

# Minkowskian open/closed conformal field theory possibly without vacuum: the Cardy case

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I Geometric setup

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II Boundary fields and open string state spaces

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III Bulk fields and closed string state space

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IV Bulk fields acting on open string state spaces

Part I

# Geometric Setup

# Closed and Open Strings

## Closed and open strings

2D CFT studies the propagation and interaction of closed strings

$$S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$$

and open strings

$$[0, \pi].$$

The corresponding Minkowski space-times are

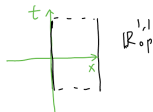
$$\mathbb{R}_{\text{cl}}^{1,1} = S^1 \times \mathbb{R}, \quad \mathbb{R}_{\text{op}}^{1,1} = [0, \pi] \times \mathbb{R}.$$

$$(x, t) \in \mathbb{R}_{\text{cl}}^{1,1} \text{ or } \mathbb{R}_{\text{op}}^{1,1},$$

$$(ds)^2 = (dt)^2 - (dx)^2,$$

$$u^+ = t + x, \quad u^- = t - x,$$

$$(ds)^2 = du^+ du^-.$$



# Conformal Diffeomorphisms

Let  $\text{Diff}_+(S^1)$  be the group of orientation-preserving diffeomorphisms of  $S^1$ , and let

$$\mathcal{G} = \widetilde{\text{Diff}_+(S^1)}$$

be its universal cover. Here

$$\text{Cf}^+(\mathbb{R}_{\text{cl}}^{1,1}), \text{Cf}^+(\mathbb{R}_{\text{op}}^{1,1})$$

denote orientation-preserving, time-direction-preserving conformal diffeomorphism groups. Elements  $F \in \mathcal{G}$  can be viewed as smooth functions  $F : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$F(x + 2\pi) = F(x) + 2\pi, \quad F'(x) > 0.$$

In light-cone coordinates, the relevant actions are

$$(g_1, g_2)(u^+, u^-) = (g_1 u^+, g_2 u^-) \quad \text{on } \mathbb{R}_{\text{cl}}^{1,1},$$

$$g(u^+, u^-) = (g u^+, g u^-) \quad \text{on } \mathbb{R}_{\text{op}}^{1,1}.$$

Both actions are well-defined. This allows us to classify  $\text{Cf}^+(\mathbb{R}_{\text{cl}}^{1,1})$  and  $\text{Cf}^+(\mathbb{R}_{\text{op}}^{1,1})$ .

The action of  $\mathcal{G} \times \mathcal{G}$  on  $\mathbb{R}_{\text{cl}}^{1,1}$  gives the covering exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{G} \times \mathcal{G} \longrightarrow \text{Cf}^+(\mathbb{R}_{\text{cl}}^{1,1}) \longrightarrow 1.$$

### Kernel

The subgroup  $\mathbb{Z}$  is generated by

$$(\rho(2\pi), \rho(-2\pi)) \in \mathcal{G} \times \mathcal{G},$$

where

$$\rho : \mathbb{R} \longrightarrow \mathcal{G}$$

is the one-parameter rotation subgroup.

Let  $\mathcal{G}$  act on  $\mathbb{R}_{\text{op}}^{1,1}$  by

$$g(u^+, u^-) = (gu^+, gu^-).$$

### Classification

The action is well-defined and gives

$$\mathcal{G} \simeq \text{Cf}^+(\mathbb{R}_{\text{op}}^{1,1}).$$

- Closed CFT uses  $\mathcal{G} \times \mathcal{G}$ .
- Open CFT uses  $\mathcal{G}$ .
- With central charge  $c$ , we use the central extension  $\mathcal{G}_c$ .

Part II

# Boundary Fields and Open String State Spaces

# Adapting Fuchs–Schweigert–Runkel

Adapting Fuchs–Schweigert–Runkel to the operator algebraic approach of CFT, we expect a 2D open/closed CFT with central charge  $c$  as follows.

- Boundary conditions are labeled by indices

$$i, j, k, \dots$$

- The open-string state space with left and right boundary conditions  $i, j$  is

$$\mathcal{H}_{\text{op}(i,j)}.$$

- Boundary operators preserving a boundary condition  $i$  form a generally non-local net

$$\mathcal{B}_{\text{op}(i)} : \tilde{I} \in \tilde{\mathcal{J}} \longmapsto \mathcal{B}_{\text{op}(i)}(\tilde{I}) \subset \mathfrak{L}(\mathcal{H}_{\text{op}(i,i)}).$$

where  $\mathcal{B}_{\text{op}(i)}(\tilde{I})$  is a von Neumann algebra (i.e. a  $*$ -subalgebra closed under strong operator topology).

## Arg-valued intervals

$$\tilde{\mathcal{J}} = \{\tilde{I} : \tilde{I} \text{ is a component over } I \subset S^1\}$$

Let

$$\mathcal{J} = \{I \subset S^1 : I \text{ is non-empty, open, and non-dense}\}.$$

Let

$$\tilde{\mathcal{J}} = \{\tilde{I} : \tilde{I} \text{ is a component of } I \text{ in } \mathbb{R} \rightarrow \mathbb{R}/2\pi\mathbb{Z} = S^1\}.$$

Equivalently,

$$\tilde{\mathcal{J}} = \{\text{open intervals in } \mathbb{R} \text{ of length } 0 < |\tilde{I}| < 2\pi\}.$$

### Non-local boundary net

For a boundary condition  $i$ ,

$$\mathcal{B}_{\text{op}(i)} : \tilde{I} \in \tilde{\mathcal{J}} \mapsto \mathcal{B}_{\text{op}(i)}(\tilde{I}) \subset \mathfrak{L}(\mathcal{H}_{\text{op}(i,i)})$$

where  $\mathcal{B}_{\text{op}(i)}(\tilde{I})$  is a von Neumann algebra.

# Conformal Covariance

The boundary net is conformally covariant. Let  $\mathcal{G}_c$  be the central extension of  $\mathcal{G}$  with central charge  $c$ .

## Covariance

There is a strongly continuous unitary representation of  $\mathcal{G}_c$  on  $\mathcal{H}_{\text{op}(i,i)}$ , indeed on  $\mathcal{H}_{\text{op}(i,j)}$ , such that

$$g \mathcal{B}_{\text{op}(i)}(\tilde{I}) g^{-1} = \mathcal{B}_{\text{op}(i)}(g\tilde{I}), \quad \tilde{I} \in \tilde{\mathcal{J}}.$$

The same convention applies to the right action of  $\mathcal{B}_{\text{op}(i)}$ .

# Left and Right Actions

On the open-string state space  $\mathcal{H}_{\text{op}(i,j)}$ , we have a left action of  $\mathcal{B}_{\text{op}(i)}$  and a right action of  $\mathcal{B}_{\text{op}(j)}$ :

$$\mathcal{B}_{\text{op}(i)}(\tilde{I}) \curvearrowright \mathcal{H}_{\text{op}(i,j)} \curvearrowleft \mathcal{B}_{\text{op}(j)}(\tilde{J}),$$

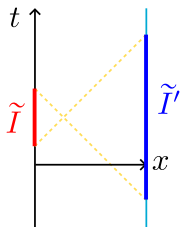
where the first one is the left action and the second one is the right action.

## Boundary-boundary locality

If  $\tilde{J}$  is clockwise to  $\tilde{I}$ , i.e.  $0 < \arg z - \arg \zeta < 2\pi$  for each  $z \in \tilde{I}$  and  $\zeta \in \tilde{J}$ , then

$$[\mathcal{B}_{\text{op}(i)}(\tilde{I}), \mathcal{B}_{\text{op}(j)}(\tilde{J})] = 0.$$

# Boundary-Boundary Haag Duality



For  $\mathcal{H}_{\text{op}(i,j)}$ , the nets  $\mathcal{B}_{\text{op}(i)}$  and  $\mathcal{B}_{\text{op}(j)}$  act on the left and right boundaries.

## Boundary-boundary Haag duality

The following von Neumann algebras are commutants of each other:

$$\mathcal{B}_{\text{op}(i)}(\tilde{I}), \quad \mathcal{B}_{\text{op}(j)}(\tilde{I}').$$

This is called the **boundary-boundary Haag duality**.

Here,  $\tilde{I}'$  is the largest element in  $\tilde{\mathcal{J}}$  clockwise to  $\tilde{I}$ . In particular, if  $\tilde{\mathcal{J}}$  is clockwise to  $\tilde{I}$ , then

$$[\mathcal{B}_{\text{op}(i)}(\tilde{I}), \mathcal{B}_{\text{op}(j)}(\tilde{\mathcal{J}})] = 0.$$

# Morita Equivalent Boundary Conditions

That  $\mathcal{B}_{\text{op}(i)}$  and  $\mathcal{B}_{\text{op}(j)}$  are commutants of each other on  $\mathcal{H}_{\text{op}(i,j)}$  implies that  $\mathcal{B}_{\text{op}(i)}$  and  $\mathcal{B}_{\text{op}(j)}$  are **Morita equivalent**.

This would imply that they produce the same closed CFT. This still needs to be proved in the operator algebra framework.

## Conjugation

$$\overline{\mathcal{H}_{\text{op}(i,j)}} = \mathcal{H}_{\text{op}(j,i)}.$$

## Composition of open strings

$$\mathcal{H}_{\text{op}(i,j)} \boxtimes_{\mathcal{B}_{\text{op}(j)}} \mathcal{H}_{\text{op}(j,k)} \simeq \mathcal{H}_{\text{op}(i,k)}.$$

Here  $\boxtimes$  denotes the Connes fusion.

# Cardy-Type CFT

We say that the CFT is of Cardy type if there is a boundary condition  $0$  such that

$$(\mathcal{B}_{\text{op}(0)}, \mathcal{H}_{\text{op}(0,0)})$$

is a local conformal net

$$(\mathcal{A}, \mathcal{H}_0).$$

This means  $(\mathcal{A}, \mathcal{H}_0)$  is a non-local net satisfying the following conditions:

① single-valuedness on  $S^1$ ;

② locality:

$$I \cap J = \emptyset \quad \Rightarrow \quad [\mathcal{A}(I), \mathcal{A}(J)] = 0;$$

③ a vacuum vector  $\Omega \in \mathcal{H}_0$  fixed by the Mobius group;

④ and some other properties.

# The Local Conformal Net $(\mathcal{A}, \mathcal{H}_0)$

The pair  $(\mathcal{A}, \mathcal{H}_0)$  is a non-local net satisfying the following extra conditions.

## Single-valuedness

If  $\tilde{I}_1, \tilde{I}_2 \in \tilde{\mathcal{J}}$  project to the same interval  $I \in \mathcal{J}$ , then

$$\mathcal{A}(\tilde{I}_1) = \mathcal{A}(\tilde{I}_2).$$

Thus we write

$$I \in \mathcal{J} \mapsto \mathcal{A}(I) \subset \mathfrak{L}(\mathcal{H}_0).$$

## Locality and vacuum

$$I \cap J = \emptyset \quad \Rightarrow \quad [\mathcal{A}(I), \mathcal{A}(J)] = 0.$$

There is a vacuum vector  $\Omega \in \mathcal{H}_0$  fixed by the action of

$$\text{Mob} \subset \mathcal{G},$$

the subgroup of linear fractional transformations preserving  $S^1$ .

# The Cardy-type CFT

Now restrict to the Cardy-type CFT. Set

$$\mathcal{H}_i := \mathcal{H}_{\text{op}(i,0)}.$$

Then  $\mathcal{H}_i$  is a right  $\mathcal{A}$ -module.

## Open-string state space

Since one expects

$$\mathcal{H}_{\text{op}(i,j)} = \mathcal{H}_{\text{op}(i,0)} \boxtimes_{\mathcal{B}_{\text{op}(0)}} \mathcal{H}_{\text{op}(0,j)} = \mathcal{H}_{\text{op}(i,0)} \boxtimes_{\mathcal{B}_{\text{op}(0)}} \overline{\mathcal{H}_{\text{op}(j,0)}},$$

we have

$$\mathcal{H}_{\text{op}(i,j)} = \mathcal{H}_i \boxtimes_{\mathcal{A}} \overline{\mathcal{H}_j}.$$

# Boundary Conditions Labeled by $\mathcal{A}$ -Modules

The boundary conditions in the Cardy case are labeled by right  $\mathcal{A}$ -modules, equivalently solitonic  $\mathcal{A}$ -modules.

## Local modules

For simplicity, in this talk we restrict to local  $\mathcal{A}$ -modules:

$$\text{left action} = \text{right action.}$$

We simply call them  $\mathcal{A}$ -modules.

## Example

If  $\mathcal{A}$  is the Virasoro conformal net, then  $\mathcal{A}$ -modules correspond to unitary representations of the Virasoro algebra.

## Part III

# Bulk Fields and Closed String State Space

# Closed CFT

For each local  $\mathcal{A}$ -module  $\mathcal{H}_i$ , there is a canonical  $\sigma$ -twisted  $\mathcal{A}^{\otimes 2}$ -module structure on the same Hilbert space, where

$$\sigma \in \text{Aut}(\mathcal{A}^{\otimes 2})$$

is the permutation of the two tensor factors.

## Longo–Xu

We denote this twisted module by

$$\mathcal{H}_{\sqrt{i}}.$$

This is the Longo–Xu result (04); a vertex-algebra version appeared in Barron–Dong–Mason (02).

# The State Space for the Closed CFT

The state space for the closed CFT is

$$\mathcal{H}_{\text{cl}} = \mathcal{H}_{\sqrt{0}} \boxtimes_{\mathcal{A}^{\otimes 2}} \overline{\mathcal{H}_{\sqrt{0}}},$$

as an untwisted  $\mathcal{A}^{\otimes 2}$ -module.

## Bulk net

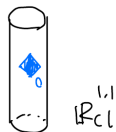
We have a net of von Neumann algebras

$$\mathcal{B}_{\text{cl}} : O \longmapsto \mathcal{B}_{\text{cl}}(O) \subset \mathfrak{L}(\mathcal{H}_{\text{cl}}),$$

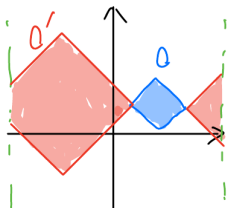
where  $O$  is a double cone in  $\mathbb{R}_{\text{cl}}^{1,1}$ .

The net satisfies properties such as

$\mathcal{G}_c \times \mathcal{G}_c$ -covariance,    locality,    **bulk-bulk Haag duality**.



# Bulk-Bulk Haag Duality



## Single double cone

If  $O'$  is the spacelike complement of  $O$ , then

$$\mathcal{B}_{\text{cl}}(O)' = \mathcal{B}_{\text{cl}}(O').$$

## Multi double-cones

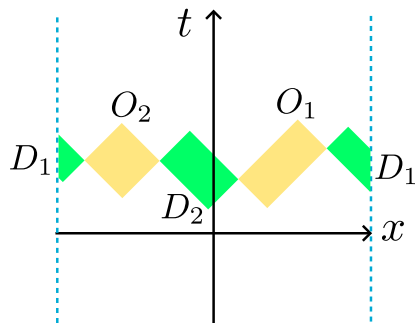
If  $O_1, \dots, O_n$  are mutually spacelike separated and

$$(O_1 \cup \dots \cup O_n)' = D_1 \cup \dots \cup D_n,$$

then

$$(\mathcal{B}_{\text{cl}}(O_1) \vee \dots \vee \mathcal{B}_{\text{cl}}(O_n))' = \mathcal{B}_{\text{cl}}(D_1) \vee \dots \vee \mathcal{B}_{\text{cl}}(D_n).$$

# Two and Multi Double-Cones



## Two double-cones version

In the configuration shown on the left,

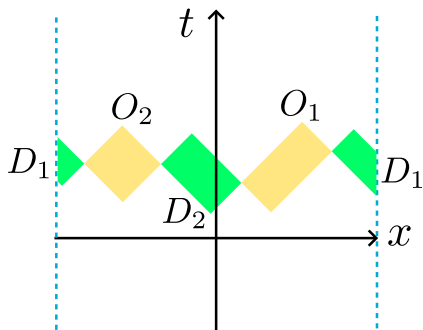
$$(O_1 \cup O_2)' = D_1 \cup D_2,$$

and

$$\begin{aligned} (\mathcal{B}_{\text{cl}}(O_1) \vee \mathcal{B}_{\text{cl}}(O_2))' \\ = \mathcal{B}_{\text{cl}}(D_1) \vee \mathcal{B}_{\text{cl}}(D_2). \end{aligned}$$

The same statement holds for multi double-cones.

# Modular Invariance



In Euclidean CFT, this version of Haag duality corresponds to **modular invariance**.

Multi double-cone version also holds.

## Part IV

# Bulk Fields Acting on Open String State Spaces

## $\mathcal{B}_{\text{cl}}$ Acts on $\mathcal{H}_{\text{op}(i,j)}$

The closed net restricts to the open-string state space

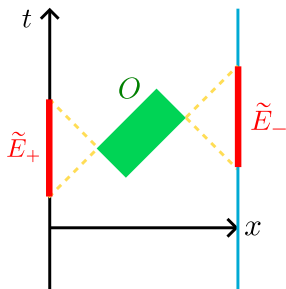
$$\mathcal{H}_{\text{op}(i,j)} = \mathcal{H}_i \boxtimes_{\mathcal{A}} \overline{\mathcal{H}_j}.$$

Thus, for each double cone  $O \subset \text{Int}\mathbb{R}_{\text{op}}^{1,1} = (0, \pi) \times \mathbb{R}$ , we get an action

$$\mathcal{B}_{\text{cl}}(O) \curvearrowright \mathcal{H}_{\text{op}(i,j)}.$$

The action satisfies, among other things, the **bulk-boundary Haag duality**.

# Bulk-Boundary Haag Duality



Let  $\tilde{E}_+$  and  $\tilde{E}_-$  be the two boundary intervals in the spacelike complement of the bulk double cone  $O$ , as in the picture.

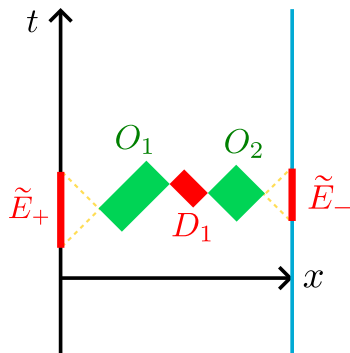
## Bulk-boundary Haag duality

$$\mathcal{B}_{\text{cl}}(O) \quad \text{and} \quad \mathcal{B}_{\text{OP}(i)}(\tilde{E}_+) \vee \mathcal{B}_{\text{OP}(j)}(\tilde{E}_-)$$

are commutants of each other.

This corresponds to the **Cardy consistency condition** in Euclidean CFT.

# Multi Double-Cones in the Strip



The multi double-cone version holds.

## Theorem

For any conformal net  $\mathcal{A}$ , there exists the open/closed CFT satisfying the previously mentioned properties, whose boundary conditions are labeled by local  $\mathcal{A}$ -modules.

**Expectations.** We expect the construction to generalize to:

- non-local  $\mathcal{A}$ -modules;
- or even general non-local nets.

We expect that the open/closed CFT for  $\mathcal{A} = \text{Virasoro conformal net of } c \geq 25$  is the Minkowskian open/closed Liouville CFT.

More generally, Toda CFTs are expected to be Cardy-type CFTs associated with principal  $\mathcal{W}(\mathfrak{g})$  conformal nets at Toda central charge.

# Three Haag Dualities

Haag duality	Spacetime	Interpretation
Bulk-bulk	$\mathbb{R}_{cl}^{1,1}$	Modular invariance in Euclidean spacetimes
Boundary-boundary	$\mathbb{R}_{op}^{1,1}$	Morita equivalence of boundary nets
Bulk-boundary	$\mathbb{R}_{op}^{1,1}$	Cardy consistency condition in Euclidean spacetimes

