Unitarity in conformal blocks and diagonal full-boundary conformal field theory

Bin Gui
Tsinghua University

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Based on arXiv:2306.11856 and a story behind it

Unitary VOAs

- Unitary vertex operator algebras (VOAs) are certain natural structures appearing in unitary 2d conformmal field theory (CFT).
- Roughly speaking, a unitary VOA is a collection of field operators $\varphi(z)$ "acting on" an inner product vector space $\mathbb{V}$. Each $\varphi(z) = \sum_{n\in\mathbb{Z}} \varphi_n z^{-n-1}$ is holomorphic with respect to $z \in \mathbb{C}^\times$. The field operators satisfy a "locality" condition. There is a 1-1 correspondence between φ and $v = \lim_{z\to 0} \varphi(z)\Omega$ where Ω is the vacuum vector. So we write $\varphi(z)$ as $Y(v,z) = \sum_{n\in\mathbb{Z}} Y(v)_n z^{-n-1}$.

I always assume that $\mathbb{V}$ is "strongly-rational" (which implies that $\mathbb{V} \in \mathrm{Irr}(\mathbb{V})$, $\mathrm{Irr}(\mathbb{V})$ is finite, all $\mathbb{V}$-modules are semi-simple, and $\mathrm{Mod}(\mathbb{V})$ is a modular tensor category (Huang 08)). I assume that $\mathbb{V}$ is unitary, and that each irreducible $\mathbb{V}$-module admits a (necessarily unique up to $\mathbb{R}_{>0}$-multiplication) unitary structure.

- If $(\mathbb{W}, Y_{\mathbb{W}})$ is a unitary $\mathbb{V}$-module, then the complex conjugate $\overline{\mathbb{W}} \subset \mathbb{W}^*$ admits a unitary $\mathbb{V}$-module structure $(\overline{\mathbb{W}}, Y_{\overline{\mathbb{W}}})$, called the dual/contragredient module. We have a canonical antiunitary map $\mathtt{C} : \mathbb{W} \to \overline{\mathbb{W}}, w \mapsto \overline{w} = \langle \cdot | w \rangle \in \mathbb{W}^*$.
- There is a canonical unitary equivalence $\mathbb{V} \simeq \overline{\mathbb{V}}$. Identify $\mathbb{V}$ with $\overline{\mathbb{V}}$. Then $\mathtt{C} : \mathbb{V} \to \mathbb{V}$ is the PCT operator (fixing Ω) which is closely related to the modular operator in Tomita-Takesaki theory.

Conformal blocks and unitarity

- Conformal blocks are "chiral halves" of the correlation functions of full (and boundary) CFT. (Or rather, correlation functions are "doubles" of conformal blocks.) They are also crucial to the construction of $\mathrm{Mod}(\mathbb{V})$ and the full-boundary CFTs extending $\mathbb{V}$.
- A hard question is to define natural inner products on the spaces of conformal blocks. Solving this unitarity problem will imply the unitarity of the MTC $\mathrm{Mod}(\mathbb{V})$.
- The goal of this talk is to explain the geometric intuition behind the unitary structures of conformal blocks. I will argue that the correct way to understand this geometric picture of unitarity is through a particular model of full-boundary CFT called diagonal CFT.

Full-boundary (i.e. open-closed) CFT

- We have a fixed state space $\mathbb{X}$ associated to closed strings S^1, state space $\mathbb{A}$ associated to open strings $S^1_+ = \{e^{\mathbf{i}t} \in S^1 : 0 \leqslant t \leqslant \pi\}$, and $\mathbb{X}^{\otimes n}, \mathbb{A}^{\otimes n}$ associated to $S^1 \sqcup \cdots \sqcup S^1, S^1_+ \sqcup \cdots \sqcup S^1_+$.
- In general, $\mathbb{A}$ is a $\mathbb{V}$-module and $\mathbb{X}$ is a $\mathbb{V} \otimes \mathbb{V}$-module. More precisely, $\mathbb{A}$ is a non-local extension of $\mathbb{V}$ (a Frobenius algebra in $\mathrm{Mod}(\mathbb{V})$), and $\mathbb{X}$ is the center $Z(\mathbb{A})$, which is a commutative Frobenius algebra in $\mathrm{Mod}(\mathbb{V}) \boxtimes \mathrm{Mod}(\mathbb{V})^{\mathrm{rev}}$. (04-15. AQFT: Bischoff, Carpi, Kawahigashi, Longo, Rehren. VOA: Huang, Kong. TQFT: Fjelstad, Fuchs, Runkel, Schweigert.)
- Diagonal CFT: $\mathbb{A} = \mathbb{V}$ and $\mathbb{X} = \bigoplus_{\mathbb{W} \in \mathrm{Irr}(\mathbb{V})} \mathbb{W} \otimes \overline{\mathbb{W}}$. (Note that $\mathbb{V} \otimes \mathbb{V} \subset \mathbb{X}$.) Note that there is a canonical isomorphism of unitary $\mathbb{V} \otimes \mathbb{V}$-modules $\mathbb{X} \simeq \overline{\mathbb{X}}$.

Worldsheet $\mathfrak{X}$ and correlation function $\Phi_{\mathfrak{X}}$

Let $\mathfrak{X} = (y_\star|C|x_\bullet) = (y_1, \ldots, y_M|C|x_1, \ldots, x_N)$ where C is a (possibly disconnected) compact Riemann surface and $x_\bullet, y_\star$ are $M+N$ distinct marked points on C. For each x_i, choose a local coordinate η_i, i.e. an injective holomorphic map from a neighborhood of x_i to $\mathbb{C}$ such that $\eta_i(x_i) = 0$. And choose μ_j for y_j.

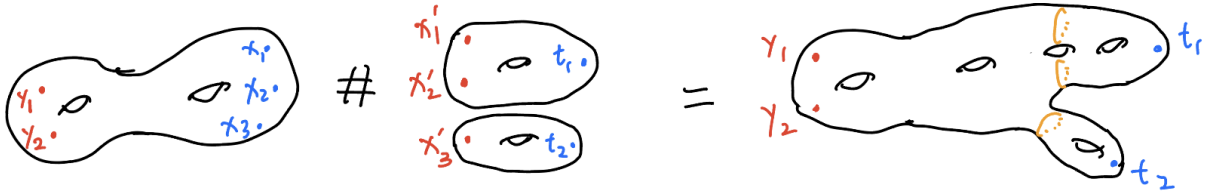
- It would be helpful to regard $\mathfrak{X}$ as a compact Riemann surface Σ with $M+N$ boundary strings by removing each $\eta_i^{-1}(\mathbb{D}_1)$ and $\mu_j^{-1}(\mathbb{D}_1)$ where $\mathbb{D}_1 = \{|z| < 1\}$. The circles around $x_\bullet$ resp. $y_\star$ are incoming resp. outgoing closed strings.

For each $\mathfrak{X}$ we have (bounded) linear functional $\Phi_{\mathfrak{X}} : \mathbb{X}^{\otimes M} \otimes \mathbb{X}^{\otimes N} \to \mathbb{C}$ (unique up to $\mathbb{R}_{>0}$-multiplication), called the correlation function.

Sewing property for correlation functions

If $\mathfrak{X} = (y_\bigstar|C|x_\bullet)$ and $\mathfrak{X}' = (x'_\bullet|C'|t_\heartsuit)$ have $M+N$ and $N+L$ marked points and local coordinated $\mu_\bigstar, \eta_\bullet$ and $\xi_\bullet, \varpi_\heartsuit$, we can sew $\mathfrak{X}$ and $\mathfrak{X}'$ to get $\mathfrak{X}\#\mathfrak{X}' = (y_\bigstar|C\#C'|t_\heartsuit)$ with $M+L$ marked points.

- Sewing procedure: Remove small discs U_i around x_i and U'_i around x'_i for all i, and glue the remaining part by identifying p near U_i and p' near U'_i whenever $\eta_i(p) = 1/\xi_i(p')$.



Sewing property: $\Phi_{\mathfrak{X}\#\mathfrak{X}'} : \mathbb{X}^{\otimes M} \otimes \mathbb{X}^{\otimes L} \to \mathbb{C}, \alpha \otimes \beta \mapsto ?$ equals the contraction $\Phi_{\mathfrak{X}}\#\Phi_{\mathfrak{X}'}(\alpha \otimes \beta) = \sum_n \Phi_{\mathfrak{X}}(\alpha \otimes e_n)\Phi_{\mathfrak{X}'}(\overline{e_n} \otimes \beta)$ where $\{e_n\}$ is a "homogeneous" orthnormal basis of $\mathbb{X}^{\otimes N}$. (Note that the canonical inner product on $\mathbb{X} = \bigoplus_{\mathbb{W}\in\mathrm{Irr}(\mathbb{V})} \mathbb{W} \otimes \overline{\mathbb{W}}$ is given by that of $\mathbb{W}$.)

Diagonal full CFT and conformal blocks

Recall $\mathbb{X} = \bigoplus \mathbb{W} \otimes \overline{\mathbb{W}}$ and $\mathfrak{X} = (y_\star|C|x_\bullet) = (y_1, \ldots, y_M|C|x_1 \ldots, x_N)$.

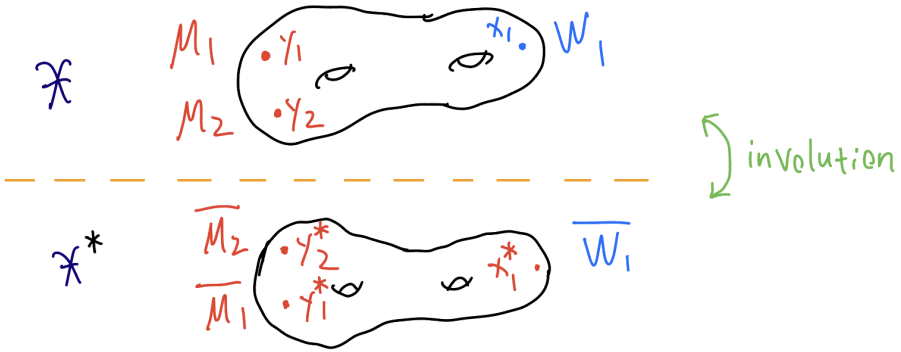
- The restriction of $\Phi_{\mathfrak{X}} : \mathbb{X}^{\otimes M} \otimes \mathbb{X}^{\otimes N} \to \mathbb{C}$ to $\left(\bigotimes_{j=1}^{m} \mathbb{M}_j\right) \otimes \left(\bigotimes_{i=1}^{n} \mathbb{W}_i\right) \otimes \left(\bigotimes_{j=1}^{m} \overline{\mathbb{M}_j}\right) \otimes \left(\bigotimes_{i=1}^{n} \overline{\mathbb{W}_i}\right)$ is $\sum_{\varphi,\psi} \varphi \otimes \psi^*$ where $\varphi, \psi : \left(\bigotimes_{j=1}^{m} \mathbb{M}_j\right) \otimes \left(\bigotimes_{i=1}^{n} \mathbb{W}_i\right) \to \mathbb{C}$ are linear functionals called conformal blocks (CB) associated to $\mathfrak{X}$ and $\mathbb{M}_\star, \mathbb{W}_\bullet$.
- The rigorous definition of conformal blocks is due to Zhu (94) and E.Frenkel&Ben-Zvi (04) and does not distinguish between incoming and outgoing marked points.
- $\psi^*(\overline{m_1} \otimes \cdots \otimes \overline{m_M} \otimes \overline{w_1} \otimes \cdots \otimes \overline{w_N}) = \overline{\psi(m_1 \otimes \cdots \otimes m_M \otimes w_1 \otimes w_N)}$ is called the conjugate CB of ψ. It is a CB associated to the complex conjugate $\mathfrak{X}^* = (y_\star^*|C^*|x_\bullet^*)$ of $\mathfrak{X}$.

Correlation functions are CB on doubles of worldsheets

Take worldsheet $\mathfrak{X} =$ 

for example.

- It is better to view the (restricted) correlation function $\Phi_{\mathfrak{X}} = \sum_{\varphi,\psi} \varphi \otimes \psi^*$ as a CB for $\mathbb{M}_1 \otimes \mathbb{M}_2 \otimes \mathbb{W}_1 \otimes \overline{\mathbb{M}_1} \otimes \overline{\mathbb{M}_2} \otimes \overline{\mathbb{W}_1}$ and the "double" $\mathfrak{D}(\mathfrak{X}) = \mathfrak{X} \sqcup \mathfrak{X}^*$. (So everything could be understood in terms of CB!)



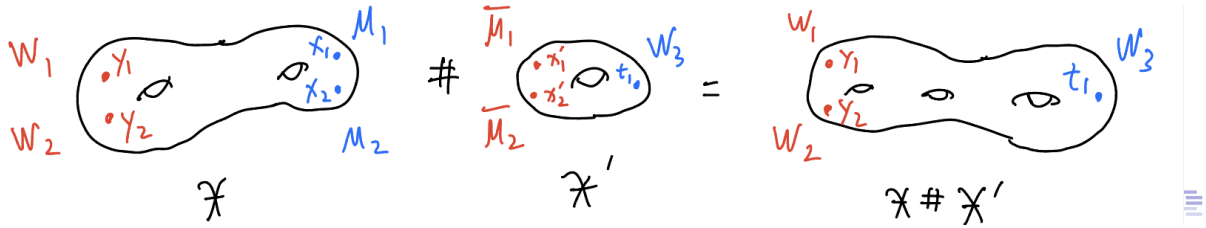
So what's good about CB?

Important properties about CB

- The decomposition $\Phi_{\mathfrak{X}} = \sum_{\varphi,\psi} \varphi \otimes \psi^*$ can be such that φ, ψ vary holomorphically w.r.t. the moduli of $\mathfrak{X}$. It is realized canonically by the Virasoro operators. (E.g. KZ equation if $\mathfrak{X}$ has genus 0.)
- The space $CB_{\mathbb{V},\mathfrak{X}}(\mathbb{M}_\star \otimes \mathbb{W}_\bullet)$ of conformal blocks has finite dimension (called (higher genus) fusion rule) which is independent of the moduli (i.e. the complex structure and the locations of marked points) of $\mathfrak{X}$.
- Sewing theorem: We have a linear isomorphism

$$\mathfrak{S} : \bigoplus_{\mathbb{M}_1,\mathbb{M}_2 \in \mathrm{Irr}(\mathbb{V})} CB_{\mathfrak{X}}(\mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \mathbb{M}_1 \otimes \mathbb{M}_2) \otimes CB_{\mathfrak{X}'}(\overline{\mathbb{M}_1} \otimes \overline{\mathbb{M}_2} \otimes \mathbb{W}_3) \longrightarrow CB_{\mathfrak{X}\#\mathfrak{X}'}(\mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \mathbb{W}_3)$$

defined by contraction $\varphi_1 \otimes \varphi_2 \mapsto \varphi_1 \# \varphi_2$. $\mathfrak{S}$ is called the sewing isomorphism.



CB unitarity $\Leftrightarrow$ rigorous construction of diagonal full CFT

CB unitarity conjecture: There exist inner products on spaces of CB satisfying several conditions including the (projective) unitarity of sewing and moduli-variation. More precisely:

- The sewing isomorphism $\mathfrak{S}$ is (projectively) unitary.
- The variation of CB w.r.t. the moduli of $\mathfrak{X}$ does not change the inner product of $CB_{\mathfrak{X}}$ (up to $\cdot\mathbb{R}_{>0}$) .

This conjecture should be formulated in the setting of diagonal CFT: Physics suggests that we define $\Phi_{\mathfrak{X}} = \sum_{\varphi\in\mathrm{ONB}(CB_{\mathfrak{X}})} \varphi\otimes\varphi^*$ (where ONB denotes orthonormal basis under this inner product). Then

- Unitarity of sewing $\Longleftrightarrow \Phi_{\mathfrak{X}_1}\#\Phi_{\mathfrak{X}_2} = \Phi_{\mathfrak{X}_1\#\mathfrak{X}_2}$ (up to $\cdot\mathbb{R}_{>0}$) .
- Unitarity of moduli-variation $\Longleftrightarrow$ Varying $\Phi_{\mathfrak{X}}$ w.r.t. the moduli of $\mathfrak{X}$ by varying CB still gives a unique (up to $\cdot\mathbb{R}_{>0}$) correlation function.

Worldsheets and their doubles in full-boundary CFT

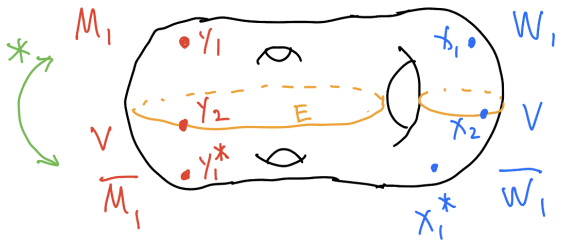
Let $\mathfrak{D}(\mathfrak{X}) = (y_1, \ldots, y_M | C, * | x_1, \ldots, x_N)$ where $y_\bigstar, x_\bullet$ are distinct incoming and outgoing marked points, equipped with local coordinates $\mu_\bigstar, \eta_\bullet$. The involution $* : C \to C$ is an anti-biholomorphism satisfying:

- The (possibly empty) physical boundary $E = \{p \in C : p^* = p\}$ is a real curve giving $C \backslash E = C_+ \sqcup C_-$ where $* : C_+ \to C_-$ is an anti-biholomorphism of open subsets of C.
- $*$ preserves the sets $\{y_\bigstar\}$ and $\{x_\bullet\}$. So $x_i^* = x_{i*}$, $y_j^* = y_{j*}$ for some $1 \leqslant i^* \leqslant N, 1 \leqslant j^* \leqslant M$.
- η_{i*} equals η_i^* where $\eta_i^*(p) = \overline{\eta_i(p^*)}$. Similarly, $\mu_{j*} = \mu_j^*$.

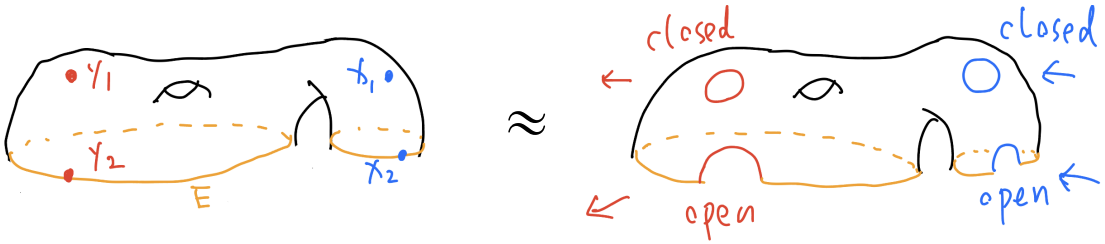
Let $\mathfrak{X}$ be $C_+ \cup E$ together with all the marked points on it and their local coordinates. Then $\mathfrak{X}$ is viewed as a worldsheet of full-boundary CFT. $\mathfrak{D}(\mathfrak{X})$ is the double of $\mathfrak{X}$. E is viewed as the lines swept by the boundaries of open strings.

Correlation functions on $\mathfrak{X}$ are CB on the double $\mathfrak{D}(\mathfrak{X})$

$$\mathfrak{D}(\mathfrak{X}) = \mathfrak{X}\#\mathfrak{X}^* =$$



$$\mathfrak{X} =$$



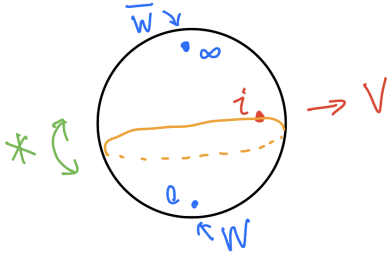
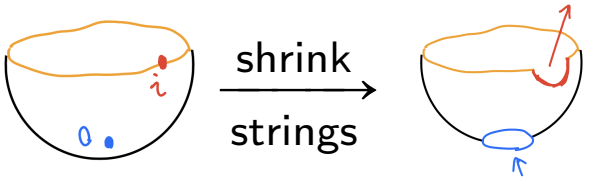
The state spaces are $\mathbb{X} = \bigoplus \mathbb{W} \otimes \overline{\mathbb{W}}$ for x_1, y_1 and $\mathbb{V}$ for x_2, y_2. Restrict $\mathbb{X}$ to $\mathbb{W}_1 \otimes \overline{\mathbb{W}}_1$ for x_1 and $\mathbb{M}_1 \otimes \overline{\mathbb{M}_1}$ for y_1. Then the correlation function $\Phi_{\mathfrak{X}} : \mathbb{M}_1 \otimes \overline{\mathbb{M}}_1 \otimes \mathbb{V} \otimes \mathbb{W}_1 \otimes \overline{\mathbb{W}_1} \otimes \mathbb{V} \to \mathbb{C}$ is a CB for $\mathfrak{D}(\mathfrak{X})$.

- The directions (in or out) of open strings do not affect $\Phi_{\mathfrak{X}}$.
- Changing an incoming closed string (say x_1) to outgoing one results in multiplying the restricted $\Phi_{\mathfrak{X}}$ by the entry $S_{\mathbb{V},\mathbb{W}_1} > 0$ of the modular S-matrix. And $1/S_{\mathbb{V},\mathbb{M}_1}$ if changing y_1 from out to in.

Examples

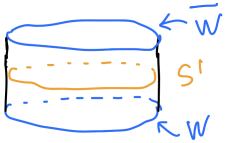
Let $\zeta : z \in \mathbb{P}^1 \mapsto z \in \mathbb{P}^1$ be the standard coordinate of $\mathbb{C}$. Let $\mathbb{W} \in \mathrm{Irr}(\mathbb{V})$.

- Let $\mathfrak{D}(\mathfrak{P}) = (\mathbf{i}|\mathbb{P}^1, *|0, \infty)$ with local coordinates $\varpi = \frac{\mathbf{i}(\zeta-\mathbf{i})}{\zeta+\mathbf{i}}$, ζ, $1/\zeta$, and $z^* = 1/\overline{z}$. Then $\varpi^* = \varpi$, $\zeta^* = 1/\zeta$, $(1/\zeta)^* = \zeta$. So

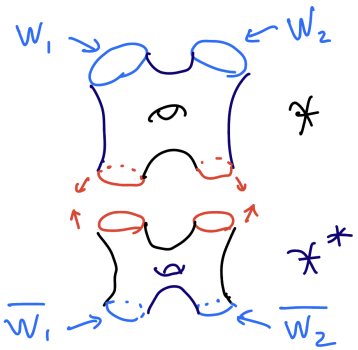
  $\mathfrak{D}(\mathfrak{P}) =$  and $\mathfrak{P} =$ 

. The correlation function $\Phi_{\mathfrak{P}} : \mathbb{V} \otimes \mathbb{W} \otimes \overline{\mathbb{W}} \to \mathbb{C}$ is given by

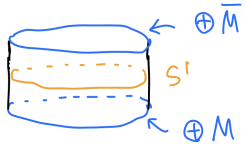
$$\Phi_{\mathfrak{P}}(v \otimes w_1 \otimes \overline{w_2}) = \langle Y_{\mathbb{W}}(2^{L_0} e^{-\mathbf{i}L_1} v, \mathbf{i}) w_1 | w_2 \rangle$$

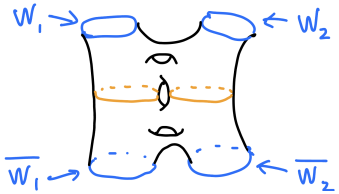
- $\mathfrak{D}(\mathfrak{A}) = (\mathbb{P}^1, *|0, \infty) =$  with local coordinates $\zeta, 1/\zeta$ and $z^* = 1/\overline{z}$ is a thin annulus around the physical boundary S^1. Then $\Phi_{\mathfrak{A}} : \mathbb{W} \otimes \overline{\mathbb{W}} \to \mathbb{C}$ sends $w_1 \otimes \overline{w_2}$ to $\langle w_1 | w_2 \rangle$.

Inner products on spaces of CB

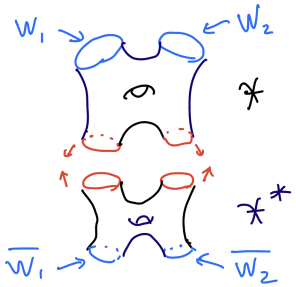
$\mathfrak{D}(\mathfrak{X}) =$ 

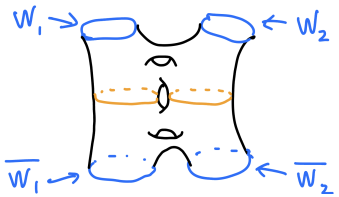
is the double of a full CFT worldsheet $\mathfrak{X}$. Recall that we want $\Phi_{\mathfrak{X}} = \bigoplus_{\mathbb{M}_1,\mathbb{M}_2\in\mathrm{Irr}(\mathbb{V})} \bigoplus_{\varphi\in\mathrm{ONB}(CB_{\mathfrak{X}}(\mathbb{W}_1\otimes\mathbb{W}_2\otimes\mathbb{M}_1\otimes\mathbb{M}_2))} \varphi\otimes\varphi^*$. Glue the four red circles with the blue circles of $\mathfrak{D}(\mathfrak{A}) \sqcup \mathfrak{D}(\mathfrak{A})$ where

$\mathfrak{D}(\mathfrak{A}) =$ 

. We get $\mathfrak{D}(\mathfrak{X})\#\mathfrak{D}(\mathfrak{A}) = \mathfrak{D}(\widetilde{\mathfrak{X}}) =$ 

(where $\widetilde{\mathfrak{X}}$ is almost equal to $\mathfrak{X}$ except that its outgoing strings (red) are turned to physical boundaries (orange)). This implies the formula for $\Phi_{\widetilde{\mathfrak{X}}}$, which we now write in an amusing way:

$\mathfrak{D}(\mathfrak{X}) =$ 

and $\mathfrak{D}(\mathfrak{X})\#\mathfrak{D}(\mathfrak{A}) = \mathfrak{D}(\widetilde{\mathfrak{X}}) =$ 

. Define the **fusion product**

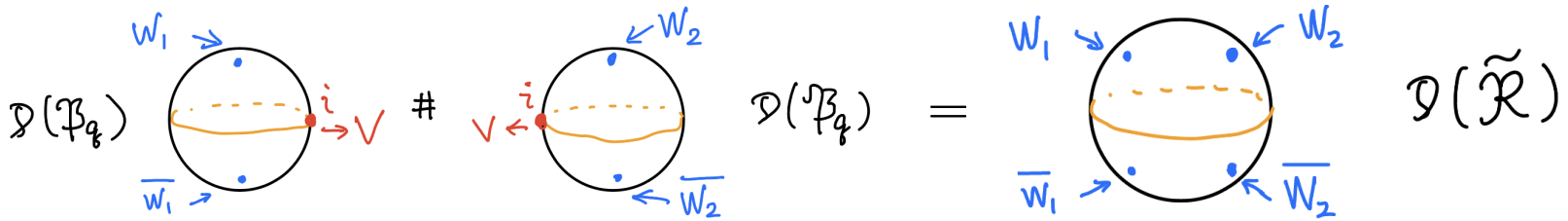
$$\boxtimes_{\mathfrak{X}}(\mathbb{W}_1 \otimes \mathbb{W}_2) = \bigoplus_{\mathbb{M}_1,\mathbb{M}_2} \overline{\mathbb{M}_1} \otimes \overline{\mathbb{M}_2} \otimes CB_{\mathfrak{X}}(\mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \mathbb{M}_1 \otimes \mathbb{M}_2)^*$$

Then $\Upsilon_{\mathfrak{X}} : \mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \overline{\boxtimes_{\mathfrak{X}}(\mathbb{W}_1 \otimes \mathbb{W}_2)} \to \mathbb{C}$ sending $w_1 \otimes w_2 \otimes m_1 \otimes m_2 \otimes \varphi$ to $\varphi(w_1 \otimes w_2 \otimes m_1 \otimes m_2)$ is a CB associated to $\mathfrak{X}$ and can be viewed as a bounded linear map $\mathcal{Y}_{\mathfrak{X}} : \mathbb{W}_1 \otimes \mathbb{W}_2 \to \mathcal{H}(\boxtimes_{\mathfrak{X}}(\mathbb{W}_1 \otimes \mathbb{W}_2))$ where $\mathcal{H}(\cdots)$ is the Hilbert space completion of $\cdots$. Defining inner products on $CB_{\mathfrak{X}}$ amounts to defining unitary structures on the $\mathbb{V} \otimes \mathbb{V}$-module $\boxtimes_{\mathfrak{X}}(\mathbb{W}_1 \otimes \mathbb{W}_2)$, which must satisfy $\langle \mathcal{Y}_{\mathfrak{X}}(w_1 \otimes w_2) | \mathcal{Y}_{\mathfrak{X}}(w_1' \otimes w_2') \rangle = \Phi_{\widetilde{\mathfrak{X}}}(w_1 \otimes w_2 \otimes \overline{w_1'} \otimes \overline{w_2'})$.

- What if there is another obvious way to compute $\Phi_{\widetilde{\mathfrak{X}}}$? Then this will tell us how to define the unitary structure on $\boxtimes_{\mathfrak{X}}(\mathbb{W}_1 \otimes \mathbb{W}_2)$! Let's see an example:

Inner products on spaces of CB for trinions

- Let $0 < q < 1$. Take $\mathfrak{D}(\mathfrak{P}_q) = (\mathbf{i}|\mathbb{P}^1, *|0, \infty)$ with local coordinates $q^{-1}\varpi = \frac{\mathbf{i}(\zeta-\mathbf{i})}{q(\zeta+\mathbf{i})}$, ζ, $1/\zeta$, and $z^* = 1/\overline{z}$. Then $\Phi_{\mathfrak{P}_q} : \mathbb{V} \otimes \mathbb{W} \otimes \overline{\mathbb{W}} \to \mathbb{C}$ satisfies $\Phi_{\mathfrak{P}_q}(v \otimes w \otimes \overline{w'}) = \langle Y_{\mathbb{W}}(2^{L_0} e^{-\mathbf{i}L_1} q^{L_0} v, \mathbf{i}) w | w' \rangle$.
- The sewing of two pieces of $\mathfrak{D}(\mathfrak{P}_q)$ along the two $\mathbf{i}$ is $\mathfrak{D}(\widetilde{\mathfrak{R}})$:

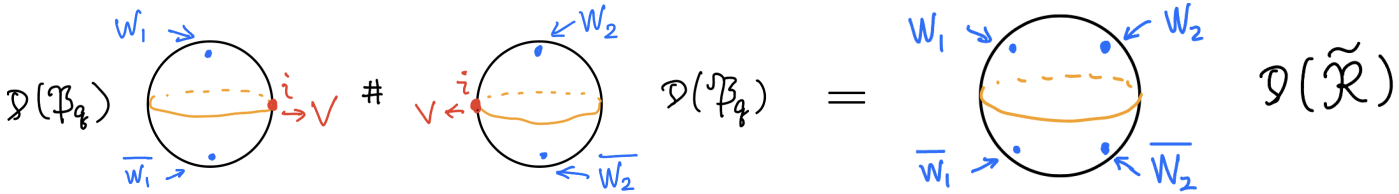


So $\Phi_{\widetilde{\mathfrak{R}}} : \mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \overline{\mathbb{W}_1} \otimes \overline{\mathbb{W}_2} \to \mathbb{C}$ equals the constraction

$$\Phi_{\mathfrak{P}_q} \# \Phi_{\mathfrak{P}_q} = \Phi_{\mathfrak{P}_q}(- \otimes w_1 \otimes \overline{w_1'}) \cdot \Phi_{\mathfrak{P}_q}(- \otimes w_2 \otimes \overline{w_2'})$$

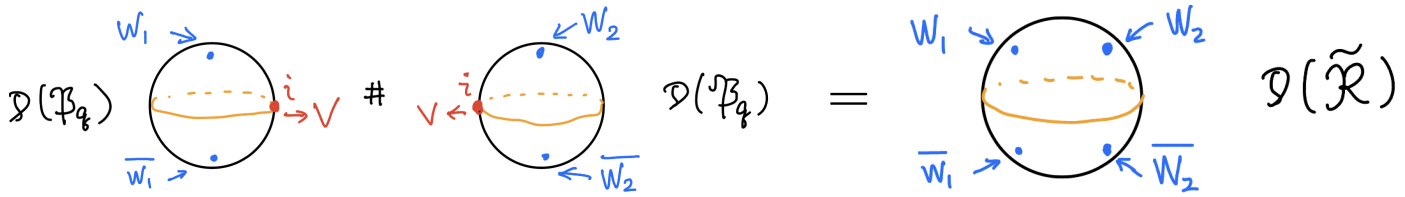
- Choose arbitrary inner products on CB. By the sewing theorem, there is a (bounded) $A \in \mathrm{End}_{\mathbb{V}}(\boxtimes_{\mathfrak{R}}(\mathbb{W}_1 \otimes \mathbb{W}_2))$ satisfying

$$\langle A \cdot \mathcal{Y}_{\mathfrak{R}}(w_1 \otimes w_2) | \mathcal{Y}_{\mathfrak{R}}(w_1' \otimes w_2') \rangle = \Phi_{\mathfrak{P}_q}(- \otimes w_1 \otimes \overline{w_1'}) \cdot \Phi_{\mathfrak{P}_q}(- \otimes w_2 \otimes \overline{w_2'})$$



$$\langle A \cdot \mathcal{Y}_{\mathfrak{R}}(w_1 \otimes w_2) | \mathcal{Y}_{\mathfrak{R}}(w_1' \otimes w_2') \rangle = \Phi_{\mathfrak{P}_q}(- \otimes w_1 \otimes \overline{w_1'}) \cdot \Phi_{\mathfrak{P}_q}(- \otimes w_2 \otimes \overline{w_2'})$$

- $A \in \mathrm{End}_{\mathbb{V}}(\boxtimes_{\mathfrak{R}}(\mathbb{W}_1 \otimes \mathbb{W}_2))$ is invertible by the rigidity of $\mathrm{Mod}(\mathbb{V})$ (Huang 08). It is not hard to show that $A^* = A$.
- If one can prove $A \geqslant 0$, then one can choose the unique unitary structure on the module $\boxtimes_{\mathfrak{R}}(\mathbb{W}_1 \otimes \mathbb{W}_2)$ such that $A = 1$. This gives the desired inner products on spaces of CB for trinions. Since we want the gluing isomorphism $\mathfrak{G}$ to be projectively unitary, we can use this to define the desired inner products on spaces of CB for all $\mathfrak{X}$.



$$\langle A \cdot \mathcal{Y}_{\mathfrak{R}}(w_1 \otimes w_2) | \mathcal{Y}_{\mathfrak{R}}(w_1' \otimes w_2') \rangle = \Phi_{\mathfrak{P}_q}(- \otimes w_1 \otimes \overline{w_1'}) \cdot \Phi_{\mathfrak{P}_q}(- \otimes w_2 \otimes \overline{w_2'})$$

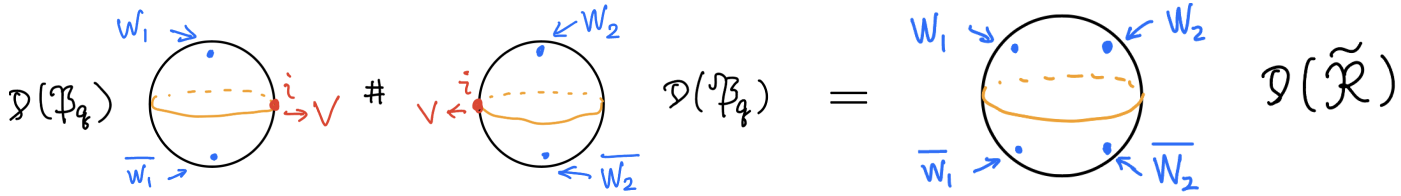
- That $A \geqslant 0$ is equivalent to the geometric positivity property:

$$\sum_{i,j=1}^{n} \Phi_{\mathfrak{P}_q}(- \otimes w_{1,i} \otimes \overline{w_{1,j}}) \cdot \Phi_{\mathfrak{P}_q}(- \otimes w_{2,i} \otimes \overline{w_{2,j}}) \geqslant 0$$

  (Proof: The LHS is $\langle A\xi | \xi \rangle$ where $\xi = \sum_i \mathcal{Y}_{\mathfrak{R}}(w_{1,i} \otimes w_{2,i})$ form a dense subspace of $\mathcal{H}(\boxtimes_{\mathfrak{R}}(\mathbb{W}_1 \otimes \mathbb{W}_2))$.)

- When $\mathbb{W}_1$ or $\mathbb{W}_2$ is a simple current so that $\boxtimes_{\mathfrak{R}}(\mathbb{W}_1 \otimes \mathbb{W}_2)$ is irreducible and hence $A \in \mathbb{R}$, the geometric positivity is easy to prove. (Proof: If not, then $A \in \mathbb{R}_{\leqslant 0}$, so the above contraction is $\leqslant 0$ for all q. But one computes that when $q \to 0$, it converges to a number > 0 .)

See the arXiv paper for details of the proof.

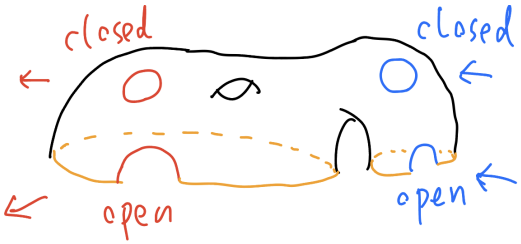


Geo. Pos.: $$\sum_{i,j=1}^{n} \Phi_{\mathfrak{P}_q}(- \otimes w_{1,i} \otimes \overline{w_{1,j}}) \cdot \Phi_{\mathfrak{P}_q}(- \otimes w_{2,i} \otimes \overline{w_{2,j}}) \geqslant 0$$

- When $\mathbb{W}_1, \mathbb{W}_2$ are not simple currents, the above method fails.
- There is a version of algebraic positivity (G. 19) defined by certain brading/fusion matrices for intertwining operators (conformal blocks associated to $(\infty|\mathbb{P}^1|0, z)$ with local coordinates $1/\zeta, \zeta, \zeta - z$). This positivity was proved for many well known examples, e.g. all WZW models (G., Tener 19), using methods from subfactors.
- It was proved in the arXiv paper that the algebraic and the geometric positivities are equivalent.
- Subfactor methods do not apply directly to many simple current examples (e.g. cyclic orbifolds of a unitary holomorphic VOA). So these two methods complement each other.

Conjectures

- Give a rigorous construction of unitary diagonal full-boundary CFT. The CB unitarity conjecture will be its consequence.

Take $\mathfrak{X} =$ 

for example. Recall $\mathbb{X} = \bigoplus \mathbb{W} \otimes \mathbb{W}$.

Recall $\mathbb{V} \simeq \overline{\mathbb{V}}$ and $\mathbb{X} \simeq \overline{\mathbb{X}}$. Then $\Phi_{\mathfrak{X}} : \mathbb{X} \otimes \mathbb{V} \otimes \mathbb{X} \otimes \mathbb{V} \to \mathbb{C}$ gives a bounded linear map between the Hilbert space completions $T_{\mathfrak{X}} : \mathcal{H}(\mathbb{X}) \otimes \mathcal{H}(\mathbb{V}) \to \mathcal{H}(\mathbb{X}) \otimes \mathcal{H}(\mathbb{V})$ whose source and target are resp. the Hilbert space for the incoming and outgoing strings.

- Conjecture: $T_{\mathfrak{X}}$ is **completely positive** in the following sense:

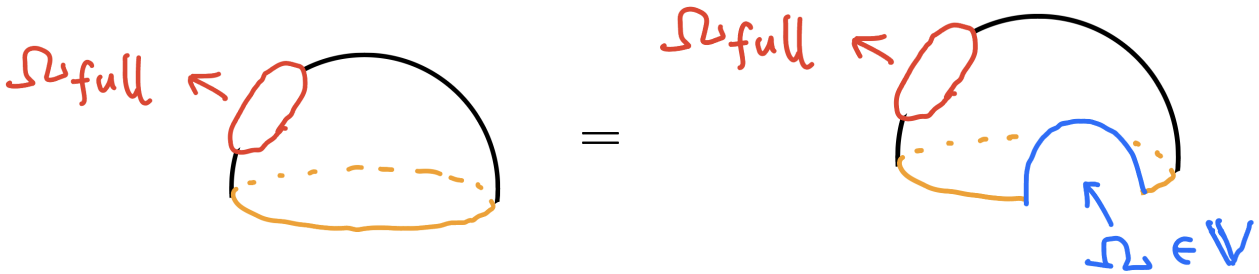
Complete positivity

- Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with cyclic separating vector Ω. Let J be the modular operator. Then the natural cone $\mathcal{H}^\natural = \{xJx\Omega : x \in \mathcal{H}\}^{\mathrm{cl}}$ is a convex cone and is self dual in the sense that for each $\xi \in \mathcal{H}$ we have $\langle \xi | \mathcal{H}^\natural \rangle \geqslant 0$ iff $\xi \in \mathcal{H}^\natural$.
  (Connes 74, Haagerup 75. Cf. Takesaki's book IX.1)
- For $(\mathcal{M}_i, \mathcal{H}_i, \Omega_i)$ where $i = 1, 2$, we understand $\mathcal{H}_1 \otimes \mathcal{H}_2$ as $(\mathcal{M}_1 \otimes \mathcal{M}_2, \mathcal{H}_1 \otimes \mathcal{H}_2, \Omega_1 \otimes \Omega_2)$.
- The VN algebra of the Hilbert space $M_n(\mathbb{C})$ is the left multiplication matrices. Ω is 1.
- We say that a bounded linear $T : \mathcal{H}_1 \to \mathcal{H}_2$ is positive if $T\mathcal{H}_1^\natural \subset \mathcal{H}_2^\natural$. We say that T is completely positive if $T \otimes 1 : \mathcal{H}_1 \otimes M_n(\mathbb{C}) \to \mathcal{H}_2 \otimes M_n(\mathbb{C})$ is positive for all n.

Let $\mathcal{A}$ be the conformal net for $\mathbb{V}$ defined by $\mathcal{A}(I) = \mathrm{VN}\{Y(v,f) : v \in \mathbb{V}, f \in C_c^\infty(I)\}$ where $I \subset S^1$ is an open interval (Carpi-Kawahigashi-Longo-Weiner 18).

Take $S^1_+ = S^1 \cap \{a + b\mathbf{i} : b > 0\}$. Recall $\mathbb{X} = \bigoplus_{\mathbb{W}} \mathbb{W} \otimes \overline{\mathbb{W}}$ and hence $\mathcal{H}(\mathbb{X}) = \bigoplus_{\mathbb{W}} \mathcal{H}(\mathbb{W}) \otimes \mathcal{H}(\overline{\mathbb{W}})$.

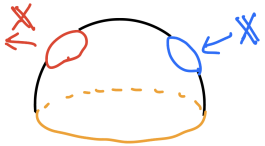
- Take $(\mathcal{A}(S^1_+), \mathcal{H}(\mathbb{V}), \Omega)$ and $(\bigoplus_{\mathbb{W}} B(\mathcal{H}(\mathbb{W})) \otimes 1, \mathcal{H}(\mathbb{X}), \Omega_{\mathrm{full}})$. Here $\Omega_{\mathrm{full}} = \bigoplus_{\mathbb{W}} S_{\mathbb{V},\mathbb{W}} \sum_i q^{L_0} e_{\mathbb{W},i} \otimes \overline{e_{\mathbb{W},i}}$ where $0 < q < 1$ and $\{e_{\mathbb{W},i}\}$ is a homogeneous ONB of $\mathbb{W}$. (The standard cone is independent of the choice of q.) We conjecture the complete positivity of $T_{\mathfrak{X}}$.



Let $\mathcal{A}$ be the conformal net for $\mathbb{V}$ defined by $\mathcal{A}(I) = \mathrm{VN}\{Y(v,f) : v \in \mathbb{V}, f \in C_c^\infty(I)\}$ where $I \subset S^1$ is an open interval (Carpi-Kawahigashi-Longo-Weiner 18). $\mathcal{H}(\mathbb{X}) = \bigoplus_{\mathbb{W}} \mathcal{H}(\mathbb{W}) \otimes \mathcal{H}(\overline{\mathbb{W}})$.

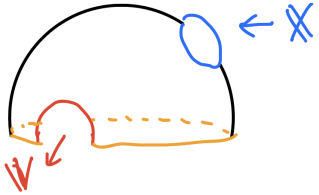
- Take $(\mathcal{A}(S^1_+), \mathcal{H}(\mathbb{V}), \Omega)$ and $(\bigoplus_{\mathbb{W}} B(\mathcal{H}(\mathbb{W})) \otimes 1, \mathcal{H}(\mathbb{X}), \Omega_{\mathrm{full}})$. Here $\Omega_{\mathrm{full}} = \bigoplus_{\mathbb{W}} S_{\mathbb{V},\mathbb{W}} \sum_i q^{L_0} e_{\mathbb{W},i} \otimes \overline{e_{\mathbb{W},i}}$ where $0 < q < 1$ and $\{e_{\mathbb{W},i}\}$ is a homogeneous ONB of $\mathbb{W}$. We conjecture the complete positivity of $T_{\mathfrak{X}}$.

Why this conjecture?

- The closed cone $(\mathcal{H}(\mathbb{X}) \otimes M_n(\mathbb{C}))^\natural$ is generated by (i.e. has dense subset $\mathbb{Z}_+$-spanned by) $\sum_{i,j=1}^n w_i \otimes \overline{w}_j \otimes e_{i,j}$ where $\mathbb{W} \in \mathrm{Irr}(\mathbb{V})$ and $w_i, w_j \in \mathbb{W}$. Take $\widetilde{\mathfrak{R}} =$ 

. Then the complete positivity of $T_{\widetilde{\mathfrak{R}}} : \mathcal{H}(\mathbb{X}) \to \mathcal{H}(\mathbb{X})$ is equivalent to the geometric positivity
$$\sum_{i,j=1}^n \Phi_{\mathfrak{P}_q}(- \otimes w_{1,i} \otimes \overline{w_{1,j}}) \cdot \Phi_{\mathfrak{P}_q}(- \otimes w_{2,i} \otimes \overline{w_{2,j}}) \geqslant 0.$$

Why this conjecture?

- $\mathcal{H}(\mathbb{X})^\natural$ is generated by $w \otimes \overline{w}$ where $w \in \mathbb{W}$. Take $\xi = v \otimes \overline{v}$ where $v \in \mathbb{V}$ for simplicity.
  Fix $\gamma \in S^1$. Take $\mathfrak{D}(\mathfrak{X}) = (\infty|\mathbb{P}^1, *|\gamma, \overline{\gamma})$ with coordinates $1/\zeta, \zeta - \gamma, \zeta - \overline{\gamma}$ and $z^* = \overline{z}$. Then $\mathfrak{X} =$ 

. Then $T_{\mathfrak{X}}$ defined on $\mathcal{H}(\mathbb{X})$ satisfies

$$T_{\mathfrak{X}}(\xi) = Y(v,\gamma)Y(\overline{v},\overline{\gamma})\Omega = Y(v,\gamma)\overline{Y(v,\gamma)\Omega}$$

understood in terms of analytic continuation. By Bisognano-Wichmann property, we have

$$T_{\mathfrak{X}}(\xi) = Y(v,\gamma)JY(v,\gamma)\Omega$$

which has formal similarity with $xJx\Omega$ in the definition of $\mathcal{H}(\mathbb{V})^\natural$.

Happy birthday, Yasu!