otimes \mathbb{M}')$ and some $\mathbb{V}$-module $\mathbb{M}$.
- Surprisingly, the answer depends: **Yes** if sewing two spheres and get a new sphere (Huang-Lepowsky-Zhang's tensor product theory 07). **No** if one is sewing a sphere to get a torus (the pseudo-trace theory).

Irrational sewing-factorization

- Key observation: whether the factorization property holds or not depends not on the genus, but on the geometry of sewing.
- In 2021 I proved (arXiv:2111.04662) that there is a 1-1 correspondece between **(a)** conformal blocks associated to permutation-twisted $\mathbb{V}^{\otimes n}$-modules and a pointed sphere with local coordinates $\mathfrak{P}$ **(b)** conformal blocks associated to (untwisted) $\mathbb{V}$-modules and a (possibly disconnected) n-fold branched covering $\mathfrak{X}$ of $\mathfrak{P}$.
- Moreover, I showed that at least when $\mathbb{V}$ is rational, the sewing-factorization in (a) corresponds to that in (b). From the proof, it is clear that even if $\mathbb{V}$ is irrational, one can translate the sewing-factorization in (a) (i.e. the permutation-twisted Huang-Lepowsky-Zhang's theory) to (b) to obtain an irrational sewing-factorization theorem for certain higher genus case.

- The permutation twisted/untwisted correspondence translates genus-0 sewing-factorization theorem to higher genus ones for disjoint sewing:



Self-sewing and disjoint sewing

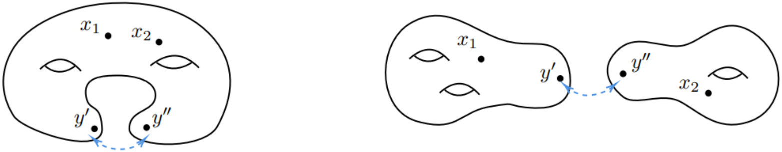


Figure 0.1.1 Self-sewing and disjoint sewing

- Since the factorization property does not hold for self-sewing, we shall only consider disjoint sewing.

Conformal blocks

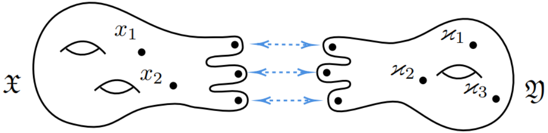
But we shall consider disjoint sewing along several pairs of points: Fix

$$\mathfrak{X} = (y'_1, \ldots, y'_M | C_1 | x_1, \ldots, x_N) \qquad \mathfrak{Y} = (y''_1, \ldots, y''_M | C_2 | \varkappa_1, \ldots, \varkappa_K)$$

$\mathfrak{X}$ is viewed as having N-incoming points and M outgoing points, and $\mathfrak{Y}$ similarly. Choose local coordinates $\eta_i, \mu_k, \xi_j, \varpi_j$ at each $x_i, \varkappa_k, y'_j, y''_j$. Then we can sew $\mathfrak{X}$ and $\mathfrak{Y}$ along the M pairs of points (y'_j, y''_j) $(1 \leqslant j \leqslant M)$ using reasonably large parameters $q_1, \ldots, q_M \neq 0$ and get

$$\mathfrak{X} \#_{q_\bullet} \mathfrak{Y} = (\mathcal{C}_{q_\bullet} | x_1, \ldots, x_N, \varkappa_1, \ldots, \varkappa_K)$$

with local coordinates $\eta_1, \ldots, \eta_N, \mu_1, \ldots, \mu_K$.



Dual fusion products

$$\mathfrak{X} = (y'_1, \dots, y'_M | C_1 | x_1, \dots, x_N) \qquad \mathfrak{Y} = (y''_1, \dots, y''_M | C_2 | \varkappa_1, \dots, \varkappa_K)$$

Associate a $\mathbb{V}^{\otimes N}$-module $\mathbb{W}$ to the incoming points $x_\bullet$ of $\mathfrak{X}$. Note that if a $\mathbb{V}^{\otimes M}$-module $\mathbb{X}$ is associated to $y'_\star$, then we can define the space of conformal blocks $\mathscr{T}^*_{\mathfrak{X}}(\mathbb{W} \otimes \mathbb{X})$ in the same way as before.

Theorem (G.-Zhang 23)

There is a unique (up to equivalence) pair $(\boxbslash_{\mathfrak{X}}(\mathbb{W}), \daleth)$ where $\boxbslash_{\mathfrak{X}}(\mathbb{W})$ is a $\mathbb{V}^{\otimes M}$-module and $\daleth \in \mathscr{T}^*_{\mathfrak{X}}(\mathbb{W} \otimes \boxbslash_{\mathfrak{X}}(\mathbb{W}))$ satisfying the universal property: For each $\mathbb{V}^{\otimes M}$-module $\mathbb{X}$ and $\Gamma \in \mathscr{T}^*_{\mathfrak{X}}(\mathbb{W} \otimes \mathbb{X})$ there is a unique morphism $T : \mathbb{X} \to \boxbslash_{\mathfrak{X}}(\mathbb{W})$ such that $\Gamma = \daleth \circ (\mathbf{1} \otimes T)$. In particular,

$$\dim \mathrm{Hom}_{\mathbb{V}^{\otimes M}}(\mathbb{X}, \boxbslash_{\mathfrak{X}}(\mathbb{W})) = \dim \mathscr{T}^*_{\mathfrak{X}}(\mathbb{W} \otimes \mathbb{X})$$

Dual fusion products

- $(\boxbslash_{\mathfrak{X}}(\mathbb{W}), \daleth)$, or simply $\boxbslash_{\mathfrak{X}}(\mathbb{W})$, is called the dual fusion product of $\mathbb{W}$ along $\mathfrak{X}$. The previous theorem actually holds in a more general setting without assuming $\mathbb{V}$ to be C_2-cofinite.
- Assuming that $\mathbb{V}$ is C_2-cofinite, if $\mathfrak{P} = (\infty|\mathbb{P}^1|0, z)$ and $\mathbb{W} = \mathbb{W}_1 \otimes \mathbb{W}_2$ where $\mathbb{W}_i$ is a $\mathbb{V}$-module, our $\boxbslash_{\mathfrak{P}}(\mathbb{W})$ agrees with Huang-Lepowsky-Zhang's $P(z)$-dual fusion product $\mathbb{W}_1 \boxbslash_{P(z)} \mathbb{W}_2$. If $\mathfrak{X}$ is $\mathfrak{Q} = (0, \infty|\mathbb{P}^1|z)$, our $\boxbslash_{\mathfrak{Q}}(\mathbb{W})$ agrees with Li's regular representation of $\mathbb{W}$. If $\mathbb{V}$ is not C_2-cofinite, there is a small difference.
- Our definition of $\boxbslash_{\mathfrak{X}}(\mathbb{W})$ *as a vector space* is due to Liang Kong and Hao Zheng. Elements of $\boxbslash_{\mathfrak{X}}(\mathbb{W})$ are called partial conformal blocks. Our construction and verification of the $\mathbb{V}^{\otimes M}$-module structure on $\boxbslash_{\mathfrak{X}}(\mathbb{W})$ relies on a method called the propagation of partial conformal blocks.

Irrational sewing-factorization theorem

$$\mathfrak{X} = (y_1', \dots, y_M' | C_1 | x_1, \dots, x_N) \qquad \mathfrak{Y} = (y_1'', \dots, y_M'' | C_2 | \varkappa_1, \dots, \varkappa_K)$$

Let $\mathbb{W}$ and $\mathbb{M}$ be modules of $\mathbb{V}^{\otimes M}$ and $\mathbb{V}^{\otimes K}$. Let $(\boxbslash_{\mathfrak{X}}(\mathbb{W}), \gimel)$ and $(\boxbslash_{\mathfrak{Y}}(\mathbb{M}), \daleth)$ be dual fusion products. Let $\boxtimes_{\mathfrak{Y}}(\mathbb{M})$ be the contragredient of $\boxbslash_{\mathfrak{Y}}(\mathbb{M})$, called the fusion product of $\mathbb{M}$. Let $L_i(n)$ be the Virasoro operator of the i-th tensor component.

Theorem (G.-Zhang, 24)

Let $\mathbb{V}$ be C_2-cofinite. There is a well-defined linear isomorphism

$$\begin{aligned} \Psi_{q_\bullet} : \mathrm{Hom}_{\mathbb{V}^{\otimes M}}\big(\boxtimes_{\mathfrak{Y}}(\mathbb{M}), \boxbslash_{\mathfrak{X}}(\mathbb{W})\big) &\rightarrow \mathscr{T}^*_{\mathfrak{X}\#_{q_\bullet}\mathfrak{Y}}(\mathbb{W}\otimes\mathbb{M}) \\ T &\mapsto \mathcal{S}_{q_\bullet}\big((\gimel\circ T)\otimes\daleth\big) \end{aligned}$$

where

$$\mathcal{S}_{q_\bullet}\big((\gimel\circ T)\otimes\daleth\big)(w\otimes m) = \gimel\big(w\otimes T(-)\big)\cdot\daleth\big(m\otimes q_1^{L_1(0)}\cdots q_M^{L_M(0)}-\big)$$

Irrational sewing-factorization theorem

$$\mathfrak{X} = (y'_1, \dots, y'_M | C_1 | x_1, \dots, x_N) \qquad \mathfrak{Y} = (y''_1, \dots, y''_M | C_2 | \varkappa_1, \dots, \varkappa_K)$$

Let $\mathbb{W}$ and $\mathbb{M}$ be modules of $\mathbb{V}^{\otimes M}$ and $\mathbb{V}^{\otimes K}$. Let $(\boxbslash_{\mathfrak{X}}(\mathbb{W}), \gimel)$ and $(\boxbslash_{\mathfrak{Y}}(\mathbb{M}), \daleth)$ be dual fusion products. Let $\boxtimes_{\mathfrak{Y}}(\mathbb{M})$ be the contragredient $\mathbb{V}^{\otimes M}$-module of $\boxbslash_{\mathfrak{Y}}(\mathbb{M})$, called the fusion product of $\mathbb{M}$. Equivalent to the previous theorem, we have:

Corollary (G.-Zhang, 24)

Let $\mathbb{V}$ be C_2-cofinite. There is a well-defined linear isomorphism

$$\Psi_{q_\bullet} : \mathscr{T}^*_{\mathfrak{X}}\big(\mathbb{W} \otimes \boxtimes_{\mathfrak{Y}}(\mathbb{M})\big) \to \mathscr{T}^*_{\mathfrak{X}\#_{q_\bullet}\mathfrak{Y}}(\mathbb{W} \otimes \mathbb{M})$$
$$\psi \mapsto \mathcal{S}_{q_\bullet}(\psi \otimes \daleth)$$

where $\mathcal{S}_{q_\bullet}(\psi \otimes \daleth)(w \otimes m) = \psi\big(w \otimes -\big) \cdot \daleth\big(m \otimes q_1^{L_1(0)} \cdots q_M^{L_M(0)} -\big)$

Genus 1 sewing-factorization theorem

Let $\mathbb{W}$ be a $\mathbb{V}$-module. Recall that $\mathfrak{Q} = (0, \infty|\mathbb{P}^1|z)$ and the $\mathbb{V}^{\otimes 2}$-module $\boxbslash_{\mathfrak{Q}}(\mathbb{W})$ is Li's regular representation. $\boxtimes_{\mathfrak{Q}}(\mathbb{W})$ is its contragredient.

Corollary

Let $0 < |q| < |z| < 1$, $\mathfrak{T} = (\mathbb{T}^1|z)$ where $\mathbb{T}^1$ is the torus obtained by gluing the boundaries of the annulus $\{\gamma \in \mathbb{C} : |q| \leqslant |\gamma| \leqslant 1\}$ such that $e^{\mathbf{i}\theta}$ is glued with $qe^{\mathbf{i}\theta}$. The local coordinate at z is $\zeta - z$ where ζ is the standard coordinate of $\mathbb{C}$. Then we have isomorphisms

$$\mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\boxtimes_{\mathfrak{Q}}(\mathbb{W}), \boxbslash_{\mathfrak{Q}}(\mathbb{V})) \simeq \mathrm{Hom}_{\mathbb{V}^{\otimes 2}}(\boxtimes_{\mathfrak{Q}}(\mathbb{V}), \boxbslash_{\mathfrak{Q}}(\mathbb{W})) \simeq \mathscr{T}_{\mathfrak{T}}^*(\mathbb{W})$$

defined by sewing.

Question: Prove that the above theorem is equivalent to the pseudo-trace version of modular invariance theorem by Miyamoto, Arike-Nagatomo, and Huang. This will give a geometric interpretation of pseudo-traces.