Über adjungierte Funktionaloperatoren*
On Adjoint Functional Operators

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Introduction

1. Like several earlier papers,[1] this paper is devoted to operators on Hilbert space. The notation and concept formation follow the author's papers cited in note 1; in particular, the terminology to be used is developed in E., pp. 63–78. Accordingly, an operator A is a function defined on a subset of the Hilbert space $\mathfrak{H}$, with values in $\mathfrak{H}$. In general, nothing further is required of an operator.

The following definitions are essential (cf. the work just cited):

I. An operator A is *defined everywhere* if its domain is all of $\mathfrak{H}$.

II. A is an *extension* of B if the domain of B is a subset of the domain of A, and A and B agree on the former; that is, $Af = Bf$ there. If their domains are equal, then $A = B$; otherwise, A is a *proper extension* of B.

III. A is *linear* if, together with $f_1, \ldots, f_n$, all $a_1 f_1 + \cdots + a_n f_n$ belong to its domain,[2] and

$$A(a_1 f_1 + \cdots + a_n f_n) = a_1 A f_1 + \cdots + a_n A f_n.$$

IV. A is *closed* if, for every sequence $f_1, f_2, \ldots$ in its domain such that f_n converges to some f and Af_n converges to some f^* as $n \to \infty$, the element f also belongs to the domain,[3] and $Af = f^*$.

The operators to be considered will in general not be defined everywhere. A certain minimum extent of the domain will, however, be desirable, namely the

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[1] *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren* [*General Eigenvalue Theory of Hermitian Functional Operators*], *Math. Annalen* **102**, no. 1 (1929), pp. 49–131; *Zur Algebra der Funktionaloperationen und Theorie der normalen Funktionaloperatoren* [*On the Algebra of Functional Operations and the Theory of Normal Functional Operators*], *Math. Annalen* **102**, no. 3 (1929), pp. 370–427; *Zur Theorie der unbeschränkten Matrizen* [*On the Theory of Unbounded Matrices*], *Journal für Math.* **161**, no. 4 (1929), pp. 208–236; *Über Funktionen von Funktionaloperatoren* [*On Functions of Functional Operators*], *Annals of Math.* **32**, no. 2 (1931), pp. 191–226. These papers will be cited repeatedly below, abbreviated respectively as E., A., M., and F.

[2] That is, the domain is a linear manifold.

[3] The conclusion that the domain is a closed set would follow from this only if A were continuous. Conversely, continuity would follow from closedness only if $\mathfrak{H}$ were compact, which it is not. (Cf. E., p. 70, especially n. 29.)

following:

V. A is a $*$-*operator* if the closed linear manifold spanned by its domain is $\mathfrak{H}$,[4] that is, if no $f \neq 0$ is orthogonal to its domain.

We now consider the condition

$$* \qquad (f, Ag) = (f^*, g),$$

where f and f^* are regarded as fixed and g ranges over the entire domain of A. For a given f, there need not be an f^* satisfying $*$; for certain f, however, there is one, for example $f = 0$, $f^* = 0$. The fact that, whenever $*$ is solvable, it has only one solution f^* is characteristic of $*$-operators.[5] For these, and only for these, the following definition can therefore be made:

VI. If A is a $*$-operator, then A^* is the following operator: its domain consists of those f for which $*$ is solvable, and in that case $A^* f = f^*$.

It is immediately apparent that A^*, which, following the usual notion that it generalizes,[6] is called the *adjoint* of A, is linear and closed. In general, however, we do not know whether it is defined anywhere other than at 0, still less whether it is a $*$-operator. It is evident that A^{**}, if it can be formed (that is, if A^* is a $*$-operator), is an extension of A.

These notions permit a simpler description of certain concepts from E.: a Hermitian operator A is a $*$-operator for which A^* is an extension of A; a hypermaximal operator is one for which $A^* = A$ (E., p. 70, Definition 6, and p. 72, Definitions 8 and 9).[7]

2. We shall show that a $*$-operator A can be extended to a closed linear operator B if and only if A^* is also a $*$-operator. Among all such B there is then a smallest one, that is, one of which every other is an extension; denote it by $\widetilde{A}$. We have

$$\widetilde{A}^* = A^*, \qquad \widetilde{A} = A^{**}.$$

Another criterion is as follows. Suppose there is a sequence of linear aggregates

$$a_1^{(\nu)} f_1^{(\nu)} + \cdots + a_{n_\nu}^{(\nu)} f_{n_\nu}^{(\nu)} \qquad (\nu = 1, 2, \ldots;\ f_1^{(\nu)}, \ldots, f_{n_\nu}^{(\nu)} \text{ in the domain})$$

such that the aggregates themselves tend to 0, while

$$a_1^{(\nu)} A f_1^{(\nu)} + \cdots + a_{n_\nu}^{(\nu)} A f_{n_\nu}^{(\nu)} \longrightarrow f^* \qquad (\nu \to \infty).$$

[4]This was required in E., p. 70, Definition 6, for Hermitian operators, and in A., pp. 405–406, Definitions 5 and 6, for normal operators.

[5]For from one solution f^* one obtains all the others as $f^* + h$, where h ranges over all elements of $\mathfrak{H}$ orthogonal to the domain of A.

[6]Cf. E., pp. 111–112, especially n. 64; A., p. 372, n. 12, and p. 405, Definition 5.

[7]The equation $A^* = A$ agrees with the term "self-adjoint" used by M. Stone, *Proc. Nat. Acad.* **16** (1930), pp. 173–174, for hypermaximal operators.

Then $f^* = 0$. If A is linear, this has the simpler form: if $f_1, f_2, \ldots$ lie in the domain, $f_n \longrightarrow 0$, and $Af_n \longrightarrow f^*$ as $n \longrightarrow \infty$, then $f^* = 0$.[8]

These operators are of particular interest:

VII. A is a $**$-*operator* if both A and A^* are $*$-operators.

If A is a $**$-operator, then $\widetilde{A}$, since it extends A and $\widetilde{A}^* = A^*$, is likewise a $*$-operator. Thus $\widetilde{A}$ is linear and closed, is a $**$-operator, extends A, and has the same adjoint as A. We therefore obtain a complete survey of all $**$-operators by considering only the closed linear ones among them. By what was just said, these in turn are identical with the closed linear $*$-operators.

Henceforth we shall therefore always consider closed linear $*$-operators. If A is such an operator, then, by the foregoing, A^* is one as well, and

$$A^{**} = A \qquad \left(\text{since } \widetilde{A} = A\right).$$

Now consider the operator A^*A, defined as follows: if f lies in the domain of A and Af lies in the domain of A^*, then A^*Af is defined and is equal to $A^*(Af)$.

From the definition it is clear that A^*A0 is defined and equals 0, and also that A^*A is linear; its closedness is likewise easy to prove.[9] If A^*A is a $*$-operator, it is plainly Hermitian and definite.[10] For the moment, however, there is no way to see whether the domain of A^*A contains any elements other than 0, or even whether it is a $*$-operator. This would be clear only if A and A^* were defined everywhere; but, as we shall show by adapting a familiar theorem of Toeplitz, that happens only when A and A^* are continuous.

Nevertheless, we shall succeed in proving in full generality that A^*A is a $*$-operator, indeed a hypermaximal one. Replacing A by A^* shows that the same holds for AA^*.

It is useful to illustrate these relations by some examples. Realize Hilbert space as the space of all measurable functions $f(x)$ on the interval $0 \le x \le 1$ for which

$$\int_0^1 |f(x)|^2\, dx < \infty$$

[8] Continuity does not follow from this, because the existence of the limit f^* was assumed. Cf. note 3.

[9] From $f_n \longrightarrow f$ and $A^*Af_n \longrightarrow f^*$ it follows that, for $m, n \longrightarrow \infty$, $A^*Af_m - A^*Af_n \longrightarrow 0$, and hence

$$\begin{aligned}&(A^*Af_m - A^*Af_n, f_m - f_n)\\ &\qquad = (Af_m - Af_n, Af_m - Af_n) = \|Af_m - Af_n\|^2 \longrightarrow 0.\end{aligned}$$

Thus Af_m has a limit, say f'. Since A is closed, Af is defined and equals f'; since A^* is closed, A^*f' is defined and equals f^* (put $Af_n = f'_n$). Consequently A^*Af is defined and equals f^*.

[10] $(A^*Af, f) = (Af, Af) = \|Af\|^2$ is real and nonnegative; cf. E., p. 70, Definition 6, and p. 74, Definition 10.

(cf., for example, E., pp. 108–109). Let $\mathfrak{A}$ be the set of all continuous differentiable functions $f(x)$ for which

$$\int_0^1 |f'(x)|^2\, dx < \infty,$$

and which satisfy certain boundary conditions to be specified immediately; define A on $\mathfrak{A}$ by $Af = if'$. We allow the following boundary conditions:

1. No boundary conditions.
2. $f(1) : f(0) = C$, where C is a prescribed number.
3. $f(1) = f(0) = 0$.

Correspondingly, we speak of

$$\mathfrak{A}_1, A_1; \qquad \mathfrak{A}_{2,C}, A_{2,C}; \qquad \mathfrak{A}_3, A_3.$$

The operators $A_1, A_{2,C}, A_3$ are linear, nonclosed $*$-operators; A_1 extends $A_{2,C}$, and the latter extends A_3.[T1] No $A_{2,C}$ is extended by any other $A_{2,D}$. From

$$\begin{aligned}(Af, g) - (f, Ag) &= i \int_0^1 \left(f'(x)\overline{g(x)} + f(x)\overline{g'(x)} \right) dx \\ &= i\left(f(x)\overline{g(x)}\right)_0^1 \\ &= i\left(f(1)\overline{g(1)} - f(0)\overline{g(0)} \right)\end{aligned}$$

it follows that A_1^* is an extension of A_3, likewise $A_{2,C}^*$ of $A_{2,1/\overline{C}}$, and A_3^* of A_1. Thus these three adjoints are $*$-operators, and hence $A_1, A_{2,C}, A_3$ are $**$-operators. Consequently,

$$A_1^* = \widetilde{A}_1^*, \qquad A_{2,C}^* = \widetilde{A}_{2,C}^*, \qquad A_3^* = \widetilde{A}_3^*$$

extend, respectively,

$$\widetilde{A}_3, \qquad \widetilde{A}_{2,1/\overline{C}}, \qquad \widetilde{A}_1.$$

It is not difficult to show–though we shall not enter into the matter more closely here–that the respective operators are in fact equal. It follows that $\widetilde{A}_3$ is Hermitian but not hypermaximal; $\widetilde{A}_{2,C}$ is Hermitian only when $C = 1/\overline{C}$, that is, $|C| = 1$, but is then also hypermaximal; $\widetilde{A}_1$ is not Hermitian.[T2] On the other hand, the three

[T1] *Translator's note.* The source states the two extension relations in the opposite direction. The domains just listed give $\mathfrak{A}_3 \subset \mathfrak{A}_{2,C} \subset \mathfrak{A}_1$, so the direction used here follows Definition II.

[T2] *Translator's note.* The source interchanges the subscripts 1 and 3 in this sentence. The adjoint relations immediately above, together with the boundary conditions, give the corrected order used here.

operators

$$D_1 = \widetilde{A}_1^*\widetilde{A}_1 = \widetilde{A}_3\widetilde{A}_1,$$
$$D_{2,C} = \widetilde{A}_{2,C}^*\widetilde{A}_{2,C} = \widetilde{A}_{2,1/\overline{C}}\widetilde{A}_{2,C},$$
$$D_3 = \widetilde{A}_3^*\widetilde{A}_3 = \widetilde{A}_1\widetilde{A}_3$$

are all hypermaximal. For all three,[T3] $D_j f = -f''$ ($j = 1, 2, 3$); only the domains differ, their boundary conditions being:

1. For D_1: $f'(1) = f'(0) = 0$.
2. For $D_{2,C}$: $f(1) : f(0) = C, \quad f'(1) : f'(0) = 1/\overline{C}$.
3. For D_3: $f(1) = f(0) = 0$.

These are indeed all admissible boundary conditions for the eigenvalue problem

$$-f'' = \lambda f,$$

which is equivalent to hypermaximality (cf. E., p. 49, n. 2, and p. 61).

More general or more complicated examples (Sturm–Liouville problems, partial differential equations, and so forth) exhibit the same behavior.

3. Since A^*A and AA^* are Hermitian, definite, and hypermaximal, there exists (as will be shown) one and only one Hermitian, definite, hypermaximal operator B, respectively C, such that

$$B^2 = A^*A, \qquad C^2 = AA^*.$$

It will be shown that B has the same domain as A and that always $\|Bf\| = \|Af\|$; the same relation holds between C and A^*. One can then construct an operator W which is essentially length-preserving (cf. below) and for which

$$A = WB = CW, \qquad A^* = BW^* = W^*C,$$
$$C = WBW^*, \qquad B = W^*CW.$$

As one sees, this is a decomposition of A and A^* analogous to the decomposition of a complex number z and its conjugate into a real factor ≥ 0 (the absolute value) and a factor of absolute value 1. Indeed, in many respects the hypermaximal operators B and C behave like real numbers; definiteness corresponds to ≥ 0, and length preservation to absolute value 1. The only striking point is that two different operators, B and C, represent the absolute value; they are, however, carried into one another by the length-preserving mapping W.[11]

[11]For completely continuous operators, E. Schmidt investigated the eigenvalue problems belonging to B and C; cf. *Math. Ann.* **63** (1907), pp. 433–476, §§ 14–16.

[T3]*Translator's note.* The source prints $Af = -f''$ here; the three operators under discussion are $D_1, D_{2,C}, D_3$, so $D_j f = -f''$ is intended.

For operators A, A^* defined everywhere (and hence continuous; cf. above), this decomposition into B, C, W is trivial. It is noteworthy, however, that it persists for every closed linear $*$-operator A, and that B and C always turn out to be hypermaximal. Thus, for example, if A is Hermitian and definite but not maximal, so that A^* is a proper extension of A, and hence $A^*A = B^2$ is an extension of A^2, B is nevertheless hypermaximal. Thus $B \neq A$, nor is B an extension of A, since it has the same domain!

On the basis of these results it is easy to show that $B = C$ is characteristic of the normal operators A, that is, of those admitting a complex eigenvalue representation (cf. A., p. 406, Definition 6, and pp. 410–416, Theorems 17–19). Equivalently,

$$B^2 = C^2, \qquad \text{or} \qquad A^*A = AA^*.$$

It is well known that this condition characterizes normality in finite-dimensional space[12] and for completely continuous operators on Hilbert space.[13] The author further showed that, in Hilbert space, it characterizes all continuous operators (E., pp. 115–116; A., p. 406; and the work cited above), but for general A he obtained there a condition of a different form. We now recognize $A^*A = AA^*$ as characteristic in general; indeed, it is enough that A^*A be an extension of AA^*, or conversely, since both operators are hypermaximal.

4. If A is any linear operator, then on its domain $\mathfrak{A}$ we can introduce a new metric by defining

$$\|f\|_A = \|Af\|.$$

For $\mathfrak{A}$ to be dense in $\mathfrak{H}$, A must be a $*$-operator; for it to be topologically complete in the new metric (at least relative to $\mathfrak{H}$),[14] A must be closed.[15] Thus we are led back to our class of operators: linear, closed, and $*$.

Two operators A_1, A_2 determine the same metric if they have the same domain and $\|A_1 f\| = \|A_2 f\|$ there. We then call them *metrically equal* and write

$$A_1 \underset{m}{=} A_2.$$

[12] Frobenius, *Journ. f. Math.* **84** (1877), pp. 51–54.

[13] Hellinger and Toeplitz, *Enzykl. d. Math. Wiss.*, II.C.13, pp. 1562–1563.

[14] That is, $\|f_m - f_n\|_A \to 0$ as $m, n \to \infty$ is to imply the existence in $\mathfrak{A}$ of a $\|\cdot\|_A$-limit of $f_1, f_2, \ldots$, insofar as a $\|\cdot\|$-limit exists in $\mathfrak{H}$. If $\|Af\| \geq \varepsilon\|f\|$ always holds for some fixed $\varepsilon > 0$, the latter restriction is superfluous.

[15] This consideration of different metrizations in $\mathfrak{H}$, as well as of topological completeness, was suggested by Mr. K. Friedrichs in a lecture that he gave to the Göttingen Mathematical Society in June 1931.

If the domain of A_1 is a subset of that of A_2, and $\|A_1 f\| = \|A_2 f\|$ on the former, then A_2 is called a *metric extension* of A_1. If the domains are equal, then $A_1 \underset{m}{=} A_2$; otherwise, A_2 is a *proper metric extension* of A_1.

As we have seen, every A is metrically equal to a Hermitian, definite, hypermaximal B, and one easily shows that B is uniquely determined by this property (cf. below).

On the other hand, we shall show that every discontinuous operator A has proper metric extensions, even when it is unextendible in the ordinary sense, that is, maximal. Thus there is no maximality here!

5. Our method will rest essentially on considering the space of all pairs $\{f, g\}$, where f, g are arbitrary elements of $\mathfrak{H}$. If we define

$$a\{f, g\} = \{af, ag\},$$
$$\{f, g\} + \{h, k\} = \{f + h, g + k\},$$
$$(\!(\{f, g\}, \{h, k\})\!) = (f, h) + (g, k),$$

so that

$$\|\{f, g\}\|^2 = \|f\|^2 + \|g\|^2,$$

this is a Hilbert space, for the characteristic conditions A–E from E., pp. 64–66, hold for it just as they do for $\mathfrak{H}$. It stands in approximately the same relation to $\mathfrak{H}$ as the plane does to the line; we shall denote it by H. Every operator A on $\mathfrak{H}$ can be represented in H by the set of all pairs $\{f, Af\}$. This is the analogue of graphically representing, in the plane, functions defined on the real line. We shall call this set the *image* of A, and denote it by B_A.

It is immediately apparent that $A = B$ means $B_A = B_B$; that A is an extension of B means that B_A, as a set, contains B_B; that A is linear means that B_A is a linear manifold; and that A is closed means that B_A, as a set, is closed.

We shall obtain our results by studying the graphs B_A in H. Indeed, the whole theory of operators could be developed in H much more systematically than is done here; we shall not, however, pursue this point.

I. The Operators A, A^*, A^*A.

1. As explained in Introduction 4, we study the Hilbert space H and the graphs B_A in H, in place of the operators A in $\mathfrak{H}$. We introduce in H the further operator

$$U\{f, g\} = \{ig, -if\}.$$

It is Hermitian and unitary, that is, $U = U^* = U^{-1}$ (and hence $U^2 = 1$). Since $(f, Ag) = (f^*, g)$ can be written as

$$(\!(\{f, f^*\}, U\{g, Ag\})\!) = 0,$$

the relation $A^* f = f^*$ says that $\{f, f^*\}$ is orthogonal to UB_A. Thus B_{A^*} is the set of all elements of H orthogonal to UB_A; it is therefore a closed linear manifold, and A^* is linear and closed.

The existence of closed linear extensions of A means that there are closed linear manifolds B_B containing B_A. Now a subset of H can be a graph B_B provided that it never contains two elements $\{f, f_1^*\}$, $\{f, f_2^*\}$ with $f_1^* \neq f_2^*$; equivalently, since the subsets in question are linear manifolds, provided that it never contains an element $\{0, f^*\}$ with $f^* \neq 0$. Hence one need only require that the smallest closed linear manifold containing B_A, namely the closed linear manifold spanned by B_A, contain no $\{0, f^*\}$ with $f^* \neq 0$. Replacing these notions by their definitions gives:

Theorem 1. *The $*$-operator A, with domain $\mathfrak{A}$, has a closed linear extension B if and only if the following condition holds. If all*

$$f_1^{(n)}, \dots, f_{\nu_n}^{(n)} \qquad (n = 1, 2, \dots)$$

belong to $\mathfrak{A}$, and if

$$a_1^{(n)} f_1^{(n)} + \cdots + a_{\nu_n}^{(n)} f_{\nu_n}^{(n)} \longrightarrow 0, \qquad a_1^{(n)} A f_1^{(n)} + \cdots + a_{\nu_n}^{(n)} A f_{\nu_n}^{(n)} \longrightarrow f^* \quad (n \to \infty),$$

then $f^* = 0$.[16] *Moreover, among all such B there is a smallest one, of which all the others are extensions: namely, the operator for which B_B is the closed linear manifold spanned by B_A.*

Another criterion is obtained as follows. If A^* is also a $*$-operator, then A^{**} exists; it is linear and closed and is plainly an extension of A, so that $\widetilde{A}$ exists. Conversely, if $\widetilde{A}$ exists, then $B_{\widetilde{A}}$ is the closed linear manifold spanned by B_A; correspondingly, $UB_{\widetilde{A}}$ and UB_A have the same orthogonal elements in H, and hence $\widetilde{A}^* = A^*$. Furthermore, $B_{\widetilde{A}^*} = B_{A^*}$ is the orthogonal complement of $UB_{\widetilde{A}}$. Here two closed linear manifolds are involved; see E., p. 74. Since U is unitary, $U^2 = 1$, and orthogonal complementarity is a symmetric relation, $B_{\widetilde{A}}$ is therefore the orthogonal complement of UB_{A^*}. In the terminology of Introduction 1, this says

[16] The strong convergence used throughout here could equally well be replaced by weak convergence (see, for example, A., pp. 378–381), since for linear manifolds (in this case B_A and B_B) strong closedness and weak closedness are equivalent, as E. Schmidt showed in *Rendiconti del Circolo Matematico di Palermo*, **25** (1908), pp. 57–73.

that Definition VI there, applied to A^{**}, yields a unique operator, namely $\widetilde{A}$. Thus A^* is itself a $*$-operator and $A^{**} = \widetilde{A}$. We have proved:

Theorem 2. *The following condition is also necessary and sufficient for the existence of $\widetilde{A}$: A^* is itself a $*$-operator (A was already assumed to be one). In that case*

$$\widetilde{A}^* = A^*, \qquad \widetilde{A} = A^{**}.$$

It should be noted that no relation whatever between the domains of A and A^* has been required or asserted.

2. Let A now be a closed linear $*$-operator. By Theorem 2, A^* is one as well. Since UB_A and B_{A^*} are orthogonal complements in H, every element of H can be represented, in one and only one way, as the sum of an element of UB_A and an element of B_{A^*} (see, for example, E., p. 74, Theorem 13). Choose the element of H to be $\{0, -if\}$. Then

$$\{0, -if\} = \{iAg, -ig\} + \{h, A^*h\}$$

has a unique solution, with g in the domain of A and h in that of A^*. Thus

$$iAg + h = 0, \qquad -ig + A^*h = -if.$$

It follows[T4] that $h = -iAg$, equivalently $Ag = ih$, so Ag also belongs to the domain of A^*, and consequently

$$-if = -ig + A^*h = -i(g + A^*Ag) = -i(A^*A + 1)g, \qquad (A^*A + 1)g = f.$$

The operator $A^*A + 1$ is therefore defined for sufficiently many g that its range is the whole Hilbert space.

Its inverse D, defined by $Df = g$, is thus linear and everywhere defined. Moreover,

$$\begin{aligned}(f, Df) &= (A^*Ag + g, g)\\ &= (A^*Ag, g) + (g, g)\\ &= (Ag, Ag) + (g, g) = \|Ag\|^2 + \|g\|^2.\end{aligned}$$

First, this quantity is real, so D is Hermitian. Second, it is nonnegative and vanishes only when $\|g\|^2 = 0$, that is, when $g = 0$ and $f = (A^*A + 1)g = 0$; hence D is definite, and $Df \neq 0$ when $f \neq 0$. Third, it is at least $\|g\|^2 = \|Df\|^2$, and therefore

$$\|Df\|^2 \leq (f, Df) \leq \|f\|\,\|Df\|, \qquad \|Df\| \leq \|f\|.$$

[T4] *Translator's note.* The source prints $Ag = -ih$. The first displayed equation gives $h = -iAg$, which is also the sign used in the following calculation.

Thus D is continuous. To summarize: D is a Hermitian, definite, continuous operator; $Df \neq 0$ whenever $f \neq 0$; its entire spectrum lies between 0 and 1,[17] and

$$A^*A + 1 = D^{-1}.$$

It is now easy to infer the hypermaximality of A^*A+1. For example, if f is orthogonal to every Dk, then Df is orthogonal to every k, because D is Hermitian and everywhere defined. Hence $Df = 0$ and $f = 0$. Thus the orthogonal complement of the closed linear manifold spanned by the range of D, in $\mathfrak{H}$, is (0); that closed linear manifold is consequently all of $\mathfrak{H}$. The range of D is the domain of A^*A+1, and hence also the domain of A^*A, so A^*A is a $*$-operator. By Introduction 2, especially note 10, A^*A is Hermitian and definite, and therefore so is A^*A+1. Since $A^*A + 1$ assumes every value in $\mathfrak{H}$, E., p. 102, Theorem 41, applies: $A^*A + 1$ is hypermaximal, and consequently A^*A is hypermaximal as well. Thus:

Theorem 3. *Let A be a closed linear $*$-operator. Then A^* is also a closed linear $*$-operator, and A^*A is Hermitian, definite, and hypermaximal.*

3. The domain $\mathfrak{A}_1$ of A^*A is contained in the domain $\mathfrak{A}$ of A. Denote by A_1 the restriction of A to $\mathfrak{A}_1$. Plainly, A_1 is linear and a $*$-operator, but need not be closed. Since A is an extension of A_1, A_1^* is an extension of A^*, and is therefore also a $*$-operator; hence A_1 is even a $**$-operator. We shall prove that $\widetilde{A_1} = A$, that is, that B_A is the closed linear manifold spanned by B_{A_1} in H.

Equivalently, the difference between B_A and the closed linear manifold spanned by B_{A_1} is (0). In other words, the only element of B_A orthogonal to that manifold—or, equivalently, to B_{A_1}—is 0; see E., p. 74, Definition 11. An element of B_A has the form $\{k, Ak\}$. If it is orthogonal to B_{A_1}, then in particular it is orthogonal to the elements $\{g, Ag\}$ constructed at the beginning of I.2. Hence $\{iAk, -ik\}$ is orthogonal to $\{iAg, -ig\}$; here A^*Ag was defined because $g \in \mathfrak{A}_1$. The element $\{iAk, -ik\}$, which belongs to UB_A, is all the more orthogonal to the element $\{h, A^*h\}$ used there, since the latter belongs to B_{A^*}. It is therefore orthogonal to their sum, $\{0, -if\}$. The orthogonality in H of $\{iAk, -ik\}$ and $\{0, -if\}$ means that k and f are orthogonal in $\mathfrak{H}$. Since this holds for every f, we have $k = 0$ and $\{k, Ak\} = \{0, 0\}$. This proves the assertion:

[17] The last assertion follows from

$$0 \leq (f, Df) \leq \|f\|^2,$$

by calculations similar to those in A., p. 418.

Theorem 4. *Let the domain of A be $\mathfrak{A}$, and let the domain of A^*A be $\mathfrak{A}_1 \subset \mathfrak{A}$. Denote by A_1 the restriction of A to $\mathfrak{A}_1$. Then A_1 is a linear $**$-operator, though it need not be closed. Nevertheless,*

$$\widetilde{A_1} = A.$$

4. If B is a Hermitian hypermaximal operator, it has an associated resolution of the identity $E(\lambda)$, by E., p. 91, Definition 17, and p. 92, Theorem 36. If $E(\lambda)$ is constant, and hence equal to 0, for $\lambda < 0$, the formulas there give $(Bf, f) \geq 0$ for every f; thus B is definite. If, on the other hand, $E(\lambda) \neq 0$ for some $\lambda < 0$, then for some such $\lambda = \lambda_0 < 0$ there is an $f \neq 0$ with $E(\lambda_0)f = f$, and therefore $E(\lambda)f = f$ for $\lambda \geq \lambda_0$, since $E(\lambda_0)$ is contained in $E(\lambda)$; compare the discussion of projection operators in E. From the same formulas one then obtains

$$(Bf, f) \leq \lambda_0 \|f\|^2 < 0,$$

so B is not definite. Thus B is definite if and only if $E(\lambda)$ is constant, and hence 0, for $\lambda < 0$.

Now consider the family[T5]

$$F(\mu) = \begin{cases} 0, & \mu < 0, \\ E(\sqrt{\mu}), & \mu \geq 0. \end{cases}$$

It too is plainly a resolution of the identity, and its associated Hermitian hypermaximal operator B' is definite. If $B'f$ is defined, then, by definition,

$$\int \mu^2 \, d\|F(\mu)f\|^2 = \int \mu^2 \, d\|E(\sqrt{\mu})f\|^2 = \int \lambda^4 \, d\|E(\lambda)f\|^2$$

is finite. Hence $\int \lambda^2 \, d\|E(\lambda)f\|^2$ is finite as well, for $\int d\|E(\lambda)f\|^2$ is also finite and

$$\lambda^2 \leq \tfrac{1}{2} + \tfrac{1}{2}\lambda^4.$$

Thus Bf is defined. The same calculation carried out in F., p. 206, to prove formula e) there shows that

$$\int \lambda^2 \, d\|E(\lambda)Bf\|^2 = \int \lambda^4 \, d\|E(\lambda)f\|^2.$$

[T5] *Translator's note.* The source prints $E(\sqrt{\lambda})$ here; the subsequent calculation and the definition as a function of μ require $E(\sqrt{\mu})$, used above.

This is finite, so $B(Bf)$, and hence $B^2 f$, is defined. One more calculation of the same kind gives

$$(B^2 f, g) = (B(Bf), g) = \int \lambda\, d(E(\lambda)Bf, g) = \int \lambda^2\, d(E(\lambda)f, g)$$
$$= \int \mu\, d(E(\sqrt{\mu})f, g) = \int \mu\, d(F(\mu)f, g) = (B'f, g).$$

Thus $B^2 f = B'f$. Hence B^2 is an extension of the hypermaximal operator B', and therefore $B^2 = B'$.

Rearranging the statement slightly, we may formulate it as follows:

Theorem 5. *Let B and B' be Hermitian, hypermaximal, definite operators, and let $E(\lambda)$ and $F(\lambda)$, respectively, be their resolutions of the identity; thus $E(\lambda) = F(\lambda) = 0$ for $\lambda < 0$. Then*

$$B^2 = B'$$

is equivalent to

$$E(\lambda) = F(\lambda^2) \qquad \textit{for every } \lambda \geq 0.$$

Consequently, if B' is given and B is not, the equation $B^2 = B'$ has exactly one solution B of this kind; it will be denoted by $\sqrt{B'}$.

We now apply Theorem 5 to $B' = A^*A$, and set

$$B = \sqrt{A^*A}.$$

The domain of $B^2 = A^*A$ was denoted by $\mathfrak{A}_1$; let A_1 be the restriction of A to this domain, and let B_1 be the restriction of B to it. The domain of B contains that of B^2, namely $\mathfrak{A}_1$. By Theorem 4,[T6] $\widetilde{A}_1 = A$; replacing A there by B, and using $B^* = B$ by hypermaximality, gives $\widetilde{B}_1 = B$.

The operators A_1 and B_1 have the same domain, and on that domain

$$\|A_1 f\|^2 = (A_1 f, A_1 f) = (Af, Af) = (A^*Af, f),$$
$$\|B_1 f\|^2 = (B_1 f, B_1 f) = (Bf, Bf) = (B^2 f, f).$$

Thus $\|A_1 f\| = \|B_1 f\|$. Hence A_1 and B_1 are metrically equal (see Introduction 4), and it follows that their closures $\widetilde{A}_1$ and $\widetilde{B}_1$[18] are metrically equal as well; that is,

[18] Let C be a linear $**$-operator (by Theorem 4, A_1 and B_1 are such operators). Then B_C is a linear manifold and $B_{\widetilde{C}}$ is its closure. In other words, $\widetilde{C}$ is defined as follows: if $f_1, f_2, \ldots$ belong to the

[T6] *Translator's note.* The source prints "Theorem 3" here, but the assertion $\widetilde{A}_1 = A$ is Theorem 4, which is plainly intended.

A and B are metrically equal. Substitution into the definition of metric equality yields:

Theorem 6. *Let A be as in Theorems 2 and 3, and let*

$$B = \sqrt{A^*A} \qquad \textit{(see Theorem 5)}.$$

Thus B is Hermitian, definite, and hypermaximal. Then A and B are metrically equal: they have the same domain $\mathfrak{A}$, and throughout that domain

$$\|Af\| = \|Bf\|.$$

5. Denote the range of A by $\mathfrak{W}$, and denote the closures of the linear manifolds $\mathfrak{A}$ and $\mathfrak{W}$, respectively, by

$$\overline{\mathfrak{A}} = \mathfrak{H}, \qquad \overline{\mathfrak{W}}.$$

Further, let $\mathfrak{N}$ be the set of all f such that $Af = 0$. It is contained in $\mathfrak{A}$, and hence in $\overline{\mathfrak{A}}$, and, since A is linear and closed, it is a closed linear manifold (all these sets being in $\mathfrak{H}$). For A^*, denote the corresponding sets by

$$\mathfrak{A}', \quad \mathfrak{W}', \quad \overline{\mathfrak{A}}' = \mathfrak{H}, \quad \overline{\mathfrak{W}}', \quad \mathfrak{N}'.$$

For f to be orthogonal to $\overline{\mathfrak{W}}$ is the same as for f to be orthogonal to $\mathfrak{W}$, that is, to every Ag. Thus $(f, Ag) = 0$ for every g, which says precisely that A^*f is defined and equal to 0, or that $f \in \mathfrak{N}'$. Therefore $\mathfrak{N}' = \mathfrak{H} - \overline{\mathfrak{W}}$. Similarly, $\mathfrak{N} = \mathfrak{H} - \overline{\mathfrak{W}}'$. In summary,

$$\mathfrak{N} \subset \mathfrak{A}, \qquad \mathfrak{N} = \mathfrak{H} - \overline{\mathfrak{W}}', \qquad \mathfrak{N}' \subset \mathfrak{A}', \qquad \mathfrak{N}' = \mathfrak{H} - \overline{\mathfrak{W}}.$$

By Theorem 6,[T7] the set on which $Af = 0$ is identical with the set on which $Bf = 0$; hence $\mathfrak{N}$ is also the corresponding set for B. Let $\mathfrak{B}$ be the range of B, a linear manifold, and let $\overline{\mathfrak{B}}$ be its closure, a closed linear manifold. The same argument gives

$$\mathfrak{N} = \mathfrak{H} - \overline{\mathfrak{B}},$$

domain of C, if $f_n \to f$ as $n \to \infty$, and if the Cf_n also have a limit, that is,

$$\|C(f_m - f_n)\| \longrightarrow 0 \qquad (m, n \to \infty),$$

then f belongs to the domain of $\widetilde{C}$, and $\widetilde{C}f$ is the limit of the Cf_n. This formulation makes it clear that, when C is replaced by a metrically equal operator, $\widetilde{C}$ is likewise replaced by a metrically equal operator.

[T7] *Translator's note.* The source prints "Theorem 4" here; the equality of the two null sets follows from the metric equality asserted in Theorem 6.

because $B^* = B$ by hypermaximality. Consequently,

$$\overline{\mathfrak{B}} = \overline{\mathfrak{W}}'.$$

Replace A by A^*, and hence A^* by $A^{**} = A$. By Theorem 5 and Theorem 6, this produces the Hermitian, definite, hypermaximal operators AA^* and

$$C = \sqrt{AA^*}.$$

By Theorem 6, A^* and C are metrically equal: they have the same domain $\mathfrak{A}'$, and on it

$$\|A^* f\| = \|C f\|.$$

For C, let $\mathfrak{B}'$ be its range and $\overline{\mathfrak{B}}'$ its closure. Then

$$\overline{\mathfrak{B}}' = \overline{\mathfrak{W}}.$$

By Theorem 6,[T8] the correspondence

$$Bf \Longleftrightarrow Af$$

maps $\mathfrak{B}$ one-to-one, linearly, and isometrically onto $\mathfrak{W}$. It can therefore be extended, again one-to-one, linearly, and isometrically, to the closures

$$\overline{\mathfrak{B}} = \overline{\mathfrak{W}}' \quad \text{and} \quad \overline{\mathfrak{W}},$$

that is, from $\mathfrak{H} - \mathfrak{N}$ onto $\mathfrak{H} - \mathfrak{N}'$. Denote the operator of this isometric mapping by V. Its domain is $\mathfrak{H} - \mathfrak{N}$, and its range is $\mathfrak{H} - \mathfrak{N}'$; compare E., p. 70, Definition 6, and p. 71, Theorem 10.

Similarly,

$$Cf \Longleftrightarrow A^* f$$

is a mapping of $\mathfrak{B}'$ onto $\mathfrak{W}'$, and it can be extended to $\overline{\mathfrak{B}}' = \overline{\mathfrak{W}}$ and $\overline{\mathfrak{W}}'$, that is, from $\mathfrak{H} - \mathfrak{N}'$ onto $\mathfrak{H} - \mathfrak{N}$. Denote its isometric operator by V'; its domain is $\mathfrak{H} - \mathfrak{N}'$, and its range is $\mathfrak{H} - \mathfrak{N}$. By the definitions of V and V',

$$A = VB, \qquad A^* = V'C.$$

The way in which V, respectively V', is defined outside the range of B, respectively C, that is, outside $\mathfrak{H} - \mathfrak{N}$, respectively $\mathfrak{H} - \mathfrak{N}'$, is immaterial. We may therefore

[T8] *Translator's note.* The source again prints "Theorem 4" here; the isometry of the correspondence is the conclusion of Theorem 6.

extend them to everywhere-defined linear operators W, respectively W', which vanish on $\mathfrak{N}$, respectively $\mathfrak{N}'$. Then again

$$A = WB, \qquad A^* = W'C.$$

Let E, respectively E', denote the projection operators onto $\mathfrak{H}-\mathfrak{N}$, respectively $\mathfrak{H}-\mathfrak{N}'$. Then

$$W = VE, \qquad W' = V'E',$$

and therefore immediately

$$\|Wf\| = \|Ef\|, \qquad \|W'f\| = \|E'f\|.$$

This proves that W and W' are continuous. It also gives

◎ $$W^*W = E, \qquad W'^*W' = E'.$$ [19]

Since W maps[T9] $\mathfrak{H}-\mathfrak{N}$ onto $\mathfrak{H}-\mathfrak{N}'$, while $W^*W = E$ is the identity mapping of $\mathfrak{H}-\mathfrak{N}$ onto itself, W^* effects the inverse, likewise isometric, mapping from $\mathfrak{H}-\mathfrak{N}'$ onto $\mathfrak{H}-\mathfrak{N}$. Hence $\|W^*f\| = \|f\|$ on $\mathfrak{H}-\mathfrak{N}'$. On $\mathfrak{N}'$, since it is orthogonal to the range $\mathfrak{H}-\mathfrak{N}'$ of W, one has $W^*f = 0$. Thus, in general, $W^*E'f = W^*f$, and therefore

$$\|W^*f\| = \|E'f\|.$$

It follows (compare note 19) that

◎ $$WW^* = E',$$

and likewise

◎ $$W'W'^* = E.$$

Left multiplication of $A = WB$ by W^* gives, since $EB = B$ (the values of B lie in $\mathfrak{H}-\mathfrak{N}$),

* $$B = W^*A.$$

[19] Indeed,

$$\|Wf\|^2 = (Wf, Wf) = (W^*Wf, f), \qquad \|Ef\|^2 = (Ef, f),$$

and hence $W^*W = E$; similarly $W'^*W' = E'$.

[T9] *Translator's note.* The source prints W' here. The mapping described is W, as required by $W^*W = E$ and by the domain and range stated in the same sentence.

Likewise, left multiplication of $A^* = W'C$ by W'^* gives

$$C = W'^* A^*. \tag{*}$$

Taking adjoints in these four relations yields, after a simple discussion,

$$A^* = BW^*, \qquad A = CW'^*, \tag{\#}$$

$$B = A^*W, \qquad C = AW'. \tag{\#}$$

We now form the operator WBW^*.[20] Since W is defined everywhere, this product is defined whenever W^*f belongs to $\mathfrak{A}$. Since $\mathfrak{N} \subset \mathfrak{A}$, the projection onto $\mathfrak{N}$ of every element of $\mathfrak{A}$ belongs to $\mathfrak{A}$, and so does its projection onto $\mathfrak{H} - \mathfrak{N}$. Thus the part of $\mathfrak{A}$ lying in $\mathfrak{H} - \mathfrak{N}$ is everywhere dense there; equivalently, the part of $\mathfrak{A}$ lying in the range of W^*, which is $\mathfrak{H} - \mathfrak{N}$, is everywhere dense there. Since W^* maps $\mathfrak{H} - \mathfrak{N}'$ one-to-one and isometrically onto $\mathfrak{H} - \mathfrak{N}$, those f in $\mathfrak{H} - \mathfrak{N}'$ for which $W^*f \in \mathfrak{A}$ are everywhere dense in that subspace. For $f \in \mathfrak{N}'$ one has $W^*f = 0$, which also belongs to $\mathfrak{A}$. Hence the f just described are everywhere dense in $\mathfrak{H}$, and WBW^* is a $*$-operator.

It is Hermitian and definite just as B is. It is also hypermaximal. Indeed, suppose that, for every g for which the expression is defined,

$$(f, WBW^*g) = (f^*, g),$$

or equivalently

$$(W^*f, BW^*g) = (f^*, g).$$

Put $g = Wh$, where $h \in \mathfrak{H} - \mathfrak{N}$ and Bh is defined. Then $W^*g = W^*Wh = h$, and therefore

$$(W^*f, Bh) = (f^*, Wh) = (W^*f^*, h).$$

For every $h \in \mathfrak{N}$, Bh is defined and equals 0; the right-hand side also vanishes because W^*f^* belongs to $\mathfrak{H} - \mathfrak{N}$. Consequently,

$$(W^*f, Bh) = (W^*f^*, h)$$

holds for every h in the domain of B.

[20] Operator products are consistently understood as follows: RSf is defined when Sf and $R(Sf)$ are defined, and it is equal to $R(Sf)$; correspondingly for $QRS, PQRS, \ldots$. This multiplication is plainly associative.

Since B is hypermaximal, $BW^* f$ is defined and equals $W^* f^*$. Thus $WBW^* f = WW^* f^* = f^*$, provided $f^* \in \mathfrak{H} - \mathfrak{N}'$.[T10] This last condition follows from the original equation: for every $g \in \mathfrak{N}'$, $W^* g = 0$, so its left-hand side is 0, and hence $(f^*, g) = 0$.

Now, using the $*$, $\circledcirc$, and $\#$ formulas[20], one obtains

$$\begin{aligned}(WBW^*)^2 &= WBW^* \cdot WBW^* = WB \cdot W^*W \cdot BW^* \\ &= WB \cdot E \cdot BW^* = WB \cdot EB \cdot W^* \\ &= WB \cdot BW^* = A \cdot A^* = C^2.\end{aligned}$$

Therefore, by Theorem 5,[T11]

$$WBW^* = C.$$

By $\#$, this says $WA^* = C$, while $*$ says $W'^* A^* = C$. Thus $Wf = W'^* f$ throughout the range $\mathfrak{W}'$ of A^*, and hence also throughout $\overline{\mathfrak{W}}' = \mathfrak{H} - \mathfrak{N}$. On $\mathfrak{N}$ the equality holds as well, since both sides vanish. Consequently $W = W'^*$; taking adjoints gives $W^* = W'$.

We may summarize the results as follows.

Theorem 7. *Let A be as in Theorems 2 and 3, and set*

$$B = \sqrt{A^*A}, \qquad C = \sqrt{AA^*}.$$

(Compare Theorem 6, applied to A and to A^*.) Thus B and C are Hermitian, definite, and hypermaximal. Let the domains of A, A^*, B, C be, respectively, the linear manifolds

$$\mathfrak{A}, \quad \mathfrak{A}', \quad \mathfrak{A}, \quad \mathfrak{A}',$$

and let their ranges be, respectively,

$$\mathfrak{W}, \quad \mathfrak{W}', \quad \mathfrak{B}, \quad \mathfrak{B}',$$

with an overline denoting closure. Let the respective sets (closed linear manifolds) on which $Af = 0, A^*f = 0, Bf = 0, Cf = 0$ be

$$\mathfrak{N}, \quad \mathfrak{N}', \quad \mathfrak{N}, \quad \mathfrak{N}'.$$

[T10] *Translator's note.* The source prints $f^* \in \mathfrak{N}'$; the argument immediately following shows that f^* is orthogonal to $\mathfrak{N}'$, hence belongs to $\mathfrak{H} - \mathfrak{N}'$.

[T11] *Translator's note.* The source prints "Theorem 4" both here and in the parenthetical reference in Theorem 7. Here the intended result is plainly Theorem 5, the uniqueness of the definite hypermaximal square root; in Theorem 7 the intended result is Theorem 6, applied to A and A^*.

Then

$$\mathfrak{N} \subset \mathfrak{A} \subset \overline{\mathfrak{A}} = \mathfrak{H}, \qquad \mathfrak{N}' \subset \mathfrak{A}' \subset \overline{\mathfrak{A}}' = \mathfrak{H},$$
$$\overline{\mathfrak{W}}' = \overline{\mathfrak{B}} = \mathfrak{H} - \mathfrak{N}, \qquad \overline{\mathfrak{W}} = \overline{\mathfrak{B}}' = \mathfrak{H} - \mathfrak{N}'.$$

Let E, E' be the projection operators onto $\mathfrak{H} - \mathfrak{N}$ and $\mathfrak{H} - \mathfrak{N}'$, respectively. There exists an everywhere-defined continuous operator W which maps $\mathfrak{B}$ one-to-one, linearly, and isometrically onto $\mathfrak{W}$, and therefore maps $\overline{\mathfrak{B}} = \mathfrak{H} - \mathfrak{N}$ onto $\overline{\mathfrak{W}} = \mathfrak{H} - \mathfrak{N}'$. Likewise W^* maps $\mathfrak{B}'$ onto $\mathfrak{W}'$, and therefore maps $\overline{\mathfrak{B}}' = \mathfrak{H} - \mathfrak{N}'$ onto $\overline{\mathfrak{W}}' = \mathfrak{H} - \mathfrak{N}$; on these latter subspaces W^* and W are inverse to one another. The operator W vanishes on $\mathfrak{N}$, and W^* on $\mathfrak{N}'$. Moreover,

$$W^*W = E, \qquad WW^* = E'.$$

The following relations hold[20]*:*

$$A = WB = CW, \qquad A^* = BW^* = W^*C,$$
$$B = W^*A = A^*W, \qquad C = AW^* = WA^*,$$

and consequently

$$C = WBW^*, \qquad B = W^*CW.$$

It should be noted that B vanishes on $\mathfrak{N}$, and C on $\mathfrak{N}'$, so that they are of interest only on $\mathfrak{H}-\mathfrak{N}$ and $\mathfrak{H}-\mathfrak{N}'$, respectively. On these subspaces, however, they pass into one another under the mutually inverse isometric mappings between $\mathfrak{H} - \mathfrak{N}$ and $\mathfrak{H} - \mathfrak{N}'$ effected by W and W^*. This is the generalization of E. Schmidt's theorem mentioned in Introduction 3 and note 11, respectively.

If 0 is a (point-)eigenvalue of neither A nor A^*, that is, if $Af = 0$ and $A^*f = 0$ each imply $f = 0$, then

$$\mathfrak{N} = \mathfrak{N}' = (0), \qquad E = E' = 1,$$

and hence $W^*W = WW^* = 1$; that is, W is unitary and $W^* = W^{-1}$. In this case the formulas of Theorem 7 are still more transparent.

II. Applications.

1. If $A^*A = AA^*$, that is, if $B = C$, then Theorem 7 gives

$$A = WB = BW, \qquad A^* = BW^* = W^*B,$$

and also[T12]

$$\mathfrak{N} = \mathfrak{N}', \qquad E = E', \qquad W^*W = WW^* = E.$$

Thus all the operators B, W, W^* commute with one another. From this the normality of A (compare Introduction 3) may be deduced in various ways. For example, by the methods of E., p. 115 (before Theorem 11*), one could give a simultaneous spectral representation for B, W, W^*, and hence for A, A^*. We prefer to use the definition in A., p. 116, Definition 6, and argue as follows. The operators A, A^* have the same domains as B, C, respectively, and hence the same domain as one another; thus, in the sense of A., p. 405, Definition 5, they form a conjugate pair. In the notation used there, if $E(\lambda)$ also denotes the resolution of the identity belonging to B, then

$$(A)'' = (WB)'' \subseteq (W, B)'' = \big(W, W^*, \text{all } E(\lambda)\big)''.$$

This set is Abelian, since it is the set generated by $W, W^*, E(\lambda)$; compare A., p. 389, above. Indeed, W, W^* commute with B, and therefore with all $E(\lambda)$, by A., p. 408, Theorem 13.

Conversely, suppose that A is normal. Then the closed operators A, A^* have the same domain (A., p. 416, Theorem 21). If A^*Af and AA^*g are defined, then

$$(f, AA^*g) = (A^*f, A^*g)^{21} = (Af, Ag) = (A^*Af, g).$$

By the hypermaximality of AA^*, it follows that AA^*f is defined and equal to A^*Af. Thus AA^* is an extension of A^*A; but the latter too is hypermaximal, and consequently $A^*A = AA^*$. Thus:

Theorem 8. *Let A be as in Theorems 2 and 3, and let*

$$B = \sqrt{A^*A}, \qquad C = \sqrt{AA^*}.$$

For A to be normal it is necessary and sufficient that

$$A^*A = AA^*,$$

or, equivalently, that $B = C$. Since A^*A and AA^* are both hypermaximal, it even suffices that one be an extension of the other.

[21] According to A., at the place just cited, whenever Ah is defined, $\|Ah\| = \|A^*h\|$, and hence $(Ah, Ah) = (A^*h, A^*h)$. From this one concludes, whenever Af and Ag are defined, by considering $\frac{f \pm g}{2}$ and $\frac{f \pm ig}{2}$, that $(Af, Ag) = (A^*f, A^*g)$ as well.

[T12] *Translator's note.* The source prints $E = F$; in the notation of Theorem 7, E' is intended.

Suppose that A, A^* have the same domain and that $\|Af\| = \|A^*f\|$ throughout it. The same then holds for B, C. The correspondence

$$Bf \Longleftrightarrow Cf$$

maps $\mathfrak{B}$ one-to-one, linearly, and isometrically onto $\mathfrak{B}'$. It may therefore be extended in the same way to

$$\overline{\mathfrak{B}} = \mathfrak{H} - \mathfrak{N}, \qquad \overline{\mathfrak{B}}' = \mathfrak{H} - \mathfrak{N}';$$

but $\mathfrak{N} = \mathfrak{N}'$, and consequently $\overline{\mathfrak{B}} = \overline{\mathfrak{B}}'$. Denote this operator by V_0. It therefore maps $\overline{\mathfrak{B}} = \overline{\mathfrak{B}}' = \mathfrak{H} - \mathfrak{N}$ onto itself. Extend it to a linear operator W_0 which vanishes on $\mathfrak{N}$. As in Theorem 6,

$$C = V_0 B = W_0 B, \qquad W_0^* W_0 = W_0 W_0^* = E.$$

Taking adjoints gives $C = BW_0^*$. Thus W_0^*, within $\mathfrak{H} - \mathfrak{N}$, carries the domain of C into that of B, that is, carries the domain of B into itself. The same is true of its inverse W_0 on $\mathfrak{H} - \mathfrak{N}$. Hence Bf and $Cf = W_0 Bf$, both of which lie in $\mathfrak{H} - \mathfrak{N}$, belong simultaneously to the common domain of B, C. In other words, B^2 and C^2 have the same domain.

On that domain one always has

$$(B^2 f, f) = (Bf, Bf) = \|Bf\|^2, \qquad (C^2 f, f) = (Cf, Cf) = \|Cf\|^2.$$

Since $(B^2 f, f) = (C^2 f, f)$, it follows that $B^2 = C^2$, and therefore $B = C$. Thus:

Theorem 9. *The following is also a necessary and sufficient condition for the normality of A: A and A^* have the same domain, and throughout it*

$$\|Af\| = \|A^*f\|.$$[22]

2. We now turn to the notions of metric equality and metric extension of operators defined in Introduction 4.

From Theorem 6, together with the argument used in the proof of Theorem 9, which shows that any two Hermitian, definite, hypermaximal operators B, C that are metrically equal must actually be equal (for the relation of B, C to A, A^* was not used there), we immediately obtain:

[22]In A., p. 423, above, it was shown that it is not sufficient for this equality to hold merely on some everywhere dense set.

Theorem 10. *Every closed linear $*$-operator A is metrically equal to one and only one Hermitian, definite, hypermaximal operator B, namely the operator B of Theorem 6:*

$$B = \sqrt{A^*A}.$$

Every discontinuous A has proper metric extensions. By Theorem 10, we may assume A to be Hermitian, definite, and hypermaximal. By E., p. 108, Theorem 49, A is a proper extension of a closed linear Hermitian operator A', which can in fact be chosen in continuum-many distinct ways. Consequently A'^*, which is a $*$-operator because A' is a $**$-operator, is a proper extension of A. Thus:

Theorem 11. *Let A be as in Theorem 10. If A is discontinuous, then it has proper metric extensions (indeed, continuum-many distinct ones).*

If A is continuous, then its domain is closed; since it must be everywhere dense, the domain is then equal to $\mathfrak{H}$. From this and Theorem 11 it follows that:

Theorem 12. *Let A be as in Theorem 10. Then A is continuous if and only if it is defined everywhere.*

As noted in Introduction 2, this generalizes Toeplitz's theorem[23,T13]. This theorem can incidentally also be proved directly.

There are also very general theorems on the simultaneous metric extension of several operators, for example of operators whose domains have no points in common (compare M., p. 232, Theorem 18). The author intends to discuss these shortly.

[23] E. Hellinger and O. Toeplitz, "Grundlagen für eine Theorie der unendlichen Matrizen," *Mathematische Annalen*, **69** (1910), pp. 289–330, especially p. 321; compare also E., pp. 107–108, Theorem 48, γ.

[T13] *Translator's note.* Source note 23 prints the year as 1911; the cited volume and article were published in 1910.