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ON RINGS OF OPERATORS

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Introduction

1. The problems discussed in this paper arose naturally in continuation of the work begun in a paper of one of us ((18), chiefly parts I and II). Their solution seems to be essential for the further advance of abstract operator theory in Hilbert space under several aspects. First, the formal calculus with operator-rings leads to them. Second, our attempts to generalise the theory of unitary group-representations essentially beyond their classical frame have always been blocked by the unsolved questions connected with these problems. Third, various aspects of the quantum mechanical formalism suggest strongly the elucidation of this subject. Fourth, the knowledge obtained in these investigations gives an approach to a class of abstract algebras without a finite basis, which seems to differ essentially from all types hitherto investigated.

The results which we shall obtain throw light on an entirely new side of operator theory; they lead to a new notion of linear dimensionality (which, under certain conditions, has a continuous range of numerical values); and indicate a way out of the paradoxes of unbounded operator theory.

In the four following §§ of this Introduction we will give a somewhat more detailed outline of the four aspects of our problems, which were enumerated above.

2. We consider a linear space $\mathfrak{H}$ with an inner product (that is a Hilbert space or a finite dimensional Euclidean space, cf. (b) in §1.1), and in it the ring **B** of all bounded operators (cf. (f) in §1.1). Subsets **M** of **B** for which $A, B \in \mathbf{M}$ implies $\alpha A, A^*, A + B, AB \in \mathbf{M}$ (α is any complex number, A^* is the adjoint of A, cf. the

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notation in §1.1), and which are closed in a suitable topology of $\mathbf{B}$ are called rings. (All these notions are discussed in detail in (18).) We will chiefly consider rings $\mathbf{M}$ which contain the unit operator 1: $1 \in \mathbf{M}$. If $\mathbf{M}, \mathbf{N}$ are given rings, denote the smallest ring containing $\mathbf{M}, \mathbf{N}$ by $R(\mathbf{M}, \mathbf{N})$, and the greatest ring contained in $\mathbf{M}$ and $\mathbf{N}$ by $\mathbf{M} \cdot \mathbf{N}$ (this, by the way, happens to be the set theoretical intersection of $\mathbf{M}$ and $\mathbf{N}$).

The operations $R(\mathbf{M}, \mathbf{N})$ and $\mathbf{M} \cdot \mathbf{N}$ obey both the commutative and the associative law, therefore each of them could be analogised with addition or with multiplication. Owing to $R(\mathbf{M}, \mathbf{M}) = \mathbf{M} \cdot \mathbf{M} = \mathbf{M}$ the analogy is closer with the "Boolean algebras" (as they occur in set theory or in logics), than with algebra proper. As the analogue of the distributive law, which connects addition and multiplication, does not hold in general, this analogy is not perfect. (The distributive law would be, according to how we identify $R(\mathbf{M}, \mathbf{N})$ and $\mathbf{M} \cdot \mathbf{N}$ with addition and multiplication, either

$$R(\mathbf{M} \cdot \mathbf{N}, \mathbf{M} \cdot \mathbf{P}) = \mathbf{M} \cdot R(\mathbf{N}, \mathbf{P})$$

or

$$R(\mathbf{M}, \mathbf{N}) \cdot R(\mathbf{M}, \mathbf{P}) = R(\mathbf{M}, \mathbf{N} \cdot \mathbf{P}).$$

None of these equations holds in general.) Thus it is somewhat arbitrary how we carry out these identifications. We choose to make $R(\mathbf{M}, \mathbf{N})$ correspond to addition and $\mathbf{M} \cdot \mathbf{N}$ to multiplication.

In this notation the rings $\mathbf{M}$ (and similarly the rings with $1 \in \mathbf{M}$) form a lattice. (Cf. (1, 9, 12a).) We use the terminology of (9), and therefore write in this § $\mathbf{M} \cup \mathbf{N}$ for $R(\mathbf{M}, \mathbf{N})$ and $\mathbf{M} \cap \mathbf{N}$ for $\mathbf{M} \cdot \mathbf{N}$. One verifies immediately the lattice postulates I–IV (cf. (1, p. 422)) for these operations.

The lattice R of all rings with $1 \in \mathbf{M}$ contains a smallest element: the set $(\alpha 1)$ of all operators of the form $\alpha 1$; and a greatest element: the set $\mathbf{B}$ of all bounded operators in $\mathfrak{H}$. We write in this §, 0 and 1 for $(\alpha 1)$ and $\mathbf{B}$.

Now R has a property which brings it nearer to the Boolean algebras than lattices usually are: there exists an operation $\mathbf{M}'$ in R which dualises addition and multiplication; that is, for which

$$(\mathbf{M} \cup \mathbf{N})' = \mathbf{M}' \cap \mathbf{N}', \tag{D_1}$$

$$(\mathbf{M} \cap \mathbf{N})' = \mathbf{M}' \cup \mathbf{N}'. \tag{D_2}$$

To this end we define $\mathbf{M}'$ as the set of those A which commute with all $B \in \mathbf{M}$ (cf. (18, p. 374)); then $\mathbf{M}' \in R$ and

$$\mathbf{M} = \mathbf{M}''. \tag{D_3}$$

(Cf. (18, p. 397).) Now (D_1) is obvious; and (D_2) follows from (D_1) by replacing $\mathbf{M}, \mathbf{N}$ by $\mathbf{M}', \mathbf{N}'$, applying $'$ to the equation, and using (D_3).

In the analogy with Boolean algebras (or logics) in which $\mathbf{M} \cup \mathbf{N}$ corresponds to the sum (resp. "or") and $\mathbf{M} \cap \mathbf{N}$ to the product (resp. "and"), $\mathbf{M}'$ should correspond to the complement (resp. "no").

Another lattice of such a structure consists of all closed linear manifolds of $\mathfrak{H}$. Denoting it by M we define: If $\mathfrak{M}, \mathfrak{N}$ are two closed linear manifolds in $\mathfrak{H}$, then $\mathfrak{M} \cup \mathfrak{N}$ is the smallest closed linear manifold containing them (cf. (d) in §1.1, where it is denoted by $[\mathfrak{M}, \mathfrak{N}]$); $\mathfrak{M} \cap \mathfrak{N}$ is the greatest closed linear manifold contained in them (their set theoretical intersection, $\mathfrak{M} \cdot \mathfrak{N}$); 0 is the smallest element of M: the set (0) consisting of 0 alone; 1 is the greatest element of M: $\mathfrak{H}$. (We use these notations in this § only.) Then we may define $\mathfrak{M}'$ as the set of all elements of $\mathfrak{H}$ which are orthogonal to all elements of $\mathfrak{M}$: $\mathfrak{H} \ominus \mathfrak{M}$ (cf. (16, p. 74)). One verifies again the lattice postulates I–IV (cf. (1, p. 422)) for M, while the distributive law (postulate VI, eod., p. 453) does not hold in general. (Postulate V, eod., p. 445, holds, thus M is a B-lattice, cf. eod.) (D_1)–(D_3) too hold.

Following the analogy with the complement in Boolean algebras (and "no" in logics) further, we are led to expect

$$\mathbf{M} \cup \mathbf{M}' = 1, \tag{D_4}$$

$$\mathbf{M} \cap \mathbf{M}' = 0. \tag{D_5}$$

(In the lattice M (D_4), (D_5) hold. In fact there is a quite intimate connection between M and a certain type of logics: The probability-logics of quantum mechanics. Cf. (20, pp. 130–134).) These equations, however, are not true in general in R. For instance, if $\mathbf{M}$ is Abelian (cf. (18, p. 374)), then $\mathbf{M} \subset \mathbf{M}'$, and thus $\mathbf{M} \cup \mathbf{M}' = \mathbf{M}'$, $\mathbf{M} \cap \mathbf{M}' = \mathbf{M}$. For this reason it is of interest to find out, for which particular elements $\mathbf{M}$ of R (D_4), (D_5) hold. As $'$ dualises (D_4) and (D_5) (apply $'$ and use (D_1)–(D_3)), they are equivalent. Thus it suffices to discuss one of them. In our original notations they read:

$$R(\mathbf{M}, \mathbf{M}') = \mathbf{B}, \tag{$\overline{D}_4$}$$

$$\mathbf{M} \cdot \mathbf{M}' = (\alpha 1). \tag{$\overline{D}_5$}$$

The theory of these equations will be the chief subject of this paper (cf. §3.1).

3. Let $\mathfrak{H}$ be the n-dimensional Euclidean space $\mathfrak{E}_n$, $n = 1, 2, \ldots$. In this case it is easy to detail the meaning of $(\overline{D}_5)$.

Assume $\mathbf{M} \cdot \mathbf{M}' = (\alpha 1)$. Let $\mathbf{M}^u$ be the set of all unitary elements of $\mathbf{M}$, then $\mathbf{M}$ is the ring generated by $\mathbf{M}^u$ (cf. (18, p. 392)). Thus $\mathbf{M}' = (\mathbf{M}^u)'$; and as $\mathbf{M}^u$ is a group of unitary matrices, $\mathbf{M}$ consists of all linear aggregates of elements of $\mathbf{M}^u$. Now apply the theory of unitary group representations to $\mathbf{M}^u$.

As $\mathbf{M}^u$ consists of unitary matrices, it is completely reducible (cf. (14, pp. 11–12, footnote 1)). Introduce a coordinate system in $\mathfrak{H} = \mathfrak{E}_n$, in which $\mathbf{M}^u$ appears completely reduced. Let the irreducible parts of $\mathbf{M}^u$ correspond to the sets of coordinates

$$p_1 + \cdots + p_{s-1} + 1, \ldots, p_1 + \cdots + p_{s-1} + p_s$$

for $s = 1, \ldots, r$, where $r = 1, 2, \ldots$; $p_1, \ldots, p_r = 1, 2, \ldots$; $p_1 + \cdots + p_r = n$, and denote them by $I_1, \ldots, I_r$. Denote the number of those which are equivalent to I_1 by q $(= 1, 2, \ldots, r)$, and arrange the I_s so that these should be $I_1, \ldots, I_q$. By a further coordinate transformation we can even make $I_1, \ldots, I_q$ identical. Besides $p_1 = \cdots = p_q = p$. Thus every element of $\mathbf{M}^u$ looks like this: It consists of r $(\geq q)$ matrices succeeding each other along the main diagonal, the q first ones being identical and of degree p. The same is true for their linear aggregates, the elements of $\mathbf{M}$.

The matrices which occur in the q first diagonal minors form an irreducible system, inequivalent to those which occur in the $r - q$ other diagonal minors. So the theorems of Burnside, Frobenius, and I. Schur (cf. (3, 8); or (14, p. 412)) apply. Therefore these matrices are perfectly arbitrary, and independent of the $r - q$ other ones. Thus we can prescribe the q first ones to be equal to the unit matrix, and the $r - q$ others to vanish. So an element E_0 of $\mathbf{M}$ results, which commutes obviously with all elements of $\mathbf{M}$; thus $E_0 \in \mathbf{M} \cdot \mathbf{M}'$. As $\mathbf{M} \cdot \mathbf{M}' = (\alpha 1)$, we have $E_0 = \alpha 1$. This clearly necessitates $\alpha = 1$, and so the $r - q$ vanishing diagonal minors in E_0 cannot be present. Therefore $r - q = 0$, $q = r$, $n = p_1 + \cdots + p_q = pq$.

If we write the general element A of $\mathbf{M}$ as a matrix: $A = \{a_{\mu,\nu}\}$, $\mu, \nu = 1, \ldots, n$, then our discussions have characterised A as follows: Put

$$\mu = p(s-1) + \sigma, \qquad \nu = p(t-1) + \tau,$$

where $s, t = 1, \dots, q$; $\sigma, \tau = 1, \dots, p$ (remember $n = pq$), then

$$a_{\mu,\nu} = \begin{cases} 0, & s \neq t, \\ \text{a function of } \sigma, \tau \text{ alone}, & s = t. \end{cases}$$

If we replace the index μ $(= 1, \dots, n)$ by the two indices σ $(= 1, \dots, p)$ and s $(= 1, \dots, q)$, and similarly ν by τ and t, then we have

$$A = \{a_{\sigma,s,\tau,t}\}_{\substack{\sigma,\tau=1,\dots,p \\ s,t=1,\dots,q}} \quad \text{with} \quad a_{\sigma,s,\tau,t} = \delta_{s,t} b_{\sigma,\tau}. \tag{S}$$

Here $\delta_{s,t}$ is the Weierstrass–Kronecker symbol, and $b_{\sigma,\tau}$ is arbitrary.

Conversely: If $\mathbf{M}$ consists of all

$$\{\delta_{s,t} b_{\sigma,\tau}\}_{\substack{\sigma,\tau=1,\dots,p \\ s,t=1,\dots,q}},$$

then it is a ring, $\mathbf{M}'$ consists of all

$$\{\delta_{\sigma,\tau} c_{s,t}\}_{\substack{\sigma,\tau=1,\dots,p \\ s,t=1,\dots,q}},$$

and so $\mathbf{M} \cdot \mathbf{M}'$ of all $\{\delta_{\sigma,\tau}\delta_{s,t}\alpha\} = \alpha 1$. Thus (S) gives the general solution of ($\overline{D}_5$) for $\mathfrak{H} = \mathfrak{E}_n$.

In other words: The general solution of ($\overline{D}_5$) obtains, by replacing the one coordinate index $\mu = 1, \dots, n$ in $\mathfrak{H} = \mathfrak{E}_n$ by two independent coordinate indices $\sigma = 1, \dots, p$ and $s = 1, \dots, q$ $(n = pq)$, and collecting in $\mathbf{M}$ all linear operators A which operate on σ alone, while $\mathbf{M}'$ consists of all those which operate on s alone.

This is the effect of the application of the classical Burnside–Frobenius–I. Schur theory of unitary group representations. Thus the solving of ($\overline{D}_5$) connects directly with the fundamental facts about unitary representations.

If the analogous situation held in Hilbert space, we would be led to expect that if $\mathfrak{H}$ is Hilbert space, the solutions of ($\overline{D}_5$) can be brought in the following form: $\mathfrak{H}$ is isomorphic to the functional space of all functions $f(x, y)$ in two independent variables x, y, ($\iint |f(x, y)|^2 \, dx \, dy$ finite); the ranges of x and of y are continuous or discrete, in the latter case $\int \cdots dx$ resp. $\int \cdots dy$ is meant to indicate a sum. (Cf. (16, pp. 69 and 108–111).) $\mathbf{M}$ consists of all operators which operate on x alone, and $\mathbf{M}'$ of all those which operate on y alone. (Cf. §3.2, where this is more precisely discussed.)

The essentially different character of Hilbert space, compared with the spaces $\mathfrak{E}_n$, $n = 1, 2, \ldots$, is reflected in the fact that this is not so. We will see that if $\mathfrak{H}$ is Hilbert space, $(\overline{D}_5)$ has further solutions. (Cf. in particular §8.6 and §13.2.) Their properties seem to be of great importance for the general theory of operators.

The general importance of the solutions of $(\overline{D}_5)$ for operator theory follows from this fact too, that their knowledge allows a characterisation and classification of all rings of operators. This will be discussed in a subsequent paper of the second author.

4. Another interpretation of $(\overline{D}_5)$ is suggested by quantum mechanics. The operators of $\mathfrak{H}$ correspond there to all observable quantities which occur in a mechanical system $\mathfrak{S}$. (Cf. (6, pp. 55–60), and (20, p. 167). We restrict ourselves to bounded operators, which correspond to those observables which have a bounded range. Thus $\mathbf{B}$ corresponds to the totality of these observables.) Now if $\mathfrak{S}$ can be decomposed into two parts $\mathfrak{S}_1, \mathfrak{S}_2$ and if we denote the set of the operators which correspond to observables situated entirely in $\mathfrak{S}_1$ or in $\mathfrak{S}_2$ by $\mathbf{M}_1$ resp. $\mathbf{M}_2$, then we see:

(1) $\mathbf{M}_1, \mathbf{M}_2$ are rings, and 1 (which corresponds to the "constant" observable 1) belongs to both $\mathbf{M}_1, \mathbf{M}_2$.
(2) If $A \in \mathbf{M}_1$, $B \in \mathbf{M}_2$ then the measurements of the observables of A and B do not interfere (being in different parts of $\mathfrak{S}$); therefore A, B commute (cf. (6, pp. 11–14 and 76), or (20, pp. 117–121)). Thus $\mathbf{M}_2 \subset \mathbf{M}_1'$.
(3) As $\mathfrak{S}$ is the sum of $\mathfrak{S}_1, \mathfrak{S}_2$ therefore $R(\mathbf{M}_1, \mathbf{M}_2) = \mathbf{B}$.

(1)–(3) describe the problem of "factorising" $\mathbf{B}$ which is discussed in more detail in §3.1; it leads to our old problem: As $\mathbf{M}_1' \supset \mathbf{M}_2$ therefore

$$R(\mathbf{M}_1, \mathbf{M}_1') \supset R(\mathbf{M}_1, \mathbf{M}_2) = \mathbf{B},$$

so $R(\mathbf{M}_1, \mathbf{M}_1') = \mathbf{B}$, that is precisely $(\overline{D}_4)$, which, as we know, is equivalent to $(\overline{D}_5)$. Conversely: If $\mathbf{M}$ fulfills $(\overline{D}_4)$ (that is $(\overline{D}_5)$), then $\mathbf{M}_1 = \mathbf{M}$, $\mathbf{M}_2 = \mathbf{M}'$ satisfy (1)–(3). (Cf. §3.1 for more details.)

Thus our problem of solving $(\overline{D}_5)$ corresponds to the quantum mechanical problem of dividing a system $\mathfrak{S}$ into two subsystems $\mathfrak{S}_1, \mathfrak{S}_2$; and in particular the solutions $\mathbf{M}$ of $(\overline{D}_5)$ correspond to the complete rings of all observables of suitable quantum mechanical systems.

This interpretation of $(\overline{D}_5)$ suggests of course strongly the surmise formulated at the end of §2.2: It should be possible to describe $\mathfrak{H}$ as (isomorphic to) the space of all two variable functions $f(x, y)$, ($\iint |f(x, y)|^2\, dx\, dy$ finite), **M** operating on x only, and **M′** on y only. In this case $\mathfrak{S}_1$, $\mathfrak{S}_2$ would be explicitly given: $\mathfrak{S}_1$ being described by the coordinate x, and $\mathfrak{S}_2$ by the coordinate y.

The fact that the surmise of §2.2 is not true, is therefore the more remarkable; particularly so because certain features of the "exceptional" rings **M** seem to make them even better suited for quantum mechanical purposes than the customary **B**. We will now discuss these properties of **M**.

5. The full system of solutions of ($\overline{D}_5$) will be discussed in §8.3–§8.4. While we refer to those sections for a complete discussion, we would like to call attention here to the following: If the surmise formulated at the end of §2.2 holds, then **M** is obviously isomorphic to the ring of all operators on x. That is, according to whether x has a finite or infinite range, **M** is isomorphic to the ring **B** of either a finite dimensional Euclidean space $\mathfrak{E}_n$, $n = 1, 2, \ldots$, or of a Hilbert space $\mathfrak{H}'$. In the systematic notation used in §8.4 these possibilities are labeled as cases (I_n), $n = 1, 2, \ldots$, respectively, (I_∞). (In the last case the range of x may be either continuous or discontinuous, but it is well known, that this does not affect the character of the Hilbert space $\mathfrak{H}'$ at all.) As we mentioned, these cases do not exhaust all possibilities for **M**: further cases, labeled (II_1), (II_∞), (III_∞) may occur. (All of them, except (III_∞) certainly exist; while the existence of (III_∞) is as yet undecided. Cf. §13.3.)

The cases (I_n), $n = 1, 2, \ldots$, are of course the simplest ones, and they are the ones on which our notions as to what operators should look like have been developed. This is true for general operator theory as well as for quantum mechanics. It is therefore of particular importance to compare the other cases (I_∞)–(III_∞) under the following aspects: Which one has the most properties in common with (I_n), the ring of all matrices of n rows and n columns, and which one can be considered as the limiting case of (I_n) for $n \to \infty$? The investigations of operators in Hilbert space have always been carried on with the idea that the **B** of Hilbert space, that is (I_∞), is the natural limiting case. We think however that our results indicate that there is more point in assigning this rôle to (II_1). There are two chief reasons for this assertion: The existence of a trace, and the behavior of unbounded operators. Chaps. XV and XVI of this paper have been devoted to the purpose of deriving these common features of (I_n) and (II_1). In what follows we will make a few qualitative remarks on the subject.

The cases (I_n) and (II_1) are characterised by this property: It is possible to define for all Hermitian operators $A \in \mathbf{M}$ a function $T(A)$ with finite real values, which has the formal properties of the trace (cf. §15.4–§15.5), and there exists only one such function $T(A)$. In the cases (I_n), where the operators $A \in \mathbf{M}$ correspond to matrices $\{a_{\mu,\nu}\}_{\mu,\nu=1,\ldots,n}$, obviously

$$T(A) = \frac{1}{n}\sum_{\mu=1}^{n} a_{\mu,\mu}$$

(the "normalising factor" $1/n$ is necessary, because we require $T(1) = 1$), and it is well known, that in case (I_∞) (that is: for all bounded infinite matrices $\{a_{\mu,\nu}\}_{\mu,\nu=1,2,\ldots}$) no such function $T(A)$ can be defined. (The expression which is formed in analogy to $(1/n)\sum_{\mu=1}^{n} a_{\mu,\mu}$ will not converge in general.) Now we proved, as mentioned above, that the only further case where such a $T(A)$ exists and is unique, is (II_1).

Considering the immediate applicability of $T(A)$ to quantum mechanics (it is the "a priori" expectation value of the observable A, which is correctly normalised here, but cannot be in (I_∞), cf. (20, pp. 165, 169)), it is significant that its existence and uniqueness is a characteristic of (II_1).

A more general problem which will be discussed by the second named author in a forthcoming paper arises now from the following motive: It would be desirable to characterise those abstract algebras, in which one and only one function $T(A)$ with the properties of the trace (as discussed above) exists. (They should be abstract, that is no connection with the operator rings should be postulated.) The result is, that all algebras which meet these requirements are isomorphic to an operator ring $\mathbf{M}$ in a (Euclidean or Hilbert) space $\mathfrak{H}$, which solves $(\overline{D}_5)$ and belongs to cases (I_n), $n = 1, 2, \ldots$, or (II_1).

Returning to case (II_1) let us point out, that the analogy with (I_n) goes so far, that theorems of the following type hold: Every system of linear equations (which can of course be written as one operatorial equation $Af = 0$ where $A \in \mathbf{M}$ is the "coefficient matrix," and $f \in \mathfrak{H}$ represents the variables) has a rank. In case (I_n) this is a number $0, 1, \ldots, n-1, n$, but we replace it by its n-th part: $0, 1/n, \ldots, (n-1)/n, 1$. Now in case (II_1) it is any number α, $0 \leq \alpha \leq 1$. Two adjoint systems of equations $Af = 0$ and $A^*f = 0$ (cf. §16.1) have the same rank. (Observe the analogy with (I_n), and the contrast with (I_∞)!) Similarly a dimensionality for subspaces of $\mathfrak{H}$ (linear, closed sets) can be defined, which has the values $0, 1/n, \ldots, (n-1)/n, 1$ (in the customary normalisation $0, 1, \ldots, n-1, n$)

in case (I_n), and all values α, $0 \le \alpha \le 1$ in case (II_1).

With this notion of dimensionality even the "minimax principle" (cf. (5, pp. 26–29)) can be proved, which thus labels the proper values of all Hermitian operators (in **M**), even in continuous spectra! So it makes sense to speak about the "proper value No. α of the operator A (from below)" for any $\alpha \ge 0$, $\alpha \le 1$, even for irrational ones! (Cf. §15.2.)

6. Let us now consider the unbounded operators. While **M** consists prima facie of bounded operators only, there is a natural way to extend it, so as to include unbounded ones too: An arbitrary (not necessarily bounded) operator X is said to belong to **M**, if it is invariant under all unitary transformations $U' \in \mathbf{M}'$; we denote the set of all those linear, closed operators X with an everywhere dense domain which belong to **M**, by $U(\mathbf{M})$. (Cf. §4.2 and §16.4; as to the notions of linearity and closure cf. (16, p. 70). The bounded elements of $U(\mathbf{M})$ form precisely **M**.)

While in general Hermitian operators possess a resolution of unity if and only if they are hypermaximal (cf. (16, p. 92), or (15, Theorem 9.3, p. 339)) this is much simpler in cases (I_n) and (II_1): Here every Hermitian operator $A \in U(\mathbf{M})$ is hypermaximal, and has therefore a (unique) resolution of unity. (In case (I_n) this is due to the trivial reason, that every linear operator is bounded in that case. But in case (II_1) unbounded operators exist, in the same way as they do in (I_∞)!)

The operators in $U(\mathbf{B})$ ($\mathfrak{H}$ a Hilbert space) behave very pathologically from the point of view of operator algebra: Thus if $A, B \in U(\mathbf{B})$, then $A + B$ and AB cannot be formed in general, and the entire mechanism of replacing operators by their extension leads to very paradoxic results. (Cf. (17, pp. 230–234), where these conditions are discussed in detail.) This applies, of course, to every $U(\mathbf{M})$, **M** in case (I_∞). In cases (I_n) and (II_1) again it is not so: The algebra of $U(\mathbf{M})$ works without any difficulties, and if A is an extension of B, $A, B \in U(\mathbf{M})$, then $A = B$. (Cf. for details §16.3 and §16.4.)

7. A detailed account of the content of this paper is given in the table of contents which follows. The main results are summed up in the Theorems I–XV, while the essential problems are formulated as Problems. It will be pointed out after each problem to what extent it is solved and where the solutions are to be found. The auxiliary considerations, which lead to the Theorems, are grouped into Lemmas.

The notations are explained in §1.1, where the chief references too are given. These, as well as all other quotations, refer to the bibliography, which follows after the table of contents.

Various continuations of the investigations begun in this paper, which we propose to undertake in several subsequent papers, are indicated throughout the text. (Cf. also note at the end of this paper.)

Here however we would like to mention that very similar methods to those used in this paper permit us to discuss the corresponding problems in the non-separable analogues of Hilbert space (cf. (11, 13)). The analogy to Cantor's theory of alephs, which we stress in Chapters VI and VII, becomes then even more apparent. This subject, too, will be treated at a later occasion.

Contents

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Part I: Preparatory Considerations

Chapter I: Notations

1.1. In what follows we will have to assume that the reader is familiar with the unitary–orthogonal–geometrical discussion of the elementary properties of Hilbert space, as contained in the treatise (16, particularly pp. 63–78) or in the remarkable exposition of the subject in (15, Chapter I). Besides we will have to make frequent use of some other papers of one of us, namely (18), (21) and (22), which is a variant to a theorem of (18).

Certain notations will be used on this account subsequently, without further references to their meaning. They are as follows:
(a) $x \in S$ means that x is an element of the set S, $S \subset T$ or $T \supset S$ means that S is a subset of T, (including $S = T$). The set of all elements x with a certain property $\epsilon(x)$, will be denoted by $(x; \epsilon(x))$; the set of all $f(x)$ with x having the property $\epsilon(x)$, by $(f(x); \epsilon(x))$; the set theoretical sum of the sets $S, T, \ldots$, plus the elements $x_0, y_0, \ldots$ by $(S, T, \ldots, x_0, y_0, \ldots)$. The set theoretical product (common part) of the sets $S, T, \ldots$ is denoted by $S \cdot T \cdots$.

A symbol η, such that $x \; \eta \; S$ has a meaning similar to that of $x \in S$, will be defined in §4.2, Definition 4.2.1.
(b) A linear space with a linear product, and which is separable and complete, will be denoted by $\mathfrak{H}$. (We will use affixes and suffixes if more than one such space occurs.) In other words: $\mathfrak{H}$ is a space in which operations $\alpha \cdot f$, $f + g$, (f, g) satisfying the conditions A, B, C, E, of (16, pp. 64–66) are given. Condition D (loc. cit.) is explicitly excepted. Thus $\mathfrak{H}$ is either a Hilbert space or a finite dimensional Euclidean space accordingly as D is fulfilled or not. We only consider $1, 2, \ldots,$ dimensional Euclidean spaces, excluding explicitly the 0-dimensional case where $\mathfrak{H} = (0)$.

(c) We denote complex or real numbers by $\alpha, \epsilon, a, x, C, D$; integers by $i, j, k, m, n, p, q, s, t, u, v$ and N. Elements of $\mathfrak{H}$ are denoted by f, g, φ, ψ, sometimes (in direct products) by Φ, Ψ.
(d) Arbitrary subsets of $\mathfrak{H}$ are denoted by $\mathfrak{S}$, linear subsets by $\mathfrak{A}$, linear and closed subsets by $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{P}$, $\mathfrak{Q}$. As the latter are again spaces of the type of $\mathfrak{H}$, their symbols sometimes replace $\mathfrak{H}$.

The smallest linear and the smallest closed linear set containing certain sets or elements are denoted by $(\dots)$ and $[\dots]$ respectively (instead of $(\dots)$, cf. (a)). The set of all elements of $\mathfrak{N}$ which are orthogonal to $\mathfrak{M}$ is denoted by $\mathfrak{N} - \mathfrak{M}$.
(e) Operators in an $\mathfrak{H}$ will be denoted by A, B, X, Y. For certain special kinds of operators, the following letters will be used: E, F, G and P for projection operators, U, V, and W for unitary and partially isometric operators (cf. (16, p. 70)).
(f) The ring of all bounded, everywhere defined, linear and closed operators in $\mathfrak{H}$ is $\mathbf{B}$, its subrings are $\mathbf{M}, \mathbf{N}$, (cf. (18, p. 388)).

Arbitrary subsets of $\mathbf{B}$ are S, the smallest ring containing S is $R(S)$, the ring of all A which commute with X and X^* for every $X \in S$ is S'.
(g) As discussed in (18, pp. 378–388), two different topologies can be used in $\mathfrak{H}$, and four in $\mathbf{B}$, “Strong” and “Weak” in $\mathfrak{H}$ and $\mathbf{B}$, “uniform” in $\mathbf{B}$ and finally there is the “strongest” topology in $\mathbf{B}$. If nothing in particular is said, we always mean the “strong” topology in $\mathfrak{H}$; otherwise the topology to be used will be specified. The properties of these topologies are not all of the customary type and they are of some importance. Details are given in (18) and (22) loc. cit.
(h) In part IV, a group $\mathfrak{G}$ and a space S, will be considered, where $\mathfrak{G}$ consists of one-to-one mappings of S on itself. We will denote the elements of $\mathfrak{G}$ by a, b, and c; those of S by x, y; the unit, the inverse induced by $a \in \mathfrak{G}$ in S are $1, a^{-1}$. These notations will however only be used in Part IV.

The results of operator theory which we will use most frequently (apart from the general theory of Hilbert space and the spectral form of hypermaximal operators) are these: Theorems 1, 5, 9 in (18), Theorems 5, 6, in (19), and Theorem 7 in (21).

Chapter II: Direct Products

2.1. An operation which generates a new space $\mathfrak{H}$ with the help of n given ones, $\mathfrak{H}_1, \dots, \mathfrak{H}_n$ is the *direct multiplication*. As it is one of the essential elements of the situations which we are going to discuss, and as a general abstract treatment (valid for Hilbert spaces too and not using special coordinate systems) of this notion does not occur in the literature, we will give a systematic discussion.

Definition 2.1.1. Let $n = 1, 2, \ldots$ spaces $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$ be given. Consider all functionals $\Phi(f_1, \ldots, f_n)$, which are defined for all systems

$$f_i \in \mathfrak{H}_i, \quad i = 1, \ldots, n,$$

and have complex values, and which are conjugate linear in each f_i:

$$\text{(i)} \quad \Phi(\ldots, \alpha f_i, \ldots) = \overline{\alpha}\Phi(\ldots, f_i, \ldots),$$
$$\text{(ii)} \quad \Phi(\ldots, f_i + g_i, \ldots) = \Phi(\ldots, f_i, \ldots) + \Phi(\ldots, g_i, \ldots).$$

Call their set $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$.

Definition 2.1.2. If a system $f_i^0 \in \mathfrak{H}_i$, $i = 1, \ldots, n$, is given, form the functional

$$\Phi(f_1, \ldots, f_n) = \prod_{i=1}^{n} (f_i^0, f_i).$$

Clearly $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$. Define

$$\overline{\Phi} = \prod_{i=1}^{n} \otimes f_i^0 = f_1^0 \otimes \cdots \otimes f_n^0.$$

Definition 2.1.3. Consider all finite linear aggregates of the form

$$\overline{\Phi} = \sum_{\nu=1}^{p} \prod_{i=1}^{n} \otimes f_{i,\nu}^0, \qquad p = 0, 1, \ldots, \quad f_{i,\nu}^0 \in \mathfrak{H}_i.$$

Call their set $\prod{}'^{n}_{i=1} \otimes \mathfrak{H}_i$. (Clearly $\prod{}'^{n}_{i=1} \otimes \mathfrak{H}_i \subset \prod_{i=1}^{n} \otimes \mathfrak{H}_i$.)

Lemma 2.1.1. If $\overline{\Phi}, \overline{\Psi} \in \prod{}'^{n}_{i=1} \otimes \mathfrak{H}_i$, that is

$$\overline{\Phi} = \sum_{\nu=1}^{p} \prod_{i=1}^{n} \otimes f_{i,\nu}^0; \qquad \overline{\Psi} = \sum_{\mu=1}^{q} \prod_{i=1}^{n} \otimes g_{i,\mu}^0,$$

then form

$$(\overline{\Phi}, \overline{\Psi}) = \sum_{\nu=1}^{p} \sum_{\mu=1}^{q} \prod_{i=1}^{n} (f_{i,\nu}^0, g_{i,\mu}^0).$$

This expression depends on $\overline{\Phi}, \overline{\Psi}$ only, but not on the particular decomposition used.

Proof: Owing to the linearity it suffices to consider the case when $\overline{\Phi} = 0$ or $\overline{\Psi} = 0$; and owing to the symmetry in $\overline{\Phi}, \overline{\Psi}$, we may assume $\overline{\Psi} = 0$. We will show that each addend of the sum $\sum_{\nu=1}^{p}$ in $(\overline{\Phi}, \overline{\Psi})$ vanishes separately. Indeed:

$$\sum_{\mu=1}^{q} \prod_{i=1}^{n} (f_{i,\nu}^{0}, g_{i,\mu}^{0}) = \overline{\Psi}(f_{1,\nu}^{0}, \ldots, f_{n,\nu}^{0}) = 0.$$

Lemma 2.1.2. If $\overline{\Phi} \in \prod'^{n}_{i=1} \otimes \mathfrak{H}_i$, then $(\overline{\Phi}, \overline{\Phi}) > 0$ for $\overline{\Phi} \neq 0$.

Proof: Consider first any $\overline{\Phi} \in \prod'^{n}_{i=1} \otimes \mathfrak{H}_i$. Then

$$(\overline{\Phi}, \overline{\Phi}) = \sum_{\mu,\nu=1}^{p} \prod_{i=1}^{n} (f_{i,\mu}^{0}, f_{i,\nu}^{0}).$$

For any complex $x_1, \ldots, x_p$,

$$\sum_{\mu,\nu=1}^{p} (f_{i,\mu}^{0}, f_{i,\nu}^{0}) x_\mu \overline{x_\nu} = \left(\sum_{\mu=1}^{p} x_\mu f_{i,\mu}^{0}, \sum_{\nu=1}^{p} x_\nu f_{i,\nu}^{0} \right) = \left\| \sum_{\mu=1}^{p} x_\mu f_{i,\mu}^{0} \right\|^2 \geq 0;$$

therefore each matrix $((f_{i,\mu}^{0}, f_{i,\nu}^{0}))_{\mu,\nu=1,\ldots,p}$ (p dimensional!) is semidefinite. Thus it is the sum of (at most p) semidefinite matrices of rank 1, that is of the form $((\alpha_\mu^i \overline{\alpha_\nu^i}))_{\mu,\nu=1,\ldots,p}$. Thus $(\overline{\Phi}, \overline{\Phi})$ is a sum of terms

$$\begin{aligned} \sum_{\mu,\nu=1}^{p} \prod_{i=1}^{n} \alpha_\mu^i \overline{\alpha_\nu^i} &= \sum_{\mu,\nu=1}^{p} \prod_{i=1}^{n} \alpha_\mu^i \cdot \overline{\prod_{i=1}^{n} \alpha_\nu^i} \\ &= \sum_{\mu=1}^{p} \prod_{i=1}^{n} \alpha_\mu^i \cdot \overline{\sum_{\nu=1}^{p} \prod_{i=1}^{n} \alpha_\nu^i} = \left| \sum_{\mu=1}^{p} \prod_{i=1}^{n} \alpha_\mu^i \right|^2 \geq 0. \end{aligned}$$

So we have $(\overline{\Phi}, \overline{\Phi}) \geq 0$.

From this, Schwarz's inequality

$$|(\overline{\Phi}, \overline{\Psi})| \leq \sqrt{(\overline{\Phi}, \overline{\Phi}) \cdot (\overline{\Psi}, \overline{\Psi})}$$

follows literally as in (16, p. 64). Thus $(\overline{\Phi},\overline{\Phi}) = 0$ implies $(\overline{\Phi},\overline{\Psi}) = 0$ for every $\overline{\Psi} \in \prod'^{n}_{i=1} \otimes\mathfrak{H}_i$. Put $\overline{\Psi} = \prod^{n}_{i=1} \otimes f_i$, then

$$\overline{\Phi}(f_1, \ldots, f_n) = \left(\overline{\Phi}, \prod_{i=1}^{n} \otimes f_i\right) = 0,$$

and so $\overline{\Phi} = 0$. Thus $\overline{\Phi} \neq 0$ implies $(\overline{\Phi},\overline{\Phi}) > 0$.

Lemma 2.1.3. With the above definition of $(\overline{\Phi},\overline{\Psi})$, $\prod'^{n}_{i=1} \otimes\mathfrak{H}_i$ is a linear space with an inner product, that is, it satisfies the conditions A, B, in (16, p. 64). Thus it can be metrized by defining

$$\text{Distance}(\overline{\Phi},\overline{\Psi}) = \|\overline{\Phi} - \overline{\Psi}\|, \qquad \text{where} \qquad \|\overline{\Phi}\| = \sqrt{(\overline{\Phi},\overline{\Phi})}.$$

Proof: This follows immediately from Lemmas 2.1.1 and 2.1.2, remembering (16, pp. 64–65).

Lemma 2.1.4. We have

$$\left\|\prod_{i=1}^{n} \otimes f_i\right\| = \prod_{i=1}^{n} \|f_i\|,$$

and for every $\overline{\Phi} \in \prod'^{n}_{i=1} \otimes\mathfrak{H}_i$,

$$\overline{\Phi}(f_1, \ldots, f_n) = \left(\overline{\Phi}, \prod_{i=1}^{n} \otimes f_i\right), \qquad |\overline{\Phi}(f_1, \ldots, f_n)| \leq \|\overline{\Phi}\| \cdot \prod_{i=1}^{n} \|f_i\|.$$

Proof: The first equation follows directly from the definition of $\|\cdot\cdot\cdot\|$ in $\prod'^{n}_{i=1} \otimes\mathfrak{H}_i$; while the second is obvious, and the third results from Schwarz's inequality:

$$|\overline{\Phi}(f_1, \ldots, f_n)| = \left|\left(\overline{\Phi}, \prod_{i=1}^{n} \otimes f_i\right)\right| \leq \|\overline{\Phi}\| \cdot \left\|\prod_{i=1}^{n} \otimes f_i\right\| = \|\overline{\Phi}\| \cdot \prod_{i=1}^{n} \|f_i\|.$$

Definition 2.1.4. Consider those functionals $\Phi \in \prod^{n}_{i=1} \otimes\mathfrak{H}_i$, for which a sequence $\overline{\Phi}_1, \overline{\Phi}_2, \ldots \in \prod'^{n}_{i=1} \otimes\mathfrak{H}_i$ exists, so that

(i) $\lim\limits_{r\to\infty} \overline{\Phi}_r(f_1, \ldots, f_n) = \Phi(f_1, \ldots, f_n)$ for all $f_i \in \mathfrak{H}_i$,

(ii) $\lim\limits_{r,s\to\infty} \|\overline{\Phi}_r - \overline{\Phi}_s\| = 0$.

We write for this briefly $\Phi \sim (\overline{\Phi}_1, \overline{\Phi}_2, \ldots)$. Call their set

$$\prod_{i=1}^{n} \otimes \mathfrak{H}_i = \mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_n,$$

the *direct product* of $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$.

Lemma 2.1.5. Every sequence $\overline{\Phi}_1, \overline{\Phi}_2, \ldots$ from $\prod'^{n}_{i=1} \otimes \mathfrak{H}_i$ satisfying (ii) in Definition 2.1.4 belongs to precisely one $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ in the sense of (i), (ii) eod.

If $\Phi \sim (\overline{\Phi}_1, \overline{\Phi}_2, \ldots)$, then all other sequences with $\Phi \sim (\overline{\Psi}_1, \overline{\Psi}_2, \ldots)$ are characterised by $\lim_{r\to\infty} \|\overline{\Phi}_r - \overline{\Psi}_r\| = 0$.

Proof: We have (by Lemma 2.1.4)

$$|\overline{\Phi}_r(f_1, \ldots, f_n) - \overline{\Phi}_s(f_1, \ldots, f_n)| \leq \|\overline{\Phi}_r - \overline{\Phi}_s\| \prod_{i=1}^{n} \|f_i\|.$$

Thus $\lim_{r,s\to\infty} |\overline{\Phi}_r(f_1, \ldots, f_n) - \overline{\Phi}_s(f_1, \ldots, f_n)| = 0$, so $\lim_{r\to\infty} \overline{\Phi}_r(f_1, \ldots, f_n)$ exists. Call it $\Phi(f_1, \ldots, f_n)$. Clearly $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$. Thus (i), (ii) hold; therefore $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$; $\Phi \sim (\overline{\Phi}_1, \overline{\Phi}_2, \ldots)$. Φ is unique by (i).

As to the second statement, we can assume, owing to the linearity, that $\Phi = \overline{\Phi}_1 = \overline{\Phi}_2 = \cdots = 0$. If $\lim_{r\to\infty} \|\overline{\Psi}_r\| = 0$ then we find, as above, $0 = \lim_{r\to\infty} \overline{\Psi}_r(f_1, \ldots, f_n)$. And $\|\overline{\Psi}_r - \overline{\Psi}_s\| \leq \|\overline{\Psi}_r\| + \|\overline{\Psi}_s\|$, thus $\lim_{r,s\to\infty} \|\overline{\Psi}_r - \overline{\Psi}_s\| = 0$. Thus $0 \sim (\overline{\Psi}_1, \overline{\Psi}_2, \ldots)$.

Assume conversely $0 \sim (\overline{\Psi}_1, \overline{\Psi}_2, \ldots)$. If we did not have $\lim_{r\to\infty} \|\overline{\Psi}_r\| = 0$, we would have $\|\overline{\Psi}_r\| \geq a > 0$ for a fixed $a > 0$ and infinitely many r's. Considering a suitable subsequence, we obtain it for all r's. Now $\lim_{r\to\infty} \overline{\Psi}_r(f_1, \ldots, f_n) = 0$ implies $\lim_{r\to\infty} (\overline{\Psi}_r, \overline{\Psi}^0) = 0$ for any fixed $\overline{\Psi}^0 \in \prod'^{n}_{i=1} \otimes \mathfrak{H}_i$. Put $\overline{\Psi}^0 = \overline{\Psi}_{r_0}$, where r_0 is so great that $r, s \geq r_0$ imply $\|\overline{\Psi}_r - \overline{\Psi}_s\| \leq a/2$. Then we have for $r \geq r_0$

$$\begin{aligned}
|(\overline{\Psi}_r, \overline{\Psi}_{r_0})| &\geq |(\overline{\Psi}_{r_0}, \overline{\Psi}_{r_0})| - |(\overline{\Psi}_{r_0} - \overline{\Psi}_r, \overline{\Psi}_{r_0})| \\
&\geq \|\overline{\Psi}_{r_0}\|^2 - \|\overline{\Psi}_{r_0} - \overline{\Psi}_r\| \cdot \|\overline{\Psi}_{r_0}\| \\
&\geq \|\overline{\Psi}_{r_0}\|^2 - \frac{a}{2}\|\overline{\Psi}_{r_0}\| = \left(\|\overline{\Psi}_{r_0}\| - \frac{a}{2}\right)\|\overline{\Psi}_{r_0}\| \\
&\geq \left(a - \frac{a}{2}\right) a = \frac{a^2}{2},
\end{aligned}$$

contradicting $\lim_{r\to\infty} (\overline{\Psi}_r, \overline{\Psi}_{r_0}) = 0$.

Lemma 2.1.6. If $\Phi, \Psi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$, that is

$$\Phi \sim (\overline{\Phi}_1, \overline{\Phi}_2, \ldots), \qquad \Psi \sim (\overline{\Psi}_1, \overline{\Psi}_2, \ldots), \qquad \overline{\Psi}_r, \overline{\Phi}_r \in \prod_{i=1}^{n}{}' \otimes \mathfrak{H}_i,$$

then $\lim_{r\to\infty}(\overline{\Phi}_r, \overline{\Psi}_r)$ exists. Denote it by (Φ, Ψ). This quantity depends on Φ, Ψ only, and not on the particular representation used. If $\overline{\Phi}, \overline{\Psi} \in \prod'^{n}_{i=1} \otimes \mathfrak{H}_i$, it agrees with the previous definition.

Proof: $\lim_{r,s\to\infty} \|\overline{\Phi}_r - \overline{\Phi}_s\| = 0$, $\lim_{r,s\to\infty} \|\overline{\Psi}_r - \overline{\Psi}_s\| = 0$ imply by Schwarz's inequality $\lim_{r,s\to\infty} |(\overline{\Phi}_r, \overline{\Psi}_r) - (\overline{\Phi}_s, \overline{\Psi}_s)| = 0$, thus $\lim_{r\to\infty}(\overline{\Phi}_r, \overline{\Psi}_r)$ exists. If further $\Phi \sim (\overline{\Phi}'_1, \overline{\Phi}'_2, \ldots)$, $\Psi \sim (\overline{\Psi}'_1, \overline{\Psi}'_2, \ldots)$, then we have by Lemma 2.1.5 $\lim_{r\to\infty} \|\overline{\Phi}_r - \overline{\Phi}'_r\| = 0$, $\lim_{r\to\infty} \|\overline{\Psi}_r - \overline{\Psi}'_r\| = 0$ and so

$$\lim_{r\to\infty}(\overline{\Phi}_r, \overline{\Psi}_r) = \lim_{r\to\infty}(\overline{\Phi}'_r, \overline{\Psi}'_r).$$

If $\overline{\Phi}, \overline{\Psi} \in \prod'^{n}_{i=1} \otimes \mathfrak{H}_i$, then $\Phi \sim (\overline{\Phi}, \overline{\Phi}, \ldots)$, $\Psi \sim (\overline{\Psi}, \overline{\Psi}, \ldots)$ show that the new (Φ, Ψ) agrees with the old one.

Lemma 2.1.7. If $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$, then $(\Phi, \Phi) > 0$ for $\Phi \neq 0$.

Proof: $(\Phi, \Phi) \geq 0$ follows by continuity from Lemma 2.1.2. If $(\Phi, \Phi) = 0$, then for $\Phi \sim (\overline{\Phi}_1, \overline{\Phi}_2, \ldots)$, $\overline{\Phi}_r \in \prod'^{n}_{i=1} \otimes \mathfrak{H}_i$, $\lim_{r\to\infty}(\overline{\Phi}_r, \overline{\Phi}_r) = 0$,

$$\lim_{r\to\infty} \|\overline{\Phi}_r\| = 0,$$

and thus by Lemma 2.1.5 $\Phi = 0$. So $\Phi \neq 0$ implies $(\Phi, \Phi) > 0$.

Lemma 2.1.8. With the above definition of (Φ, Ψ), $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ is a linear space with an inner product, that is it satisfies the conditions A, B in (16, p. 64). Thus it can be metrized by defining.

$$\text{Distance}(\Phi, \Psi) = \|\Phi - \Psi\|, \qquad \text{where} \qquad \|\Phi\| = (\Phi, \Phi)^{\frac{1}{2}}.$$

Proof: This follows immediately from Lemmas 2.1.6 and 2.1.7, remembering (16, pp. 64–65).

Lemma 2.1.9. $\prod_{i=1}^{n} \otimes f_i$ is a continuous function of the $f_i \in \mathfrak{H}_i$. (We use now the metric, strong, topology in all these spaces.)

Proof: Put $a = \max_{i=1,\ldots,n}\|f_i^0\|$, and assume that all $\|f_i - f_i^0\| \le \epsilon$ for an $\epsilon > 0$. Then

$$\begin{aligned}
&\left\|\prod_{i=1}^{n} \otimes f_i - \prod_{i=1}^{n} \otimes f_i^0\right\| \\
&\quad = \left\|\sum_{j=1}^{n}\left(\prod_{i=1}^{j} \otimes f_i \otimes \prod_{i=j+1}^{n} \otimes f_i^0 - \prod_{i=1}^{j-1} \otimes f_i \otimes \prod_{i=j}^{n} \otimes f_i^0\right)\right\| \\
&\quad = \left\|\sum_{j=1}^{n}\prod_{i=1}^{j-1} \otimes f_i \otimes (f_j - f_j^0) \otimes \prod_{i=j+1}^{n} \otimes f_i^0\right\| \\
&\quad \le \sum_{j=1}^{n}\left\|\prod_{i=1}^{j-1} \otimes f_i \otimes (f_j - f_j^0) \otimes \prod_{i=j+1}^{n} \otimes f_i^0\right\| \\
&\quad \le \sum_{j=1}^{n}\prod_{i=1}^{j-1}\|f_i\| \cdot \|f_j - f_j^0\| \cdot \prod_{i=j+1}^{n}\|f_i^0\| \\
&\quad \le \sum_{j=1}^{n}(a+\epsilon)^{j-1}\epsilon a^{n-j} = (a+\epsilon)^n - a^n.
\end{aligned}$$

Lemma 2.1.10. Lemma 2.1.4 holds for $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ too.

Proof: Follows by continuity.

Lemma 2.1.11. If $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$, $\overline{\Phi}_1, \overline{\Phi}_2, \ldots \in \prod{}'^{n}_{i=1} \otimes \mathfrak{H}_i$ then

$$\Phi \sim (\overline{\Phi}_1, \overline{\Phi}_2, \ldots)$$

is equivalent to $\lim_{r\to\infty}\|\Phi - \overline{\Phi}_r\| = 0$.

Proof: If $\lim_{r\to\infty}\|\Phi - \overline{\Phi}_r\| = 0$, then Lemma 2.1.10 gives Definition 2.1.4, (i), and $\|\overline{\Phi}_r - \overline{\Phi}_s\| \le \|\Phi - \overline{\Phi}_r\| + \|\Phi - \overline{\Phi}_s\|$ gives Definition 2.1.4, (ii). Thus $\Phi \sim (\overline{\Phi}_1, \overline{\Phi}_2, \ldots)$. Conversely, if $\Phi \sim (\overline{\Phi}_1, \overline{\Phi}_2, \ldots)$, then by definition, $\|\Phi - \overline{\Phi}_s\| = \lim_{r\to\infty}\|\overline{\Phi}_r - \overline{\Phi}_s\|$. Thus $\lim_{r,s\to\infty}\|\overline{\Phi}_r - \overline{\Phi}_s\| = 0$ implies $\lim_{s\to\infty}\|\Phi - \overline{\Phi}_s\| = 0$.

Lemma 2.1.12. $\prod{}'^{n}_{i=1} \otimes \mathfrak{H}_i$, $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ are both separable spaces, that is they satisfy condition D in (16, p. 65).

Proof: $\prod{}'^{n}_{i=1} \otimes \mathfrak{H}_i$ is dense in $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ by Lemma 2.1.11, so it suffices to show that $\prod{}'^{n}_{i=1} \otimes \mathfrak{H}_i$ is separable. By Definition 2.1.3 the separability of the set of all $\prod_{i=1}^{n} \otimes f_i$,

$f_i \in \mathfrak{H}_i$, suffices; and this follows, by Lemma 2.1.9 from the separability of the spaces $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$.

Lemma 2.1.13. $\prod_{i=1}^n \otimes \mathfrak{H}_i$ is topologically complete, that is it satisfies condition E in (16, p. 65).

Proof: Assume $\Phi_1, \Phi_2, \ldots \in \prod_{i=1}^n \otimes \mathfrak{H}_i$, $\lim_{r,s\to\infty} \|\Phi_r - \Phi_s\| = 0$. For each Φ_r choose a $\overline{\Phi}_r^0 \in \prod'^n_{i=1} \otimes \mathfrak{H}_i$ with $\|\Phi_r - \overline{\Phi}_r^0\| \leq 1/r$; thus $\lim_{r\to\infty} \|\Phi_r - \overline{\Phi}_r^0\| = 0$. So $\lim_{r,s\to\infty} \|\overline{\Phi}_r^0 - \overline{\Phi}_s^0\| = 0$, and for $\Phi \sim (\overline{\Phi}_1^0, \overline{\Phi}_2^0, \ldots)$, $\lim_{r\to\infty} \|\Phi - \overline{\Phi}_r^0\| = 0$ (by Lemma 2.1.11), and thus $\lim_{r\to\infty} \|\Phi - \Phi_r\| = 0$, too.

This discussion can be summed up as follows:

Theorem I. The direct product of $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$, $\prod_{i=1}^n \otimes \mathfrak{H}_i$, arises by the topological completion of the set $\prod'^n_{i=1} \otimes \mathfrak{H}_i$ of all finite linear aggregates of expressions $\prod_{i=1}^n \otimes f_i^0$, $f_i^0 \in \mathfrak{H}_i$. It is a set of functionals $\Phi = \Phi(f_1, \ldots, f_n)$ in the sense of Definition 2.1.1, thus

$$\prod_{i=1}^n \otimes \mathfrak{H}_i \subset \prod_{i=1}^n \boldsymbol{\otimes} \mathfrak{H}_i$$

and it is a space satisfying conditions A, B, C, E of (16, pp. 64–65) (cf. our §1.1, (b)). Further essential properties are given in Lemmas 2.1.1, 2.1.4, 2.1.9, 2.1.10, 2.1.11.

If $n = 1$, $\prod_{i=1}^n \otimes \mathfrak{H}_i$ coincides not only in our notation with $\mathfrak{H}_1$, but in reality too.

In this case we may write f_1^0 for $\prod_{i=1}^n \otimes f_i^0$, or even better f^0. The linear aggregates of f^0's are f^0's again, so $\prod'^n_{i=1} \otimes \mathfrak{H}_i$ coincides with $\mathfrak{H}_1$. Thus already $\prod'^n_{i=1} \otimes \mathfrak{H}_i$ is complete, and $\prod_{i=1}^n \otimes \mathfrak{H}_i$ contains no further elements.

Note that in this correspondence of the one-variable functionals Φ to the $f^0 \in \mathfrak{H}$, $\Phi(f)$ becomes simply (f^0, f). A well known theorem of M. Fréchet and F. Riesz implies therefore, that in this case the $\Phi \in \prod_{i=1}^n \otimes \mathfrak{H}_i$ are characterised by $\|\Phi(f)\| \leq C \cdot \|f\|$ for a suitable constant C. (Cf. for instance (16, p. 94).)

This latter point is remarkable, because for $n > 1$ the condition

$$|\Phi(f_1, \ldots, f_n)| \leq C \cdot \prod_{i=1}^n \|f_i\|$$

is necessary (cf. Lemma 2.1.10) but not sufficient for the $\Phi \in \prod_{i=1}^n \otimes \mathfrak{H}_i$, as we will show after Lemma 2.2.2.

2.2. It is of interest to discuss the relationship between $\prod_{i=1}^n \otimes \mathfrak{H}_i$ and a set of complete normalised orthogonal sets in each $\mathfrak{H}_i$.

Lemma 2.2.1. Let a complete normalised orthogonal set $\varphi_{i,1}, \varphi_{i,2}, \ldots$ be given in each $\mathfrak{H}_i$, $i = 1, \ldots, n$. Then the $\prod_{i=1}^n \otimes \varphi_{i,t_i}$, $t_1, t_2, \ldots, t_n = 1, 2, \ldots$ form a complete normalised orthogonal set in $\prod_{i=1}^n \otimes \mathfrak{H}_i$.

Proof: By definition

$$\left(\prod_{i=1}^n \otimes \varphi_{i,t_i}, \prod_{i=1}^n \otimes \varphi_{i,u_i} \right) = \prod_{i=1}^n (\varphi_{i,t_i}, \varphi_{i,u_i}) = \prod_{i=1}^n \delta_{t_i,u_i}$$

proving the normalised and orthogonal character. The linear aggregates of the $\varphi_{i,t}$ are dense in $\mathfrak{H}_i$, thus those of the $\prod_{i=1}^n \otimes \varphi_{i,t_i}$ are dense in $\prod{}'^{n}_{i=1} \otimes \mathfrak{H}_i$ (by Lemma 2.1.9), and therefore in $\prod_{i=1}^n \otimes \mathfrak{H}_i$. This proves the completeness too.

Lemma 2.2.2. In order that an n-variable functional $\Phi \in \prod_{i=1}^n \boldsymbol{\otimes} \mathfrak{H}_i$ (cf. Definition 2.1.1) should belong to $\prod_{i=1}^n \otimes \mathfrak{H}_i$

$$|\Phi(f_1, \ldots, f_n)| \leq C \prod_{i=1}^n \|f_i\|$$

(for a suitable constant C) is necessary. If this is the case, Φ is completely characterised by the numerical values

$$\Phi(\varphi_{1,t_1}, \ldots, \varphi_{n,t_n}) = a_{t_1,\ldots,t_n} \quad \text{for all } t_1, \ldots, t_n = 1, 2, \ldots.$$

The finiteness of $\sum_{t_1,\ldots,t_n=1}^{\infty} |a_{t_1,\ldots,t_n}|^2$ is then characteristic for $\Phi \in \prod_{i=1}^n \otimes \mathfrak{H}_i$. With this restriction the $a_{t_1,\ldots,t_n}$ can even be prescribed arbitrarily, and a $\Phi \in \prod_{i=1}^n \otimes \mathfrak{H}_i$ with these $a_{t_1,\ldots,t_n}$ will exist.

Proof: The necessity of the first condition follows from Lemma 2.1.10. It implies, owing to the linearity, that $\Phi(f_1, \ldots, f_n)$ is continuous in the $f_1, \ldots, f_n$. Thus the values $\Phi(\varphi_{1,t_1}, \ldots, \varphi_{n,t_n})$ determine $\Phi(f_1, \ldots, f_n)$ if every f_i is a limit of linear aggregates of $\varphi_{i,1}, \varphi_{i,2}, \ldots$; that is always.

By Lemma 2.1.10,

$$\left(\Phi, \prod_{i=1}^n \otimes \varphi_{i,t_i} \right) = \Phi(\varphi_{1,t_1}, \ldots, \varphi_{n,t_n}) = a_{t_1,\ldots,t_n},$$

therefore $\sum_{t_1,\ldots,t_n=1}^{\infty} |a_{t_1,\ldots,t_n}|^2$ must be finite, by Lemma 2.2.1 and (16, p. 66). Conversely if a system $a_{t_1,\ldots,t_n}$ with a finite $\sum_{t_1,\ldots,t_n=1}^{\infty} |a_{t_1,\ldots,t_n}|^2$ is given, then

$$\Phi = \sum_{t_1,\ldots,t_n=1}^{\infty} a_{t_1,t_2,\ldots,t_n} \prod_{i=1}^n \otimes \varphi_{i,t_i}$$

exists and is $\in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ by Lemma 2.2.1 and (16, p. 67), and for the same reason

$$\Phi(\varphi_{1,t_1}, \ldots, \varphi_{n,t_n}) = \left(\Phi, \prod_{i=1}^{n} \otimes \varphi_{i,t_i}\right) = a_{t_1,\ldots,t_n}.$$

We now see that in the case $n = 2$, the preliminary condition on Φ,

$$|\Phi(f_1, f_2)| \leq C\|f_1\| \cdot \|f_2\|,$$

means that the matrix $(a_{t_1,t_2})_{t_1,t_2=1,2,\ldots}$ associated with Φ is bounded in the sense of Hilbert's theory; while $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ means the finiteness of

$$\sum_{t_1,t_2=1}^{\infty} |a_{t_1,t_2}|^2,$$

that is that the matrix belongs to E. Schmidt's class of matrices. Thus while the preliminary condition is already characteristic for $n = 1$, it is not for $n = 2$. If $\mathfrak{H}_1 = \mathfrak{H}_2 = \mathfrak{H}$ is a Hilbert space, and I a conjugation (passing to the complex conjugate, cf. (15, p. 357)) in $\mathfrak{H}$, then $\Phi(f_1, f_2) = (If_1, f_2)$ is an example for the opposite; because with $\varphi_{1,t} = \varphi_{2,t}$ we obtain $a_{t_1,t_2} = \delta_{t_1,t_2}$.

Lemma 2.2.3. If in the notation of Lemma 2.2.2 $\Phi, \Psi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ corresponds to $a_{t_1,\ldots,t_n}$ resp. $b_{t_1,\ldots,t_n}$, then

$$(\Phi, \Psi) = \sum_{t_1,\ldots,t_n=1}^{\infty} a_{t_1,\ldots,t_n} \overline{b_{t_1,\ldots,t_n}},$$

the series being absolutely convergent.

Proof: The last formula of the proof of Lemma 2.2.2, together with Lemma 2.2.1 and (16, p. 66), give this.

Lemmas 2.2.2 and 2.2.3 show that the dimension of $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ is the product of the dimensions of all $\mathfrak{H}_i$, that is: 0 if at least one of them is 0; ∞ if none of them is 0 but at least one ∞; the finite product if all are finite. From now on we assume that all $\mathfrak{H}_i$ have dimensions $\neq 0$, that is that $\mathfrak{H}_i \neq (0)$.

2.3. We define now certain operators in $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$. We will denote the ring of all bounded, everywhere defined, linear and closed operators in $\mathfrak{H}_i$ by $\mathbf{B}_i$, $i = 1, \ldots, n$ the same in $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ by $\mathbf{B}\otimes$. (Cf. (17, p. 370).)

Lemma 2.3.1. If an operator $A_i \in \mathbf{B}_i$ and a $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ is given, then a unique $\Phi^* \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ with

$$\Phi^*(f_1, \ldots, f_i, \ldots, f_n) = \Phi(f_1, \ldots, A_i^* f_i, \ldots, f_n)$$

exists. (A_i^* is the adjoint of A_i). If $\|A_i f_i\| \leq D\|f_i\|$ for all $f_i \in \mathfrak{H}_i$, then $\|\Phi^*\| \leq D\|\Phi\|$.
Proof: We may assume $i = n$. The uniqueness of Φ^* and its existence in $\prod_{i=1}^{n} \otimes \mathfrak{H}_i$ are obvious.

A C with $|\Phi(f_1, \ldots, f_n)|^2 \leq C^2 \prod_{i=1}^{n} \|f_i\|^2$ exists, therefore there exists for any given $f_1, \ldots, f_{n-1}$ an $f^0 = f^0(f_1, \ldots, f_{n-1}) \in \mathfrak{H}_n$ with $\Phi(f_1, \ldots, f_n) = (f^0, f_n)$, and $\|f^0\| \leq C \prod_{i=1}^{n-1} \|f_i\|$ (cf. (16, p. 94)). Now

$$\Phi^*(f_1, \ldots, f_n) = \Phi(f_1, \ldots, A_n^* f_n) = (f^0, A_n^* f_n) = (A_n f^0, f_n).$$

Furthermore A_n is bounded, therefore a D with $\|A_n f\| \leq D\|f\|$ exists. So we have:

$$\begin{aligned} |\Phi^*(f_1, \ldots, f_n)| &= |(A_n f^0, f_n)| \leq \|A_n f^0\| \cdot \|f_n\| \\ &\leq D\|f^0\| \cdot \|f_n\| \leq CD \prod_{i=1}^{n} \|f_i\|, \end{aligned}$$

and

$$\begin{aligned} &\sum_{t_1, \ldots, t_n = 1}^{\infty} |\Phi^*(\varphi_{1,t_1}, \ldots, \varphi_{n,t_n})|^2 \\ &= \sum_{t_1, \ldots, t_n = 1}^{\infty} |(A_n f^0(\varphi_{1,t_1}, \ldots, \varphi_{n-1,t_{n-1}}), \varphi_{n,t_n})|^2 \\ &= \sum_{t_1, \ldots, t_{n-1} = 1}^{\infty} \|A_n f^0(\varphi_{1,t_1}, \ldots, \varphi_{n-1,t_{n-1}})\|^2 \\ &\leq D^2 \sum_{t_1, \ldots, t_{n-1} = 1}^{\infty} \|f^0(\varphi_{1,t_1}, \ldots, \varphi_{n-1,t_{n-1}})\|^2 \\ &\leq D^2 \sum_{t_1, \ldots, t_n = 1}^{\infty} |(f^0(\varphi_{1,t_1}, \ldots, \varphi_{n-1,t_{n-1}}), \varphi_{n,t_n})|^2 \\ &= D^2 \sum_{t_1, \ldots, t_n = 1}^{\infty} |\Phi(\varphi_{1,t_1}, \ldots, \varphi_{n,t_n})|^2. \end{aligned}$$

Thus $\Phi \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ implies $\Phi^* \in \prod_{i=1}^{n} \otimes \mathfrak{H}_i$ (by Lemma 2.2.2), and we have $\|\Phi^*\| \leq D\|\Phi\|$ (by Lemma 2.2.3).

Lemma 2.3.2. If we define an operator $A_i^{(i)}$ by $A_i^{(i)}\Phi = \Phi^*$ in the sense of Lemma 2.3.1, then $A_i^{(i)} \in \mathbf{B}\otimes$.

Proof: $A_i^{(i)}$ is everywhere defined and linear by Lemma 2.3.1; and if we choose D with $\|A_i f\| \leq D\|f\|$, then $\|A_i^{(i)}\Phi\| \leq D\|\Phi\|$ by the same Lemma. Thus $A_i^{(i)}$ is bounded and therefore it is closed, too.

Lemma 2.3.3.

$$A_i^{(i)} \prod_{j=1}^{n} \otimes f_j = \prod_{j=1}^{i-1} \otimes f_j \otimes A_i f_i \otimes \prod_{j=i+1}^{n} \otimes f_j.$$

Proof: Follows immediately from the definition.

Lemma 2.3.4. The correspondence $A_i \sim A_i^{(i)}$ is a (one to one) isomorphism for the operations $\alpha A_i, A_i^*, A_i + B_i$, and $A_i B_i$.

Proof: This is obvious for αA_i and $A_i + B_i$; for A_i^* and $A_i B_i$ it follows by applying these operators to the $\prod_{j=1}^{n} \otimes f_j$ as the limits of linear aggregates of $\prod_{j=1}^{n} \otimes f_j$ cover all $\prod_{j=1}^{n} \otimes \mathfrak{H}_j$.

That $A_i = B_i$ implies $A_i^{(i)} = B_i^{(i)}$, is clear; the converse is again seen by considering the $\prod_{j=1}^{n} \otimes f_j$.

Definition 2.3.1. Call the set of all $A_i^{(i)}$ if A_i runs over all $\mathbf{B}_i$, $\mathbf{B}_i^{(i)}$.

Lemma 2.3.5.

$$\mathbf{B}_i^{(i)} = (\mathbf{B}_j^{(j)}; j \neq i)';$$

that is: $\mathbf{B}_i^{(i)}$ is the set of all operators in $\mathbf{B}\otimes$ which commute with all elements of all $\mathbf{B}_j^{(j)}$'s $j \neq i$.

Proof: It is obvious that for $j \neq i$, $A_i^{(i)} A_j^{(j)} \Phi = A_j^{(j)} A_i^{(i)} \Phi$ if $\Phi = \prod_{k=1}^{n} \otimes f_k$. Thus this holds for all $\Phi \in \prod_{k=1}^{n} \otimes \mathfrak{H}_k$, $A_j^{(j)}$ and $A_i^{(i)}$ commute. In other words: $\mathbf{B}_i^{(i)} \subset (\mathbf{B}_j^{(j)}; j \neq i)'$.

Assume now $A^0 \in (\mathbf{B}_j^{(j)}; j \neq i)'$. Then A^0 commutes with every $A_j^{(j)}$, $j \neq i$. Introduce again the complete normalised orthogonal systems $\varphi_{j,1}, \varphi_{j,2}, \ldots$ in $\mathfrak{H}_j$, $j = 1, \ldots, n$. The projection $P_{[\varphi_{j,t}]}$ (cf. (16, p. 74)) belongs to $\mathbf{B}_j$, it transforms $\varphi_{j,u}$ into $\delta_{u,t}\varphi_{j,t}$. Thus $P_{[\varphi_{j,t}]}^{(j)} \in \mathbf{B}_j^{(j)}$ transforms $\prod_{k=1}^{n} \otimes \varphi_{k,t_k}$ into $\delta_{t_j,t} \prod_{k=1}^{n} \otimes \varphi_{k,t_k}$; and as this is a complete normalised orthogonal system, this means that $P^{(j)}_{[\varphi_{j,t}]}$ is the projection of

$$\left[\prod_{k=1}^{n} \otimes\varphi_{k,t_k},\ t_j = t\right]$$

(the smallest linear and closed set containing all $\prod_{k=1}^{n} \otimes\varphi_{k,t_k}$ with $t_j = t$). Now A^0 commutes with $P^{(j)}_{[\varphi_{j,t}]}$. Therefore $P^{(j)}_{[\varphi_{j,t}]}\Phi = \Phi$ implies $P^{(j)}_{[\varphi_{j,t}]}A^0\Phi = A^0\Phi$. The former is the case for

$$\Phi = \prod_{k=1}^{i-1} \otimes\varphi_{k,t_k} \otimes f_i \otimes \prod_{k=i+1}^{n} \otimes\varphi_{k,t_k}$$

and $t = t_j$. Thus $A^0\Phi$ when written as a $\prod_{k=1}^{n} \otimes\varphi_{k,u_k}$ series, $u_1, \ldots, u_n = 1, 2, \ldots$ has terms with $u_j = t_j$ only. As this holds for every $j \neq i$ we have

$$A^0\Phi = \prod_{k=1}^{i-1} \otimes\varphi_{k,t_k} \otimes f_i^* \otimes \prod_{k=i+1}^{n} \otimes\varphi_{k,t_k}.$$

Thus f_i^* is determined uniquely by f_i and the t_k, $k \neq i$. Consideration of the operator $P^{(j)}_{[\varphi_{j,t}+\varphi_{j,s}]}$, $j \neq i$, $t \neq s$, shows however, that $t_j = t$ and $t_j = s$ give the same. So the value of t_j does not matter. Thus $f_i^* = A_i f_i$ for a fixed operator A_i in $\mathfrak{H}_i$. It is clear that A_i is everywhere defined and linear; and $\|A^0\Phi\| \leq D\|\Phi\|$ implies $\|A_i f_i\| \leq D\|f_i\|$ so that A_i is bounded and therefore closed. So $A_i \in \mathbf{B}_i$. The definition of A_i gives $A^0\Phi = A_i^{(i)}\Phi$ for all $\Phi = \prod_{i=1}^{n} \otimes\varphi_{i,t_i}$, thus for all Φ. So we have $A^0 = A_i^{(i)} \in \mathbf{B}_i^{(i)}$.

This proves $\mathbf{B}_i^{(i)} \supset (\mathbf{B}_j^{(j)}; j \neq i)'$, completing the proof of $\mathbf{B}_i^{(i)} = (\mathbf{B}_j^{(j)}; j \neq i)'$.

Lemma 2.3.6. $\mathbf{B}_i^{(i)}$ is a ring and contains the operator 1.

Proof: This follows from (18, p. 397), because $\mathbf{B}_i^{(i)}$ has the form S' (by Lemma 2.3.5).

Lemma 2.3.7. Let S_i be a set $\subset \mathbf{B}_i$, and $S_i^{(i)}$ the set of all $A_i^{(i)}$, $A_i \in S_i$, $S_i^{(i)} \subset \mathbf{B}\otimes$. Then S_i is a ring if and only if $S_i^{(i)}$ is one.

Proof: Define a ring by its characteristics established in (22, §3). The only notions entering there are the operations $\alpha A, A^*, A + B, AB$, and the strongest topology, as defined in (22, §2). Now all these are invariant under $A_i \sim A_i^{(i)}$: the algebraic operations by Lemma 2.3.4, and the strongest topology by (22, §4). Besides closure in $\mathbf{B}_i^{(i)}$ and in $\mathbf{B}\otimes$ is the same thing, because $\mathbf{B}_i^{(i)}$ is a ring by Lemma 2.3.6, thus weakly closed in $\mathbf{B}\otimes$, and a fortiori in the strongest sense. This completes the proof.

Lemma 2.3.8.

$$R(\mathbf{B}_1^{(1)}, \ldots, \mathbf{B}_n^{(n)}) = \mathbf{B}\otimes.$$

Proof: We could prove this by a direct construction, but the following indirect proof is quicker: The same considerations as the ones we made in the proof of Lemma 2.3.5 (but now without the exceptional i) show that

$$(\mathbf{B}_j^{(j)}; j = 1, \ldots, n)' = (\alpha 1).$$

So

$$R(\mathbf{B}_1^{(1)}, \ldots, \mathbf{B}_n^{(n)})' = (\mathbf{B}_1^{(1)}, \ldots, \mathbf{B}_n^{(n)})' = (\alpha 1),$$

and by (18, p. 397),

$$R(\mathbf{B}_1^{(1)}, \ldots, \mathbf{B}_n^{(n)}) = (\alpha 1)' = \mathbf{B}\otimes.$$

We will frequently use the above quoted Theorem 8 from (18, p. 397): That if $\mathbf{M}$ is a ring which contains 1, then $\mathbf{M} = \mathbf{M}''$. No particular references will therefore be made to it.

The results of the foregone discussion can be summed up as follows:

Theorem II. The sets $\mathbf{B}_i^{(i)}$, $i = 1, \ldots, n$, defined in Definition 2.3.1, are rings which contain 1; $\mathbf{B}_i^{(i)}$ being isomorphic to $\mathbf{B}_i$. Any two $\mathbf{B}_i^{(i)}$ commute, and

$$R(\mathbf{B}_1^{(1)}, \ldots, \mathbf{B}_n^{(n)}) = \mathbf{B}\otimes.$$

Further essential properties are given in Lemmas 2.3.5, and 2.3.7.

2.4. The asymmetric aspect of the case $n = 2$ is of importance.

Form $\prod_{i=1}^{2} \otimes \mathfrak{H}_i = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, and introduce a (finite or infinite) complete normalised orthogonal system in $\mathfrak{H}_1$: $\varphi_1, \varphi_2, \ldots$. Now define

Definition 2.4.1. If $\Phi \in \mathfrak{H}_1 \otimes \mathfrak{H}_2$, form the $f^0(\varphi_t) \in \mathfrak{H}_2$, $t = 1, 2, \ldots$ as in the proof of Lemma 2.3.1. Denote them by $f^{(t)}$, $t = 1, 2, \ldots$ resp., and write

$$\Phi \sim \langle f^{(1)}, f^{(2)}, \ldots \rangle = \langle f^{(t)} \rangle_{t=1,2,\ldots}.$$

Lemma 2.4.1. The $f^{(t)}$, $t = 1, 2, \ldots$ determine $\Phi \sim \langle f^{(1)}, f^{(2)}, \ldots \rangle$ uniquely. If dimension $\mathfrak{H}_1$ is finite, the $f^{(t)}$ can be prescribed arbitrarily; if it is infinite, then the only restriction is that $\sum_{t=1}^{\infty} \|f^{(t)}\|^2$ must be finite.

If $\Phi \sim \langle f^{(1)}, f^{(2)}, \ldots\rangle$, $\Psi \sim \langle g^{(1)}, g^{(2)}, \ldots\rangle$, then $(\Phi, \Psi) = \sum_t (f^{(t)}, g^{(t)})$. The sum is absolutely convergent, if infinite at all.

Proof: Let $\psi_1, \psi_2, \ldots$ be a complete normalised orthogonal system in $\mathfrak{H}_2$; form, as in Lemma 2.2.2,

$$\Phi(\varphi_{t_1}, \psi_{t_2}) = (f^{(t_1)}, \psi_{t_2}) = a_{t_1,t_2}.$$

Φ is uniquely determined by the a_{t_1,t_2} and so by the $f^{(t_1)}$. For given a_{t_1,t_2}'s Φ exists if $\sum_{t_1,t_2} |a_{t_1,t_2}|^2$ is finite; but if the a_{t_1,t_2} are defined by the $f^{(t_1)}$: $a_{t_1,t_2} = (f^{(t_1)}, \psi_{t_2})$ then

$$\sum_{t_2} |a_{t_1,t_2}|^2 = \|f^{(t_1)}\|^2,$$

therefore the condition is, that $\sum_{t_1} \|f^{(t_1)}\|^2$ is finite.

If the dimension of $\mathfrak{H}_1$ is finite, the number of the t_1's is, and the condition is void. If the dimension of $\mathfrak{H}_1$ is infinite, we obtain the condition: $\sum_{t_1=1}^{\infty} \|f^{(t_1)}\|^2$ is finite. Finally Lemma 2.2.3 gives:

$$(\Phi, \Psi) = \sum_{t_1,t_2} (f^{(t_1)}, \psi_{t_2})\overline{(g^{(t_1)}, \psi_{t_2})} = \sum_{t_1} (f^{(t_1)}, g^{(t_1)}),$$

all sums being absolutely convergent.

The operators in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ can be expressed in a simple normal form, in which only operators from $\mathfrak{H}_2$ occur:

Definition 2.4.2. If A^0 is an operator in $\mathbf{B}\otimes$ (for $\mathfrak{H}_1 \otimes \mathfrak{H}_2$), then form

$$A^0\langle 0, \ldots, 0, f, 0, \ldots\rangle = \langle f_1^*, f_2^*, \ldots\rangle \quad \text{(the } f \text{ stands in place no. } t\text{)}.$$

Every f_s^* defines an operator $A_{t,s}$ by $f_s^* = A_{t,s} f_t$. Write

$$A^0 \sim \langle A_{t,s}\rangle_{t,s=1,2,\ldots}.$$

Lemma 2.4.2. All $A_{t,s}$ belong to $\mathbf{B}_2$ (for $\mathfrak{H}_2$), they determine A^0 uniquely. If dimension of $\mathfrak{H}_1$ is finite, the $A_{t,s}$ can be prescribed arbitrarily; if it is infinite, they cannot.

Proof: $A_{t,s}$ is clearly everywhere defined and linear. A^0 is bounded, so a C with $\|A^0\Phi\| \le C\|\Phi\|$ exists. Then

$$\begin{aligned}\|A_{t,s} f\|^2 &\le \sum_{s'} \|A_{t,s'} f\|^2 = \|A^0\langle 0, 0, \ldots, 0, f, 0, \ldots\rangle\|^2 \\ &\le C^2 \|\langle 0, \ldots, 0, f, 0, \ldots\rangle\|^2 = C^2\|f\|^2, \qquad \|A_{t,s} f\| \le C\|f\|.\end{aligned}$$

Thus $A_{t,s}$ is bounded and closed too. It is clear from Definition 2.4.2, that the $A_{t,s}$ determine A^0 uniquely for all $\Phi = \langle 0, \ldots, 0, f, 0, \ldots \rangle$, and therefore for all Φ.

If dimension of $\mathfrak{H}_1$ is finite, say p, then the formula in Definition 2.4.2 certainly defines an operator A^0, which is everywhere defined and linear. Each $A_{t,s}$ is bounded, say $\|A_{t,s} f\| \leq C_{t,s}\|f\|$. Then

$$A^0\langle f_1, \ldots, f_p \rangle = \left\langle \sum_{t=1}^{p} A_{t,1} f_t, \ldots, \sum_{t=1}^{p} A_{t,p} f_t \right\rangle.$$

$$\begin{aligned}
\|A^0\Phi\|^2 = \|A^0\langle f_1, \ldots, f_p\rangle\|^2 &= \sum_{s=1}^{p} \left\| \sum_{t=1}^{p} A_{t,s} f_t \right\|^2 \\
&\leq \sum_{s=1}^{p} p \sum_{t=1}^{p} \|A_{t,s} f_t\|^2 \\
&\leq \sum_{s=1}^{p} p \sum_{t=1}^{p} C_{t,s}^2 \|f_t\|^2 \leq C^2 \sum_{t=1}^{p} \|f_t\|^2 = C^2\|\Phi\|^2,
\end{aligned}$$

where $C^2 = p \max_{t=1,\ldots,p} \sum_{s=1}^{p} C_{t,s}^2$, and so $\|A^0\Phi\| \leq C\|\Phi\|$. Thus A^0 is bounded and closed too.

It would be easy to formulate the restrictions arising if the dimension of $\mathfrak{H}_1$ is infinite, but they are of a very implicit type. In fact, even if $\mathfrak{H}_2$ is one dimensional, they express that the numerical matrix $\langle a_{t,s} \rangle_{t,s=1,2,\ldots}$ is bounded in the sense of Hilbert's theory.

The rules of computation for this representation $A^0 \sim \langle A_{t,s} \rangle_{t,s=1,2,\ldots}$ are very similar to those of common matrix-algebra.

Lemma 2.4.3. If $A^0, B^0, C^0 \in \mathbf{B}\otimes$, and $A^0 \sim \langle A_{t,s} \rangle_{t,s=1,2,\ldots}$, $B^0 \sim \langle B_{t,s} \rangle_{t,s=1,2,\ldots}$, $C^0 \sim \langle C_{t,s} \rangle_{t,s=1,2,\ldots}$ then the following rules of computation hold:

(i) If $C^0 = \alpha A^0$, then $C_{t,s} = \alpha A_{t,s}$.

(ii) If $C^0 = (A^0)^*$, then $C_{t,s} = A_{s,t}^*$.

(iii) If $C^0 = A^0 + B^0$, then $C_{t,s} = A_{t,s} + B_{t,s}$.

(iv) If $C^0 = A^0 B^0$, then $C_{t,s} = \sum_{n=1}^{\infty} A_{n,s} B_{t,n}$. The $\sum_{n=1}^{\infty}$ converges in the sense of the strong topology of $\mathbf{B}_2$, irrespectively of the order in which the $n = 1, 2, \ldots$ are gone through.

Proof: (i), (iii) are obvious. Denoting $\langle 0, \ldots, 0, f, 0, \ldots \rangle$ (the f stands in place no. t) by $f^{(t)}$, we see: $(A^0 f^{(t)}, g^{(s)}) = (A_{t,s} f, g)$. Therefore in case (ii) we have on one hand $(C^0 f^{(t)}, g^{(s)}) = (C_{t,s} f, g)$ and on the other

$$(C^0 f^{(t)}, g^{(s)}) = \overline{(A^0 g^{(s)}, f^{(t)})} = \overline{(A_{s,t} g, f)} = (A^*_{s,t} f, g)$$

thus proving $C_{t,s} = A^*_{s,t}$.

Consider finally case (iv). We have $C^0 f^{(t)} = \langle C_{t,s} f\rangle_{s=1,2,\ldots}$.

On the other hand

$$C^0 f^{(t)} = A^0 B^0 f^{(t)} = A^0 \langle B_{t,n} f\rangle_{n=1,2,\ldots} = A^0 \sum_{n=1}^{\infty} (B_{t,n} f)^{(n)}$$

where the $\sum_{n=1}^{\infty}$ is strongly convergent, irrespectively of the order in which the $n = 1, 2, \ldots$ are gone through. So we have further (as A^0 is continuous):

$$\begin{aligned} C^0 f^{(t)} &= \sum_{n=1}^{\infty} A^0 (B_{t,n} f)^{(n)} = \sum_{n=1}^{\infty} \langle A_{n,s} B_{t,n} f\rangle_{s=1,2,\ldots} \\ &= \left\langle \sum_{n=1}^{\infty} A_{n,s} B_{t,n} f \right\rangle_{s=1,2,\ldots} \end{aligned}$$

with the same kind of convergence. This proves $C_{t,s} = \sum_{n=1}^{\infty} A_{n,s} B_{t,n}$. We now investigate the correspondences $A_i \sim A_i^{(i)}$.

Lemma 2.4.4. If $A \in \mathbf{B}_2$, then $A^{(2)} \in \mathbf{B}\otimes$ is

$$A^{(2)} \sim \langle \delta_{t,s} A\rangle_{t,s=1,2,\ldots}.$$

Proof: Obviously $\varphi_t \otimes f \sim \langle 0, \ldots, 0, f, 0, \ldots\rangle$ (f in place no. t), therefore by Lemma 2.3.3

$$A^{(2)} \langle 0, \ldots, 0, f, 0, \ldots\rangle = \langle 0, \ldots, 0, Af, 0, \ldots\rangle.$$

Thus $A_{t,s} = 0$ if $s \neq t$ and $= A$ if $s = t$, in other words $A_{t,s} = \delta_{t,s} A$.

Lemma 2.4.5. If S is a set $\subset \mathbf{B}_2$, then the set $S^{(2)}$ of all $A^{(2)}$, $A^{(2)} \in S$, consists of all $A^0 \sim \langle \delta_{t,s} A\rangle_{t,s=1,2,\ldots}$, $A \in S$ (every $A \in S$ generates an A^0!); and the set $S^{(2)\prime}$ consists of all $A^0 \sim \langle A_{t,s}\rangle_{t,s=1,2,\ldots}$, $A_{t,s} \in S'$.

Proof: The first statement follows immediately from Lemma 2.4.4. As to the second, observe that

$$\langle \delta_{t,s} A\rangle \langle A_{t,s}\rangle = \langle A A_{t,s}\rangle; \qquad \langle A_{t,s}\rangle \langle \delta_{t,s} A\rangle = \langle A_{t,s} A\rangle;$$

thus $\langle A_{t,s}\rangle$ commutes with $A^{(2)} \sim \langle \delta_{t,s}A\rangle$ if and only if every $A_{t,s}$ commutes with A. Furthermore $A^{(2)*} = A^{*(2)}$ (by Lemma 2.3.4, $\langle(\delta_{s,t}A)^*\rangle = \langle\delta_{t,s}A^*\rangle$). These facts together establish the equivalence of $\langle A_{t,s}\rangle \in S^{(2)\prime}$ and of all $A_{t,s} \in S'$.

Lemma 2.4.6. $\mathbf{B}_2^{(2)}$ is the set of all $A^0 \sim \langle\delta_{t,s}A\rangle_{t,s=1,2,\ldots}$, $A \in \mathbf{B}_2$ (every $A \in \mathbf{B}_2$ generates an A^0!); $\mathbf{B}_1^{(1)}$ is the set of all $A^0 \sim \langle\alpha_{t,s}1\rangle_{t,s=1,2,\ldots}$.

Proof: Follows from Lemma 2.4.5 by putting $S = \mathbf{B}_2$, considering that $\mathbf{B}_2^{(2)\prime} = \mathbf{B}_1^{(1)}$ (by Lemma 2.3.5).

In those applications in which the aspect of this § will be used, the important object will always be $\mathfrak{H}_2$, while $\mathfrak{H}_1$ will only play a rôle in so far as its dimension may be prescribed. If dimension of $\mathfrak{H}_1$ is $N = 1, 2, \ldots, \infty$, and if we write $\mathfrak{H}$ for $\mathfrak{H}_2$, we will use the abbreviated notation $N \otimes \mathfrak{H}$ for $\mathfrak{H}_1 \otimes \mathfrak{H}_2$.

Chapter III: Factorisation of B

3.1. We now formulate the two notions which play the central rôle in all our discussions (chiefly the second one). We consider a space $\mathfrak{H}$ and the ring $\mathbf{B}$ of everywhere defined, bounded, linear and closed operators in $\mathfrak{H}$.

Definition 3.1.1. A system of $n(= 1, 2, \ldots)$ subrings $\mathbf{M}_1, \ldots, \mathbf{M}_n \neq (0)$ of $\mathbf{B}$ is called a *factorisation*, if

(i) $\mathbf{M}_i \subset \mathbf{M}_j'$, that is every element of $\mathbf{M}_i$ commutes with every element of $\mathbf{M}_j$, for $i \neq j$,

(ii) $R(\mathbf{M}_1, \ldots, \mathbf{M}_n) = \mathbf{B}$.

Definition 3.1.2. A subring $\mathbf{M}$ of $\mathbf{B}$ is called a *factor*, if

$$\mathbf{M} \cdot \mathbf{M}' = (\alpha 1),$$

that is, if the center of $\mathbf{M}$ (the set of all those elements of $\mathbf{M}$ which commute with every element of $\mathbf{M}$) consists of the numerical multiples of 1 alone.

The connection between these two notions is given by the following Lemmas:

Lemma 3.1.1. *If $\mathbf{M}_1, \ldots, \mathbf{M}_n$ is a factorisation, then every $\mathbf{M}_i$ is a factor.*

Proof: We have

$$\mathbf{M}_i \cdot \mathbf{M}_i' \subset \left(\prod_{\substack{j=1\\ j\neq i}}^{n} \mathbf{M}_j'\right) \cdot \mathbf{M}_i' = \prod_{j=1}^{n} \mathbf{M}_j' = (\mathbf{M}_1, \ldots, \mathbf{M}_n)'$$
$$= \big(R(\mathbf{M}_1, \ldots, \mathbf{M}_n)\big)' = \mathbf{B}' = (\alpha 1).$$

By (18, p. 393), $\mathbf{M}_i \cdot \mathbf{M}'_i$ contains a projection E_0; thus $E_0 = \alpha 1$, that is $E_0 = 0$ or 1. $E_0 = 0$ would imply $\mathbf{M}_i = (0)$ (cf. eod.: if $A \in \mathbf{M}_i$, then $AE_0 = A$), thus $E_0 = 1$. Therefore $\mathbf{M}_i \cdot \mathbf{M}'_i = (\alpha 1)$.

Lemma 3.1.2. *$\mathbf{M}$ is a factor if and only if $\mathbf{M}, \mathbf{M}'$ is a factorisation.*

Proof: If $\mathbf{M}, \mathbf{M}'$ is a factorisation then $\mathbf{M}$ is a factor by Lemma 3.1.1. If conversely $\mathbf{M}$ is a factor, then $1 \in \mathbf{M} \cdot \mathbf{M}'$, and so $\mathbf{M}, \mathbf{M}' \neq (0)$; further $1 \in \mathbf{M} \subset R(\mathbf{M}, \mathbf{M}')$, and so $\mathbf{M} = \mathbf{M}''$ and $R(\mathbf{M}, \mathbf{M}') = (R(\mathbf{M}, \mathbf{M}'))''$. Now

$$(R(\mathbf{M}, \mathbf{M}'))' = (\mathbf{M}, \mathbf{M}')' = \mathbf{M}' \cdot \mathbf{M}'' = \mathbf{M}' \cdot \mathbf{M} = (\alpha 1).$$

$R(\mathbf{M}, \mathbf{M}') = (\alpha 1)' = \mathbf{B}$. Finally $\mathbf{M}$ and $\mathbf{M}'$ commute. Thus $\mathbf{M}, \mathbf{M}'$ is a factorisation.

These Lemmas prove that if $\mathbf{M}$ is a factor, then $\mathbf{M}'$ is one too. If $\mathbf{M}_1$ is a factor then it is a ring and $1 \in \mathbf{M}_1$; thus $\mathbf{M}_1 = \mathbf{M}''_1$; therefore $\mathbf{M}'_1 = \mathbf{M}_2$ implies $\mathbf{M}'_2 = \mathbf{M}_1$. Therefore we can define:

Definition 3.1.3. If $\mathbf{M}_1, \mathbf{M}_2$ is a factorisation, then $\mathbf{M}_1$ and $\mathbf{M}_2$ are called *coupled factors* if $\mathbf{M}'_1 = \mathbf{M}_2$ or (which is equivalent to it) $\mathbf{M}'_2 = \mathbf{M}_1$. The following problem arises immediately:

Problem 1. *Is every factorisation $\mathbf{M}_1, \mathbf{M}_2$ made up of coupled factors?*

We will see that the answer is affirmative in a certain special case (cf. Lemma 3.2.4) but negative in general (cf. §13.4). This problem is of some interest in the case $n > 2$ too, because if $\mathbf{M}_1, \ldots, \mathbf{M}_n$ is a factorisation, then

$$R(\mathbf{M}_1, \ldots, \mathbf{M}_k), \qquad R(\mathbf{M}_{k+1}, \ldots, \mathbf{M}_n), \qquad 1 \leq k \leq n-1,$$

is one too.

3.2. A simple example of a factorisation is this one: If

$$\mathfrak{H} = \prod_{i=1}^{n} \otimes \mathfrak{H}_i = \mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_n,$$

then $\mathbf{B}_1^{(1)}, \ldots, \mathbf{B}_n^{(n)}$ form a factorisation by Theorem II. Thus every $\mathbf{B}_i^{(i)}$ is a factor. This suggests the following definitions:

Definition 3.2.1. A factorisation $\mathbf{M}_1, \ldots, \mathbf{M}_n$ is a *direct factorisation*, if $\mathfrak{H}$ is isomorphic to a direct product $\mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_n$, so that $\mathbf{M}_1, \ldots, \mathbf{M}_n$ becomes $\mathbf{B}_1^{(1)}, \ldots, \mathbf{B}_n^{(n)}$ resp.

Definition 3.2.2. A factor $\mathbf{M}$ is a *direct factor*, if $\mathfrak{H}$ is isomorphic to a direct product $\mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_n$, so that $\mathbf{M}$ becomes a $\mathbf{B}_i^{(i)}$, $i = 1, \ldots, n$.

The following remarks lie near:

Lemma 3.2.1. *We can restrict ourselves in Definition 3.2.1 to the case where $n = 2$, $i = 1$ (or just as well $i = 2$).*

Proof: Lemmas 2.2.2 and 2.2.3 show, that $\prod_{j=1}^{n} \otimes \mathfrak{H}_j$ is isomorphic to

$$\mathfrak{H}_i \otimes \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} \otimes \mathfrak{H}_j \right) \quad \text{and to} \quad \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} \otimes \mathfrak{H}_j \right) \otimes \mathfrak{H}_i .$$

Lemma 3.2.2. *If $\mathbf{M}_1, \ldots, \mathbf{M}_n$ is a direct factorisation, then every $\mathbf{M}_i$ is a direct factor.*

Proof: Obvious.

Lemma 3.2.3. *$\mathbf{M}$ is a direct factor if and only if $\mathbf{M}, \mathbf{M}'$ is a direct factorisation.*

Proof: If $\mathbf{M}, \mathbf{M}'$ is a direct factorisation, then $\mathbf{M}$ is a direct factor by Lemma 3.2.2. If conversely $\mathbf{M}$ is a direct factor, then we may assume by Lemma 3.2.1 that $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, $\mathbf{M} = \mathbf{B}_1^{(1)}$. Then by Lemma 2.3.5 $\mathbf{M}' = (\mathbf{B}_1^{(1)})' = \mathbf{B}_2^{(2)}$, proving the statement.

Lemmas 3.2.2 and 3.2.3 are perfect analogs of Lemmas 3.1.1 and 3.1.2. We now solve Problem 1 for direct factors.

Lemma 3.2.4. *If $\mathbf{M}_1, \mathbf{M}_2$ is a factorisation, and either $\mathbf{M}_1$ or $\mathbf{M}_2$ is a direct factor, then both are it, they are coupled, and $\mathbf{M}_1, \mathbf{M}_2$ is a direct factorisation.*

Proof: Owing to the symmetry we may assume that $\mathbf{M}_1$ is a direct factor, and thus that $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, $\mathbf{M}_1 = \mathbf{B}_1^{(1)}$. Now $\mathbf{M}_2 \subset \mathbf{M}_1' = (\mathbf{B}_1^{(1)})' = \mathbf{B}_2^{(2)}$. Besides

$$\mathbf{M}_2' \cdot \mathbf{B}_2^{(2)} = \mathbf{M}_2' \cdot \mathbf{M}_1' = (\mathbf{M}_1, \mathbf{M}_2)' = (R(\mathbf{M}_1, \mathbf{M}_2))' = \mathbf{B}' = (\alpha 1).$$

Now consider the mapping $A^{(2)} \leftrightarrow A_2$ of $\mathbf{B}_2^{(2)}$ on $\mathbf{B}_2$ (belonging to $\mathfrak{H}_2$). It carries the ring $\mathbf{M}_2 \subset \mathbf{B}_2^{(2)}$ into a ring $\mathbf{N} \subset \mathbf{B}_2$ by Lemma 2.3.4 and the ring $\mathbf{M}_2' \cdot \mathbf{B}_2^{(2)}$ into $\mathbf{N}'$ by Lemma 2.3.7. So $\mathbf{N}' = (\alpha 1)$, and therefore $\mathbf{N} = (\alpha 1)' = \mathbf{B}_2$, $\mathbf{M}_2 = \mathbf{B}_2^{(2)}$. Thus $\mathbf{M}_2 = \mathbf{M}_1'$, $\mathbf{M}_1 = \mathbf{B}_1^{(1)}$, $\mathbf{M}_2 = \mathbf{B}_2^{(2)}$, proving all statements.

We have thus a general picture of the formal properties of factorisations and factors, general, direct, and coupled. A further point which is of interest in this connection will be elucidated by Lemma 5.4.1.

The decisive question is now this:

Problem 2. *Is every factorisation direct? Is every factor direct?*

We will see in §8.4 and §13.2, that the answer to the second and therefore (by Lemma 3.2.2) to both questions is no. We will therefore have to investigate and characterise the non-direct factors.

Chapter IV: Auxiliary Theorems

4.1. The theorem which follows has some interest of its own. We therefore prove it in its general form, although we will mostly need its extremely special Corollary only.

Theorem III. *Let* $\mathbf{M}$ *be a factor and let operators* $A_1, \dots, A_n \in \mathbf{M}$, $A'_1, \dots, A'_n \in \mathbf{M}'$ *be given. We have*

$$\sum_{i=1}^{n} A_i A'_i = 0,$$

if and only if a matrix $(a_{i,j})_{i,j=1,\dots,n}$ *exists, such that*

$$\sum_{i=1}^{n} a_{i,j} A_i = 0, \qquad \sum_{j=1}^{n} a_{i,j} A'_j = A'_i.$$

Proof: The condition is obviously sufficient: If it is valid, $\sum_{i,j=1}^{n} a_{i,j} A_i A'_j$ gives, when first summed over i, 0; and when first summed over j, $\sum_{i=1}^{n} A_i A'_i$. In order to prove its necessity, assume $\sum_{i=1}^{n} A_i A'_i = 0$.

Form the space $n \otimes \mathfrak{H}$ of all systems $\langle f_1, \dots, f_n \rangle$ in $\mathfrak{H}$, with the inner product

$$(\langle f_1, \dots, f_n \rangle, \langle g_1, \dots, g_n \rangle) = \sum_{i=1}^{n} (f_i, g_i)$$

(cf. §2.4). Let $\overline{\mathfrak{M}}$ be the set of all those $\langle f_1, \dots, f_n \rangle$, for which $\sum_{i=1}^{n} A_i X f_i = 0$ for every $X \in \mathbf{M}$. $\overline{\mathfrak{M}}$ is obviously linear and closed. Let $\overline{E}$ be the projection of $\overline{\mathfrak{M}}$. $\overline{E}$ can be written as an operator matrix $\langle E_{r,s} \rangle_{r,s=1,\dots,n}$.

Assume now $\langle f_1, \dots, f_n \rangle \in \overline{\mathfrak{M}}$. If $X_0 \in \mathbf{M}$, then for every $X \in \mathbf{M}$

$$\sum_{i=1}^{n} A_i X X_0 f_i = 0,$$

thus $\langle X_0 f_1, \dots, X_0 f_n\rangle \in \overline{\mathfrak{M}}$. If $X_0' \in \mathbf{M}'$, then for every $X \in \mathbf{M}$,

$$\sum_{i=1}^{n} A_i X X_0' f_i = X_0' \sum_{i=1}^{n} A_i X f_i = 0,$$

thus $\langle X_0' f_1, \dots, X_0' f_n\rangle \in \overline{\mathfrak{M}}$. Thus $\Phi \in \overline{\mathfrak{M}}$ and $X_0 \in \mathbf{M}$ or $\mathbf{M}'$ imply for $X_0^{(2)} = \langle \delta_{r,s} X_0\rangle_{r,s=1,\dots,n}$, $X_0^{(2)}\Phi \in \overline{\mathfrak{M}}$. As every $\overline{E}\Phi \in \overline{\mathfrak{M}}$, we have $X_0^{(2)}\overline{E}\Phi \in \overline{\mathfrak{M}}$,

$$\overline{E} X_0^{(2)} \overline{E}\Phi = X_0^{(2)}\overline{E}\Phi,$$

that is $\overline{E} X_0^{(2)} \overline{E} = X_0^{(2)}\overline{E}$. Replacing X_0 by X_0^*, and thus $X_0^{(2)}$ by $X_0^{*(2)} = X_0^{(2)*}$, and applying $*$ gives $\overline{E} X_0^{(2)} \overline{E} = \overline{E} X_0^{(2)}$. Thus $X_0^{(2)}\overline{E} = \overline{E} X_0^{(2)}$, or in other words $X_0 E_{r,s} = E_{r,s} X_0$. Thus $E_{r,s}$ commutes with every $X_0 \in \mathbf{M}$ or $\mathbf{M}'$, $E_{r,s} \in \mathbf{M}' \cdot \mathbf{M}'' = \mathbf{M}' \cdot \mathbf{M} = (\alpha 1)$, that is $E_{r,s} = a_{r,s} 1$.

Consider $\langle A_1' f, \dots, A_n' f\rangle$ for an arbitrary $f \in \mathfrak{H}$. If $X \in \mathbf{M}$, then

$$\sum_{i=1}^{n} A_i X A_i' f = \sum_{i=1}^{n} A_i A_i' X f = 0.$$

Thus $\langle A_1' f, \dots, A_n' f\rangle \in \overline{\mathfrak{M}}$,

$$\overline{E}\langle A_1' f, \dots, A_n' f\rangle = \langle A_1' f, \dots, A_n' f\rangle.$$

That is:

$$A_i' f = \sum_{j=1}^{n} E_{i,j} A_j' f = \sum_{j=1}^{n} a_{i,j} A_j' f, \qquad \sum_{j=1}^{n} a_{i,j} A_j' = A_i'.$$

Consider next $\langle A_1^* f, \dots, A_n^* f\rangle$ for an arbitrary $f \in \mathfrak{H}$. Further choose an $\langle f_1, \dots, f_n\rangle \in \overline{\mathfrak{M}}$. Then

$$\big(\langle A_1^* f, \dots, A_n^* f\rangle, \langle f_1, \dots, f_n\rangle\big) = \sum_{i=1}^{n} (A_i^* f, f_i) = \sum_{i=1}^{n} (f, A_i f_i)$$
$$= \left(f, \sum_{i=1}^{n} A_i f_i\right) = 0.$$

Thus $\langle A_1^* f, \ldots, A_n^* f\rangle$ is orthogonal to $\overline{\mathfrak{M}}$, and therefore

$$\overline{E}\langle A_1^* f, \ldots, A_n^* f\rangle = 0.$$

That is:

$$0 = \sum_{j=1}^{n} E_{i,j} A_j^* f = \sum_{j=1}^{n} a_{i,j} A_j^* f, \qquad \sum_{j=1}^{n} a_{i,j} A_j^* = 0.$$

Applying $*$ gives $\sum_{j=1}^{n} \overline{a}_{i,j} A_j = 0$. $\overline{E}$ is Hermitian, this implies obviously $E_{r,s}^* = E_{s,r}$, $\overline{a}_{r,s} = a_{s,r}$. Thus the last equation becomes $\sum_{j=1}^{n} a_{j,i} A_j = 0$. So the proof is completed.

Corollary: *For $A \in \mathbf{M}$, $A' \in \mathbf{M}'$ we have $AA' = 0$ if and only if $A = 0$ or $A' = 0$.*

Proof: Put $n = 1$, $a_{1,1} = \alpha$: $AA' = 0$ means $\alpha A = 0$, $(1-\alpha)A' = 0$. This means for $\alpha = 0$, $\alpha = 1$, $\alpha \neq 0, 1$, that $A' = 0$ resp. $A = 0$ resp. both.

4.2. We define

Definition 4.2.1. Let $\mathbf{M}$ be a ring which contains 1. A subset $\mathfrak{S}$ of $\mathfrak{H}$ or an arbitrary operator Z in $\mathfrak{H}$ (not necessarily $\in \mathbf{B}$!) are said to *belong* to the ring $\mathbf{M}$ in symbols $\mathfrak{S}\ \eta\ \mathbf{M}$ resp. $Z\ \eta\ \mathbf{M}$ (we use the η in order to distinguish this from the element-relation $\in$), if they are invariant under every unitary $U \in \mathbf{M}'$: that is, $U\mathfrak{S} = \mathfrak{S}$ resp. $UZU^{-1} = Z$ for all $U \in \mathbf{M}'$.

The relationship between η and $\in$ is easily established.

Lemma 4.2.1. *If $\mathfrak{M}$ is a linear closed set, then $\mathfrak{M}\ \eta\ \mathbf{M}$ is equivalent to $P_{\mathfrak{M}} \in \mathbf{M}$ (cf. (16, p. 74)). If $Z \in \mathbf{B}$, then $Z\ \eta\ \mathbf{M}$ is equivalent to $Z \in \mathbf{M}$.*

Proof: $\mathfrak{M}\ \eta\ \mathbf{M}$ means $UP_{\mathfrak{M}}U^{-1} = P_{U\mathfrak{M}} = P_{\mathfrak{M}}$, $UP_{\mathfrak{M}} = P_{\mathfrak{M}}U$ if U is unitary and $\in \mathbf{M}'$, thus $P_{\mathfrak{M}}\ \eta\ \mathbf{M}$. Thus it suffices to prove the second statement.

If $Z \in \mathbf{B}$ then $Z\ \eta\ \mathbf{M}$ means $UZ = ZU$ for all unitary $U \in \mathbf{M}'$ that is for $U \in \mathbf{M}'^u$ (cf. (18, p. 392)). That is:

$$Z \in (\mathbf{M}'^u)' = (R(\mathbf{M}'^u))' = \mathbf{M}'' = \mathbf{M}.$$

Lemma 4.2.2. *The conditions of Definition 4.2.1 can be replaced by $U\mathfrak{S} \subset \mathfrak{S}$ and $UZ \subset ZU$. The latter means that Z commutes with U in the sense of (18, p. 404).*

Proof: Replacing U by U^{-1} (which is unitary and $\in \mathbf{M}'$ too, $U^{-1} = U^*$) and multiplying left resp. on both sides with U gives $\mathfrak{S} \subset U\mathfrak{S}$ and $ZU \subset UZ$. Thus $\mathfrak{S} = U\mathfrak{S}$ and $UZU^{-1} = Z$, $UZ = ZU$.

4.3. We will now define a class of operators which is a generalisation of the unitary ones.

Definition 4.3.1. An operator $U \in \mathbf{B}$ is *partially isometric* if it maps a linear closed set $\mathfrak{E}$ isometrically on another linear closed set $\mathfrak{F}$, while it maps $\mathfrak{H} - \mathfrak{E}$ on 0. That is: $f \in \mathfrak{E}$ implies $Uf \in \mathfrak{F}$ and $\|Uf\| = \|f\|$, the range of U is all $\mathfrak{F}$, $f \in \mathfrak{H} - \mathfrak{E}$ implies $Uf = 0$. $\mathfrak{E}$ and $\mathfrak{F}$ are the *initial* resp. *final* sets of U.

A unitary operator U is obviously such a partially isometric one, with $\mathfrak{H}$ as initial and final set.

Lemma 4.3.1. *$P_{\mathfrak{E}} = U^*U$, $P_{\mathfrak{F}} = UU^*$, U^* is partially isometric too, with interchanged initial and final sets: $\mathfrak{F}$, $\mathfrak{E}$; as a mapping of $\mathfrak{F}$ on $\mathfrak{E}$, U^* is the inverse of the mapping of $\mathfrak{E}$ on $\mathfrak{F}$, U.*

Proof: If $f, g \in \mathfrak{E}$, then substitution of $h = (f \pm g)/2, (f \pm ig)/2$ into $\|Uh\|^2 = \|h\|^2$ gives $(Uf, Ug) = (f, g)$, that is $(Uf, Ug) = (P_{\mathfrak{E}}f, P_{\mathfrak{E}}g)$ (cf. (16, p. 71)). If f or $g \in \mathfrak{H} - \mathfrak{E}$, the latter equation remains valid, as both sides vanish. Owing to the linearity, it holds therefore for all f, g. We can write it as $(U^*Uf, g) = (P_{\mathfrak{E}}f, g)$. So $U^*U = P_{\mathfrak{E}}$. If the character of U^* were already established, substitution of U^* would give $UU^* = P_{\mathfrak{F}}$. So we need only to discuss U^*.

For $f \in \mathfrak{E}$, $U^*Uf = P_{\mathfrak{E}}f = f$, so U^* maps $\mathfrak{F}$ to $\mathfrak{E}$ inversely to U. Thus this mapping is isometric in $\mathfrak{F}$. Assume now $f \in \mathfrak{H} - \mathfrak{F}$. Then $(f, Ug) = 0$ whenever $Ug \in \mathfrak{F}$, but this is the case for $g \in \mathfrak{E}$ and for $g \in \mathfrak{H} - \mathfrak{E}$ (then $Ug = 0$), thus for all g. So $(U^*f, g) = (f, Ug) = 0$, $U^*f = 0$. This completes the proof of U^*'s being partially isometric with the initial and final sets $\mathfrak{F}$, $\mathfrak{E}$ resp., and its being inverse to U there.

Lemma 4.3.2. *Any of the four following conditions is characteristic for partially isometric operators. $UU^*U = U$, $U^*UU^* = U^*$, U^*U is a projection, UU^* is a projection.*

Proof: Owing to the symmetry between U and U^* it suffices to consider the first and the third conditions. If $UU^*U = U$ then $U^*UU^*U = U^*U$, $(U^*U)^2 = U^*U$, thus U^*U is a projection; so we need only to prove that the first condition is necessary, and that the third one is sufficient.

If U is partially isometric, we have always $Uf \in \mathfrak{F}$, so $P_{\mathfrak{F}}Uf = Uf$, $P_{\mathfrak{F}}U = U$; but $UU^* = P_{\mathfrak{F}}$, so $UU^*U = U$.

If U^*U is a projection, put $U^*U = P_{\mathfrak{E}}$. Thus for $f \in \mathfrak{E}$, $\|Uf\|^2 = (U^*Uf, f) = (P_{\mathfrak{E}}f, f) = \|f\|^2$, Uf is isometric in $\mathfrak{E}$. The image $\mathfrak{F}$ of $\mathfrak{E}$ is therefore isomorphic to $\mathfrak{E}$, thus a linear closed set (that is: a linear and topologically complete space) too. For $f \in \mathfrak{H} - \mathfrak{E}$, $\|Uf\|^2 = (U^*Uf, f) = (P_{\mathfrak{E}}f, f) = 0$. Thus U is partially isometric.

Lemma 4.3.3. *Let $U_1, U_2, \dots$ be a (finite or infinite) sequence of partially isometric operators, with the resp. initial sets $\mathfrak{E}_1, \mathfrak{E}_2, \dots$ and the resp. final sets $\mathfrak{F}_1, \mathfrak{F}_2, \dots$; let the $\mathfrak{E}_i$ be mutually orthogonal and let the $\mathfrak{F}_i$ be mutually orthogonal. Then the sum $\sum_i U_i$ is strongly convergent (if infinite at all), it is partially isometric, and its initial and final sets are $[\mathfrak{E}_i; i = 1, 2, \dots]$ and $[\mathfrak{F}_i; i = 1, 2, \dots]$.*

Proof: The strong convergence of $\sum_i U_i$ means the same thing as that of each $\sum_i U_i f$. As $U_i f \in \mathfrak{F}_i$, the addends are mutually orthogonal, thus the above convergence is equivalent to the finiteness of $\sum_i \|U_i f\|^2$ (cf. (16, p. 67)). Now

$$\sum_i \|U_i f\|^2 = \sum_i (U_i^* U_i f, f) = \sum_i (P_{\mathfrak{E}_i} f, f) = \left(P_{[\mathfrak{E}_i; i=1,2,\dots]} f, f\right),$$

and thus finite. Replacing U_i by U_i^*, we see that $\sum_i U_i^*$ is strongly convergent too.

So

$$\left(\sum_i U_i\right)^* \left(\sum_i U_i\right) = \sum_i U_i^* \sum_j U_j = \sum_{i,j} U_i^* U_j.$$

If $i \neq j$, then $U_j f \in \mathfrak{F}_j \subset \mathfrak{H} - \mathfrak{F}_i$, $U_i^* U_j f = 0$, $U_i^* U_j = 0$. So our above expression is equal to

$$\sum_i U_i^* U_i = \sum_i P_{\mathfrak{E}_i} = P_{[\mathfrak{E}_i; i=1,2,\dots]}.$$

Thus $\sum_i U_i$ is a partially isometric operator by Lemma 4.3.2 and its initial set is $[\mathfrak{E}_i; i = 1, 2, \dots]$, by Lemma 4.3.1. Replacing U_i by U_i^*, we see that the final set is

$$[\mathfrak{F}_i; i = 1, 2, \dots].$$

4.4. A construction which is described in (21, Theorem 7, p. 307), will be of importance in all our discussions. This Theorem applies to all linear, closed operators A with an everywhere dense domain, and leads to some operators of which we will consider B and W. B is self adjoint, and definite, and W is obviously partially isometric, its initial and final sets being

$$[\text{Range } B] = \mathfrak{H} - (f; Bf = 0) = \mathfrak{H} - (f; Af = 0) = [\text{Range } A^*]$$

and

$$\mathfrak{H} - (f; A^* f = 0) = [\text{Range } A]$$

resp. We have

$$A = WB, \qquad A^* = BW^*, \qquad B^2 = A^* A.$$

Cf. loc. cit. (21).

We define:

Definition 4.4.1. If A is linear, closed, and has an everywhere dense domain, then the above described W, B are its *canonical decomposition*.

We are interested in the relation between rings and the canonical decomposition.

Lemma 4.4.1. *If* $\mathbf{M}$ *is a ring containing* 1, $A \ \eta \ \mathbf{M}$, *and* W, B *the canonical decomposition of* A, *then* $W \in \mathbf{M}$, $B \ \eta \ \mathbf{M}$.

Proof: W, B are uniquely determined by their properties which follow:

$$B \text{ is self adjoint and definite;} \qquad B^2 = A^*A.$$
$$A = WB; \qquad Bf = 0 \text{ implies } Wf = 0.$$

In fact: The two first properties determine B (cf. (21, p. 303)); and the two last ones determine W on Range B and on $(f; Bf = 0)$, therefore on

$$[[\text{Range}\, B], (f; Bf = 0)] = [\mathfrak{H} - (f; Af = 0), (f; Af = 0)] = \mathfrak{H},$$

that is they determine W completely.

Now if U is unitary and W, B is the canonical decomposition of A, then UWU^{-1}, UBU^{-1} is the one of UAU^{-1}. Thus $UAU^{-1} = A$ implies $UWU^{-1} = W$, $UBU^{-1} = B$. If $A \ \eta \ \mathbf{M}$ then let U run over all $\mathbf{M}'$, this gives $W, B \ \eta \ \mathbf{M}$. As $W \in \mathbf{B}$, we even have $W \in \mathbf{M}$.

Chapter V: Direct Factors. Ring Isomorphisms

5.1. We define

Definition 5.1.1. If $\mathbf{M}$ is a ring containing 1, denote $[Af; A \in \mathbf{M}']$ by $\mathfrak{M}_f^{\mathbf{M}'}$. Write $E_f^{\mathbf{M}'}$ for $P_{\mathfrak{M}_f^{\mathbf{M}'}}$.

Lemma 5.1.1. $f \in \mathfrak{M}_f^{\mathbf{M}'} \ \eta \ \mathbf{M}$; *whenever* $f \in \mathfrak{M} \ \eta \ \mathbf{M}$ *then* $\mathfrak{M}_f^{\mathbf{M}'} \subset \mathfrak{M}$.

Proof: $A = 1$ gives $f \in \mathfrak{M}_f^{\mathbf{M}'}$. If $U \in \mathbf{M}'$, then

$$U(Af; A \in \mathbf{M}') = (UAf; A \in \mathbf{M}') \subset (Bf; B \in \mathbf{M}');$$

that is $U\mathfrak{M}_f^{\mathbf{M}'} \subset \mathfrak{M}_f^{\mathbf{M}'}$. Thus $\mathfrak{M}_f^{\mathbf{M}'} \ \eta \ \mathbf{M}$ (by Lemma 4.2.2). Conversely: Assume $f \in \mathfrak{M} \ \eta \ \mathbf{M}$, and consider those A for which Af and A^*f both $\in \mathfrak{M}$ for all $f \in \mathfrak{M}$.

They clearly form a ring (use the strong topology; for instance by (22, §3)), and contain all unitary $U \in \mathbf{M}'$, that is all $(\mathbf{M}')^u$. Thus they contain all $R((\mathbf{M}')^u) = \mathbf{M}'$ (cf. (18, p. 392)), and so

$$(Af; A \in \mathbf{M}') \subset \mathfrak{M}, \qquad [Af; A \in \mathbf{M}'] \subset \mathfrak{M}, \qquad \mathfrak{M}_f^{\mathbf{M}'} \subset \mathfrak{M}.$$

Definition 5.1.2. $\mathfrak{M}\ \eta\ \mathbf{M}$ is *minimal* (with respect to $\mathbf{M}$), if $\mathfrak{M} \neq (0)$, and $\mathfrak{N}\ \eta\ \mathbf{M}$, $\mathfrak{N} \subset \mathfrak{M}$ implies $\mathfrak{N} = (0)$ or $\mathfrak{M}$. Replacing $\mathfrak{M}, \mathfrak{N}$ by $E = P_{\mathfrak{M}}$, $F = P_{\mathfrak{N}}$ we can say (cf. Lemma 4.2.1 and (16, p. 76)): projection $E \in \mathbf{M}$ is minimal (with respect to $\mathbf{M}$), if $E \neq 0$, and if for every projection $F \in \mathbf{M}$, $F \leq E$ implies $F = 0$ or $F = E$.

Lemma 5.1.2. *$\mathfrak{M}\ \eta\ \mathbf{M}$ is minimal if and only if $\mathfrak{M} \neq (0)$, and for every $f \in \mathfrak{M}$, $f \neq 0$, $\mathfrak{M}_f^{\mathbf{M}'} = \mathfrak{M}$.*

Proof: Assume first that $\mathfrak{M}\ \eta\ \mathbf{M}$ is minimal. Then $\mathfrak{M} \neq (0)$. If $f \in \mathfrak{M}$, $f \neq 0$, then $\mathfrak{M}_f^{\mathbf{M}'}\ \eta\ \mathbf{M}$; as $f \in \mathfrak{M}_f^{\mathbf{M}'}$, $\mathfrak{M}_f^{\mathbf{M}'} \neq 0$ and as $f \in \mathfrak{M}$, $\mathfrak{M}_f^{\mathbf{M}'} \subset \mathfrak{M}$. Thus $\mathfrak{M}_f^{\mathbf{M}'} = \mathfrak{M}$ by definition.

Assume now conversely that our condition is fulfilled. If $\mathfrak{N} \subset \mathfrak{M}$, $\mathfrak{N}\ \eta\ \mathbf{M}$, then either $\mathfrak{N} = (0)$, or an $f \in \mathfrak{N}$, $f \neq 0$ exists. Then $\mathfrak{M}_f^{\mathbf{M}'} \subset \mathfrak{N}$; but as $f \in \mathfrak{M}$, $f \neq 0$, $\mathfrak{M}_f^{\mathbf{M}'} = \mathfrak{M}$; so $\mathfrak{M} \subset \mathfrak{N} \subset \mathfrak{M}$, $\mathfrak{N} = \mathfrak{M}$.

Lemma 5.1.3. *If $\mathbf{M}$ is a factor, then $\mathfrak{M}_f^{\mathbf{M}'}$ is minimal if and only if $f \neq 0$ and $\mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}} = (\alpha f)$.*

Proof: Assume first that $f \neq 0$ and $\mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}} = (\alpha f)$. Assume that $\mathfrak{M}_f^{\mathbf{M}'}$ is not minimal. As $f \neq 0$, $\mathfrak{M}_f^{\mathbf{M}'} \neq (0)$, there must exist an $\mathfrak{N}\ \eta\ \mathbf{M}$ with $\mathfrak{N} \subset \mathfrak{M}_f^{\mathbf{M}'}$, $\mathfrak{N} \neq 0, \mathfrak{M}_f^{\mathbf{M}'}$. Put $\mathfrak{P} = \mathfrak{M}_f^{\mathbf{M}'} - \mathfrak{N}$; then $\mathfrak{N}, \mathfrak{P}$ are orthogonal, both $\subset \mathfrak{M}_f^{\mathbf{M}'}$, both $\neq 0$.

Thus for $F = P_{\mathfrak{N}}$, $G = P_{\mathfrak{P}}$ we have: $F, G \in \mathbf{M}$, $FG = 0$, $F, G \neq 0$. As $E_f^{\mathbf{M}} = P_{\mathfrak{M}_f^{\mathbf{M}}} \in \mathbf{M}'$ and $\neq 0$ too, the Corollary to Theorem III gives

$$P_{\mathfrak{N}} P_{\mathfrak{M}_f^{\mathbf{M}}} = F \cdot P_{\mathfrak{M}_f^{\mathbf{M}}} \neq 0, \qquad P_{\mathfrak{P}} P_{\mathfrak{M}_f^{\mathbf{M}}} = G \cdot P_{\mathfrak{M}_f^{\mathbf{M}}} \neq 0.$$

So $\mathfrak{N} \cdot \mathfrak{M}_f^{\mathbf{M}}$ and $\mathfrak{P} \cdot \mathfrak{M}_f^{\mathbf{M}}$ are both $\neq (0)$. But they are $\subset \mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}} = (\alpha f)$, so they are both $= (\alpha f)$. Thus $f \in \mathfrak{N} \cdot \mathfrak{M}_f^{\mathbf{M}}$ and $\mathfrak{P} \cdot \mathfrak{M}_f^{\mathbf{M}}$ and a fortiori $f \in \mathfrak{N}$ and $\mathfrak{P}$. As $\mathfrak{N}, \mathfrak{P}$ are orthogonal, this would imply $f = 0$, contradicting our original assumption.

Assume now conversely that $\mathfrak{M}_f^{\mathbf{M}'}$ is minimal. As $\mathfrak{M}_f^{\mathbf{M}'} \neq (0)$, we have at any rate $f \neq 0$.

Put $E = E_f^{\mathbf{M}'}$, $E_f^{\mathbf{M}} = E'$. As $E \in \mathbf{M}$, $E' \in \mathbf{M}'$, therefore E, E' commute; therefore the E-image of $\mathfrak{M}_f^{\mathbf{M}}$ is $\mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}}$. Thus (remember $Ef = f$!)

$$\begin{aligned}\mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}} &= [EAf, A \in \mathbf{M}] = [EAEf, A \in \mathbf{M}] \\ &= [Bf, B \in \overline{\mathbf{M}}],\end{aligned}$$

where $\overline{\mathbf{M}}$ is the set of all $B \in \mathbf{M}$ with $EB = BE = B$ ($\overline{\mathbf{M}}$ is obviously a ring, thus $\overline{\mathbf{M}} = R(\overline{\mathbf{M}}^p)$ (cf. (18, p. 392))). $F' \in \overline{\mathbf{M}}^p$ means: F' is a projection, and $\in \mathbf{M}$, and so $EF' = F'E = F'$, $F' \in \mathbf{M}$. So $F' \in \mathbf{M}$, $F' \leq E$, and as E is minimal, $F' = 0$ or E. So $\overline{\mathbf{M}} = R(0, E) = (\alpha E)$. Therefore

$$\mathfrak{M}_f^{\mathbf{M}'} \cdot \mathfrak{M}_f^{\mathbf{M}} = [Bf, B \in (\alpha E)] = [\alpha Ef] = [\alpha f] = (\alpha f).$$

Lemma 5.1.4. *If* $\mathbf{M}$ *is a factor, then* $\mathfrak{M}_f^{\mathbf{M}'}$ *is minimal (with respect to* $\mathbf{M}$*) if and only if* $\mathfrak{M}_f^{\mathbf{M}}$ *is minimal (with respect to* $\mathbf{M}'$*).*

Proof: The criterium of Lemma 5.1.3 is obviously symmetric in $\mathbf{M}$ and $\mathbf{M}'$, as $\mathbf{M} = \mathbf{M}''$.

Definition 5.1.3. An $f \neq 0$ is *minimal* (with respect to $\mathbf{M}, \mathbf{M}'$) if its $\mathfrak{M}_f^{\mathbf{M}'}$, or, which is equivalent, its $\mathfrak{M}_f^{\mathbf{M}}$ is minimal.

Lemma 5.1.5. *If* $\mathbf{M}$ *is a factor, and if* f *is minimal, then every* $g \neq 0$, $g \in \mathfrak{M}_f^{\mathbf{M}'}$ *or* $\mathfrak{M}_f^{\mathbf{M}}$ *is minimal.*

Proof: $\mathfrak{M}_f^{\mathbf{M}'}$ is minimal, and since $g \neq 0$, $g \in \mathfrak{M}_f^{\mathbf{M}'}$ implies by Lemma 5.1.2, $\mathfrak{M}_g^{\mathbf{M}'} = \mathfrak{M}_f^{\mathbf{M}'}$, g is too. $g \neq 0$, $g \in \mathfrak{M}_f^{\mathbf{M}}$ is taken care of by symmetry.

5.2. Various sorts of ring isomorphisms are possible. They are defined as follows:

Definition 5.2.1. Let $\mathfrak{H}_{\mathrm{I}}$ and $\mathfrak{H}_{\mathrm{II}}$ be two spaces, $\mathbf{B}_{\mathrm{I}}$ and $\mathbf{B}_{\mathrm{II}}$ their operator rings, $\mathbf{M}_{\mathrm{I}} \subset \mathbf{B}_{\mathrm{I}}$ and $\mathbf{M}_{\mathrm{II}} \subset \mathbf{B}_{\mathrm{II}}$ rings in $\mathfrak{H}_{\mathrm{I}}$ resp. $\mathfrak{H}_{\mathrm{II}}$ which contain 1. If V is any set of operations and properties defined in both $\mathbf{B}_{\mathrm{I}}$ and $\mathbf{B}_{\mathrm{II}}$, then $\mathbf{M}_{\mathrm{I}}$ and $\mathbf{M}_{\mathrm{II}}$ are called *V-ring-isomorphic*, if a one-to-one mapping of $\mathbf{M}_{\mathrm{I}}$ on $\mathbf{M}_{\mathrm{II}}$ exists which leaves the operations and properties of V invariant.

If $V = (\alpha A, A^*, A + B, AB$, strongest closure$)$, we speak of *full ring-isomorphism*; if $V = (\alpha A, A^*, A + B, AB)$, of *algebraic ring-isomorphism*.

It is clear that every isomorphism of the spaces $\mathfrak{H}_{\mathrm{I}}$, $\mathfrak{H}_{\mathrm{II}}$ generates full ring isomorphisms (in particular one between $\mathbf{B}_{\mathrm{I}}$ and $\mathbf{B}_{\mathrm{II}}$). But the really interesting cases are such ring isomorphisms between an $\mathbf{M}_{\mathrm{I}}$ and an $\mathbf{M}_{\mathrm{II}}$, which do not originate from spatial isomorphisms.

As our methods are invariant with respect to the spaces $\mathfrak{H}$, all notions we considered are invariant under spatial isomorphisms. But it is not so with ring isomorphisms. For instance the operation $\mathbf{M}'$ is not invariant even under full ring isomorphisms. (Let $\mathfrak{H}_{\mathrm{I}}$, $\mathfrak{H}_{\mathrm{II}}$ be two finite dimensional Euclidean spaces of different dimensions. Put $\mathbf{M}_{\mathrm{I}} = \mathbf{M}_{\mathrm{II}} = (\alpha 1)$, then $\mathbf{M}'_{\mathrm{I}} = \mathbf{B}_{\mathrm{I}}$, $\mathbf{M}'_{\mathrm{II}} = \mathbf{B}_{\mathrm{II}}$; thus $\mathbf{M}_{\mathrm{I}}$, $\mathbf{M}_{\mathrm{II}}$ are fully ring isomorphic, while $\mathbf{M}'_{\mathrm{I}}$, $\mathbf{M}'_{\mathrm{II}}$ are not.) It is therefore of importance to show that some fundamental notions are invariant under certain types of ring isomorphism.

Lemma 5.2.1. *Every (A^*, AB)-ring-isomorphism leaves the following notions invariant: To be 0; to be 1; to be a projection; to be partially isometric; for two operators: to commute; for two projections E, F: $F \leq E$; for a projection: to be minimal.*

Proof: $A = 0$ means $AX = A$ for all $X \in \mathbf{M}_{\mathrm{I}}$; $A = 1$ means $AX = X$ for all $X \in \mathbf{M}_{\mathrm{I}}$; A is a projection if $A^* = A$, $AA = A$; A is partially isometric if $AA^*A = A$; commutativity means $AB = BA$; $F \leq E$ means $EF = F$; minimality can be expressed as a combination of the foregoing statements.

Lemma 5.2.2. *Every $(\alpha A, A^*, AB)$-ring-isomorphism leaves the notion of a factor (as applied to the rings $\mathbf{M}_{\mathrm{I}}$, $\mathbf{M}_{\mathrm{II}}$ themselves) invariant.*

Proof: As the rings $\mathbf{M}$ under consideration contain 1, being a factor means $\mathbf{M} \cdot \mathbf{M}' = (\alpha 1)$, that is: If $A \in \mathbf{M}$ is such that for every $B \in \mathbf{M}$, $AB = BA$, then $A = \alpha 1$. The first half is invariant under (A^*, AB)-ring-isomorphisms, and so is the 1 (cf. Lemma 5.2.1), while the equation $A = \alpha 1$ is then invariant under (αA)-ring-isomorphisms.

5.3. We are now able to give a simple criterium for direct factors.

Lemma 5.3.1. *The three following conditions are equivalent for a factor* $\mathbf{M}$*:*

(i) $\mathbf{M}$ *contains a minimal projection E (with respect to* $\mathbf{M}$*).*

(ii) $\mathbf{M}'$ *contains a minimal projection E' (with respect to* $\mathbf{M}'$*).*

(iii) *There exists a minimal $f \in \mathfrak{H}$ (with respect to $\mathbf{M}, \mathbf{M}'$).*

Proof: (iii) implies (i). (i) implies (iii) by Lemma 5.1.2. Similarly (ii) and (iii) are equivalent.

Lemma 5.3.2. *The conditions of Lemma 5.3.1 are fulfilled for every direct factor* $\mathbf{M}$.

Proof: If $\mathbf{M}$ is a direct factor in the $\mathbf{B}$ of $\mathfrak{H}$ then $\mathfrak{H}$ is isomorphic to $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, so that $\mathbf{M}$ becomes $\mathbf{B}_1^{(1)}$. $\mathbf{B}_1^{(1)}$ is algebraically (even fully) ring-isomorphic to $\mathbf{B}_1$ of $\mathfrak{H}_1$ by Lemma 2.3.4 (resp. (22, §4)); and the condition of Lemma 5.3.1, in its form (i), invariant under algebraic ring-isomorphisms by Lemma 5.2.1. Thus we need only to consider $\mathbf{B}_1$ in $\mathfrak{H}_1$. Now if φ is any element $\neq 0$ of $\mathfrak{H}_1$, then $P_{[\varphi]} \in \mathbf{B}_1$ and obviously minimal.

We begin now to discuss the sufficiency of these conditions.

Lemma 5.3.3. *If a factor $\mathbf{M}$ fulfills the conditions of Lemma 5.3.1, then every $\mathfrak{M}\ \eta\ \mathbf{M}$, $\mathfrak{M} \neq 0$, contains a minimal f with $\|f\| = 1$.*

Proof: A minimal f_0 exists, of course $f_0 \neq 0$. Thus $E^{\mathbf{M}}_{f_0} \neq 0$; besides $E^{\mathbf{M}}_{f_0} \in \mathbf{M}'$. Now $P_{\mathfrak{M}} \neq 0$, $P_{\mathfrak{M}} \in \mathbf{M}$, so by the Corollary of Theorem III,

$$P_{\mathfrak{M}} \cdot E^{\mathbf{M}}_{f_0} \neq 0, \qquad \mathfrak{M}^{\mathbf{M}}_{f_0} \cdot \mathfrak{M} \neq (0).$$

Thus an $f \in \mathfrak{M}^{\mathbf{M}}_{f_0} \cdot \mathfrak{M}$ with $f \neq 0$ exists, multiplying it with $1/\|f\|$ makes $\|f\| = 1$. As $f \in \mathfrak{M}^{\mathbf{M}}_{f_0}$, $f \neq 0$ this f is minimal by Lemma 5.1.5; and besides $f \in \mathfrak{M}$.

Lemma 5.3.4. *If a factor $\mathbf{M}$ fulfills the conditions of Lemma 5.3.1, then a finite or infinite normalised orthogonal system $\varphi_1, \varphi_2, \ldots$ exists, such that*

(i) *every φ_n is minimal;*
(ii) *$\mathfrak{M}^{\mathbf{M}'}_{\varphi_1}, \mathfrak{M}^{\mathbf{M}'}_{\varphi_2}, \ldots$ are mutually orthogonal;*
(iii) *$[\mathfrak{M}^{\mathbf{M}'}_{\varphi_1}, \mathfrak{M}^{\mathbf{M}'}_{\varphi_2}, \ldots] = \mathfrak{H}$.*

Proof: Let Ω be the first uncountable Cantor ordinal number. Define for all $\alpha < \Omega$, a φ_α as follows: If $\alpha < \Omega$, and all φ_β, $\beta < \alpha$ are already defined then form

$$\mathfrak{H} - [\mathfrak{M}^{\mathbf{M}'}_{\varphi_\beta};\ \beta < \alpha].$$

As this is obviously $\eta\ \mathbf{M}$, it will contain a minimal f with $\|f\| = 1$ if it is $\neq (0)$. Let then φ_α be some such f; otherwise let φ_α (and all φ_γ, $\gamma \geq \alpha$) be undefined.

Let the first α for which φ_α is undefined, be $\bar{\alpha} \leq \Omega$. If $\bar{\alpha} < \Omega$, we must have

$$\mathfrak{H} - [\mathfrak{M}^{\mathbf{M}'}_{\varphi_\beta};\ \beta < \bar{\alpha}] = 0, \qquad [\mathfrak{M}^{\mathbf{M}'}_{\varphi_\beta};\ \beta < \bar{\alpha}] = \mathfrak{H}.$$

Consider now all φ_α, $\alpha < \bar{\alpha}$. Always $\|\varphi_\alpha\| = 1$. If $\alpha, \beta < \bar{\alpha}$, assume $\alpha > \beta$. Then $A', B' \in \mathbf{M}'$ imply $A'^*B' \in \mathbf{M}'$, $A'^*B'\varphi_\beta \in \mathfrak{M}^{\mathbf{M}'}_{\varphi_\beta}$ orthogonal to φ_α, thus $B'\varphi_\beta$ orthogonal to $A'\varphi_\alpha$ (because $(A'^*B'\varphi_\beta, \varphi_\alpha) = (B'\varphi_\beta, A'\varphi_\alpha)$). So $(A'\varphi_\alpha; A' \in \mathbf{M}')$ is orthogonal to $(B'\varphi_\beta; B' \in \mathbf{M}')$, $[A'\varphi_\alpha; A' \in \mathbf{M}']$ to $[B'\varphi_\beta; B' \in \mathbf{M}']$ and so $\mathfrak{M}^{\mathbf{M}'}_{\varphi_\alpha}$ to $\mathfrak{M}^{\mathbf{M}'}_{\varphi_\beta}$. This is symmetric in α, β so it holds for $\alpha < \beta$ too, that is whenever $\alpha \neq \beta$.

Thus $\varphi_\alpha, \varphi_\beta$ are orthogonal if $\alpha \neq \beta$, that is the φ_α, $\alpha < \bar{\alpha}$, form a normalised orthogonal set. It must therefore be finite or countably infinite (cf. (16, p. 66)), thus excluding $\bar{\alpha} = \Omega$. Thus $\bar{\alpha} < \Omega$, and so $[\mathfrak{M}^{\mathbf{M}'}_{\varphi_\alpha}; \alpha < \bar{\alpha}] = \mathfrak{H}$.

If $\bar{\alpha}$ is finite, $\mathfrak{H} \neq (0)$ excludes $\bar{\alpha} = 1$, so $\bar{\alpha} = n + 1$, $n = 1, 2, \ldots$, and α runs over $1, \ldots, n$. If $\bar{\alpha}$ is infinite then $\alpha < \bar{\alpha}$ has the same aleph as $\alpha < \omega$, and may therefore (by a re-indexing of the φ_α) be replaced by it; so we may assume that α runs over $1, 2, \ldots$.

This completes the proof.

Lemma 5.3.5. *If a factor* $\mathbf{M}$ *fulfills the conditions of* Lemma 5.3.1 *then a complete normalised orthogonal set* $f_{m,n}$, $m = 1, 2, \ldots$; $n = 1, 2, \ldots$ *exists (both ranges for* m *and for* n *may be finite or infinite), such that:*

(i) *every* $f_{m,n}$ *is minimal,*

(ii) $\mathfrak{M}^{\mathbf{M}'}_{f_{m,n}} = [f_{m,1}, f_{m,2}, \ldots]$, $\mathfrak{M}^{\mathbf{M}}_{f_{m,n}} = [f_{1,n}, f_{2,n}, \ldots]$.

Proof: If we replace $\mathbf{M}$ by $\mathbf{M}'$, condition (i) of Lemma 5.3.1 becomes (ii), so $\mathbf{M}'$ fulfills it too. Apply therefore Lemma 5.3.4 to $\mathbf{M}$ and to $\mathbf{M}'$, obtaining the two normalised orthogonal systems $\varphi_1, \varphi_2, \ldots$ and $\psi_1, \psi_2, \ldots$. As $E^{\mathbf{M}'}_{\varphi_m} \in \mathbf{M}$ and $\neq 0$, $E^{\mathbf{M}}_{\psi_n} \in \mathbf{M}'$ and $\neq 0$, we have by the Corollary to Theorem III

$$P_{\mathfrak{M}^{\mathbf{M}'}_{\varphi_m} \cdot \mathfrak{M}^{\mathbf{M}}_{\psi_n}} = E^{\mathbf{M}'}_{\varphi_m} \cdot E^{\mathbf{M}}_{\psi_n} \neq 0, \qquad \mathfrak{M}^{\mathbf{M}'}_{\varphi_m} \cdot \mathfrak{M}^{\mathbf{M}}_{\psi_n} \neq (0).$$

Choose an $f_{m,n} \in \mathfrak{M}^{\mathbf{M}'}_{\varphi_m} \cdot \mathfrak{M}^{\mathbf{M}}_{\psi_n}$ which is $\neq 0$; multiplying it with $1/\|f_{m,n}\|$ makes $\|f_{m,n}\| = 1$.

If $m \neq p$, $f_{m,n} \in \mathfrak{M}^{\mathbf{M}'}_{\varphi_m}$, $f_{p,q} \in \mathfrak{M}^{\mathbf{M}'}_{\varphi_p}$; if $n \neq q$, $f_{m,n} \in \mathfrak{M}^{\mathbf{M}}_{\psi_n}$, $f_{p,q} \in \mathfrak{M}^{\mathbf{M}}_{\psi_q}$ both imply $(f_{m,n}, f_{p,q}) = 0$. Thus the $f_{m,n}$ form a normalised orthogonal system.

$\mathfrak{M}^{\mathbf{M}'}_{\varphi_m}$ is minimal and $f_{m,n}$ belongs to it and is $\neq 0$, therefore Lemma 5.1.5 applies: $f_{m,n}$ is minimal too, and $\mathfrak{M}^{\mathbf{M}'}_{\varphi_m} = \mathfrak{M}^{\mathbf{M}'}_{f_{m,n}}$. Similarly $\mathfrak{M}^{\mathbf{M}}_{\psi_n} = \mathfrak{M}^{\mathbf{M}}_{f_{m,n}}$. As $f_{m,n}$ is minimal, Lemma 5.1.3 gives $\mathfrak{M}^{\mathbf{M}'}_{f_{m,n}} \cdot \mathfrak{M}^{\mathbf{M}}_{f_{m,n}} = (\alpha f_{m,n})$, that is

$$\mathfrak{M}^{\mathbf{M}'}_{\varphi_m} \cdot \mathfrak{M}^{\mathbf{M}}_{\psi_n} = (\alpha f_{m,n}).$$

Thus

$$\sum_{m,n} P_{[f_{m,n}]} = \sum_{m,n} E^{\mathbf{M}'}_{\varphi_m} \cdot E^{\mathbf{M}}_{\psi_n} = \sum_m E^{\mathbf{M}'}_{\varphi_m} \sum_n E^{\mathbf{M}}_{\psi_n} = 1 \cdot 1 = 1;$$

that is: the normalised orthogonal system of all $f_{m,n}$ is complete.

$f_{m,n} \in \mathfrak{M}^{\mathbf{M}'}_{f_{m,n}} = \mathfrak{M}^{\mathbf{M}'}_{\varphi_m}$, and if $p \neq m$, then $f_{p,n} \in \mathfrak{M}^{\mathbf{M}'}_{\varphi_p}$ is orthogonal to $\mathfrak{M}^{\mathbf{M}'}_{\varphi_m}$. This proves $\mathfrak{M}^{\mathbf{M}'}_{\varphi_m} = [f_{m,1}, f_{m,2}, \ldots]$. Similarly $\mathfrak{M}^{\mathbf{M}}_{\psi_n} = [f_{1,n}, f_{2,n}, \ldots]$.

Thus all parts of our Lemma are proved.

Lemma 5.3.6. *If a factor* $\mathbf{M}$ *fulfills the conditions of Lemma 5.3.1 we can choose the* $f_{m,n}$ *of Lemma 5.3.5 so that we have:*

There is a unique partially isometric operator $U \in \mathbf{M}$ *which has the initial and final sets* $[f_{m,1}, f_{m,2}, \ldots]$ *resp.* $[f_{p,1}, f_{p,2}, \ldots]$ *and for which* $Uf_{m,n} = f_{p,n}$, $Uf_{r,n} = 0$, *if* $r \neq m$. *Denote it by* $U_{m,p}$.

If an $A \in \mathbf{M}$ *has the properties*

$$P_{[f_{p,1}, f_{p,2}, \ldots]}A = AP_{[f_{m,1}, f_{m,2}, \ldots]} = A$$

then $A = \alpha U_{m,p}$.

Proof: Assume first $A \in \mathbf{M}$ and

$$P_{[f_{p,1}, f_{p,2}, \ldots]}A = AP_{[f_{m,1}, f_{m,2}, \ldots]} = A$$

and the same for B. Then applying a $*$ gives:

$$P_{[f_{m,1}, f_{m,2}, \ldots]}B^* = B^*P_{[f_{p,1}, f_{p,2}, \ldots]} = B^*$$

and so

$$P_{[f_{m,1}, f_{m,2}, \ldots]}B^*A = B^*AP_{[f_{m,1}, f_{m,2}, \ldots]} = B^*A.$$

Now $B^*A \in \mathbf{M}$ and $P_{[f_{m,1}, f_{m,2}, \ldots]} = E^{\mathbf{M}'}_{f_{m,n}}$ is minimal with respect to $\mathbf{M}$, so the argument made at the end of the proof of Lemma 5.1.3 applies

$$B^*A = \beta P_{[f_{m,1}, f_{m,2}, \ldots]}.$$

Put first $A = B$, then $B^*B = \beta_1 P_{[f_{m,1}, f_{m,2}, \ldots]}$. As the left side is Hermitian and definite, $\beta_1 \geq 0$. If $B \neq 0$, then even $\beta_1 \neq 0$ and we have

$$(\beta_1^{-1/2}B)^*(\beta_1^{-1/2}B) = P_{[f_{m,1}, f_{m,2}, \ldots]}.$$

Thus $\beta_1^{-1/2}B$ is a partially isometric operator with the initial set $[f_{m,1}, f_{m,2}, \ldots]$ (cf. Lemmas 4.3.1 and 4.3.2). Replacing B by B^* we interchange m and p, so $\beta_2^{-1/2}B^*$ is partially isometric with the initial set $[f_{p,1}, f_{p,2}, \ldots]$ and $\beta_2^{-1/2}B$ with the final set $[f_{p,1}, f_{p,2}, \ldots]$. As $\beta_1^{-1/2}B$, $\beta_2^{-1/2}B$ are both partially isometric, we have $|\beta_1^{-1/2}| = |\beta_2^{-1/2}|$, that is $\beta_1 = \beta_2$. So $\beta_1^{-1/2}B$ has the initial and final sets $[f_{m,1}, f_{m,2}, \ldots]$ resp. $[f_{p,1}, f_{p,2}, \ldots]$.

Return now to the pair A, B. If $B \neq 0$, then we have

$$A = P_{[f_{p,1}, f_{p,2},\dots]}A = \frac{1}{\beta_1}BB^*A = \frac{\beta}{\beta_1}BP_{[f_{m,1}, f_{m,2},\dots]} = \frac{\beta}{\beta_1}B.$$

So we see: either $B = 0$ or $A = \alpha B$.

Consider now $f_{m,n}$ and $f_{p,n}$. As $\mathfrak{M}^{\mathbf{M}}_{f_{m,n}} = [f_{1,n}, f_{2,n}, \dots]$ we have $f_{p,n} \in \mathfrak{M}^{\mathbf{M}}_{f_{m,n}}$. So an $A \in \mathbf{M}$ with $\|f_{p,n} - Af_{m,n}\| < 1$ exists. A fortiori

$$\left\|P_{[f_{p,1}, f_{p,2},\dots]}f_{p,n} - P_{[f_{p,1}, f_{p,2},\dots]}Af_{m,n}\right\| < 1$$

or as

$$P_{[f_{p,1}, f_{p,2},\dots]}f_{p,n} = f_{p,n}, \qquad P_{[f_{m,1}, f_{m,2},\dots]}f_{m,n} = f_{m,n}$$

we have $\|f_{p,n} - A_0f_{m,n}\| < 1$, where

$$A_0 = P_{[f_{p,1}, f_{p,2},\dots]}AP_{[f_{m,1}, f_{m,2},\dots]}.$$

By what we prove above $A_0 = 0$ or $A_0 = \beta_0 U$, where $\beta_0 \neq 0$ and U is partially isometric with the initial resp. final set $[f_{m,1}, f_{m,2}, \dots]$ resp. $[f_{p,1}, f_{p,2}, \dots]$. Now $\|f_{p,n}\| = 1$ implies $A_0f_{m,n} \neq 0$ and so the latter alternative holds.

As $A_0 \in \mathbf{M}$, $A_0f_{m,n} \in \mathfrak{M}^{\mathbf{M}}_{f_{m,n}} = \mathfrak{M}^{\mathbf{M}}_{f_{p,n}}$, as $A_0 = P_{[f_{p,1}, f_{p,2},\dots]}A_0$ therefore

$$A_0f_{m,n} \in [f_{p,1}, f_{p,2}, \dots] = \mathfrak{M}^{\mathbf{M}'}_{f_{p,n}}.$$

Thus $A_0f_{m,n} \in \mathfrak{M}^{\mathbf{M}'}_{f_{p,n}} \cdot \mathfrak{M}^{\mathbf{M}}_{f_{p,n}} = (\alpha f_{p,n})$, $A_0f_{m,n} = \alpha_0 f_{p,n}$, $Uf_{m,n} = (\alpha_0/\beta_0)f_{p,n}$. As $\|f_{m,n}\| = \|f_{p,n}\| = 1$ and $f_{m,n}$ is in the initial set of U, we have $|\alpha_0/\beta_0| = 1$. Replacing U by $(\beta_0/\alpha_0)U$ does not affect its other properties, but makes $Uf_{m,n} = f_{p,n}$. So we have:

$$P_{[f_{p,1}, f_{p,2},\dots]}U = UP_{[f_{m,1}, f_{m,2},\dots]} = U, \qquad Uf_{m,n} = f_{p,n}.$$

As U depends on m, n, p write $U = U^{(n)}_{m,p}$. The first equations determine it uniquely up to a constant factor, the last one makes this factor equal to 1. Comparing $U^{(n)}_{m,p}$, $U^{(q)}_{m,p}$ we see that the first equations still make them agree up to a constant factor:

$$U^{(n)}_{m,p} = \theta^{(n,q)}_{m,p}U^{(q)}_{m,p}$$

and $\theta^{(n,q)}_{m,p}$ is obviously of absolute value 1. On the other hand the uniqueness property gives

$$U^{(n)}_{m,p} \cdot U^{(n)}_{p,q} = U^{(n)}_{m,r}.$$

Replace every $f_{p,q}$, $p \neq 1$, by $\theta_{1,p}^{(1,q)} f_{p,q}$. This makes every $\theta_{1,p}^{(1,q)} = 1$. Now obviously

$$\theta_{1,p}^{(1,n)} \cdot \theta_{1,p}^{(n,q)} = \theta_{1,p}^{(1,q)},$$

therefore $\theta_{1,p}^{(n,q)} = 1$. Furthermore it is easy to see, that

$$\theta_{1,m}^{(n,q)} \cdot \theta_{m,p}^{(n,q)} = \theta_{1,p}^{(n,q)},$$

therefore $\theta_{m,p}^{(n,q)} = 1$. Thus $U_{m,p}^{(n)}$ does not depend on n, $U_{m,p}^{(n)} = U_{m,p}$.

$U_{m,p}$ is partially isometric, its initial and final sets are $[f_{m,1}, f_{m,2}, \ldots]$ resp. $[f_{p,1}, f_{p,2}, \ldots]$, and $U_{m,p} f_{m,n} = f_{p,n}$. If $r \neq m$, then $f_{r,n}$ is orthogonal to $[f_{m,1}, f_{m,2}, \ldots]$, and so $U_{m,p} f_{r,n} = 0$. $U_{m,p}$ is uniquely determined by these properties, because already the $U_{m,p}^{(n)}$ were.

Thus we have proved the first statement of our Lemma. The second results by replacing in our result on A, B at the beginning of this proof A, B by $A, U_{m,p}$.

Lemma 5.3.7. *If a factor* $\mathbf{M}$ *fulfills the conditions of Lemma 5.3.1, then we have with the* $U_{m,p}$ *of Lemma 5.3.6*

$$\mathbf{M} = R(U_{m,p}; m, p = 1, 2, \ldots).$$

Proof: As all $U_{m,p} \in \mathbf{M}$, we have $R(U_{m,p}; m, p = 1, 2, \ldots) \subset \mathbf{M}$. Assume now conversely $A \in \mathbf{M}$. Put

$$A_{m,p} = P_{[f_{p,1}, f_{p,2}, \ldots]} A P_{[f_{m,1}, f_{m,2}, \ldots]}.$$

Then $A_{m,p} \in \mathbf{M}$,

$$P_{[f_{p,1}, f_{p,2}, \ldots]} A_{m,p} = A_{m,p} P_{[f_{m,1}, f_{m,2}, \ldots]} = A_{m,p},$$

so by Lemma 5.3.6 $A_{m,p} = a_{m,p} U_{m,p}$. Thus $A_{m,p} \in R(U_{m',p'}; m', p' = 1, 2, \ldots)$. But

$$\begin{aligned} \sum_{m,p} A_{m,p} &= \sum_{m,p} P_{[f_{p,1}, f_{p,2}, \ldots]} A P_{[f_{m,1}, f_{m,2}, \ldots]} \\ &= \left(\sum_{p} P_{[f_{p,1}, f_{p,2}, \ldots]} \right) A \left(\sum_{m} P_{[f_{m,1}, f_{m,2}, \ldots]} \right) \\ &= 1 \cdot A \cdot 1 = A. \end{aligned}$$

If these sums are infinite at all, they are strongly convergent, cf. (16, pp. 77–78), so $A \in R(U_{m',p'}; m', p' = 1, 2, \ldots)$. This holds for any $A \in \mathbf{M}$, so

$$\mathbf{M} \subset R(U_{m,p}; m, p = 1, 2, \ldots).$$

So we have proved

$$\mathbf{M} = R(U_{m,p}; m, p = 1, 2, \ldots).$$

Lemma 5.3.8. *If a factor* **M** *fulfills the conditions of Lemma 5.3.1, then it is a direct factor.*

Proof: Consider the $f_{m,n}$ and the $U_{m,p}$ of Lemma 5.3.7; m, p have the same finite or infinite range, n has another finite or infinite range. Take two spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ with the same dimensions as there are elements in the resp. ranges of m and n; and let φ_m^0, $m = 1, 2, \ldots$, and ψ_n^0, $n = 1, 2, \ldots$, be complete, normalised orthogonal sets in $\mathfrak{H}_1$ resp. $\mathfrak{H}_2$. Then $\varphi_m^0 \otimes \psi_n^0$, $m = 1, 2, \ldots$, $n = 1, 2, \ldots$, is one in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. If we let correspond $\varphi_m^0 \otimes \psi_n^0$ to $f_{m,n}$, we obtain an isomorphism of $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and $\mathfrak{H}$.

Define in $\mathfrak{H}_1$ operators $V_{m,p}$ by

$$V_{m,p}\varphi_m^0 = \varphi_p^0, \qquad V_{m,p}\varphi_r^0 = 0 \quad \text{if } r \neq m.$$

Then (cf. Lemma 2.3.3)

$$V_{m,p}^{(1)}\varphi_m^0 \otimes \varphi_r^0 = \varphi_p^0 \otimes \varphi_r^0,$$
$$V_{m,p}^{(1)}\varphi_r^0 \otimes \varphi_q^0 = 0 \quad \text{if } r \neq m.$$

Thus the above spatial isomorphism, of $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ and $\mathfrak{H}$ transforms $V_{m,p}^{(1)}$ into $U_{m,p}$, and thus $R(V_{m,p}^{(1)}; m, p = 1, 2, \ldots)$ into $R(U_{m,p}; m, p = 1, 2, \ldots) = \mathbf{M}$.

Now consider any $A_1 \in \mathbf{B}_1$. Then

$$P_{[\varphi_p^0]}A_1P_{[\varphi_m^0]} = (A_1\varphi_m^0, \varphi_p^0)V_{m,p} = \alpha_{m,p}V_{m,p}.$$

So $P_{[\varphi_p^0]}A_1P_{[\varphi_m^0]} \in R(V_{m',p'}; m', p' = 1, 2, \ldots)$. But

$$\sum_{m,p} P_{[\varphi_p^0]}A_1P_{[\varphi_m^0]} = \left(\sum_p P_{[\varphi_p^0]}\right) A_1 \left(\sum_m P_{[\varphi_m^0]}\right) = 1 \cdot A_1 \cdot 1 = A_1,$$

(if the sums are infinite at all, they are strongly convergent), so

$$A_1 \in R(V_{m,p}; m, p = 1, 2, \ldots).$$

Thus $R(V_{m,p}; m,p = 1,2,\ldots) = \mathbf{B}_1$. Now Lemmas 2.3.6 and 2.3.7 give $R(V^{(1)}_{m,p}; m,p = 1,2,\ldots) = \mathbf{B}^{(1)}_1$. Thus our special isomorphism transforms $\mathbf{B}^{(1)}_1$ into $\mathbf{M}$, proving that $\mathbf{M}$ is a direct factor.

The results of Lemmas 5.3.1, 5.3.2 and 5.3.8 give together:

Theorem IV. *A factor $\mathbf{M}$ is direct if and only if it fulfills the (equivalent) conditions of Lemma 5.3.1.*

The condition (i) in Lemma 5.3.1 gives, together with Lemmas Lemma 5.2.1 and Lemma 5.2.2, the

Theorem V. *Every algebraic ring-isomorphism leaves invariant the notions of a factor and of a direct factor (as applied to the rings $\mathbf{M}_{\mathrm{I}}$, $\mathbf{M}_{\mathrm{II}}$ themselves).*

5.4. Theorem V makes it possible to prove the following Lemma, which belongs naturally to §3.2.

Lemma 5.4.1. *If $\mathbf{M}_1, \ldots, \mathbf{M}_n$ is a factorisation, and at least $n-1$ of its factors $\mathbf{M}_i$ are direct, then the factorisation is a direct one too.*

Proof: For $n = 1$, $R(\mathbf{M}_1) = \mathbf{B}$ means $\mathbf{M}_1 = \mathbf{B}_1$, thus $\mathbf{M}_1$ is a direct factor: Put $\mathfrak{H}_1$ a one dimensional space, $\mathfrak{H}_2 = \mathfrak{H}$; then the results of §2.4 show that

$$\mathfrak{H}_1 \otimes \mathfrak{H}_2 = \mathfrak{H}_2 = \mathfrak{H}, \qquad \mathbf{B}^{(2)}_2 = \mathbf{B}_2 = \mathbf{B} = \mathbf{M}_1.$$

So the case $n = 1$ is settled, and we must only prove the Lemma for $n = 2, 3, \ldots$, if it is already established for $n - 1$.

There is an i such that all $\mathbf{M}_j$, $j \neq i$, are direct. As a permutation of all $\mathbf{M}_j$'s does not matter, we may assume that this i is $\neq 1$, that is, that $\mathbf{M}_1$ is direct.

Then $\mathfrak{H}$ is isomorphic to $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, so that $\mathbf{M}_1$ becomes $\mathbf{B}^{(1)}_1$. Let $\mathbf{M}_2, \ldots, \mathbf{M}_n$ become $\mathbf{N}_2, \ldots, \mathbf{N}_n$. If $j \neq 1$, then $\mathbf{M}_j \subset \mathbf{M}'_1$ so $\mathbf{N}_j \subset (\mathbf{B}^{(1)}_1)' = \mathbf{B}^{(2)}_2$. The correspondence $A^{(2)} \leftrightarrow A$ maps therefore the $\mathbf{N}_j$'s on rings $\mathbf{N}^0_j \subset \mathbf{B}_2$ (cf. Lemma 2.3.7). The $\mathbf{N}_j$'s are algebraically (even fully) ring-isomorphic to the resp. $\mathbf{N}^0_j$ (cf. Lemma 2.3.4, resp. (22, §4)). Now all $\mathbf{M}_2, \ldots, \mathbf{M}_n$ are factors, and at least $n-2$ of them direct, so the same holds for $\mathbf{N}_2, \ldots, \mathbf{N}_n$ and, by Theorem V, for $\mathbf{N}^0_2, \ldots, \mathbf{N}^0_n$.

As the $\mathbf{N}_j, \mathbf{N}_k$ commute, the $\mathbf{N}^0_2, \ldots, \mathbf{N}^0_n$ do too; as $R(\mathbf{N}_2, \ldots, \mathbf{N}_n) = \mathbf{B}^{(2)}_2$, $R(\mathbf{N}^0_2, \ldots, \mathbf{N}^0_n) = \mathbf{B}_2$ by Lemma 2.3.7. Thus $\mathbf{N}^0_2, \ldots, \mathbf{N}^0_n$ is a factorisation in $\mathbf{B}_2$ of $\mathfrak{H}_2$.

So our Lemma for $n-1$ applies: $\mathbf{N}^0_2, \ldots, \mathbf{N}^0_n$ is a direct factorisation in $\mathbf{B}_2$ of $\mathfrak{H}_2$. So $\mathfrak{H}_2$ is isomorphic to $\mathfrak{H}^*_2 \otimes \cdots \otimes \mathfrak{H}^*_n$, and $\mathbf{N}^0_2, \ldots, \mathbf{N}^0_n$ become $\mathbf{B}^{*(2)}_2, \ldots, \mathbf{B}^{*(n)}_n$ resp.

Now it is obvious, that $\mathfrak{H}$ is isomorphic to $\mathfrak{H}_1 \otimes \mathfrak{H}_2^* \otimes \cdots \otimes \mathfrak{H}_n^*$ and that $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_n$ become $\mathbf{B}_1^{(1)}, \mathbf{B}_2^{*(2)}, \ldots, \mathbf{B}_n^{*(n)}$ (the $n-1$ last ones have to be taken now in the product $\mathfrak{H}_1 \otimes \mathfrak{H}_2^* \otimes \cdots \otimes \mathfrak{H}_n^*$!) resp. Thus $\mathbf{M}_1, \ldots, \mathbf{M}_n$ is a direct factorisation.

Note that now Lemma 3.2.2 shows, that all $\mathbf{M}_i$'s are direct. Note furthermore, that our Lemma would even then not have been obvious, if all n factors $\mathbf{M}_i$ had been known to be direct. $n-1$ can not be replaced in our Lemma by $n-2$: Because this would give for every factor $\mathbf{M}$, that the factorisation $\mathbf{M}, \mathbf{M}'$ ($n = 2$) is direct, and thus $\mathbf{M}$ is direct; and then by our Lemma every factorisation $\mathbf{M}_1, \ldots, \mathbf{M}_n$ would be direct. Thus the answer to both parts of Problem 2 at the end of §3.2 would be affirmative, which is not the case (cf. there and §8.4 and §13.2).

Part II: The General Problem

Chapter VI: Relative Dimensionality

6.1. In what follows $\mathbf{M}$ will always denote a factor in $\mathbf{B}$ of the space $\mathfrak{H}$. Thus $\mathbf{M}'$ is a factor too.

Definition 6.1.1. We write $\mathfrak{M} \sim \mathfrak{N}$ $(\cdots \mathbf{M})$, and for $E = P_{\mathfrak{M}}$, $F = P_{\mathfrak{N}}$, $E \sim F$ $(\cdots \mathbf{M})$, if a partially isometric $U \in \mathbf{M}$ exists, the initial and final sets of which are $\mathfrak{M}$ resp. $\mathfrak{N}$. (If no misunderstanding is possible we will omit the $(\cdots \mathbf{M})$.) We say that $\mathfrak{M}, \mathfrak{N}$ have *the same relative dimension* (with respect to $\mathbf{M}$).

In discussing this notion $\sim$, we have always the choice to speak about the linear closed sets $\mathfrak{M}$, or their projections $E = P_{\mathfrak{M}}$. We will mostly use the first mentioned terminology, remembering however the equivalence of the two.

Lemma 6.1.1. $\mathfrak{M} \sim \mathfrak{N}$ *implies* $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$. *If* $\mathfrak{M}, \mathfrak{N}, \mathfrak{P}\ \eta\ \mathbf{M}$ *then we have:*

$$\mathfrak{M} \sim \mathfrak{M},$$
$$\mathfrak{M} \sim \mathfrak{N} \textit{ implies } \mathfrak{N} \sim \mathfrak{M},$$
$$\mathfrak{M} \sim \mathfrak{N},\ \mathfrak{N} \sim \mathfrak{P} \ \textit{ imply } \mathfrak{M} \sim \mathfrak{P}.$$

Proof: If U is the partially isometric operator in question, we have $P_{\mathfrak{M}} = U^*U \in \mathbf{M}$, $P_{\mathfrak{N}} = UU^* \in \mathbf{M}$, and so $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$.

$\mathfrak{M} \sim \mathfrak{M}$ results by putting $U = P_{\mathfrak{M}}$; $\mathfrak{M} \sim \mathfrak{N}$ gives $\mathfrak{N} \sim \mathfrak{M}$ by replacing U by U^*; $\mathfrak{M} \sim \mathfrak{N}$ with U and $\mathfrak{N} \sim \mathfrak{P}$ with V gives $\mathfrak{M} \sim \mathfrak{P}$ by considering VU.

Thus $\sim$ is naturally applied to linear closed sets $\mathfrak{M} \ \eta \ \mathbf{M}$, and has there the properties of an equivalence. We may say: Two linear, closed sets $\mathfrak{M}, \mathfrak{N}$ which are invariant under all unitary elements of $\mathbf{M}'$, are of the same dimension in the usual sense of the word, if an isometric mapping U of $\mathfrak{M}$ on $\mathfrak{N}$ exists, that is a partially isometric operator U with the initial and final sets $\mathfrak{M}$ resp. $\mathfrak{N}$. But they have the same relative dimension, if even the mapping U can be chosen so as to be invariant under all unitary elements of $\mathbf{M}'$. (Cf. Definition 4.2.1 and Lemma 4.2.1.)

We now prove two Lemmas, the first of which expresses an additivity property of our $\sim$, while the second one is the analogon of the Cantor–Bernstein "equivalence theorem" of general set-theory. (In fact, it is a special case of the Kuratowski–Tarski generalisation of that theorem.)

Lemma 6.1.2. *Let $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ and $\mathfrak{N}_1, \mathfrak{N}_2, \ldots$ be two (finite or infinite) sequences of the same length; let the $\mathfrak{M}_i$ be mutually orthogonal and let the $\mathfrak{N}_i$ be mutually orthogonal. If we have $\mathfrak{M}_i \sim \mathfrak{N}_i$ for all i, then*

$$[\,\mathfrak{M}_1, \mathfrak{M}_2, \ldots\,] \sim [\,\mathfrak{N}_1, \mathfrak{N}_2, \ldots\,].$$

Proof: Let $U_i \in \mathbf{M}$ be the partially isometric mapping of $\mathfrak{M}_i$ on $\mathfrak{N}_i$, then $U = \sum_i U_i$ does the desired thing for $[\,\mathfrak{M}_1, \mathfrak{M}_2, \ldots\,]$ and $[\,\mathfrak{N}_1, \mathfrak{N}_2, \ldots\,]$ by Lemma 4.3.3.

Lemma 6.1.3. *If $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$ and $\mathfrak{N} \sim \mathfrak{M}' \subset \mathfrak{M}$, then $\mathfrak{M} \sim \mathfrak{N}$.*

Proof: Our relations imply $\mathfrak{M}, \mathfrak{N}, \mathfrak{M}', \mathfrak{N}' \ \eta \ \mathbf{M}$. Now let $U \in \mathbf{M}$ be the partially isomorphic mapping of $\mathfrak{N}$ on $\mathfrak{M}'$; it maps $\mathfrak{N}' \subset \mathfrak{N}$ on some $\mathfrak{M}'' \subset \mathfrak{M}'$. Then consideration of $UP_{\mathfrak{N}'}$ proves that $\mathfrak{N}' \sim \mathfrak{M}''$, and so $\mathfrak{M} \sim \mathfrak{M}''$.

Let $V \in \mathbf{M}$ be the partially isomorphic mapping of $\mathfrak{M}$ on $\mathfrak{M}''$. Form for every $n = 0, 1, 2, \ldots$ the V^n-images of $\mathfrak{M}$ and of $\mathfrak{M}'$, and call them $\mathfrak{M}^{(2n)}$ resp. $\mathfrak{M}^{(2n+1)}$ ($\mathfrak{M}, \mathfrak{M}', \mathfrak{M}''$ coincide with $\mathfrak{M}^{(0)}, \mathfrak{M}^{(1)}, \mathfrak{M}^{(2)}$ resp.). We have $\mathfrak{M}^{(0)} \supset \mathfrak{M}^{(1)} \supset \mathfrak{M}^{(2)}$, thus (apply V^n!)

$$\mathfrak{M}^{(2n)} \supset \mathfrak{M}^{(2n+1)} \supset \mathfrak{M}^{(2n+2)},$$

therefore

$$\mathfrak{M}^{(0)} \supset \mathfrak{M}^{(1)} \supset \mathfrak{M}^{(2)} \supset \cdots .$$

V^n is partially isometric with the initial and final sets $\mathfrak{M}^{(0)}$ resp. $\mathfrak{M}^{(2n)}$, $V^n P_{\mathfrak{M}^{(1)}}$ similarly with $\mathfrak{M}^{(1)}$ and $\mathfrak{M}^{(2n+1)}$. Thus all $\mathfrak{M}^{(p)} \ \eta \ \mathbf{M}$, $n = 0, 1, 2, \ldots$, and

$$\mathfrak{M}^{(0)} \sim \mathfrak{M}^{(2n)}, \qquad \mathfrak{M}^{(1)} \sim \mathfrak{M}^{(2n+1)},$$

that is: $\mathfrak{M}^{(p)} \sim \mathfrak{M}$ or $\mathfrak{M}'$ if p is even resp. odd.

V maps $\mathfrak{M}^{(p)}$ on $\mathfrak{M}^{(p+2)}$, $\mathfrak{M}^{(p+1)}$ on $\mathfrak{M}^{(p+3)}$, therefore $\mathfrak{M}^{(p)} - \mathfrak{M}^{(p+1)}$ on $\mathfrak{M}^{(p+2)} - \mathfrak{M}^{(p+3)}$. Using $V(P_{\mathfrak{M}^{(p)}} - P_{\mathfrak{M}^{(p+1)}})$ shows therefore, that

$$\mathfrak{M}^{(p)} - \mathfrak{M}^{(p+1)} \sim \mathfrak{M}^{(p+2)} - \mathfrak{M}^{(p+3)}.$$

Now apply Lemma 6.1.2 to the sequences

$$\mathfrak{M}^{(0)} \cdot \mathfrak{M}^{(1)} \cdot \mathfrak{M}^{(2)} \cdots, \ \mathfrak{M}^{(0)} - \mathfrak{M}^{(1)}, \ \mathfrak{M}^{(1)} - \mathfrak{M}^{(2)}, \ \mathfrak{M}^{(2)} - \mathfrak{M}^{(3)}, \ \mathfrak{M}^{(3)} - \mathfrak{M}^{(4)}, \ldots,$$

and

$$\mathfrak{M}^{(0)} \cdot \mathfrak{M}^{(1)} \cdot \mathfrak{M}^{(2)} \cdots, \ \mathfrak{M}^{(2)} - \mathfrak{M}^{(3)}, \ \mathfrak{M}^{(1)} - \mathfrak{M}^{(2)}, \ \mathfrak{M}^{(4)} - \mathfrak{M}^{(5)}, \ \mathfrak{M}^{(3)} - \mathfrak{M}^{(4)}, \ldots.$$

It gives $\mathfrak{M}^{(0)} \sim \mathfrak{M}^{(1)}$, that is $\mathfrak{M} \sim \mathfrak{M}'$, and thus $\mathfrak{M} \sim \mathfrak{N}$.

6.2. [Range X] $\sim$ [Range X^*]. Comparability theorem The following Lemma is an important means of establishing equality of relative dimension.

Lemma 6.2.1. *If X is a linear, closed operator with an everywhere dense domain, and $X \,\eta\, \mathbf{M}$, then*

$$[\,\mathrm{Range}\, X\,] \sim [\,\mathrm{Range}\, X^*\,].$$

Proof: Form the canonical decomposition of X in the sense of Definition 4.4.1. $X \,\eta\, \mathbf{M}$ implies by Lemma 4.4.1 $W \in \mathbf{M}$. W is partially isometric its initial and final sets being $[\,\mathrm{Range}\, X^*\,]$ resp. $[\,\mathrm{Range}\, X\,]$. So

$$[\,\mathrm{Range}\, X^*\,] \sim [\,\mathrm{Range}\, X\,], \qquad [\,\mathrm{Range}\, X\,] \sim [\,\mathrm{Range}\, X^*\,].$$

We now prove in two steps the analogue of the "comparability theorem" of general set-theory.

Lemma 6.2.2. *If $\mathfrak{M}, \mathfrak{N} \,\eta\, \mathbf{M}$ and $\neq (0)$, then two $\mathfrak{P}, \mathfrak{Q} \neq (0)$, $\mathfrak{P} \subset \mathfrak{M}$, $\mathfrak{Q} \subset \mathfrak{N}$, with $\mathfrak{P} \sim \mathfrak{Q}$ exist.*

Proof: Choose an $f \in \mathfrak{M}$, $f \neq 0$. Then $E_f^{\mathbf{M}} \in \mathbf{M}'$ and $\neq 0$, $P_{\mathfrak{N}} \in \mathbf{M}$ and $\neq 0$, so by the Corollary to Theorem III.

$$P_{\mathfrak{M}_f^{\mathbf{M}} \cdot \mathfrak{N}} = P_{\mathfrak{N}} \cdot E_f^{\mathbf{M}} \neq 0, \qquad \mathfrak{M}_f^{\mathbf{M}} \cdot \mathfrak{N} \neq (0).$$

Choose $g \in \mathfrak{M}_f^{\mathbf{M}} \cdot \mathfrak{N}$, $g \neq 0$, multiplying with $1/\|g\|$ makes $\|g\| = 1$. As $g \in \mathfrak{M}_f^{\mathbf{M}}$, therefore an $A \in \mathbf{M}$ with $\|g - Af\| < 1$ exists. So $\|P_{\mathfrak{N}}g - P_{\mathfrak{N}}Af\| < 1$. Now $P_{\mathfrak{N}}g = g$, $P_{\mathfrak{M}}f = f$ have the effect that

$$\|g - P_{\mathfrak{N}}AP_{\mathfrak{M}}f\| < 1.$$

As $\|g\| = 1$ this implies $P_{\mathfrak{N}}AP_{\mathfrak{M}}f \neq 0$, $P_{\mathfrak{N}}AP_{\mathfrak{M}} \neq 0$.

Now apply Lemma 6.2.1:

$$[\,\mathrm{Range}\, P_{\mathfrak{N}}AP_{\mathfrak{M}}\,] \sim [\,\mathrm{Range}(P_{\mathfrak{N}}AP_{\mathfrak{M}})^*\,].$$

As $P_{\mathfrak{N}}AP_{\mathfrak{M}} \neq 0$ and so $(P_{\mathfrak{N}}AP_{\mathfrak{M}})^* \neq 0$, both above sets are $\neq (0)$. Furthermore

$$[\,\mathrm{Range}\, P_{\mathfrak{N}}AP_{\mathfrak{M}}\,] \subset [\,\mathrm{Range}\, P_{\mathfrak{N}}\,] = \mathfrak{N},$$

$$[\,\mathrm{Range}(P_{\mathfrak{N}}AP_{\mathfrak{M}})^*\,] = [\,\mathrm{Range}\, P_{\mathfrak{M}}A^*P_{\mathfrak{N}}\,] \subset [\,\mathrm{Range}\, P_{\mathfrak{M}}\,] = \mathfrak{M}.$$

So

$$\mathfrak{P} = [\,\mathrm{Range}(P_{\mathfrak{N}}AP_{\mathfrak{M}})^*\,], \qquad \mathfrak{Q} = [\,\mathrm{Range}\, P_{\mathfrak{N}}AP_{\mathfrak{M}}\,]$$

meet our requirements.

Lemma 6.2.3. *If $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$, then either $\mathfrak{M} \sim \mathfrak{M}' \subset \mathfrak{N}$ or $\mathfrak{N} \sim \mathfrak{N}' \subset \mathfrak{M}$.*

Proof: Let Ω be the first uncountable Cantor ordinal number. Define for all $\alpha < \Omega$ a pair $\mathfrak{M}_\alpha, \mathfrak{N}_\alpha$ as follows: All $\mathfrak{M}_\alpha, \mathfrak{N}_\alpha$ will be linear, closed sets, $\eta\mathbf{M}$, $\neq (0)$, and

$$\mathfrak{M}_\alpha \subset \mathfrak{M}, \qquad \mathfrak{N}_\alpha \subset \mathfrak{N}, \qquad \mathfrak{M}_\alpha \sim \mathfrak{N}_\alpha.$$

If $\alpha < \Omega$ and all $\mathfrak{M}_\beta, \mathfrak{N}_\beta$, $\beta < \alpha$ are already defined, then form

$$\mathfrak{M} - [\,\mathfrak{M}_\beta,\ \beta < \alpha\,] \quad \text{and} \quad \mathfrak{N} - [\,\mathfrak{N}_\beta,\ \beta < \alpha\,].$$

As $\mathfrak{M}, \mathfrak{N}, \mathfrak{M}_\beta, \mathfrak{N}_\beta$ all $\eta\mathbf{M}$, these two sets are $\eta\mathbf{M}$ too. If both are $\neq 0$, then there exist two $\mathfrak{P}, \mathfrak{Q} \neq (0)$ with

$$\mathfrak{P} \subset \mathfrak{M} - [\,\mathfrak{M}_\beta,\ \beta < \alpha\,], \qquad \mathfrak{Q} \subset \mathfrak{N} - [\,\mathfrak{N}_\beta,\ \beta < \alpha\,], \qquad \mathfrak{P} \sim \mathfrak{Q}.$$

Let them be $\mathfrak{M}_\alpha = \mathfrak{P}$, $\mathfrak{N}_\alpha = \mathfrak{Q}$ for some such pair $\mathfrak{P}, \mathfrak{Q}$. As we see $\mathfrak{M}_\alpha, \mathfrak{N}_\alpha \neq (0)$, $\mathfrak{M}_\alpha \subset \mathfrak{M}$, $\mathfrak{N}_\alpha \subset \mathfrak{N}$, $\mathfrak{M}_\alpha \sim \mathfrak{N}_\alpha$ are maintained. If the above condition is not fulfilled, let $\mathfrak{M}_\alpha, \mathfrak{N}_\alpha$ (and all $\mathfrak{M}_\gamma, \mathfrak{N}_\gamma$, $\gamma \geq \alpha$) be undefined.

Let the first α for which $\mathfrak{M}_\alpha, \mathfrak{N}_\alpha$ are undefined, be $\bar{\alpha} \leq \Omega$. If $\bar{\alpha} < \Omega$, we must have

$$\mathfrak{M} - [\,\mathfrak{M}_\alpha,\ \alpha < \bar{\alpha}\,] = (0) \quad \text{or} \quad \mathfrak{N} - [\,\mathfrak{N}_\alpha,\ \alpha < \bar{\alpha}\,] = (0).$$

Consider now all $\mathfrak{M}_\alpha, \mathfrak{N}_\alpha$, $\alpha < \bar{\alpha}$. Always $\mathfrak{M}_\alpha \neq (0)$. If $\alpha, \beta < \bar{\alpha}$ assume $\alpha > \beta$. Then

$$\mathfrak{M}_\alpha \subset \mathfrak{M} - [\,\mathfrak{M}_{\beta'},\ \beta' < \alpha\,],$$

so $\mathfrak{M}_\alpha$ is orthogonal to $\mathfrak{M}_\beta$. This is symmetric in α, β, so it holds for $\alpha < \beta$ too, that is whenever $\alpha \neq \beta$. Thus all $\mathfrak{M}_\alpha$, $\alpha < \bar{\alpha}$, are $\neq (0)$ and mutually orthogonal. The same is of course true for the $\mathfrak{N}_\alpha$, $\alpha < \bar{\alpha}$.

Choose an $f_\alpha \in \mathfrak{M}_\alpha$, $f_\alpha \neq 0$, multiplying with $1/\|f_\alpha\|$ makes $\|f_\alpha\| = 1$. So the f_α, $\alpha < \bar{\alpha}$ form a normalised orthogonal set. It must therefore be finite or countably infinite (cf. (16, p. 66)), thus excluding $\bar{\alpha} = \Omega$.

If $\bar{\alpha}$ is finite, we have $\bar{\alpha} = n + 1$, $n = 0, 1, 2, \ldots$, and α runs over $1, \ldots, n$. If $\bar{\alpha}$ is infinite, then $\alpha < \bar{\alpha}$ has the same aleph as $\alpha < \omega$ and may therefore (by a re-indexing of the $\mathfrak{M}_\alpha$) be replaced by it; so we may assume that α runs over $1, 2, \ldots$.

Writing i for α, we see that we have a finite or infinite sequence of pairs $\mathfrak{M}_i, \mathfrak{N}_i$, $i = 1, 2, \ldots$; where the $\mathfrak{M}_i$ are mutually orthogonal, the $\mathfrak{N}_i$ are mutually orthogonal; $\mathfrak{M}_i \subset \mathfrak{M}$, $\mathfrak{N}_i \subset \mathfrak{N}$; $\mathfrak{M}_i \sim \mathfrak{N}_i$ and either

$$[\,\mathfrak{M}_i,\ i = 1, 2, \ldots\,] = \mathfrak{M} \quad \text{or} \quad [\,\mathfrak{N}_i,\ i = 1, 2, \ldots\,] = \mathfrak{N}.$$

Now Lemma 6.1.2 applies:

$$[\,\mathfrak{M}_i,\ i = 1, 2, \ldots\,] \sim [\,\mathfrak{N}_i,\ i = 1, 2, \ldots\,].$$

Thus if we define $\mathfrak{N}' = [\,\mathfrak{N}_i,\ i = 1, 2, \ldots\,]$ resp. $\mathfrak{M}' = [\,\mathfrak{M}_i,\ i = 1, 2, \ldots\,]$ we will have $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$ resp. $\mathfrak{N} \sim \mathfrak{M}' \subset \mathfrak{M}$. So the proof is completed.

6.3. Now we are in the position to define:

Definition 6.3.1. We write $\mathfrak{M} \lesssim \mathfrak{N}$ or $\mathfrak{N} \gtrsim \mathfrak{M}$, if $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$. We write $\mathfrak{M} < \mathfrak{N}$ or $\mathfrak{N} > \mathfrak{M}$, if $\mathfrak{M} \lesssim \mathfrak{N}$ but not $\mathfrak{M} \sim \mathfrak{N}$. (As to replacing $\mathfrak{M}, \mathfrak{N}$ by $E = P_{\mathfrak{M}}$, $F = P_{\mathfrak{N}}$, and the explanatory symbol $(\cdots \mathbf{M})$, cf. Definition 6.1.1.)

And we can prove:

Lemma 6.3.1. *If $\mathfrak{M}, \mathfrak{N}, \mathfrak{P}, \mathfrak{Q}\ \eta\ \mathbf{M}$, then the following statements hold:*

(i) *$\mathfrak{M} \lesssim \mathfrak{N}$ means $\mathfrak{M} < \mathfrak{N}$ or $\mathfrak{M} \sim \mathfrak{N}$.*

(ii) *If* $\mathfrak{M} \sim \mathfrak{P}$, $\mathfrak{N} \sim \mathfrak{Q}$, *then* $\mathfrak{M} < \mathfrak{N}$ *is equivalent to* $\mathfrak{P} < \mathfrak{Q}$.

(iii) *One and only one of the three relations* $\mathfrak{M} \gtreqless \mathfrak{N}$ *holds.*

(iv) $\mathfrak{M} < \mathfrak{N}$, $\mathfrak{N} < \mathfrak{P}$ *imply* $\mathfrak{M} < \mathfrak{P}$.

Proof: We prove the statements in a somewhat changed order.

Ad (i): Obvious.

Ad (iv): $\mathfrak{M} \lesssim \mathfrak{N}$, $\mathfrak{N} \lesssim \mathfrak{P}$ imply $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$, $\mathfrak{N} \sim \mathfrak{P}' \subset \mathfrak{P}$. If $U \in \mathbf{M}$ is the isometric mapping of $\mathfrak{N}$ on $\mathfrak{P}'$, and $\mathfrak{P}''$ the U-image of $\mathfrak{N}'$, then we have

$$\mathfrak{M} \sim \mathfrak{N}' \sim \mathfrak{P}'' \subset \mathfrak{P}' \subset \mathfrak{P},$$

so $\mathfrak{M} \lesssim \mathfrak{P}$.

Now if we had $\mathfrak{M} \sim \mathfrak{P}$, then we could write $\mathfrak{P} \sim \mathfrak{M} \subset \mathfrak{M}$, that is $\mathfrak{P} \lesssim \mathfrak{M}$; together with $\mathfrak{M} \lesssim \mathfrak{N}$ this gives $\mathfrak{P} \lesssim \mathfrak{N}$. $\mathfrak{N} \lesssim \mathfrak{P}$ and $\mathfrak{P} \lesssim \mathfrak{N}$ gives by Lemma 6.1.3 $\mathfrak{N} \sim \mathfrak{P}$, contradicting $\mathfrak{N} < \mathfrak{P}$. Thus we must have $\mathfrak{M} < \mathfrak{P}$.

Ad (ii): Assume $\mathfrak{M} < \mathfrak{N}$. As $\mathfrak{M} \sim \mathfrak{P}$, $\mathfrak{N} \sim \mathfrak{Q}$, we can write $\mathfrak{P} \lesssim \mathfrak{N}$, $\mathfrak{M} \lesssim \mathfrak{N}$, $\mathfrak{N} \lesssim \mathfrak{Q}$, implying $\mathfrak{P} \lesssim \mathfrak{Q}$ (cf. the beginning of the proof of (iv)). $\mathfrak{P} \sim \mathfrak{Q}$ would imply $\mathfrak{M} \sim \mathfrak{P}$, $\mathfrak{P} \sim \mathfrak{Q}$, $\mathfrak{Q} \sim \mathfrak{N}$ so $\mathfrak{M} \sim \mathfrak{N}$, contradicting $\mathfrak{M} < \mathfrak{N}$. Thus we must have $\mathfrak{P} < \mathfrak{Q}$. In the same way $\mathfrak{P} < \mathfrak{Q}$ implies $\mathfrak{M} < \mathfrak{N}$.

Ad (iii): By Lemma 6.2.3 we have $\mathfrak{M} \lesssim \mathfrak{N}$ or $\mathfrak{N} \lesssim \mathfrak{M}$, by (i) this means $\mathfrak{M} \gtreqless \mathfrak{N}$. $\sim$ and $>$ at once imply $\mathfrak{M} < \mathfrak{M}$ by (ii); $<$ and $\sim$ similarly; $<$ and $>$ imply again $\mathfrak{M} < \mathfrak{M}$ by (iv). Now $\mathfrak{M} < \mathfrak{M}$ is impossible as $\mathfrak{M} \sim \mathfrak{M}$. So precisely one of the three relations holds.

Summing up Lemma 6.1.1 and Lemma 6.3.1 we have:

Theorem VI. *The notions* $\mathfrak{M} \sim \mathfrak{N}$ *and* $\mathfrak{M} < \mathfrak{N}$ *(that is* $\mathfrak{N} > \mathfrak{M}$*) for* $\mathfrak{M}, \mathfrak{N} \ \eta\ \mathbf{M}$ *(cf. Definition 6.1.1 and Definition 6.3.1) have the properties of an equivalence and of a complete ordering.*

Further essential properties are given in Lemma 6.1.2 and Lemma 6.3.1.

Some immediate consequences:

Lemma 6.3.2. *Assume* $\mathfrak{M}, \mathfrak{N} \ \eta\ \mathbf{M}$. *Then we have:*

(i) $\mathfrak{M} \subset \mathfrak{N}$ *implies* $\mathfrak{M} \lesssim \mathfrak{N}$.

(ii) *Always* $(0) \lesssim \mathfrak{M} \lesssim \mathfrak{H}$.

(iii) $\mathfrak{M} \sim (0)$ *implies* $\mathfrak{M} = (0)$.

Proof: Ad (i): Write $\mathfrak{M} \sim \mathfrak{M} \subset \mathfrak{N}$.

Ad (ii): By $(0) \subset \mathfrak{M} \subset \mathfrak{H}$ and (i).

Ad (iii): $\mathfrak{M}$ is the image of (0) by some linear U so $\mathfrak{M} = (0)$.

Chapter VII: Finite and Infinite Sets

7.1. We continue along the lines which are familiar from set-theory:

Definition 7.1.1. An $\mathfrak{M}\ \eta\ \mathbf{M}$ is *finite* if $\mathfrak{M} \sim \mathfrak{N} \subset \mathfrak{M}$ implies $\mathfrak{N} = \mathfrak{M}$; it is *infinite*, if this is not the case. (As to replacing $\mathfrak{M}$ by $E = P_{\mathfrak{M}}$, and the explanatory symbol $(\cdots \mathbf{M})$, cf. Definition 6.1.1.)

Some immediate consequences:

Lemma 7.1.1. *Assume $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$. Then we have:*

(i) If $\mathfrak{M} \lesssim \mathfrak{N}$ and $\mathfrak{M}$ is infinite, then $\mathfrak{N}$ is too; if $\mathfrak{N}$ is finite, $\mathfrak{M}$ is too.

(ii) (0) is finite.

(iii) If infinite $\mathfrak{M}$'s exist at all, then $\mathfrak{H}$ is infinite.

Proof: Ad (i): The second statement follows from the first one, so we consider the first one only. As $\mathfrak{M}$ is infinite, a $\mathfrak{P}$ with $\mathfrak{M} \sim \mathfrak{P} \subsetneqq \mathfrak{M}$ exists. Then

$$[\,\mathfrak{M}, \mathfrak{N} - \mathfrak{M}\,] \sim [\,\mathfrak{P}, \mathfrak{N} - \mathfrak{M}\,] \subsetneqq [\,\mathfrak{M}, \mathfrak{N} - \mathfrak{M}\,],$$

$$\mathfrak{N} \sim [\,\mathfrak{P}, \mathfrak{N} - \mathfrak{M}\,] \subsetneqq \mathfrak{N}.$$

So $\mathfrak{N}$ is infinite too.

Ad (ii): $\mathfrak{N} \subset (0)$ alone implies $\mathfrak{N} = (0)$.

Ad (iii): If $\mathfrak{M}$ is infinite, $\mathfrak{M} \lesssim \mathfrak{H}$ implies by (i), that $\mathfrak{H}$ is infinite too.

We now begin the preparations for a quantitative comparison of sets $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$.

Lemma 7.1.2. *Assume $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$, $\mathfrak{M} \neq (0)$. Then there exists a finite or infinite sequence $\mathfrak{P}_1, \mathfrak{P}_2, \ldots$ (its length can be 0 too) and a $\mathfrak{Q}$, such that we have:*

$\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{Q}$ are mutually orthogonal linear, closed sets, all $\eta\mathbf{M}$,

$$[\,\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{Q}\,] = \mathfrak{N}, \qquad \mathfrak{M} \sim \mathfrak{P}_1 \sim \mathfrak{P}_2 \sim \cdots, \qquad \mathfrak{Q} < \mathfrak{M},$$

and if the sequence is infinite, we may even assume $\mathfrak{Q} = (0)$.

Proof: Let Ω be the first uncountable Cantor ordinal number. Define for all $\alpha < \Omega$ a $\mathfrak{P}_\alpha$ as follows: All $\mathfrak{P}_\alpha$ will be linear, closed sets,

$$\mathfrak{P}_\alpha\ \eta\ \mathbf{M}, \qquad \mathfrak{P}_\alpha \subset \mathfrak{N}, \qquad \mathfrak{M} \sim \mathfrak{P}_\alpha.$$

If $\alpha < \Omega$ and all $\mathfrak{P}_\beta$, $\beta < \alpha$ are already defined, then form $\mathfrak{N} - [\,\mathfrak{P}_\beta,\ \beta < \alpha\,]$. As $\mathfrak{N}, \mathfrak{P}_\beta$ all $\eta\mathbf{M}$, this set is $\eta\mathbf{M}$ too. If it is $\gtrsim \mathfrak{M}$, then there exists a $\mathfrak{P}\ \eta\ \mathbf{M}$ with

$$\mathfrak{P} \subset \mathfrak{N} - [\,\mathfrak{P}_\beta,\ \beta < \alpha\,], \qquad \mathfrak{M} \sim \mathfrak{P}.$$

Let then $\mathfrak{P}_\alpha$ be some such $\mathfrak{P}$. If the above condition is not fulfilled, let $\mathfrak{P}_\alpha$ (and all $\mathfrak{P}_\gamma$, $\gamma \geq \alpha$) be undefined.

Let the first α for which $\mathfrak{P}_\alpha$ is undefined, be $\bar{\alpha} \leq \Omega$. If $\bar{\alpha} < \Omega$, we must have

$$\mathfrak{N} - [\,\mathfrak{P}_\alpha,\ \alpha < \bar{\alpha}\,] < \mathfrak{M}.$$

We see in the same way as in the proof of Lemma 6.2.3, that the $\mathfrak{P}_\alpha$, $\alpha < \bar{\alpha}$, are mutually orthogonal; and as $\mathfrak{P}_\alpha \sim \mathfrak{M} \neq (0)$, therefore $\mathfrak{P}_\alpha \neq (0)$. This implies, as above, that their set is finite or countably infinite, thus excluding $\bar{\alpha} = \Omega$.

If $\bar{\alpha}$ is finite, we have $\bar{\alpha} = n + 1$, $n = 0, 1, 2, \ldots$, and α runs over $1, \ldots, n$. If $\bar{\alpha}$ is infinite, then $\alpha < \bar{\alpha}$ has the same aleph as $\alpha < \omega$, and may therefore (by a re-indexing of the $\mathfrak{P}_\alpha$) be replaced by it; so we may assume that α runs over $1, 2, \ldots$.

Writing i for α we have a finite or infinite sequence $\mathfrak{P}_i$, $i = 1, 2, \ldots$; where the $\mathfrak{P}_i$ are mutually orthogonal; $\mathfrak{M} \sim \mathfrak{P}_1 \sim \mathfrak{P}_2 \sim \cdots$; and

$$\mathfrak{Q} = \mathfrak{N} - [\,\mathfrak{P}_1, \mathfrak{P}_2, \ldots\,] < \mathfrak{M}.$$

Thus $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{Q}$ are mutually orthogonal and

$$\mathfrak{N} = [\,\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{Q}\,].$$

So the case of a finite sequence is completely settled. In the case of an infinite sequence we must still get rid of $\mathfrak{Q}$. Assume therefore that the $\mathfrak{P}_1, \mathfrak{P}_2, \ldots$ form an infinite sequence.

We have $\mathfrak{Q} < \mathfrak{M} \sim \mathfrak{P}_1$, so $\mathfrak{Q} \sim \mathfrak{Q}' \subset \mathfrak{P}_1$. Apply Lemma 6.1.2 to the two sequences $\mathfrak{Q}, \mathfrak{P}_1, \mathfrak{P}_2, \ldots$ and $\mathfrak{Q}', \mathfrak{P}_2, \mathfrak{P}_3, \ldots$. It gives

$$\begin{aligned} \mathfrak{N} &= [\,\mathfrak{Q}, \mathfrak{P}_1, \mathfrak{P}_2, \ldots\,] \sim [\,\mathfrak{Q}', \mathfrak{P}_2, \mathfrak{P}_3, \ldots\,] \\ &\subset [\,\mathfrak{P}_1, \mathfrak{P}_2, \ldots\,] \subset [\,\mathfrak{Q}, \mathfrak{P}_1, \mathfrak{P}_2, \ldots\,] = \mathfrak{N}. \end{aligned}$$

So

$$\mathfrak{N} \lesssim [\,\mathfrak{P}_1, \mathfrak{P}_2, \ldots\,] \lesssim \mathfrak{N}, \qquad [\,\mathfrak{P}_1, \mathfrak{P}_2, \ldots\,] \sim \mathfrak{N}.$$

Let $U \in \mathbf{M}$ be an isometric mapping of $[\,\mathfrak{P}_1, \mathfrak{P}_2, \ldots\,]$ on $\mathfrak{N}$, $\mathfrak{P}'_i$ the U-image of $\mathfrak{P}_i$. Then the $\mathfrak{P}'_i$ are mutually orthogonal and $[\,\mathfrak{P}'_1, \mathfrak{P}'_2, \ldots\,] = \mathfrak{N}$. Using $UP_{\mathfrak{P}_i}$ shows $\mathfrak{P}_i \sim \mathfrak{P}'_i$; so $\mathfrak{M} \sim \mathfrak{P}'_1 \sim \mathfrak{P}'_2 \sim \cdots$. Thus we can replace

$$\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{Q} \quad \text{by} \quad \mathfrak{P}'_1, \mathfrak{P}'_2, \ldots, (0).$$

This completes the proof.

Lemma 7.1.3. *Assume $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$, $\mathfrak{M} \neq (0)$, $\mathfrak{N}$ finite. Then the sequence $\mathfrak{P}_1, \mathfrak{P}_2, \ldots$ from Lemma 7.1.2 must be finite, and its length is uniquely determined by $\mathfrak{M}, \mathfrak{N}$. Call this number*

$$\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] = 0, 1, 2, \ldots .$$

Proof: If the sequence $\mathfrak{P}_1, \mathfrak{P}_2, \ldots$ was infinite, we could apply Lemma 6.1.2 to $\mathfrak{P}_1, \mathfrak{P}_2, \ldots$ and $\mathfrak{P}_2, \mathfrak{P}_3, \ldots$, giving

$$\mathfrak{N} = [\,\mathfrak{P}_1, \mathfrak{P}_2, \ldots\,] \sim [\,\mathfrak{P}_2, \mathfrak{P}_3, \ldots\,] \subsetneqq [\,\mathfrak{P}_1, \mathfrak{P}_2, \ldots\,] = \mathfrak{N}$$

($\subsetneqq$ because $\mathfrak{P}_1 \neq (0)$, as $\mathfrak{P}_1 \sim \mathfrak{M} \neq (0)$), thus $\mathfrak{N}$ infinite, contradicting the assumption.

Assume now we had two representations

$$\mathfrak{N} = [\,\mathfrak{P}_1, \ldots, \mathfrak{P}_m, \mathfrak{Q}\,] = [\,\mathfrak{P}'_1, \ldots, \mathfrak{P}'_n, \mathfrak{Q}'\,]$$

in the sense of Lemma 7.1.2, with $m \neq n$. Owing to the symmetry we may assume $m < n$, $m + 1 \leq n$.

Now $\mathfrak{Q} < \mathfrak{M} \sim \mathfrak{P}'_{m+1}$, so $\mathfrak{Q} \sim \mathfrak{Q}^* \subset \mathfrak{P}'_{m+1}$, and $\mathfrak{Q}^* \neq \mathfrak{P}'_{m+1}$. So by Lemma 6.1.2,

$$\begin{aligned}\mathfrak{N} &= [\,\mathfrak{P}_1, \ldots, \mathfrak{P}_m, \mathfrak{Q}\,] \sim [\,\mathfrak{P}'_1, \ldots, \mathfrak{P}'_m, \mathfrak{Q}^*\,] \\ &\subsetneqq [\,\mathfrak{P}'_1, \ldots, \mathfrak{P}'_m, \mathfrak{P}'_{m+1}\,] \subset [\,\mathfrak{P}'_1, \ldots, \mathfrak{P}'_n, \mathfrak{Q}'\,] = \mathfrak{N},\end{aligned}$$

and again $\mathfrak{N}$ would be infinite, contradicting the assumption.

7.2. We are now in the position to determine the chief characteristics of infinity.

Lemma 7.2.1. *Assume $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$, $\mathfrak{N}$ infinite. Then $\mathfrak{M} \lesssim \mathfrak{N}$.*

Proof: We have $\mathfrak{N} \sim \mathfrak{N}' \subsetneqq \mathfrak{N}$. Let $U \in \mathbf{M}$ be the partially isometric mapping of $\mathfrak{N}$ on $\mathfrak{N}'$. Form for every $n = 0, 1, 2, \ldots$ the U^n-image of $\mathfrak{N}$, and call them $\mathfrak{N}^{(n)}$ ($\mathfrak{N}, \mathfrak{N}'$ coincide with $\mathfrak{N}^{(0)}, \mathfrak{N}^{(1)}$ resp.). We have $\mathfrak{N}^{(0)} \supset \mathfrak{N}^{(1)}$, thus (apply U^n!) $\mathfrak{N}^{(n)} \supset \mathfrak{N}^{(n+1)}$; therefore

$$\mathfrak{N}^{(0)} \supset \mathfrak{N}^{(1)} \supset \mathfrak{N}^{(2)} \supset \cdots .$$

U^n is partially isometric with the final set $\mathfrak{N}^{(n)}$ therefore $\mathfrak{N}^{(n)}\ \eta\ \mathbf{M}$. U maps $\mathfrak{N}^{(n)}$ on $\mathfrak{N}^{(n+1)}$, $\mathfrak{N}^{(n+1)}$ on $\mathfrak{N}^{(n+2)}$, therefore $\mathfrak{N}^{(n)} - \mathfrak{N}^{(n+1)}$ on $\mathfrak{N}^{(n+1)} - \mathfrak{N}^{(n+2)}$. Using

$$U(P_{\mathfrak{N}^{(n)}} - P_{\mathfrak{N}^{(n+1)}})$$

shows therefore, that

$$\mathfrak{N}^{(n)} - \mathfrak{N}^{(n+1)} \sim \mathfrak{N}^{(n+1)} - \mathfrak{N}^{(n+2)}.$$

Put $\mathfrak{P}_n^* = \mathfrak{N}^{(n)} - \mathfrak{N}^{(n+1)}$, then we see: The $\mathfrak{P}_1^*, \mathfrak{P}_2^*, \ldots$ form an infinite sequence, are mutually orthogonal, all $\subset \mathfrak{N}^{(0)} = \mathfrak{N}$ and $\mathfrak{P}_1^* \sim \mathfrak{P}_2^* \sim \cdots$. As

$$\mathfrak{P}_1^* = \mathfrak{N}^{(0)} - \mathfrak{N}^{(1)} = \mathfrak{N} - \mathfrak{N}' \neq (0),$$

all $\mathfrak{P}_n^* \neq 0$.

Apply now the construction of Lemma 7.1.2 to $\mathfrak{P}_1^*, \mathfrak{N}$ (for $\mathfrak{M}, \mathfrak{N}$) and choose $\mathfrak{P}_\alpha$ for $\alpha = 1, 2, \ldots$ equal to $\mathfrak{P}_1^*, \mathfrak{P}_2^*, \ldots$ resp. Then an infinite $\bar{\alpha}$ will result, which will be replaced, as described in the proof of Lemma 7.1.2, by $\bar{\alpha} = \omega$. And then the $\mathfrak{Q}$ will be eliminated. So

$$\mathfrak{N} = [\, \mathfrak{P}_1, \mathfrak{P}_2, \ldots \,],$$

where the sequence is infinite, and $\mathfrak{P}_1^* \sim \mathfrak{P}_1 \sim \mathfrak{P}_2 \sim \cdots$.

Apply next the result of Lemma 7.1.2 to $\mathfrak{P}_1^*, \mathfrak{M}$ (for $\mathfrak{M}, \mathfrak{N}$), then

$$\mathfrak{M} = [\, \mathfrak{P}_1', \mathfrak{P}_2', \ldots, \mathfrak{Q}' \,],$$

where the sequence is finite or infinite, and

$$\mathfrak{P}_1^* \sim \mathfrak{P}_1' \sim \mathfrak{P}_2' \sim \cdots, \qquad \mathfrak{Q}' < \mathfrak{P}_1^*.$$

If it is infinite, we may put $\mathfrak{Q}' = (0)$, and so, using Lemma 6.1.2,

$$\mathfrak{M} = [\, \mathfrak{P}_1', \mathfrak{P}_2', \ldots \,] \sim [\, \mathfrak{P}_1, \mathfrak{P}_2, \ldots \,] = \mathfrak{N}.$$

If it is finite, we have

$$\mathfrak{M} = [\, \mathfrak{P}_1', \ldots, \mathfrak{P}_n', \mathfrak{Q}' \,] \quad \text{and} \quad \mathfrak{Q}' < \mathfrak{P}_1^* \sim \mathfrak{P}_{n+1}, \quad \mathfrak{Q}' \sim \mathfrak{Q}'' \subset \mathfrak{P}_{n+1}.$$

Thus Lemma 6.1.2 gives

$$\begin{aligned} \mathfrak{M} = [\, \mathfrak{P}_1', \ldots, \mathfrak{P}_n', \mathfrak{Q}' \,] &\sim [\, \mathfrak{P}_1, \ldots, \mathfrak{P}_n, \mathfrak{Q}'' \,] \\ &\subset [\, \mathfrak{P}_1, \ldots, \mathfrak{P}_n, \mathfrak{P}_{n+1} \,] \subset [\, \mathfrak{P}_1, \mathfrak{P}_2, \ldots \,] = \mathfrak{N}. \end{aligned}$$

So we have $\mathfrak{M} \lesssim \mathfrak{N}$ in any event.

Lemma 7.2.2. *There are two possibilities:*

(a) An $\mathfrak{M}\ \eta\ \mathbf{M}$ is infinite if and only if $\mathfrak{M} \sim \mathfrak{H}$ and finite if and only if $\mathfrak{M} < \mathfrak{H}$.

(b) All $\mathfrak{M}\ \eta\ \mathbf{M}$ are finite. In this case $\mathfrak{M} \sim \mathfrak{H}$ implies $\mathfrak{M} = \mathfrak{H}$.

Proof: If $\mathfrak{H}$ is finite, every $\mathfrak{M}$ is so, by Lemma 7.1.1, (iii). If $\mathfrak{M} \neq \mathfrak{H}$, then $\mathfrak{M} \subsetneq \mathfrak{H}$, and $\mathfrak{H} \sim \mathfrak{M}$ would imply that $\mathfrak{H}$ is infinite. Thus if $\mathfrak{H}$ is finite, case (b) holds.

If $\mathfrak{H}$ is infinite, then $\mathfrak{M} \sim \mathfrak{H}$ implies that $\mathfrak{M}$ too is infinite by Lemma 7.1.1, (i). If conversely $\mathfrak{M}$ is infinite, Lemma 7.2.1 gives if applied to $\mathfrak{H}, \mathfrak{M}$ (for $\mathfrak{M}, \mathfrak{N}$) $\mathfrak{H} \lesssim \mathfrak{M}$. By Lemma 6.3.2, (i), $\mathfrak{M} \lesssim \mathfrak{H}$, so $\mathfrak{M} \sim \mathfrak{H}$. And as we have always $\mathfrak{M} \lesssim \mathfrak{H}$ (cf. as above), therefore $\mathfrak{M} < \mathfrak{H}$ is characteristic for finite $\mathfrak{M}$'s. Thus case (a) holds.

A convenient characterisation of infinity is this:

Lemma 7.2.3. *An $\mathfrak{M}\ \eta\ \mathbf{M}$ is infinite if and only if $\mathfrak{M} \neq (0)$ and an $\mathfrak{N} \subset \mathfrak{M}$ with $\mathfrak{M} \sim \mathfrak{N} \sim \mathfrak{M} - \mathfrak{N}$ exists.*

Proof: The condition is sufficient: If $\mathfrak{N}$ is such, then

$$\mathfrak{M} - \mathfrak{N} \sim \mathfrak{M} \neq (0), \qquad \mathfrak{M} - \mathfrak{N} \neq (0), \qquad \mathfrak{N} \neq \mathfrak{M},$$

so $\mathfrak{M} \sim \mathfrak{N} \subsetneq \mathfrak{M}$.

It is necessary: If $\mathfrak{M}$ is infinite, then the proof of Lemma 7.2.1 shows that an infinite sequence $\mathfrak{P}_1, \mathfrak{P}_2, \ldots$ exists such that all $\mathfrak{P}_i$ are mutually orthogonal, $\neq (0)$,

$$\mathfrak{M} = [\,\mathfrak{P}_1, \mathfrak{P}_2, \ldots\,], \qquad \mathfrak{P}_1 \sim \mathfrak{P}_2 \sim \mathfrak{P}_3 \sim \cdots.$$

Now Lemma 6.1.2 gives

$$[\,\mathfrak{P}_1, \mathfrak{P}_2, \ldots\,] \sim [\,\mathfrak{P}_1, \mathfrak{P}_3, \ldots\,] \sim [\,\mathfrak{P}_2, \mathfrak{P}_4, \ldots\,]$$

so for $\mathfrak{N} = [\,\mathfrak{P}_1, \mathfrak{P}_3, \ldots\,]$, $\mathfrak{N} \subset \mathfrak{M}$, $\mathfrak{M} \sim \mathfrak{N} \sim \mathfrak{M} - \mathfrak{N}$. As $\mathfrak{P}_1 \neq (0)$, necessarily $\mathfrak{M} \neq (0)$.

7.3. Continuing our discussion along the lines which are customary in general set-theory, it is natural to establish the additivity of the notion of finiteness. We will prove it in five successive steps, the first Lemma having a certain interest of its own.

Lemma 7.3.1. *Assume $\mathfrak{M}, \mathfrak{N}, \mathfrak{P}\ \eta\ \mathbf{M}$; $\mathfrak{M}, \mathfrak{N}$ orthogonal; $\mathfrak{P} \subset [\mathfrak{M}, \mathfrak{N}]$. Then $\mathfrak{P}$ can be represented in the following form (all sets which follow being linear and closed):*

There exist two orthogonal sets $\mathfrak{M}', \mathfrak{M}''\ \eta\ \mathbf{M}$ and $\subset \mathfrak{M}$; two orthogonal sets $\mathfrak{N}', \mathfrak{N}''\ \eta\ \mathbf{M}$ and $\subset \mathfrak{N}$; and a linear, closed operator $A\ \eta\ \mathbf{M}$, with the following properties; Domain A is a part of and dense in $\mathfrak{M}''$; Range A is a part of and dense in $\mathfrak{N}''$; $Af = 0$ implies $f = 0$. The set $\mathfrak{P}^*$ of all $f + Af$ ($f \in$ Domain A) is linear, closed, and $\eta\mathbf{M}$.

$$\mathfrak{P}^* \sim \mathfrak{M}'' \sim \mathfrak{N}'',$$

$\mathfrak{M}', \mathfrak{N}', \mathfrak{P}^*$ are mutually orthogonal and

$$\mathfrak{P} = [\mathfrak{M}', \mathfrak{N}', \mathfrak{P}^*].$$

Proof: Put $\mathfrak{M}' = \mathfrak{M} \cdot \mathfrak{P}$, $\mathfrak{N}' = \mathfrak{N} \cdot \mathfrak{P}$, then $\mathfrak{M}', \mathfrak{N}' \; \eta \; \mathbf{M}$, $\mathfrak{M}' \subset \mathfrak{M}$, $\mathfrak{N}' \subset \mathfrak{N}$, thus $\mathfrak{M}', \mathfrak{N}'$ are orthogonal; and $\mathfrak{M}', \mathfrak{N}' \subset \mathfrak{P}$. Thus we can form $\mathfrak{P}^* = \mathfrak{P} - [\mathfrak{M}', \mathfrak{N}']$, then $\mathfrak{P}^* \, \eta \, \mathbf{M}$; $\mathfrak{P}^*$ is orthogonal to both $\mathfrak{M}', \mathfrak{N}'$ and $\mathfrak{P} = [\mathfrak{M}', \mathfrak{N}', \mathfrak{P}^*]$. Thus we must only give now the characterisation of $\mathfrak{P}^*$ given in our Lemma in terms of $\mathfrak{M}''$, $\mathfrak{N}''$, and A.

An $f \in \mathfrak{P}^*$ is $f \in [\mathfrak{M}, \mathfrak{N}]$, thus $f = g + h$, $g \in \mathfrak{M}$, $h \in \mathfrak{N}$ (cf. (16, p. 76)), g, h being clearly uniquely determined. If in such a decomposition $g = 0$, then $f = h \in \mathfrak{N}$, $f \in \mathfrak{P}^* \subset \mathfrak{P}$, so $f \in \mathfrak{N} \cdot \mathfrak{P} = \mathfrak{N}' \subset \mathfrak{P} - \mathfrak{P}^*$. As $f \in \mathfrak{P}^*$ this implies $f = 0$, $h = 0$. Similarly $h = 0$ implies $g = 0$. Owing to the linearity this means, that g determines h uniquely and conversely (as far as they belong to any $f \in \mathfrak{P}^*$ at all). Thus we can define a one-valued operator A with a one-valued inverse A^{-1} by $Ag = h$, whenever g, h belong to an $f \in \mathfrak{P}^*$ in the above way. Thus $\mathfrak{P}^*$ is the set of all $f + Af$. As it is $\eta \mathbf{M}$, linear, and closed, the same is true for A.

Put now $\mathfrak{M}'' = [\text{Domain}\, A]$, $\mathfrak{N}'' = [\text{Range}\, A]$. All we must prove is, that $\mathfrak{M}''$ is $\subset \mathfrak{M}$ and orthogonal to $\mathfrak{M}'$; that $\mathfrak{N}'' \subset \mathfrak{N}$ and orthogonal to $\mathfrak{N}'$; and that $\mathfrak{P}^* \sim \mathfrak{N}'' \sim \mathfrak{M}''$.

If $g \in \text{Domain}\, A$, then $f = g + h$, $f \in \mathfrak{P}^*$, $g \in \mathfrak{M}$, $h \in \mathfrak{N}$. So $g \in \mathfrak{M}$. f is orthogonal to $\mathfrak{M}'$, and $h \in \mathfrak{N}$ is orthogonal to $\mathfrak{M}$, so to $\mathfrak{M}'$ too. Thus $g = f - h$ is orthogonal to $\mathfrak{M}'$, $g \in \mathfrak{M} - \mathfrak{M}'$. Thus $\text{Domain}\, A \subset \mathfrak{M} - \mathfrak{M}'$, $\mathfrak{M}'' = [\text{Domain}\, A] \subset \mathfrak{M} - \mathfrak{M}'$. Similarly $\mathfrak{N}'' \subset \mathfrak{N} - \mathfrak{N}'$.

Consider now the operator $A_1 = (1 + A)P_{\mathfrak{M}''}$ (the linear operator, which agrees with $1 + A$ in $\mathfrak{M}''$ and with 0 in $\mathfrak{H} - \mathfrak{M}''$). A_1 is clearly $\eta \mathbf{M}$, linear, and closed, and its domain is everywhere dense. Now $[\text{Range}\, A_1] = [\text{Range}(1 + A)] = \mathfrak{P}^*$. On the other hand

$$\begin{aligned} [\text{Range}\, A_1^*] &= \mathfrak{H} - \{ f \, ; \, A_1 f = 0 \} \\ &= \mathfrak{H} - \{ f \, ; \, (1 + A)P_{\mathfrak{M}''} f = 0 \} \\ &= \mathfrak{H} - \{ f \, ; \, P_{\mathfrak{M}''} f = 0 \} \\ &= \mathfrak{H} - (\mathfrak{H} - \mathfrak{M}'') = \mathfrak{M}''. \end{aligned}$$

Now Lemma 6.2.1 gives, applied to A_1, $\mathfrak{P}^* \sim \mathfrak{M}''$. Similarly $\mathfrak{P}^* \sim \mathfrak{N}''$.

Thus all parts of our Lemma are proved.

Lemma 7.3.2. *Assume $\mathfrak{M}, \mathfrak{N}, \mathfrak{P} \; \eta \; \mathbf{M}$; $\mathfrak{M}, \mathfrak{N}$ orthogonal; $\mathfrak{P} \subset [\mathfrak{M}, \mathfrak{N}]$. Then either $\mathfrak{P} \lesssim \mathfrak{M}$, or $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{P} \lesssim \mathfrak{N}$.*

Proof: Apply Lemma 7.3.1 to $\mathfrak{M}, \mathfrak{N}, \mathfrak{P}$ obtaining the sets $\mathfrak{M}', \mathfrak{M}'', \mathfrak{N}', \mathfrak{N}'', \mathfrak{P}^*$ and the operator A; and to $\mathfrak{M}, \mathfrak{N}, [\mathfrak{M}, \mathfrak{N}] - \mathfrak{P}$ obtaining the sets $\overline{\mathfrak{M}}', \overline{\mathfrak{M}}'', \overline{\mathfrak{N}}', \overline{\mathfrak{N}}'', \overline{\mathfrak{P}}^*$ and the operator B. As $\mathfrak{P}, [\mathfrak{M}, \mathfrak{N}] - \mathfrak{P}$ are orthogonal, $\mathfrak{M}' = \mathfrak{M} \cdot \mathfrak{P}, \overline{\mathfrak{M}}' = \mathfrak{M} \cdot ([\mathfrak{M}, \mathfrak{N}] - \mathfrak{P})$ are too; similarly $\mathfrak{N}', \overline{\mathfrak{N}}'$. Form

$$\mathfrak{M}^0 = \mathfrak{M} - [\mathfrak{M}', \overline{\mathfrak{M}}'], \qquad \mathfrak{N}^0 = \mathfrak{N} - [\mathfrak{N}', \overline{\mathfrak{N}}'].$$

If $f \in \text{Domain}\, A$ then $f + Af \in \mathfrak{P}^* \subset \mathfrak{P}$ is orthogonal to $\overline{\mathfrak{M}}' \subset [\mathfrak{M}, \mathfrak{N}] - \mathfrak{P}$. $Af \in \mathfrak{N}$ is orthogonal too to $\overline{\mathfrak{M}}' \subset \mathfrak{M}$. Thus f is orthogonal to $\overline{\mathfrak{M}}'$. Thus all $\mathfrak{M}'' = [\text{Domain}\, A]$ is orthogonal to $\overline{\mathfrak{M}}'$. Besides we know that it is orthogonal to $\mathfrak{M}'$, and $\subset \mathfrak{M}$; so $\mathfrak{M}'' \subset \mathfrak{M} - [\mathfrak{M}', \overline{\mathfrak{M}}'] = \mathfrak{M}^0$. Similarly $\overline{\mathfrak{M}}'' \subset \mathfrak{M}^0$. Similarly $\mathfrak{N}'', \overline{\mathfrak{N}}'' \subset \mathfrak{N}^0$.

Now consider $\mathfrak{N}', \overline{\mathfrak{M}}'$. We have $\mathfrak{N}' \gtreqless \overline{\mathfrak{M}}'$ (by Theorem VI), that is either $\mathfrak{N}' \lesssim \overline{\mathfrak{M}}'$, $\mathfrak{N}' \sim \mathfrak{M}^* \subset \overline{\mathfrak{M}}'$, or $\mathfrak{N}' \gtrsim \overline{\mathfrak{M}}'$, $\overline{\mathfrak{M}}' \sim \mathfrak{M}^* \subset \mathfrak{N}'$. In the first case (by Lemma 6.1.2)

$$\mathfrak{P} = [\mathfrak{M}', \mathfrak{N}', \mathfrak{P}^*] \sim [\mathfrak{M}', \mathfrak{M}^*, \mathfrak{M}''] \subset [\mathfrak{M}', \overline{\mathfrak{M}}', \mathfrak{M}^0] = \mathfrak{M};$$

in the second case (as above)

$$[\mathfrak{M}, \mathfrak{N}] - \mathfrak{P} = [\overline{\mathfrak{M}}', \overline{\mathfrak{N}}', \overline{\mathfrak{P}}^*] \sim [\mathfrak{M}^*, \overline{\mathfrak{N}}', \overline{\mathfrak{N}}''] \subset [\mathfrak{N}', \overline{\mathfrak{N}}', \mathfrak{N}^0] = \mathfrak{N}.$$

Thus $\mathfrak{P} \lesssim \mathfrak{M}$ resp. $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{P} \lesssim \mathfrak{N}$.

Lemma 7.3.3. *Assume $\mathfrak{M}, \mathfrak{N} \,\eta\, \mathbf{M}$; $\mathfrak{M}, \mathfrak{N}$ orthogonal. If $\mathfrak{M}, \mathfrak{N}$ are finite, then $[\mathfrak{M}, \mathfrak{N}]$ is finite too.*

Proof: Assume that $[\mathfrak{M}, \mathfrak{N}]$ is infinite. Then there exists by Lemma 7.2.3 a $\mathfrak{P} \subset [\mathfrak{M}, \mathfrak{N}]$ such that $[\mathfrak{M}, \mathfrak{N}] \sim \mathfrak{P} \sim [\mathfrak{M}, \mathfrak{N}] - \mathfrak{P}$. By Lemma 7.3.2 we have $\mathfrak{P} \lesssim \mathfrak{M}$ or $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{P} \lesssim \mathfrak{N}$, so $\mathfrak{M}$ or $\mathfrak{N} \gtrsim [\mathfrak{M}, \mathfrak{N}]$. By Lemma 7.1.1, (i), then $\mathfrak{M}$ or $\mathfrak{N}$ must be infinite, contradicting our assumption.

Lemma 7.3.4. *If $\mathfrak{M}, \mathfrak{N} \,\eta\, \mathbf{M}$, then $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{N} \lesssim \mathfrak{M}$.*

Proof: $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{N}$ is obviously the set of all $P_{\mathfrak{H}-\mathfrak{N}} f$ where f runs over $[\mathfrak{M}, \mathfrak{N}]$. So it is equal to

$$\begin{aligned}
&[P_{\mathfrak{H}-\mathfrak{N}} f; f \in [\mathfrak{M}, \mathfrak{N}]] \\
&= [P_{\mathfrak{H}-\mathfrak{N}}(g + h); g \in \mathfrak{M}, h \in \mathfrak{N}] \\
&= [P_{\mathfrak{H}-\mathfrak{N}} g; g \in \mathfrak{M}, h \in \mathfrak{N}] \\
&= [P_{\mathfrak{H}-\mathfrak{N}} g; g \in \mathfrak{M}] \\
&= [P_{\mathfrak{H}-\mathfrak{N}} P_{\mathfrak{M}} g'; g' \in \mathfrak{H}] = [\text{Range}\, P_{\mathfrak{H}-\mathfrak{N}} P_{\mathfrak{M}}].
\end{aligned}$$

At the same time

$$[\text{Range}(P_{\mathfrak{H}-\mathfrak{N}}P_{\mathfrak{M}})^*] = [\text{Range}\, P_{\mathfrak{M}}P_{\mathfrak{H}-\mathfrak{N}}] \subset [\text{Range}\, P_{\mathfrak{M}}] = \mathfrak{M}.$$

Now applying Lemma 6.2.1 to $P_{\mathfrak{H}-\mathfrak{N}}P_{\mathfrak{M}}$ gives:

$$[\mathfrak{M}, \mathfrak{N}] - \mathfrak{N} = [\text{Range}\, P_{\mathfrak{H}-\mathfrak{N}}P_{\mathfrak{M}}] \sim [\text{Range}(P_{\mathfrak{H}-\mathfrak{N}}P_{\mathfrak{M}})^*] \subset \mathfrak{M}$$

and therefore $[\mathfrak{M}, \mathfrak{N}] - \mathfrak{N} \lesssim \mathfrak{M}$.

Lemma 7.3.5. *If $\mathfrak{M}_1, \ldots, \mathfrak{M}_n \;\eta\; \mathbf{M}$, and all $\mathfrak{M}_i$ are finite, then $[\mathfrak{M}_1, \ldots, \mathfrak{M}_n]$ is finite too.*

Proof: The statement is obvious for $n = 1$. Consider now $n = 2$. By Lemma 7.3.4 and Lemma 7.1.1, (i), $[\mathfrak{M}_1, \mathfrak{M}_2] - \mathfrak{M}_2$ is finite. As $[\mathfrak{M}_1, \mathfrak{M}_2] - \mathfrak{M}_2$ and $\mathfrak{M}_2$ are orthogonal, Lemma 7.3.3 applies: $[[\mathfrak{M}_1, \mathfrak{M}_2] - \mathfrak{M}_2, \mathfrak{M}_2]$, that is $[\mathfrak{M}_1, \mathfrak{M}_2]$, is finite. Thus the statement holds for $n = 2$, too.

Assume now $n = 3, 4, \ldots$, and that the statement is already established for $n - 1$. Then $[\mathfrak{M}_1, \ldots, \mathfrak{M}_{n-1}]$ is finite, and as our statement holds for $n = 2$, $[[\mathfrak{M}_1, \ldots, \mathfrak{M}_{n-1}], \mathfrak{M}_n]$, that is $[\mathfrak{M}_1, \ldots, \mathfrak{M}_n]$, is finite too. This completes the proof.

Chapter VIII: Numerical Dimensionality

8.1. We will use the information obtained in the foregoing about the relative dimensionality to define a quantitative measurement for it. We will do this by refining the number $\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right]$ in Lemma 7.1.3, which gave an integral and therefore only approximate measure of the ratio of the sets.

Our task is identical with that one which has to be met if one desires to pass from the purely geometrical calculus with intervals to the analytical one with numerical lengths: the problem which was solved by Euclid's famous algorithm. Our Lemma 7.1.3 is indeed the first step in this algorithm.

It is, however, technically preferable to use a somewhat different procedure. We will have to distinguish for a short time between the two possibilities that there is or is not a minimal $\mathfrak{M} \;\eta\; \mathbf{M}$. The first case is not really interesting, because if a minimal $\mathfrak{M} \;\eta\; \mathbf{M}$ exists, we possess already the complete characterization of $\mathbf{M}$ (as a direct factor) by Theorem IV. But as it makes no difference for the possibility of a quantitative measurement of the relative dimensionality, we will preserve the completeness of our discussion by including this case.

Definition 8.1.1. Assume $\mathfrak{M}, \mathfrak{N} \,\eta\, \mathbf{M}$, finite, and $\neq (0)$. If we have

$$\mathfrak{N} = [\mathfrak{P}_1, \ldots, \mathfrak{P}_n, \mathfrak{D}],$$

where $n = 0, 1, 2, \ldots$ (finite!); $\mathfrak{P}_1, \ldots, \mathfrak{P}_n, \mathfrak{D}$ are mutually orthogonal; $\mathfrak{P}_i \sim \mathfrak{M}$; then we use the abbreviated notation

$$\mathfrak{N} \simeq n \odot \mathfrak{M} \,\#\, \mathfrak{D}.$$

If $\mathfrak{D} = (0)$, we omit it.

Lemma 8.1.1. If $\mathfrak{M} \,\eta\, \mathbf{M}$, finite, $\neq (0)$, and not minimal, then an $\mathfrak{N} \,\eta\, \mathbf{M}$, finite, $\neq (0)$, with

$$\left[\frac{\mathfrak{M}}{\mathfrak{N}}\right] \geq 2$$

exists.

Proof: As $\mathfrak{M}$ is not minimal, an $\mathfrak{N} \,\eta\, \mathbf{M}$, $\mathfrak{N} \subset \mathfrak{M}$, $\mathfrak{N} \neq (0), \mathfrak{M}$ exists. So $\mathfrak{N}, \mathfrak{M} - \mathfrak{N} \neq (0)$. We have $\mathfrak{N} \gtreqless \mathfrak{M} - \mathfrak{N}$, that is $\mathfrak{N} \gtrsim \mathfrak{M} - \mathfrak{N}$ or $\mathfrak{N} \precsim \mathfrak{M} - \mathfrak{N}$. In the second case replace $\mathfrak{N}$ by $\mathfrak{M} - \mathfrak{N}$, so we have at any rate $\mathfrak{N} \precsim \mathfrak{M} - \mathfrak{N}$.

$\mathfrak{N} \subset \mathfrak{M}$ implies the finiteness of $\mathfrak{N}$. As $\mathfrak{N} \sim \mathfrak{N}' \subset \mathfrak{M} - \mathfrak{N}$, and $\mathfrak{N}, \mathfrak{N}'$ are orthogonal and $\subset \mathfrak{M}$, we have with $\mathfrak{P}_1 = \mathfrak{N}$, $\mathfrak{P}_2 = \mathfrak{N}'$, $\mathfrak{D} = \mathfrak{M} - [\mathfrak{N}, \mathfrak{N}']$, obviously

$$\mathfrak{M} \simeq 2 \odot \mathfrak{N} \,\#\, \mathfrak{D}.$$

Apply now Lemma 7.1.3 to $\mathfrak{N}, \mathfrak{D}$;

$$\mathfrak{D} \simeq p \odot \mathfrak{N} \,\#\, \mathfrak{D}', \qquad p = 0, 1, 2, \ldots, \qquad \mathfrak{D}' \prec \mathfrak{N}.$$

Then

$$\mathfrak{M} \simeq (2 + p) \odot \mathfrak{N} \,\#\, \mathfrak{D}', \qquad \mathfrak{D}' \prec \mathfrak{N},$$

and so $\left[\frac{\mathfrak{M}}{\mathfrak{N}}\right] = 2 + p \geq 2$.

Lemma 8.1.2. Every minimal $\mathfrak{M} \,\eta\, \mathbf{M}$ is finite and $\neq (0)$.

Proof: $\mathfrak{M} \neq (0)$ is part of the definition. $\mathfrak{M}$ infinite would imply $\mathfrak{M} \sim \mathfrak{N} \subsetneqq \mathfrak{M}$, so $\mathfrak{N} \,\eta\, \mathbf{M}$, and as $\mathfrak{M}$ is minimal, $\mathfrak{N} = (0)$. Then $\mathfrak{M} \sim (0)$, $\mathfrak{M} = (0)$ which is contradictory.

Definition 8.1.2. Certain (finite or infinite) sequences $\mathfrak{S} = (\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \ldots)$ of elements which are $\eta\mathbf{M}$, finite, $\neq (0)$, will be called *fundamental sequences*. They are these:

(α) Any minimal $\overline{\mathfrak{M}}_1 \,\eta\, \mathbf{M}$ is a fundamental sequence by itself (length one!).

(β) Any infinite sequence $\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \ldots$ of $\eta\mathbf{M}$, finite, $\neq (0)$ elements is a fundamental sequence if

$$\left[\frac{\overline{\mathfrak{M}}_i}{\overline{\mathfrak{M}}_{i+1}}\right] \geq 2 \qquad \text{for } i = 1, 2, \ldots .$$

Lemma 8.1.3. If sets $\mathfrak{M}\ \eta\ \mathbf{M}$ which are finite and $\neq (0)$ do exist at all, then there exists at least one fundamental sequence.
Proof: If a minimal $\mathfrak{M}\ \eta\ \mathbf{M}$ exists, put $\overline{\mathfrak{M}}_1 = \mathfrak{M}$. By Lemma 8.1.2 this meets the requirements of case (α). If no minimal $\mathfrak{M}\ \eta\ \mathbf{M}$ exists, choose a finite $\mathfrak{M}\ \eta\ \mathbf{M}$, $\mathfrak{M} \neq (0)$, and put $\overline{\mathfrak{M}}_1 = \mathfrak{M}$; and obtain $\overline{\mathfrak{M}}_{i+1}$ from $\overline{\mathfrak{M}}_i$ by means of Lemma 8.1.1. This meets the requirements of case (β).

We now prove two important evaluations.

Lemma 8.1.4. Assume $\mathfrak{M}, \mathfrak{N}, \mathfrak{P}\ \eta\ \mathbf{M}$ finite, and $\neq (0)$. Then

$$\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right]\left[\frac{\mathfrak{P}}{\mathfrak{N}}\right] \leq \left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] < \left(\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] + 1\right)\left(\left[\frac{\mathfrak{P}}{\mathfrak{N}}\right] + 1\right)$$

and, if $\mathfrak{N}, \mathfrak{P}$ are orthogonal,

$$\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] + \left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] \leq \left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] \leq \left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] + \left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] + 1.$$

Proof: First put

$$\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] = m, \quad \mathfrak{N} \simeq m \odot \mathfrak{M} \,\#\, \mathfrak{M}', \quad \mathfrak{M}' \prec \mathfrak{M},$$
$$\text{and} \quad \left[\frac{\mathfrak{P}}{\mathfrak{N}}\right] = n, \quad \mathfrak{P} \simeq n \odot \mathfrak{N} \,\#\, \mathfrak{N}', \quad \mathfrak{N}' \prec \mathfrak{N}.$$

This clearly implies $\mathfrak{P} \simeq mn \odot \mathfrak{M} \,\#\, \mathfrak{M}''$, from which $\left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] \geq mn$ follows as at the end of the proof of Lemma 8.1.1.

If we had however $\left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] \geq (m+1)(n+1)$ that would mean

$$\mathfrak{P} \simeq \left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] \odot \mathfrak{M} \,\#\, \mathfrak{M}''' \simeq (m+1)(n+1) \odot \mathfrak{M} \,\#\, \mathfrak{M}^{\mathrm{IV}},$$

where

$$\mathfrak{M}^{\mathrm{IV}} \simeq \left(\left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] - (m+1)(n+1)\right) \odot \mathfrak{M} \,\#\, \mathfrak{M}'''.$$

Or $(n+1) \odot \mathfrak{N}^{\circ} \simeq \mathfrak{P}^{\circ} \subset \mathfrak{P}$ where $\mathfrak{N}^{\circ} \simeq (m+1) \odot \mathfrak{M}$. Now clearly

$$\mathfrak{N} \simeq m \odot \mathfrak{M} \,\#\, \mathfrak{M}' \precsim \mathfrak{N}^{\circ},$$

so we may as well write $(n+1) \odot \mathfrak{N} \simeq \mathfrak{P}^{\circ\circ} \subset \mathfrak{P}^{\circ} \subset \mathfrak{P}$, that is

$$\mathfrak{P} \simeq (n+1) \odot \mathfrak{N} \,\#\, \mathfrak{N}''.$$

This gives, as above, $\left[\frac{\mathfrak{P}}{\mathfrak{N}}\right] \geq n+1$, contradicting $\left[\frac{\mathfrak{P}}{\mathfrak{N}}\right] = n$. So we must have

$$mn \leq \left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] < (m+1)(n+1),$$

which is the first set of inequalities.

Next assume that $\mathfrak{N}, \mathfrak{P}$ are orthogonal, and put

$$\left[\frac{\mathfrak{N}}{\mathfrak{M}}\right] = m, \quad \mathfrak{N} \simeq m \odot \mathfrak{M} \,\#\, \mathfrak{M}', \quad \mathfrak{M}' \prec \mathfrak{M};$$

and

$$\left[\frac{\mathfrak{P}}{\mathfrak{M}}\right] = p, \quad \mathfrak{P} \simeq p \odot \mathfrak{M} \,\#\, \mathfrak{M}^{\mathrm{V}}, \quad \mathfrak{M}^{\mathrm{V}} \prec \mathfrak{M}.$$

Then $[\mathfrak{N}, \mathfrak{P}] \simeq (m+p) \odot \mathfrak{M} \,\#\, [\mathfrak{M}', \mathfrak{M}^{\mathrm{V}}]$ and so, as above,

$$\left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] \geq m+p.$$

If we had, however, $\left[\frac{[\mathfrak{N},\mathfrak{P}]}{\mathfrak{M}}\right] \geq m+p+2$, that would mean

$$[\mathfrak{N}, \mathfrak{P}] \simeq \left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] \odot \mathfrak{M} \,\#\, \mathfrak{M}^{\mathrm{VI}} \simeq (m+p+2) \odot \mathfrak{M} \,\#\, \mathfrak{M}^{\mathrm{VII}},$$

where

$$\mathfrak{M}^{\mathrm{VII}} \simeq \left(\left[\frac{[\mathfrak{N}, \mathfrak{P}]}{\mathfrak{M}}\right] - (m+p+2)\right) \odot \mathfrak{M} \,\#\, \mathfrak{M}^{\mathrm{VI}}.$$

Thus $[\mathfrak{N},\mathfrak{P}] = [\overline{\mathfrak{N}},\overline{\mathfrak{P}},\mathfrak{M}^{\mathrm{VII}}]$ where $\overline{\mathfrak{N}},\overline{\mathfrak{P}},\mathfrak{M}^{\mathrm{VII}}$ are mutually orthogonal, and $\overline{\mathfrak{N}} \simeq (m+1)\odot\mathfrak{M}$, $\overline{\mathfrak{P}} \simeq (p+1)\odot\mathfrak{M}$. This implies $\mathfrak{N} \sim \overline{\mathfrak{N}}^{\circ} \subsetneqq \overline{\mathfrak{N}}$, $\mathfrak{P} \sim \overline{\mathfrak{P}}^{\circ} \subsetneqq \overline{\mathfrak{P}}$, and so

$$[\mathfrak{N},\mathfrak{P}] \sim [\overline{\mathfrak{N}}^{\circ},\overline{\mathfrak{P}}^{\circ}] \subsetneqq [\overline{\mathfrak{N}},\overline{\mathfrak{P}}] \subset [\overline{\mathfrak{N}},\overline{\mathfrak{P}},\mathfrak{M}^{\mathrm{VII}}] = [\mathfrak{N},\mathfrak{P}],$$

contradicting the finiteness of $[\mathfrak{N},\mathfrak{P}]$, which follows from the finiteness of $\mathfrak{N}$ and of $\mathfrak{P}$ (by Lemma 7.3.3 or Lemma 7.3.5). So $\left[\frac{[\mathfrak{N},\mathfrak{P}]}{\mathfrak{M}}\right] < m+p+2$ and therefore

$$m+p \leq \left[\frac{[\mathfrak{N},\mathfrak{P}]}{\mathfrak{M}}\right] \leq m+p+1,$$

which is the second set of inequalities.

These evaluations put us in position to establish the quantitative ratios of relative dimensionalities as follows.

Lemma 8.1.5. If $\mathfrak{S} = (\overline{\mathfrak{M}}_1,\overline{\mathfrak{M}}_2,\ldots)$ is a fundamental sequence, and $\mathfrak{M},\mathfrak{N}\ \eta\ \mathbf{M}$ are finite and $\neq (0)$, then

$$\lim_i \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}$$

exists. (In case (I) we mean by lim the value at $i=1$, in case (II) we mean by it $\lim_{i\to\infty}$.) It is a finite and positive real number.

Proof: In case (I) we must only prove that $\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right]$, $\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_1}\right]$ are both $\neq 0$. $\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right] = 0$ would imply $\mathfrak{N} \sim \mathfrak{D} < \overline{\mathfrak{M}}_1$ so $\mathfrak{N} \sim \mathfrak{N}' \subset \overline{\mathfrak{M}}_1$ excluding $\mathfrak{N}' = \overline{\mathfrak{M}}_1$. As $\mathfrak{N}'\ \eta\ \mathbf{M}$ and $\overline{\mathfrak{M}}_1$ minimal, this necessitates $\mathfrak{N}' = (0)$, $\mathfrak{N} \sim \mathfrak{N}' = (0)$, $\mathfrak{N} = (0)$, thus contradicting $\mathfrak{N} \neq (0)$. Therefore $\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right] \neq 0$, similarly $\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_1}\right] \neq 0$.

Consider now case (II). We have by Lemma 8.1.4

$$\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_{i+1}}\right] \geq \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right]\left[\frac{\overline{\mathfrak{M}}_i}{\overline{\mathfrak{M}}_{i+1}}\right] \geq 2\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right],$$

so either always $\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] = 0$, or $\lim_{i\to\infty}\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] = \infty$. The former would mean $\mathfrak{N} < \overline{\mathfrak{M}}_i$ for all $i = 1,2,\ldots$. Now Lemma 8.1.4 gives

$$\left[\frac{\overline{\mathfrak{M}}_i}{\overline{\mathfrak{M}}_{i+j}}\right] \geq \left[\frac{\overline{\mathfrak{M}}_i}{\overline{\mathfrak{M}}_{i+1}}\right]\cdots\left[\frac{\overline{\mathfrak{M}}_{i+j-1}}{\overline{\mathfrak{M}}_{i+j}}\right] \geq 2^j,$$

and so in particular $\left[\frac{\overline{\mathfrak{M}}_1}{\overline{\mathfrak{M}}_i}\right] \geq 2^{i-1}$ (write $1, i-1$ for i, j). So $\overline{\mathfrak{M}}_1 \simeq 2^{i-1} \odot \overline{\mathfrak{M}}_i \mathbin{\#} \mathfrak{M}'_i$, and *a fortiori* $\overline{\mathfrak{M}}_1 \simeq 2^{i-1} \odot \mathfrak{N} \mathbin{\#} \mathfrak{N}'_i$. This would imply, as we frequently concluded in the proofs of Lemma 8.1.1 and Lemma 8.1.4, $\left[\frac{\overline{\mathfrak{M}}_1}{\mathfrak{N}}\right] \geq 2^{i-1}$ for all $i = 1, 2, \ldots$, which is impossible. So we have proved

$$\lim_{i\to\infty} \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] = \infty;$$

similarly $\lim_{i\to\infty} \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right] = \infty$.

Another application of Lemma 8.1.4 gives

$$\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_{i+j}}\right] \leq \left(\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] + 1\right)\left(\left[\frac{\overline{\mathfrak{M}}_i}{\overline{\mathfrak{M}}_{i+j}}\right] + 1\right),$$

$$\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_{i+j}}\right] \geq \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]\left[\frac{\overline{\mathfrak{M}}_i}{\overline{\mathfrak{M}}_{i+j}}\right],$$

so

$$\frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_{i+j}}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_{i+j}}\right]} \leq \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] + 1}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]} \frac{\left[\frac{\overline{\mathfrak{M}}_i}{\overline{\mathfrak{M}}_{i+j}}\right] + 1}{\left[\frac{\overline{\mathfrak{M}}_i}{\overline{\mathfrak{M}}_{i+j}}\right]} \leq \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] + 1}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]} \frac{2^j + 1}{2^j}.$$

Now $j \to \infty$ gives

$$\limsup_{j\to\infty} \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_{i+j}}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_{i+j}}\right]} \leq \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] + 1}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]},$$

that is

$$\limsup_{i\to\infty} \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]} \leq \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] + 1}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]}.$$

For a sufficiently large i, $\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right] \neq 0$, so the expression on the right side is finite, and thus the $\limsup_{i\to\infty}$ on the left side is finite too. Now $i \to \infty$ gives, considering

$\lim_{i\to\infty}\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]=\infty$,

$$\limsup_{i\to\infty}\frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]}\leq\liminf_{i\to\infty}\frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right]+1}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]}=\liminf_{i\to\infty}\frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]}.$$

Thus the limit exists and is finite. It is non negative by its nature, and interchanging $\mathfrak{M},\mathfrak{N}$ proves that it cannot be 0. This completes the proof.

Lemma 8.1.6. If $\mathfrak{S}=(\overline{\mathfrak{M}}_1,\overline{\mathfrak{M}}_2,\ldots)$ is a fundamental sequence, and $\mathfrak{M},\mathfrak{N},\mathfrak{P}\ \eta\ \mathbf{M}$ are finite and $\neq(0)$, then the following statements hold:

(i) If $\mathfrak{N}\sim\mathfrak{P}$, then $\left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}=\left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}}$.

(ii) If $\mathfrak{M}\sim\mathfrak{N}$, then $\left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}}=\left(\frac{\mathfrak{P}}{\mathfrak{N}}\right)_{\mathfrak{S}}$.

(iii) $\left(\frac{\mathfrak{M}}{\mathfrak{M}}\right)_{\mathfrak{S}}=1$; $\left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}=\frac{1}{\left(\frac{\mathfrak{M}}{\mathfrak{N}}\right)_{\mathfrak{S}}}$; $\left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}}=\left(\frac{\mathfrak{P}}{\mathfrak{N}}\right)_{\mathfrak{S}}\left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}$.

(iv) If $\mathfrak{N},\mathfrak{P}$ are orthogonal, then

$$\left(\frac{[\mathfrak{N},\mathfrak{P}]}{\mathfrak{M}}\right)_{\mathfrak{S}}=\left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}+\left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}}.$$

Proof: We prove these statements in a somewhat changed order:

Ad (iii): Obvious from the definition.

Ad (i): Clearly $\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right]=\left[\frac{\mathfrak{P}}{\overline{\mathfrak{M}}_i}\right]$; this implies $\left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}=\left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}}$.

Ad (ii): Follows from (i), (iii).

Ad (iv): Consider first case (α). Then

$$\mathfrak{N}\simeq\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right]\odot\overline{\mathfrak{M}}_1\,\#\,\mathfrak{N}',\qquad\mathfrak{N}'\prec\overline{\mathfrak{M}}_1.$$

So $\mathfrak{N}'\sim\mathfrak{N}''\subset\overline{\mathfrak{M}}_1$, excluding $\mathfrak{N}''=\overline{\mathfrak{M}}_1$. As $\mathfrak{N}''\eta\mathbf{M}$ and $\overline{\mathfrak{M}}_1$ minimal, this necessitates $\mathfrak{N}''=(0)$, $\mathfrak{N}'\sim\mathfrak{N}''=(0)$, $\mathfrak{N}'=(0)$. So

$$\mathfrak{N}\simeq\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right]\odot\overline{\mathfrak{M}}_1,$$

similarly $\mathfrak{P}\simeq\left[\frac{\mathfrak{P}}{\overline{\mathfrak{M}}_1}\right]\odot\overline{\mathfrak{M}}_1$, and therefore

$$[\mathfrak{N},\mathfrak{P}]\simeq\left(\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right]+\left[\frac{\mathfrak{P}}{\overline{\mathfrak{M}}_1}\right]\right)\odot\overline{\mathfrak{M}}_1.$$

Thus

$$\left[\frac{[\mathfrak{N},\mathfrak{P}]}{\overline{\overline{\mathfrak{M}_1}}}\right] = \left[\frac{\mathfrak{N}}{\overline{\overline{\mathfrak{M}_1}}}\right] + \left[\frac{\mathfrak{P}}{\overline{\overline{\mathfrak{M}_1}}}\right].$$

This means

$$\frac{\left[\frac{[\mathfrak{N},\mathfrak{P}]}{\overline{\overline{\mathfrak{M}_1}}}\right]}{\left[\frac{\mathfrak{M}}{\overline{\overline{\mathfrak{M}_1}}}\right]} = \frac{\left[\frac{\mathfrak{N}}{\overline{\overline{\mathfrak{M}_1}}}\right]}{\left[\frac{\mathfrak{M}}{\overline{\overline{\mathfrak{M}_1}}}\right]} + \frac{\left[\frac{\mathfrak{P}}{\overline{\overline{\mathfrak{M}_1}}}\right]}{\left[\frac{\mathfrak{M}}{\overline{\overline{\mathfrak{M}_1}}}\right]},$$

and under the conditions of case (α),

$$\left(\frac{[\mathfrak{N},\mathfrak{P}]}{\mathfrak{M}}\right)_{\mathfrak{S}} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} + \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}}.$$

Consider next case (β). Then Lemma 8.1.4 gives

$$\frac{\left[\frac{\mathfrak{N}}{\overline{\overline{\mathfrak{M}_i}}}\right] + \left[\frac{\mathfrak{P}}{\overline{\overline{\mathfrak{M}_i}}}\right]}{\left[\frac{\mathfrak{M}}{\overline{\overline{\mathfrak{M}_i}}}\right]} \le \frac{\left[\frac{[\mathfrak{N},\mathfrak{P}]}{\overline{\overline{\mathfrak{M}_i}}}\right]}{\left[\frac{\mathfrak{M}}{\overline{\overline{\mathfrak{M}_i}}}\right]} \le \frac{\left[\frac{\mathfrak{N}}{\overline{\overline{\mathfrak{M}_i}}}\right] + \left[\frac{\mathfrak{P}}{\overline{\overline{\mathfrak{M}_i}}}\right] + 1}{\left[\frac{\mathfrak{M}}{\overline{\overline{\mathfrak{M}_i}}}\right]}.$$

Now if $i \longrightarrow \infty$ this becomes, considering $\lim_{i\to\infty}\left[\frac{\mathfrak{M}}{\overline{\overline{\mathfrak{M}_i}}}\right] = \infty$ (cf. the proof of Lemma 8.1.5),

$$\left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} + \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}} \le \left(\frac{[\mathfrak{N},\mathfrak{P}]}{\mathfrak{M}}\right)_{\mathfrak{S}} \le \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} + \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}},$$

that is

$$\left(\frac{[\mathfrak{N},\mathfrak{P}]}{\mathfrak{M}}\right)_{\mathfrak{S}} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}} + \left(\frac{\mathfrak{P}}{\mathfrak{M}}\right)_{\mathfrak{S}}.$$

8.2. We introduce the numerical relative dimensions by an implicit definition.

Definition 8.2.1. A real-valued function $D(\mathfrak{M})$, which is defined for all linear, closed sets $\mathfrak{M}\ \eta\ \mathbf{M}$, is a *relative dimension function*, if it has the following properties:

(i) $D(\mathfrak{M})$ is 0 if $\mathfrak{M} = (0)$; it is finite and positive if $\mathfrak{M}$ is finite and $\neq (0)$; it is ∞ if $\mathfrak{M}$ is infinite.

(ii) If $\mathfrak{M} \sim \mathfrak{N}$, then $D(\mathfrak{M}) = D(\mathfrak{N})$.

(iii) If $\mathfrak{M}, \mathfrak{N}$ are orthogonal, then $D([\mathfrak{M},\mathfrak{N}]) = D(\mathfrak{M}) + D(\mathfrak{N})$.

(As to replacing $\mathfrak{M}$ by $E = P_{\mathfrak{M}}$, and the use of the explanatory symbol $(\cdots \mathbf{M})$, cf. Definition 6.1.1. If we want to make the relationship between $\mathbf{M}$ and the function $D(\mathfrak{M})$ explicit, we will write $D_{\mathbf{M}}(\mathfrak{M})$.)

The results of §8.1 make it possible to deal exhaustively with the existence question.

Lemma 8.2.1. There always exists at least one relative dimension function $D(\mathfrak{M})$.

Proof: If no finite $\mathfrak{M}\ \eta\ \mathbf{M}$, $\mathfrak{M} \neq (0)$, exists, put

$$D(\mathfrak{M}) = \begin{cases} 0 & \text{for } \mathfrak{M} = (0), \\ \infty & \text{otherwise.} \end{cases}$$

This meets all requirements. If finite $\mathfrak{M}\ \eta\ \mathbf{M}$, $\mathfrak{M} \neq (0)$ do exist, choose one, say $\mathfrak{M}_0$. Furthermore choose a fundamental sequence $\mathfrak{S}$ by Lemma 8.1.3. Now define

$$D(\mathfrak{M}) = \begin{cases} 0, & \mathfrak{M} = (0), \\ \left(\dfrac{\mathfrak{M}}{\mathfrak{M}_0}\right)_{\mathfrak{S}}, & \mathfrak{M} \text{ finite and } \neq (0), \\ \infty, & \mathfrak{M} \text{ infinite.} \end{cases}$$

Remember that $[\mathfrak{M}, \mathfrak{N}]$ is infinite if and only if $\mathfrak{M}$ or $\mathfrak{N}$ is infinite, by Lemma 7.1.1, (i), and Lemma 7.3.3 or Lemma 7.3.5. Now (i) is obviously fulfilled, and (ii), (iii) follow from Lemma 8.1.6, (i), (iv), resp.

Lemma 8.2.2. If $D(\mathfrak{M})$ is a relative dimension function and $\mathfrak{S}$ a fundamental sequence, then we have, whenever $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$, finite and $\neq (0)$ the relation

$$\frac{D(\mathfrak{N})}{D(\mathfrak{M})} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}.$$

Proof: Distinguish the two cases (α) and (β) of Definition 8.1.2.

In the case (α) we have (cf. the proof of Lemma 8.1.6, (iv)),

$$\mathfrak{N} \simeq \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right] \odot \overline{\mathfrak{M}}_1, \qquad \mathfrak{M} \simeq \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_1}\right] \odot \overline{\mathfrak{M}}_1,$$

therefore by Definition 8.2.1

$$D(\mathfrak{N}) = \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right] D(\overline{\mathfrak{M}}_1), \qquad D(\mathfrak{M}) = \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_1}\right] D(\overline{\mathfrak{M}}_1),$$

and $D(\overline{\mathfrak{M}}_1)$ finite and positive, so

$$\frac{D(\mathfrak{N})}{D(\mathfrak{M})} = \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_1}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_1}\right]} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}.$$

In the case (β) we have

$$\mathfrak{N} \simeq \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] \odot \overline{\mathfrak{M}}_i \# \mathfrak{N}'_i, \quad \mathfrak{N}'_i \prec \overline{\mathfrak{M}}_i;$$

$$\mathfrak{M} \simeq \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right] \odot \overline{\mathfrak{M}}_i \# \mathfrak{M}'_i, \quad \mathfrak{M}'_i \prec \overline{\mathfrak{M}}_i.$$

So $\mathfrak{N}'_i \sim \mathfrak{N}''_i \subset \overline{\mathfrak{M}}_i$, implying (using Definition 8.2.1 here and below as we did above)

$$D(\overline{\mathfrak{M}}_i) = D(\mathfrak{N}''_i) + D(\overline{\mathfrak{M}}_i - \mathfrak{N}''_i) \geq D(\mathfrak{N}''_i) = D(\mathfrak{N}'_i).$$

Therefore

$$D(\mathfrak{N}) = \left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] D(\overline{\mathfrak{M}}_i) + D(\mathfrak{N}'_i) \leq \left(\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] + 1\right) D(\overline{\mathfrak{M}}_i),$$

$$D(\mathfrak{M}) = \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right] D(\overline{\mathfrak{M}}_i) + D(\mathfrak{M}'_i) \geq \left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right] D(\overline{\mathfrak{M}}_i),$$

and so

$$\frac{D(\mathfrak{N})}{D(\mathfrak{M})} \leq \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right] + 1}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]}.$$

Now $i \to \infty$ gives, remembering $\lim_{i\to\infty}\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right] = \infty$ from the proof of Lemma 8.1.5,

$$\frac{D(\mathfrak{N})}{D(\mathfrak{M})} \leq \lim_{i\to\infty} \frac{\left[\frac{\mathfrak{N}}{\overline{\mathfrak{M}}_i}\right]}{\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}_i}\right]} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}}.$$

Interchanging $\mathfrak{M}, \mathfrak{N}$ gives, considering Lemma 8.1.6, (iii), that $\geq$ holds too; so we have equality again.

Lemma 8.2.3. All relative dimension functions $D(\mathfrak{M})$ are obtained by taking one of them, say $D_0(\mathfrak{M})$, and forming all $aD_0(\mathfrak{M})$, for all finite and positive constants a.

Proof: It is clear, that every $aD_0(\mathfrak{M})$ is a relative dimension function, along with $D_0(\mathfrak{M})$.

Consider now two relative dimension functions $D_0(\mathfrak{M})$ and $D(\mathfrak{M})$. If no finite $\mathfrak{M}\ \eta\ \mathbf{M}$, $\mathfrak{M} \neq (0)$ exist, we have $D_0(\mathfrak{M}) = D(\mathfrak{M}) = 0$ resp. ∞ for $\mathfrak{M} = (0)$ resp. $\neq (0)$, so we have $D(\mathfrak{M}) = aD_0(\mathfrak{M})$ with $a = 1$. Let us therefore assume, that finite $\mathfrak{M}\,\eta\,\mathbf{M}$, $\mathfrak{M} \neq (0)$ do exist, and choose a fundamental sequence $\mathfrak{S}$ by Lemma 8.1.3. Then Lemma 8.2.2 gives for $\mathfrak{M}, \mathfrak{N}\,\eta\,\mathbf{M}$, finite and $\neq 0$,

$$\frac{D(\mathfrak{N})}{D(\mathfrak{M})} = \frac{D_0(\mathfrak{N})}{D_0(\mathfrak{M})} = \left(\frac{\mathfrak{N}}{\mathfrak{M}}\right)_{\mathfrak{S}},$$

that is

$$\frac{D(\mathfrak{N})}{D_0(\mathfrak{N})} = \frac{D(\mathfrak{M})}{D_0(\mathfrak{M})}.$$

Thus $D(\mathfrak{M})/D_0(\mathfrak{M})$ is, for these $\mathfrak{M}$, independent of $\mathfrak{M}$; therefore it is a finite, positive constant a. So we have $D(\mathfrak{M}) = aD_0(\mathfrak{M})$ for every $\mathfrak{M}\,\eta\,\mathbf{M}$ which is finite and $\neq (0)$. But this holds too, if $\mathfrak{M}$ is $= (0)$ or infinite, as then both sides become $= 0$ resp. $= \infty$. So the desired relation holds without exceptions. This completes the proof.

We may remark, on the other hand, that Lemma 8.2.2 shows too, that $\left(\frac{\mathfrak{M}}{\mathfrak{N}}\right)_{\mathfrak{S}}$ is independent of the choice of the fundamental sequence $\mathfrak{S}$. It is nevertheless dependent on its existence, whereas $D(\mathfrak{M})$ is not.

8.3. The connection between the ordering and the relative dimension functions is given by this Lemma.

Lemma 8.3.1. If $D(\mathfrak{M})$ is a relative dimension function and $\mathfrak{M}, \mathfrak{N}\,\eta\,\mathbf{M}$, then the relations $\mathfrak{M} \sim \mathfrak{N}$, $\mathfrak{M} \prec \mathfrak{N}$, $\mathfrak{M} \succ \mathfrak{N}$ are equivalent to $D(\mathfrak{M}) = D(\mathfrak{N})$, $D(\mathfrak{M}) < D(\mathfrak{N})$, $D(\mathfrak{M}) > D(\mathfrak{N})$, respectively.

Proof: $\mathfrak{M} \sim \mathfrak{N}$ implies $D(\mathfrak{M}) = D(\mathfrak{N})$ by Definition 8.2.1, (ii). Consider now $\mathfrak{M} \prec \mathfrak{N}$. Then $\mathfrak{M}$ is finite, because $\mathfrak{M}$ infinite would imply $\mathfrak{M} \succsim \mathfrak{N}$; so $D(\mathfrak{M})$ is finite too. Now $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$, and $\mathfrak{N}' = \mathfrak{N}$ is excluded. So $\mathfrak{N} - \mathfrak{N}' \neq (0)$, $D(\mathfrak{N} - \mathfrak{N}') > 0$. (Use both times Definition 8.2.1, (i).) Now Definition 8.2.1, (ii) and (iii) give

$$D(\mathfrak{N}) = D(\mathfrak{N}') + D(\mathfrak{N} - \mathfrak{N}') > D(\mathfrak{N}') = D(\mathfrak{M}),$$

$D(\mathfrak{M}) < D(\mathfrak{N})$. Interchanging $\mathfrak{M}, \mathfrak{N}$ shows that $\mathfrak{M} \succ \mathfrak{N}$ implies $D(\mathfrak{M}) > D(\mathfrak{N})$. As we have complete disjunctions on both sides, the converse is true too.

Summing up Lemma 8.2.1, Lemma 8.2.3, and Lemma 8.3.1, we have:

Theorem VII. A relative dimension function $D(\mathfrak{M})$ as characterized by Definition 8.2.1, does always exist, and it is unique, except for an arbitrary constant (real, finite, and positive) factor a. A particular choice of $D(\mathfrak{M})$, that is of a, is a normalization of the relative dimension.

If $\mathfrak{M}, \mathfrak{N} \,\eta\, \mathbf{M}$ then the relations $\mathfrak{M} \sim \mathfrak{N}$, $\mathfrak{M} < \mathfrak{N}$, $\mathfrak{M} > \mathfrak{N}$ are equivalent to $D(\mathfrak{M}) = D(\mathfrak{N})$, $D(\mathfrak{M}) < D(\mathfrak{N})$, $D(\mathfrak{M}) > D(\mathfrak{N})$, respectively.

We will use $D(\mathfrak{M})$ to obtain a detailed characterization of $\mathbf{M}$ itself. But first we establish some continuity properties of $D(\mathfrak{M})$.

Lemma 8.3.2. Let $\mathfrak{M}_1, \mathfrak{M}_2, \dots$ be a (finite or infinite) sequence, all $\mathfrak{M}_i \,\eta\, \mathbf{M}$ and mutually orthogonal. Then

$$D([\mathfrak{M}_1, \mathfrak{M}_2, \dots]) = \sum_i D(\mathfrak{M}_i).$$

Proof: Those $\mathfrak{M}_i$ which are $= (0)$ have no effect on either side of this equation, and we can omit them. So we may assume that all $\mathfrak{M}_i \neq (0)$, $D(\mathfrak{M}_i) > 0$.

Assume first that the sequence is finite, say $\mathfrak{M}_1, \dots, \mathfrak{M}_n$. For $n = 0, 1$ the statement is obvious, for $n = 2$ it coincides with Definition 8.2.1, (iii). So we need only to consider $n = 3, 4, \dots$, assuming that the statement is already proved for $n - 1$. Now as

$$[\mathfrak{M}_1, \dots, \mathfrak{M}_n] = [[\mathfrak{M}_1, \dots, \mathfrak{M}_{n-1}], \mathfrak{M}_n]$$

and as the statement holds for $n - 1$ and for two addends, it holds for n too.

Assume next that the sequence is infinite. Then we have

$$[\mathfrak{M}_1, \mathfrak{M}_2, \dots] = [\mathfrak{M}_1, \dots, \mathfrak{M}_n, [\mathfrak{M}_{n+1}, \mathfrak{M}_{n+2}, \dots]],$$

so

$$D([\mathfrak{M}_1, \mathfrak{M}_2, \dots]) = \sum_{i=1}^{n} D(\mathfrak{M}_i) + D([\mathfrak{M}_{n+1}, \mathfrak{M}_{n+2}, \dots]) \geq \sum_{i=1}^{n} D(\mathfrak{M}_i).$$

Thus $\sum_{i=1}^{n} D(\mathfrak{M}_i) \leq D([\mathfrak{M}_1, \mathfrak{M}_2, \dots])$ for every $n = 1, 2, \dots$, and therefore

$$\sum_{i=1}^{\infty} D(\mathfrak{M}_i) \leq D([\mathfrak{M}_1, \mathfrak{M}_2, \dots]).$$

Assume now, that $<$ does hold. Then $\sum_{i=1}^{\infty} D(\mathfrak{M}_i)$ is finite. Therefore $\lim_{i\to\infty} D(\mathfrak{M}_i) = 0$. So there exists for every $\epsilon > 0$ a finite $\mathfrak{N} \,\eta\, \mathbf{M}$ (in fact an $\mathfrak{M}_i$) with $0 < D(\mathfrak{N}) < \epsilon$. Choose such an $\mathfrak{N}$ for

$$\epsilon = D([\mathfrak{M}_1, \mathfrak{M}_2, \dots]) - \sum_{i=1}^{\infty} D(\mathfrak{M}_i) > 0.$$

Consider the expression

$$D([\mathfrak{M}_1, \mathfrak{M}_2, \ldots]) - \sum_{i=1}^{\infty} D(\mathfrak{M}_i).$$

It is equal to

$$\{D(\mathfrak{M}_1) + D([\mathfrak{M}_2, \mathfrak{M}_3, \ldots])\} - \sum_{i=1}^{\infty} D(\mathfrak{M}_i) = D([\mathfrak{M}_2, \mathfrak{M}_3, \ldots]) - \sum_{i=2}^{\infty} D(\mathfrak{M}_i).$$

Thus it is unchanged if we replace $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ by $\mathfrak{M}_2, \mathfrak{M}_3, \ldots$. Applying this i_0 times, we see, that it is unchanged even if we pass to $\mathfrak{M}_{i_0+1}, \mathfrak{M}_{i_0+2}, \ldots$. Now $\sum_{i=1}^{\infty} D(\mathfrak{M}_{i_0+i}) = \sum_{i_0+1}^{\infty} D(\mathfrak{M}_i)$ is arbitrarily small if i_0 is sufficiently great. So we can make it $\leq D(\mathfrak{N})$. Now write again $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ for $\mathfrak{M}_{i_0+1}, \mathfrak{M}_{i_0+2}, \ldots$. Then we see: $\sum_{i=1}^{\infty} D(\mathfrak{M}_i) \leq D(\mathfrak{N})$ and still

$$D(\mathfrak{N}) < D([\mathfrak{M}_1, \mathfrak{M}_2, \ldots]) - \sum_{i=1}^{\infty} D(\mathfrak{M}_i),$$

and *a fortiori* $D(\mathfrak{N}) < D([\mathfrak{M}_1, \mathfrak{M}_2, \ldots])$.

We will now construct a sequence $\mathfrak{N}_1, \mathfrak{N}_2, \ldots$ of mutually orthogonal sets $\mathfrak{N}_i \,\eta\, \mathbf{M}$, $\mathfrak{N}_i \subset \mathfrak{N}$ with $\mathfrak{M}_i \sim \mathfrak{N}_i$. We proceed by induction. Assume that for an $i = 1, 2, \ldots$ all $\mathfrak{N}_1, \ldots, \mathfrak{N}_{i-1}$ have already been constructed fulfilling these conditions, then we construct $\mathfrak{N}_i$ as follows. $D(\mathfrak{N}_j) = D(\mathfrak{M}_j)$, $j = 1, \ldots, i-1$, is finite, so

$$D(\mathfrak{N} - [\mathfrak{N}_1, \ldots, \mathfrak{N}_{i-1}]) = D(\mathfrak{N}) - \sum_{j=1}^{i-1} D(\mathfrak{N}_j) \geq \sum_{j=i}^{\infty} D(\mathfrak{M}_j) \geq D(\mathfrak{M}_i).$$

Thus $\mathfrak{M}_i \precsim \mathfrak{N} - [\mathfrak{N}_1, \ldots, \mathfrak{N}_{i-1}]$, and thus $\mathfrak{M}_i \sim \mathfrak{N}_i \subset \mathfrak{N} - [\mathfrak{N}_1, \ldots, \mathfrak{N}_{i-1}]$. This definition of $\mathfrak{N}_i$ meets all requirements.

Now $\mathfrak{M}_i \sim \mathfrak{N}_i$, all $\mathfrak{M}_i$ are mutually orthogonal, and all $\mathfrak{N}_i$ are too, so Lemma 6.1.2 applies:

$$[\mathfrak{M}_1, \mathfrak{M}_2, \ldots] \sim [\mathfrak{N}_1, \mathfrak{N}_2, \ldots] \subset \mathfrak{N}, \qquad [\mathfrak{M}_1, \mathfrak{M}_2, \ldots] \precsim \mathfrak{N},$$

$D([\mathfrak{M}_1, \mathfrak{M}_2, \ldots]) \leq D(\mathfrak{N})$. But on the other hand we had $D(\mathfrak{N}) < D([\mathfrak{M}_1, \mathfrak{M}_2, \ldots])$ which is a contradiction. So our original assumption concerning the $<$ must have been wrong, and there must be for the original sequence

$$\sum_{i=1}^{\infty} D(\mathfrak{M}_i) = D([\mathfrak{M}_1, \mathfrak{M}_2, \ldots]).$$

This completes the proof.

Lemma 8.3.3. Let $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ be an infinite sequence, all $\mathfrak{M}_i \,\eta\, \mathbf{M}$ and $\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \cdots$. Then

$$\lim_{i\to\infty} D(\mathfrak{M}_i) = D([\mathfrak{M}_1, \mathfrak{M}_2, \ldots]).$$

Proof: Put $\mathfrak{N}_1 = \mathfrak{M}_1$, $\mathfrak{N}_i = \mathfrak{M}_i - \mathfrak{M}_{i-1}$ for $i = 2, 3, \ldots$. Then all $\mathfrak{N}_i \,\eta\, \mathbf{M}$ and they are mutually orthogonal. So Lemma 8.3.2 applies to them and gives

$$\sum_{i=1}^{\infty} D(\mathfrak{N}_i) = D([\mathfrak{N}_1, \mathfrak{N}_2, \ldots]).$$

Now $[\mathfrak{N}_1, \ldots, \mathfrak{N}_i] = \mathfrak{M}_i$ and so $[\mathfrak{N}_1, \mathfrak{N}_2, \ldots] = [\mathfrak{M}_1, \mathfrak{M}_2, \ldots]$ and furthermore

$$\sum_{i=1}^{\infty} D(\mathfrak{N}_i) = \lim_{n\to\infty} \sum_{i=1}^{n} D(\mathfrak{N}_i) = \lim_{n\to\infty} D([\mathfrak{N}_1, \ldots, \mathfrak{N}_n]) = \lim_{n\to\infty} D(\mathfrak{M}_n).$$

Therefore we have $\lim_{i\to\infty} D(\mathfrak{M}_i) = D([\mathfrak{M}_1, \mathfrak{M}_2, \ldots])$.

Lemma 8.3.4. Let $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ be an infinite sequence, all $\mathfrak{M}_i \,\eta\, \mathbf{M}$ and $\mathfrak{M}_1 \supset \mathfrak{M}_2 \supset \cdots$. If not all $\mathfrak{M}_i$ are infinite, then

$$\lim_{i\to\infty} D(\mathfrak{M}_i) = D(\mathfrak{M}_1 \cdot \mathfrak{M}_2 \cdot \mathfrak{M}_3 \cdots).$$

Proof: Assume that $\mathfrak{M}_{i_0}$ is finite. Put $\mathfrak{M}'_i = \mathfrak{M}_{i_0} - \mathfrak{M}_{i_0+i}$. Then all $\mathfrak{M}'_i \,\eta\, \mathbf{M}$ and $\mathfrak{M}'_1 \subset \mathfrak{M}'_2 \subset \cdots$, so Lemma 8.3.3 applies to them and gives

$$\lim_{i\to\infty} D(\mathfrak{M}'_i) = D([\mathfrak{M}'_1, \mathfrak{M}'_2, \ldots]).$$

But

$$D(\mathfrak{M}'_i) = D(\mathfrak{M}_{i_0}) - D(\mathfrak{M}_{i_0+i})$$

and

$$\begin{aligned}
[\mathfrak{M}'_1, \mathfrak{M}'_2, \ldots] &= [\mathfrak{M}_{i_0} - \mathfrak{M}_{i_0+1}, \mathfrak{M}_{i_0} - \mathfrak{M}_{i_0+2}, \ldots] \\
&= \mathfrak{M}_{i_0} - (\mathfrak{M}_{i_0+1} \cdot \mathfrak{M}_{i_0+2} \cdots) \\
&= \mathfrak{M}_{i_0} - (\mathfrak{M}_1 \cdot \mathfrak{M}_2 \cdots).
\end{aligned}$$

Therefore we have

$$D(\mathfrak{M}_{i_0}) - \lim_{i\to\infty} D(\mathfrak{M}_{i_0+i}) = D(\mathfrak{M}_{i_0}) - D(\mathfrak{M}_1 \cdot \mathfrak{M}_2 \cdots),$$

so that

$$\lim_{i\to\infty} D(\mathfrak{M}_{i_0+i}) = D(\mathfrak{M}_1 \cdot \mathfrak{M}_2 \cdots),$$

or, which is the same thing, $\lim_{i\to\infty} D(\mathfrak{M}_i) = D(\mathfrak{M}_1 \cdot \mathfrak{M}_2 \cdots)$.

The following criterion is occasionally useful, because it contains a sufficient condition for relative dimension functions which makes no explicit use of the notions of finiteness and infiniteness for sets $\mathfrak{M}\ \eta\ \mathbf{M}$.

Lemma 8.3.5. If a real-valued function $D'(\mathfrak{M})$ which is defined for all linear, closed sets $\mathfrak{M}\ \eta\ \mathbf{M}$ assumes only values ≥ 0, $\leq \infty$ and if some of its values are actually $\neq 0, \infty$ then the conditions (ii), (iii) in Definition 8.2.1 are sufficient in order that $D'(\mathfrak{M})$ be a relative dimension function.

Proof: We must derive (i) in Definition 8.2.1 from these conditions. Choose an $\overline{\mathfrak{M}}\ \eta\ \mathbf{M}$ with $D'(\overline{\mathfrak{M}}) > 0, < \infty$.

As (0) is orthogonal to $\overline{\mathfrak{M}}$ and $[\overline{\mathfrak{M}}, (0)] = \overline{\mathfrak{M}}$; (iii) gives $D'(\overline{\mathfrak{M}}) + D'(0) = D'(\overline{\mathfrak{M}})$, as $D'(\overline{\mathfrak{M}})$ is finite, this means $D'(0) = 0$.

If any $\mathfrak{M}$ is infinite, then $\mathfrak{H}$ is it, too, and $\mathfrak{M} \sim \mathfrak{H}$. Then by (ii) we must only prove $D'(\mathfrak{H}) = \infty$. By Lemma 7.2.3 applied to $\mathfrak{H}$ we have $2D'(\mathfrak{H}) = D'(\mathfrak{H})$, $D'(\mathfrak{H}) = 0$ or ∞. But

$$D'(\mathfrak{H}) = D'(\overline{\mathfrak{M}}) + D'(\mathfrak{H} - \overline{\mathfrak{M}}) \geq D'(\overline{\mathfrak{M}}) > 0,$$

so $D'(\mathfrak{H}) = \infty$. This disposes of all infinite $\mathfrak{M}$'s.

If $\mathfrak{M}$ is finite and $\neq (0)$ then apply Lemma 7.1.3 to $\mathfrak{M}, \overline{\mathfrak{M}}$ and to $\overline{\mathfrak{M}}, \mathfrak{M}$ (for its $\mathfrak{M}, \mathfrak{N}$). Then (ii), (iii) give

$$D'(\mathfrak{M}) \leq \left(\left[\frac{\mathfrak{M}}{\overline{\mathfrak{M}}}\right] + 1\right) D'(\overline{\mathfrak{M}}).$$

Thus $D'(\overline{\mathfrak{M}}) < \infty$ necessitates $D'(\mathfrak{M}) < \infty$. Next

$$D'(\overline{\mathfrak{M}}) \leq \left(\left[\frac{\overline{\mathfrak{M}}}{\mathfrak{M}}\right] + 1\right) D'(\mathfrak{M}).$$

Thus $D'(\overline{\mathfrak{M}}) > 0$ necessitates $D'(\mathfrak{M}) > 0$. So $0 < D'(\mathfrak{M}) < \infty$, completing the proof.

8.4. We now undertake to find invariant characteristics of $\mathbf{M}$ in terms of its relative dimension functions $D(\mathfrak{M})$.

Lemma 8.4.1. The range Δ of $D(\mathfrak{M})$ (that is the set of all values of a given $D(\mathfrak{M})$ for all $\mathfrak{M}\ \eta\ \mathbf{M}$) has these properties:

(i) The elements of Δ are real numbers, ≥ 0, $\leq \infty$.

(ii) $0 \in \Delta$, Δ contains a greatest element $\overline{\alpha} > 0$.

(iii) $\alpha, \beta \in \Delta$ and $\alpha > \beta$ imply $\alpha - \beta \in \Delta$.

(iv) $\alpha_1, \alpha_2, \ldots \in \Delta$ and $\sum_{i=1}^{\infty} \alpha_i \leq \overline{\alpha}$ imply $\sum_{i=1}^{\infty} \alpha_i \in \Delta$.

Proof: Ad (i): Obvious.

Ad (ii): $D((0)) = 0$; $D(\mathfrak{H}) = \overline{\alpha}$, $\mathfrak{H} \neq (0)$ therefore $D(\mathfrak{H}) > 0$.

Ad (iii): β must be finite. Choose $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$ with $D(\mathfrak{M}) = \alpha$, $D(\mathfrak{N}) = \beta$. Then $\mathfrak{N} \precsim \mathfrak{M}$, $\mathfrak{N} \sim \mathfrak{N}' \subset \mathfrak{M}$, $D(\mathfrak{N}') = D(\mathfrak{N}) = \beta$ finite and so $D(\mathfrak{M} - \mathfrak{N}') = D(\mathfrak{M}) - D(\mathfrak{N}') = \alpha - \beta$.

Ad (iv): If any $\alpha_i = \infty$ then $\sum_{i=1}^{\infty} \alpha_i = \infty$, so $\sum_{i=1}^{\infty} \alpha_i = \alpha_i \in \Delta$, so we may assume that all α_i are finite. Define a sequence $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ of mutually orthogonal sets $\mathfrak{M}_i\ \eta\ \mathbf{M}$, $D(\mathfrak{M}_i) = \alpha_i$ by induction. Assume that for an $i = 1, 2, \ldots$, $\mathfrak{M}_1, \ldots, \mathfrak{M}_{i-1}$ have already been constructed fulfilling these conditions, then construct $\mathfrak{M}_i$ as follows.

$$D(\mathfrak{H} - [\mathfrak{M}_1, \ldots, \mathfrak{M}_{i-1}]) = D(\mathfrak{H}) - \sum_{j=1}^{i-1} D(\mathfrak{M}_j)$$
$$= \overline{\alpha} - \sum_{j=1}^{i-1} \alpha_j \geq \sum_{j=i}^{\infty} \alpha_j \geq \alpha_i.$$

Choose an $\mathfrak{M}_i'\ \eta\ \mathbf{M}$ with $D(\mathfrak{M}_i') = \alpha_i$; then $\mathfrak{M}_i' \precsim \mathfrak{H} - [\mathfrak{M}_1, \ldots, \mathfrak{M}_{i-1}]$, $\mathfrak{M}_i' \sim \mathfrak{M}_i \subset \mathfrak{H} - [\mathfrak{M}_1, \ldots, \mathfrak{M}_{i-1}]$. This definition of $\mathfrak{M}_i$ meets all requirements. Now by Lemma 8.3.2

$$D([\mathfrak{M}_1, \mathfrak{M}_2, \ldots]) = \sum_{i=1}^{\infty} D(\mathfrak{M}_i) = \sum_{i=1}^{\infty} \alpha_i.$$

Lemma 8.4.2. The only sets Δ which fulfill the requirements (i)–(iv) of Lemma 8.4.1 are the following ones ($\epsilon, \overline{\alpha}$ are finite):

(I_n) $\epsilon > 0$, $n = 1, 2, \ldots$; Δ consists of all $i\epsilon$, $i = 0, 1, \ldots, n$.

(I_∞) $\epsilon > 0$; Δ consists of all $i\epsilon$, $i = 0, 1, \ldots, \infty$.

(II_1) $\overline{\alpha} > 0$; Δ consists of all α, $0 \leq \alpha \leq \overline{\alpha}$.

(II$_\infty$) Δ consists of all α, $0 \leq \alpha \leq \infty$.

(III$_\infty$) Δ consists of $0, \infty$.

Proof: Δ contains 0 by (ii) and some element > 0 by (i), (ii). If it contains no element $> 0, < \infty$ then it consists of $0, \infty$ which is precisely case (III$_\infty$). Assume now that it does contain elements $> 0, < \infty$.

Assume first, that this set contains a smallest element ϵ. If $\alpha \in \Delta$ and finite, then there exists an $i = 0, 1, 2, \ldots$ with $i\epsilon \leq \alpha < (i+1)\epsilon$. If $i\epsilon \neq \alpha$ then by (iii) all $\alpha, \alpha - \epsilon, \ldots, \alpha - i\epsilon$ belong to Δ. But $0 < \alpha - i\epsilon < \epsilon$, which is impossible. So $\alpha = i\epsilon$. In particular $\overline{\alpha} = n\epsilon$, $n = 0, 1, 2, \ldots$ if it is finite, and as $\overline{\alpha} > 0$, $n \neq 0$. If $\overline{\alpha}$ is infinite we may write $\overline{\alpha} = \infty\epsilon$, so at any rate $\overline{\alpha} = n\epsilon$, $n = 1, 2, \ldots, \infty$.

For all other $\alpha \in \Delta$, $\alpha < \overline{\alpha}$ so α finite, $\alpha = i\epsilon$ and as $\alpha < \overline{\alpha}$, $i < n$, $i = 0, 1, 2, \ldots, n-1$. Conversely, if $i = 0, 1, \ldots, n-1$, then $i\epsilon < n\epsilon = \overline{\alpha}$ and (iv) with $\alpha_1 = \cdots = \alpha_i = \epsilon$, $\alpha_{i+1} = \alpha_{i+2} = \cdots = 0$ gives $i\epsilon \in \Delta$. Thus Δ consists of all $i\epsilon$, $i = 0, 1, \ldots, n$, which is case (I$_n$) if $n = 1, 2, \ldots$, and case (I$_\infty$) if $n = \infty$.

Assume next that there is no smallest element $> 0, < \infty$ in Δ. Let ϵ' be the g.r.l.b. of the elements $\epsilon > 0$; if we had $\epsilon' > 0$ then there would exist an $\alpha \in \Delta$, $\alpha > 0$ with $\alpha < 2\epsilon'$ and as $\alpha = \epsilon'$ would imply that α is the smallest element > 0, $< \infty$ in Δ, therefore $\alpha > \epsilon'$. Now there exists a $\beta \in \Delta$, $\beta > 0$ with $\beta < \alpha$ and $\beta > \epsilon'$. So $\alpha, \beta \in \Delta$, $\alpha > \beta$ and by (iii) $\alpha - \beta \in \Delta$, but $\alpha - \beta > 0$, $\alpha - \beta < 2\epsilon' - \epsilon' = \epsilon'$ which is impossible. Thus $\epsilon' = 0$ and so for every $\epsilon > 0$ an $\alpha \in \Delta$, $0 < \alpha < \epsilon$ exists.

Now assume $0 \leq \beta_1 < \beta_2 \leq \overline{\alpha}$. Choose an $\alpha \in \Delta$ with $0 < \alpha < \beta_2 - \beta_1$. Then an $i = 0, 1, 2, \ldots$ with $i\alpha \leq \beta_1 < (i+1)\alpha$ exists, so

$$(i+1)\alpha = i\alpha + \alpha < \beta_1 + \beta_2 - \beta_1 = \beta_2.$$

As $(i+1)\alpha < \overline{\alpha}$, therefore (iv) with $\alpha_1 = \cdots = \alpha_{i+1} = \alpha$, $\alpha_{i+2} = \alpha_{i+3} = \cdots = 0$ gives $(i+1)\alpha \in \Delta$. So we see: There exists a $\beta \in \Delta$ with $\beta_1 < \beta < \beta_2$.

Now consider any α with $0 < \alpha < \overline{\alpha}$ and form a sequence $\beta_1, \beta_2, \ldots$ with $0 < \beta_1 < \beta_2 < \cdots < \alpha$; $\lim_{i\to\infty} \beta_i = \alpha$. For each i choose a $\beta'_i \in \Delta$ with $\beta_i < \beta'_i < \beta_{i+1}$. So $\beta'_i > \beta'_{i-1}$, $\beta'_i - \beta'_{i-1} \in \Delta$, $i = 2, 3, \ldots$. Put $\alpha_1 = \beta'_1$, $\alpha_i = \beta'_i - \beta'_{i-1}$ for $i = 2, 3, \ldots$, and apply (iv): $\sum_{i=1}^{\infty} \alpha_i = \lim_{i\to\infty} \beta'_i = \alpha < \overline{\alpha}$ and so $\alpha \in \Delta$.

Thus Δ consists of all α with $0 \leq \alpha \leq \overline{\alpha}$ which is case (II$_1$) if $\overline{\alpha}$ is finite, and case (II$_\infty$) if $\overline{\alpha} = \infty$. These considerations exhaust all possibilities.

We now apply Lemma 8.4.1 and Lemma 8.4.2 simultaneously, remembering that we can normalise $D(\mathfrak{M})$ so as to make $\epsilon = 1$ (cases (I$_n$) and (I$_\infty$)) resp. $\overline{\alpha} = 1$ (case (II$_1$)). Then we have:

Theorem VIII. The range of $D(\mathfrak{M})$ (that is the set of all values of a given $D(\mathfrak{M})$ for all $\mathfrak{M}\ \eta\ \mathbf{M}$) is one of the following sets:

(I_n) $n = 1, 2, \ldots$. The set $0, 1, \ldots, n$.

(I_∞) The set $0, 1, \ldots, \infty$.

(II_1) The set of all α, $0 \leq \alpha \leq 1$.

(II_∞) The set of all α, $0 \leq \alpha \leq \infty$.

(III_∞) The set $0, \infty$.

In the cases (I_n), (I_∞), (II_1) we have included a normalisation of $D(\mathfrak{M})$, the *standard normalisation*. Case (II_∞) is not normalised; in case (III_∞) no normalisation is needed, because $0, \infty$ are unaltered by multiplication with a finite positive number.

We call the cases (I_n), (II_1) the *finite cases*, the cases (I_∞), (II_∞), (III_∞), the *infinite cases*. On the other hand we call the cases (I) (that is (I_n), (I_∞)) the *discrete cases*, the cases (II), (that is (II_1), (II_∞)) the *continuous cases*, and the case (III), (that is (III_∞)) the *purely infinite case*.

The following problems arise immediately:

Problem 3. *Which ones of the classes* (I_n)–(III_∞) *do really exist?*

Problem 4. *Which combinations of these classes do occur for coupled factors* $\mathbf{M}, \mathbf{M}'$*? For general factorisations* $\mathbf{M}_1, \ldots, \mathbf{M}_n$*?*

Problem 5. *To which classes do the direct factors* $\mathbf{M}$ *belong?*

Problem 6. *Are all factors* $\mathbf{M}$ *belonging to the same class ring-isomorphic, or do there exist further invariant characteristics?*

Problem 7. *Are all factorisations* $\mathbf{M}_1, \ldots, \mathbf{M}_n$ *in which each* $\mathbf{M}_i$ *belongs to a given class ring-isomorphic or even spatially isomorphic, or do there exist further invariant characteristics?*

We will be able to contribute considerable material to clarify these questions, but the only one to which we can give a complete answer is Problem 5.

8.5. We establish the invariance of the subjects of the last §§ under ring-isomorphisms.

Lemma 8.5.1. Replace the $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$ by their $E = P_{\mathfrak{M}}$, $F = P_{\mathfrak{N}}$ which are $\in \mathbf{M}$ (cf. Definition 6.1.1, Definition 7.1.1, and Definition 8.2.1). The notions $E \sim F$ $(\cdots \mathbf{M})$, finiteness and infiniteness of an E with respect to $\mathbf{M}$, $D_{\mathbf{M}}(E)$ (apart from its normalisation) are invariant under algebraic ring-isomorphisms.

Proof: $E \sim F$ means that a $U \in \mathbf{M}$ with $UU^*U = U$, $U^*U = E$, $UU^* = F$ exists.

E is infinite, if an $F \in \mathbf{M}$ with $F^2 = F^* = F$, $EF = F$, $F \neq E$, $F \sim E$ exists. $D(E)$ is characterised in Definition 8.2.1 by the sole use of the notions 0, finiteness, infiniteness, $E \sim F$ and $E + F$ with $EF = 0$. Thus all these constructions are invariant under algebraic ring-isomorphisms.

Now we can state, combining Lemma 8.5.1 and Theorem VIII:

Theorem IX. The cases (I_n)–(III_∞) of Theorem VIII are invariant under algebraic ring-isomorphisms. So is $D_{\mathbf{M}}(E)$ (we write $E = P_{\mathfrak{M}} \in \mathbf{M}$ for $\mathfrak{M}\ \eta\ \mathbf{M}$), apart from its normalisation, and even the standard normalisation (in the cases (I_n)–(II_1)).

8.6. We now locate the direct factors in this classification.

Lemma 8.6.1. $\mathbf{M}$ is a direct factor if and only if it belongs to the discrete cases: (I_n) or (I_∞). If we then use the spatial isomorphism of $\mathfrak{H}$ with $\mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_m$ where $\mathbf{M}$ becomes $\mathbf{B}_i^{(i)}$ then the corresponding $\mathfrak{H}_i$ will be an n-dimensional Euclidean space resp. a Hilbert space. If $E_i = P_{\mathfrak{M}_i}$ is a projection in $\mathfrak{H}_i$ then $D_{\mathbf{M}}(E_i^{(i)})$ is, in the standard normalisation, simply the common notion of dimension of $\mathfrak{M}_i$.

Proof: In the cases (I) an $\mathfrak{M}\ \eta\ \mathbf{M}$ with $D(\mathfrak{M}) = 1$ exists. If $\mathfrak{N}\ \eta\ \mathbf{M}$, $\mathfrak{N} \subset \mathfrak{M}$ then $D(\mathfrak{N}) \leq D(\mathfrak{M})$, $D(\mathfrak{N}) = 0, 1$. The former implies $\mathfrak{N} = (0)$, the latter $\mathfrak{N} \sim \mathfrak{M}$ and as $\mathfrak{M}$ is finite, it excludes $\mathfrak{N} \subsetneqq \mathfrak{M}$, so it implies $\mathfrak{N} = \mathfrak{M}$. Obviously $\mathfrak{M} \neq (0)$. Thus $\mathfrak{M}$ is minimal, and therefore $\mathbf{M}$ is a direct factor (cf. Theorem IV). So our condition is sufficient.

All other statements assume that $\mathbf{M}$ is a direct factor, and thus algebraically (even fully) ring-isomorphic to $\mathbf{B}_i$. Then we may replace (by Theorem IX) $\mathfrak{H}, \mathbf{M}$, by $\mathfrak{H}_i, \mathbf{B}_i$; that is, we may assume $\mathbf{M} = \mathbf{B}$.

Every one-dimensional $\mathfrak{M} = [\varphi]$, $\varphi \neq 0$ (now all $\mathfrak{M}\ \eta\ \mathbf{M}$) is $\neq (0)$ and finite ($\mathfrak{N} \subsetneqq \mathfrak{M}$ implies $\mathfrak{N} = (0)$ excluding $\mathfrak{N} \sim \mathfrak{M}$) so $D_{\mathbf{B}}([\varphi]) > 0, < \infty$. Besides $[\varphi] \sim [\psi]$ so $D_{\mathbf{B}}([\varphi]) = D_{\mathbf{B}}([\psi])$; $D_{\mathbf{B}}([\varphi])$ is a constant. Normalise $D_{\mathbf{B}}(\mathfrak{M})$ so as to have $D_{\mathbf{B}}([\varphi]) = 1$. Then Lemma 8.3.2 gives $D_{\mathbf{B}}(\mathfrak{M})$ = common notion of dimension of $\mathfrak{M}$.

Thus if $\mathfrak{H}$ is an n-dimensional Euclidean space, this $D_{\mathbf{B}}(\mathfrak{M})$ has the range $0, 1, \ldots, n$; and if it is a Hilbert space, it has the range $0, 1, \ldots, \infty$. This proves that $\mathbf{M}$ is in the cases (I_n) resp. (I_∞), and that this normalisation of $D_{\mathbf{B}}(\mathfrak{M})$ is the standard one. This completes the proof.

Lemma 8.6.1 makes it possible to solve Problem 3–Problem 7 in the discrete cases (that is for direct factors). The details are these:

The discrete cases (I_n), for every $n = 1, 2, \ldots$ and (I_∞) do occur; and for factorisations $\mathbf{M}_1, \ldots, \mathbf{M}_m$ we can prescribe any combination of the discrete cases (I_n) and (I_∞) for the $\mathbf{M}_i$'s. We need only form $\mathfrak{H} = \mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_m$ where the $\mathfrak{H}_i$ are the corresponding n-dimensional Euclidean or Hilbert spaces, and put $\mathbf{M}_i = \mathbf{B}_i^{(i)}$. Furthermore if $m-1$ of the factors $\mathbf{M}_i$ are in discrete cases, then all are (Lemma 5.4.1); and so in particular two factors $\mathbf{M}_1, \mathbf{M}_2$ (and *a fortiori* two coupled factors) are either both in discrete cases, or none of them is. The direct factors are identical with those in discrete cases. If in a factorisation $\mathbf{M}_1, \ldots, \mathbf{M}_m$ all $\mathbf{M}_i$ are in discrete cases, then they are all direct factors, so $\mathbf{M}_1, \ldots, \mathbf{M}_m$ is a direct factorisation (Lemma 5.4.1). Thus a spatial isomorphism carries $\mathfrak{H}$ into $\mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_m$ and each $\mathbf{M}_i$ into $\mathbf{B}_i^{(i)}$; therefore $\mathfrak{H}_i$ is determined by the class of $\mathbf{M}_i$ (cf. above). Thus if two factors $\mathbf{M}, \mathbf{N}$ have the same discrete class, they are fully ring-isomorphic (Lemma 2.3.4 and (22, §4), compare the factorisations $\mathbf{M}, \mathbf{M}'$ and $\mathbf{N}, \mathbf{N}'$); and if in two factorisations $\mathbf{M}_1, \ldots, \mathbf{M}_m$ and $\mathbf{N}_1, \ldots, \mathbf{N}_m$ the corresponding $\mathbf{M}_i, \mathbf{N}_i$ have the same discrete classes for $i = 1, \ldots, m$, then we have even spatial isomorphism.

Part III: Pairs of factors

Chapter IX: Qualitative Comparison of $\mathfrak{M}_f^{\mathbf{M}'}$ and $\mathfrak{M}_f^{\mathbf{M}}$

9.1. We need first a discussion of infinite sums of definite operators. The considerations which follow are based on a construction of K. Friedrichs, (7, pp. 472, 476). The situation which we will investigate is described as follows:

Definition 9.1.1. Let $A_1, A_2, \ldots$ be a sequence of everywhere defined, bounded, Hermitian, (semi-) definite operators. In other words; For each $i = 1, 2, \ldots$ an α_i exists, so that we have identically

$$0 \leq (A_i f, f) \leq \alpha_i \|f\|^2$$

for all $f \in \mathfrak{H}$ (cf. (16, pp. 73–74)).

Put $A_0 = 1$. Define $\mathfrak{A}$ as the set of $f \in \mathfrak{H}$ for which $\sum_{i=0}^{\infty}(A_i f, f)$ is finite. For any two $f, g \in \mathfrak{H}$ for which $\sum_{i=0}^{\infty}(A_i f, g)$ is absolutely convergent, call this $Q(f, g)$.

Lemma 9.1.1. $\mathfrak{A}$ is a linear set. If $f, g \in \mathfrak{A}$ then $Q(f, g)$ is defined. With $\alpha f, f+g$ as linear operations and $Q(f, g)$ as inner product $\mathfrak{A}$ fulfills the conditions A, B, E of (16, pp. 64–66). (We may however have $\mathfrak{A} = (0)$.)

Proof: As $(A_i\alpha f, \alpha f) = |\alpha|^2(A_i f, f)$ therefore $f \in \mathfrak{A}$ implies $\alpha f \in \mathfrak{A}$. As

$$\begin{aligned}(A_i(f+g), f+g) &\leq (A_i(f+g), f+g) + (A_i(f-g), f-g)\\ &= 2(A_i f, f) + 2(A_i g, g)\end{aligned}$$

therefore $f, g \in \mathfrak{A}$ imply $f + g \in \mathfrak{A}$. Thus $\mathfrak{A}$ is a linear set. We have

$$|(A_i f, g)| \leq \tfrac{1}{2}(A_i f, f) + \tfrac{1}{2}(A_i g, g),$$

(this is proved literally as Schwarz's inequality cf. (16, p. 64)), thus if $f, g \in \mathfrak{A}$ then $\sum_{i=0}^{\infty}(A_i f, g)$ is absolutely convergent, and so $Q(f, g)$ is defined.

We now verify A, B, E, loc. cit. All parts of A, B are obvious, except B, 4. This holds, as $f \neq 0$ implies

$$Q(f, f) = \sum_{i=0}^{\infty}(A_i f, f) \geq (A_0 f, f) = \|f\|^2 > 0.$$

Consider now E. We must prove: If $f_1, f_2, \ldots \in \mathfrak{A}$ and

$$\lim_{m,n\to\infty} Q(f_m - f_n, f_m - f_n) = 0$$

then there exists an $f \in \mathfrak{A}$ with $\lim_{n\to\infty} Q(f_n - f, f_n - f) = 0$.

As $Q(f_m - f_n, f_m - f_n) \geq \|f_m - f_n\|^2$ we have $\lim_{m,n\to\infty}\|f_m - f_n\| = 0$. So an $f \in \mathfrak{H}$ with $\lim_{n\to\infty}\|f_n - f\| = 0$ exists. As

$$\lim_{m,n\to\infty} Q(f_m - f_n, f_m - f_n) = 0$$

there exists for every $\epsilon > 0$ an $n_0 = n_0(\epsilon)$ so that $m, n \geq n_0$ imply

$$Q(f_m - f_n, f_m - f_n) \leq \epsilon$$

and a fortiori

$$\sum_{i=0}^{j}(A_i(f_m - f_n), f_m - f_n) \leq \epsilon.$$

Let now $n \to \infty$; as all A_i are continuous, this gives

$$\sum_{i=0}^{j}(A_i(f_m - f), f_m - f) \leq \epsilon.$$

This holds for every $j = 0, 1, 2, \ldots$, therefore

$$\sum_{i=0}^{\infty} (A_i(f_m - f), f_m - f) \leq \epsilon.$$

This proves $f_m - f \in \mathfrak{A}$, $f = f_m - (f_m - f) \in \mathfrak{A}$, and then $Q(f_m - f, f_m - f) \leq \epsilon$. So we have $f \in \mathfrak{A}$ and $\lim_{m\to\infty} Q(f_m - f, f_m - f) = 0$. This completes the proof.

Lemma 9.1.2. For every $f \in \mathfrak{H}$ there exists one and only one $f^* \in \mathfrak{A}$ so that $(f, g) = Q(f^*, g)$ for every $g \in \mathfrak{A}$. We have $\|f^*\| \leq \|f\|$.

Proof: Consider $L(g) = (f, g)$ for $g \in \mathfrak{A}$ as a functional in $\mathfrak{A}$. We have obviously 1) $L(\alpha g) = \overline{\alpha} L(g)$, 2) $L(g_1 + g_2) = L(g_1) + L(g_2)$. Furthermore we have

$$|L(g)| = |(f, g)| \leq \|f\| \cdot \|g\| \leq \|f\| \sqrt{Q(g, g)},$$

that is 3) $|L(g)| \leq a_0 \sqrt{Q(g, g)}$, where $a_0 = \|f\|$. Under these conditions the existence of an $f^* \in \mathfrak{A}$ is certain, for which $L(g) = Q(f^*, g)$ for all $g \in \mathfrak{A}$ and this f^* is unique. (Cf. (11, p. 11), and (13a, p. 34).) In other words: $(f, g) = Q(f^*, g)$, $g = f^*$ gives in particular:

$$Q(f^*, f^*) = (f, f^*) \leq \|f\| \cdot \|f^*\|;$$

but as $Q(f^*, f^*) \geq \|f^*\|^2$ this implies $\|f^*\| \leq \|f\|$.

Definition 9.1.2. Define an operator B by $Bf = f^*$, in the sense of Lemma 9.1.2.

Lemma 9.1.3. B is everywhere defined, bounded, Hermitian, (semi-) definite (in $\mathfrak{H}$); and Range $B \subset \mathfrak{A}$.

Proof: B is obviously everywhere defined. $(Bf, f) = (f^*, f) = Q(f^*, f^*)$ and

$$0 \leq Q(f^*, f^*) \leq \|f^*\| \cdot \|f\| \leq \|f\|^2,$$

thus B is bounded, Hermitian, and (semi-) definite. As $Bf = f^* \in \mathfrak{A}$ we have Range $B \subset \mathfrak{A}$.

Lemma 9.1.4. Form $B^{1/2}$ in the sense of F. Riesz (cf. in the proof). $B^{1/2}$ is everywhere defined, bounded, Hermitean, (semi-) definite. Range $B^{1/2} = \mathfrak{A}$. For $f \in \mathfrak{H} - [\mathfrak{A}]$, $B^{1/2} f = 0$, while for $f, g \in [\mathfrak{A}]$, $(f, g) = Q(B^{1/2} f, B^{1/2} g)$.

Proof: As to the definition and character of $B^{1/2}$ cf. for instance (16, p. 113).

$B^{1/2} f = 0$ implies $Bf = B^{1/2}(B^{1/2} f) = 0$; $Bf = 0$ implies $\|B^{1/2} f\|^2 = (B^{1/2} f, B^{1/2} f) = (Bf, f) = 0$, $B^{1/2} f = 0$; $Bf = 0$ means that the f^* of Lemma 9.1.2 is 0, that is $(f, g) = 0$ for $g \in \mathfrak{A}$. This means $f \in \mathfrak{H} - [\mathfrak{A}]$. So we have:

$$(f; B^{1/2} f = 0) = (f; Bf = 0) = \mathfrak{H} - [\mathfrak{A}].$$

$$[\text{Range}\, B^{1/2}] = [\text{Range}\, B] = [\mathfrak{A}].$$

Consider an $f \in [\mathfrak{A}]$. Then f is a condensation point of Range $B^{1/2}$ so a sequence $f_1, f_2, \ldots$ with $\lim_{n\to\infty} \|B^{1/2} f_n - f\| = 0$ exists. Now all $Bf_n \in \mathfrak{A}$ and

$$\begin{aligned} Q(Bf_m - Bf_n, Bf_m - Bf_n) &= (f_m - f_n, Bf_m - Bf_n) \\ &= (B^{1/2} f_m - B^{1/2} f_n, B^{1/2} f_m - B^{1/2} f_n) \\ &= \|B^{1/2} f_m - B^{1/2} f_n\|^2, \end{aligned}$$

so $\lim_{m,n\to\infty} Q(Bf_m - Bf_n, Bf_m - Bf_n) = 0$. But $\mathfrak{A}$ fulfills the completeness postulate E (for $Q(\ldots,\ldots)$ cf. Lemma 9.1.1), so an $f^0 \in \mathfrak{A}$ with

$$\lim_{n\to\infty} Q(Bf_n - f^0, Bf_n - f^0) = 0$$

exists. Now $Q(g, g) \geq \|g\|^2$ so $\lim_{n\to\infty} \|Bf_n - f^0\| = 0$. On the other hand the continuity of $B^{1/2}$ implies

$$\lim_{n\to\infty} \|Bf_n - B^{1/2} f\| = \lim_{n\to\infty} \|B^{1/2}(B^{1/2} f_n - f)\| = 0.$$

So $f^0 = B^{1/2} f$, proving $B^{1/2} f \in \mathfrak{A}$. Besides we found

$$\lim_{n\to\infty} Q(Bf_n - B^{1/2} f, Bf_n - B^{1/2} f) = 0.$$

Consider two $f, g \in [\mathfrak{A}]$. Then $B^{1/2} f, B^{1/2} g \in \mathfrak{A}$ and forming the above sequence $f_1, f_2, \ldots$ for f and its analogue $g_1, g_2, \ldots$ for g, the continuity of the inner product of $\mathfrak{A}$ in the metric of $\mathfrak{A}$ (all $Q(\ldots,\ldots)$ cf. (16, p. 65)) necessitates

$$\begin{aligned} Q(B^{1/2} f, B^{1/2} g) &= \lim_{n\to\infty} Q(Bf_n, Bg_n) = \lim_{n\to\infty} (f_n, Bg_n) \\ &= \lim_{n\to\infty} (B^{1/2} f_n, B^{1/2} g_n) = (f, g). \end{aligned}$$

Thus $B^{1/2}$ maps $[\mathfrak{A}]$ on some subset $\mathfrak{A}'$ of $\mathfrak{A}$ and this mapping is isomorphic, if we use the inner product (f, g) in $[\mathfrak{A}]$ but $Q(f, g)$ in $\mathfrak{A}'$.

Therefore it is one-to-one. Now $[\mathfrak{A}]$ is a closed subset in the topologically complete set $\mathfrak{H}$ (topology of the metric $\|f - g\| = (f - g, f - g)^{1/2}$), therefore $\mathfrak{A}'$ is complete too, and so it must be a closed subset of $\mathfrak{A}$ (topology of the metric $[Q(f - g, f - g)]^{1/2}$). In this sense $\mathfrak{A}'$ is a closed linear set in $\mathfrak{A}$. Consider therefore $\mathfrak{A} - \mathfrak{A}'$ (for the inner product $Q(f, g)$). $f \in \mathfrak{A} - \mathfrak{A}'$ means $f \in \mathfrak{A}$ and $Q(f, g) = 0$ for every $g \in \mathfrak{A}'$ that is $Q(f, B^{1/2}g') = 0$ for every $g' \in [\mathfrak{A}]$. For every g'', $g' = B^{1/2}g'' \in [\mathfrak{A}]$ so we have a fortiori $Q(f, Bg'') = 0$, $(f, g'') = 0$ for every g'', and so $f = 0$. This proves $\mathfrak{A} - \mathfrak{A}' = (0)$. Now every $f \in \mathfrak{A}$ is the sum of an element of $\mathfrak{A}'$ and one of $\mathfrak{A} - \mathfrak{A}'$ (this is the fundamental theorem on projections, which applies to spaces which fulfill, like $\mathfrak{A}$, only A., B., E., by (11, p. 14), or (13a, p. 34)); as $\mathfrak{A} - \mathfrak{A}' = (0)$ this means $f \in \mathfrak{A}'$. Thus we have proved $\mathfrak{A}' = \mathfrak{A}$. We see that $B^{1/2}$ maps $[\mathfrak{A}]$ on $\mathfrak{A}$. In $\mathfrak{H} - [\mathfrak{A}]$ (we use again the inner product (f, g) of $\mathfrak{H}$) however it is $= 0$. Therefore Range $B^{1/2} = \mathfrak{A}$.

Thus we have proved all statements of our Lemma.

Another essential property of $B^{1/2}$ is this:

Lemma 9.1.5. Let $\mathbf{M}$ be a ring which contains 1. If the above $A_1, A_2, \ldots$ are elements of $\mathbf{M}$ then $B^{1/2} \in \mathbf{M}$.

Proof: $A_n \in \mathbf{M}$, $A_n \,\eta\, \mathbf{M}$ so A_n is invariant under every unitary $U' \in \mathbf{M}'$. Therefore $\mathfrak{A}$ and $Q(f, g)$ are invariant under these and with them B, which was uniquely characterised with their help. So $B \,\eta\, \mathbf{M}$ and by Lemma 4.2.1 $B \in \mathbf{M}$. This implies $B^{1/2} \in \mathbf{M}$ for instance by (19, p. 213).

9.2. Consider now the sets $\mathfrak{M}_{f_0}^{\mathbf{M}'}$ of Definition 5.1.1.

Lemma 9.2.1. Let $\mathbf{M}$ be a ring which contains 1. If $g_0 \in \mathfrak{M}_{f_0}^{\mathbf{M}'}$ then two linear, closed operators $X', Y' \,\eta\, \mathbf{M}'$ with everywhere dense domains can be found such that $g_0 = X'Y'f_0$. (Of course $f_0 \in$ Domain Y', $Y'f_0 \in$ Domain X'. X' is even bounded.)

Proof: g_0 is a condensation point of the set of all $A'f_0$, $A' \in \mathbf{M}'$. So we can find for every $n = 1, 2, \ldots$ an $A'_n \in \mathbf{M}'$ with $\|A'_n f_0 - g_0\| \leq 1/2^n$. Thus

$$\|A'_{n+1} f_0 - A'_n f_0\| \leq 1/2^{n+1} + 1/2^n = 3/2^{n+1},$$

and so

$$\begin{aligned}&\big(2^n (A'_{n+1} - A'_n)^* (A'_{n+1} - A'_n) f_0, f_0\big)\\ &\qquad = 2^n \|(A'_{n+1} - A'_n) f_0\|^2 \leq 2^n (3/2^{n+1})^2 = 9/2^{n+2}.\end{aligned}$$

We now perform the construction of Definition 9.1.1 with $2^n(A'_{n+1} - A'_n)^*(A'_{n+1} - A'_n)$ in place of A_n, $n = 1, 2, \ldots$. We thus obtain $\mathfrak{A}$ and $Q(f, g)$, and the $B, B^{1/2}$ of Definition 9.1.2 and Lemma 9.1.4, which we will denote by $B', B'^{1/2}$. Lemma 9.1.5 gives $B'^{1/2} \in \mathbf{M}'$. Note that our above evaluation secures $f_0 \in \mathfrak{A}$.

$B'^{1/2}$ is a one-to-one mapping of $[\mathfrak{A}]$ on $\mathfrak{A}$ (even isomorphic! cf. Lemma 9.1.4). Denote its inverse (in $\mathfrak{A}$ only!) by Y'_0. As $B'^{1/2} \in \mathbf{M}'$ we have $\mathfrak{A} = \text{Range } B'^{1/2}\ \eta\ \mathbf{M}$ and so $Y'_0\ \eta\ \mathbf{M}'$; by its very nature Y'_0 is linear and closed. But Domain $Y'_0 = \mathfrak{A}$ is only dense in $[\mathfrak{A}]$. If we put $Y' = Y'_0 P_{[\mathfrak{A}]}$, then Y' is clearly $\eta \mathbf{M}'$, linear, closed, and has an everywhere dense domain.

Consider an $f \in \mathfrak{H}$. Then $B'^{1/2} f = B'^{1/2} P_{[\mathfrak{A}]} f \in \mathfrak{A}$ and $Q(B'^{1/2} f, B'^{1/2} f) = \|P_{[\mathfrak{A}]} f\|^2 \leq \|f\|^2$. But

$$Q(B'^{1/2} f, B'^{1/2} f) = \|B'^{1/2} f\|^2 + \sum_{n=1}^{\infty} 2^n \|(A'_{n+1} - A'_n) B'^{1/2} f\|^2$$
$$\geq 2^n \|(A'_{n+1} - A'_n) B'^{1/2} f\|^2$$

for every $n = 1, 2, \ldots$; so

$$\|(A'_{n+1} - A'_n) B'^{1/2} f\| \leq \frac{1}{\sqrt{2^n}} \|f\|.$$

Therefore

$$\|(A'_{n+p} - A'_n) B'^{1/2} f\| \leq \left(\frac{1}{\sqrt{2^n}} + \cdots + \frac{1}{\sqrt{2^{n+p-1}}} \right) \|f\|$$
$$\leq \frac{2 + \sqrt{2}}{\sqrt{2^n}} \|f\|,$$

or, which is the same thing,

$$\|(A'_m - A'_n) B'^{1/2} f\| \leq \frac{2 + \sqrt{2}}{\sqrt{2^{\min(m,n)}}} \|f\|.$$

Thus $\lim_{m,n\to\infty} \|(A'_m - A'_n) B'^{1/2} f\| = 0$ and therefore a unique f^* with $\lim_{n\to\infty} \|A'_n B'^{1/2} f - f^*\| = 0$ exists. We define an operator X' by $X' f = f^*$; then X' is clearly the uniform limit of the sequence $A'_1 B'^{1/2}, A'_2 B'^{1/2}, \ldots$ (cf. (18, p. 384)), thus bounded and $\in \mathbf{M}'$ too.

Now $f_0 \in \mathfrak{A}$, therefore $Y' f_0 = Y'_0 f_0 \in [\mathfrak{A}]$ and $B'^{1/2} Y' f_0 = f_0$; g_0 is the limit of the sequence $A'_n f_0 = A'_n B'^{1/2} Y' f_0$, so $g_0 = X' Y' f_0$. This completes the proof.

Note that we cannot make much use of the operator-product $X'Y'$ itself. It is not closed in general, and it is uncertain whether it possesses a single valued closure (cf. (21, p. 300)). We will see in §§16.3–16.4 how the notion of finiteness helps to bridge this gap; for the moment this is unimportant, because we are able to deal with the X', Y' separately.

9.3. We will compare the $\mathfrak{M}_f^{\mathbf{M}'}$ and the $\mathfrak{M}_f^{\mathbf{M}}$. From now on we assume that $\mathbf{M}$ is a factor.

Lemma 9.3.1. If $g_0 = X' f_0$ where $X' \,\eta\, \mathbf{M}'$ is a linear, closed operator with an everywhere dense domain, then

$$\mathfrak{M}_{g_0}^{\mathbf{M}} \precsim \mathfrak{M}_{f_0}^{\mathbf{M}} \quad (\cdots \mathbf{M}').$$

Proof: $X' \,\eta\, \mathbf{M}'$ means that X' commutes with all unitary $U \in \mathbf{M}'' = \mathbf{M}$, that is with all elements of $\mathbf{M}^{(U)}$ (cf. (18, p. 392)); as it is linear and closed however, it will even commute with those of $\mathrm{R}(\mathbf{M}^{(U)}) = \mathbf{M}$ (cf. (18, pp. 392, 405)). So we have

$$\begin{aligned}
\mathfrak{M}_{g_0}^{\mathbf{M}} &= [A g_0; A \in \mathbf{M}] = [A X' f_0; A \in \mathbf{M}] \\
&= [X' A f_0; A \in \mathbf{M}] = [X' f; f \in (A f_0, A \in \mathbf{M})] \\
&\subset [X' f; f \in \text{Domain}\, X' \cdot \mathfrak{M}_{f_0}^{\mathbf{M}}].
\end{aligned}$$

Put $[\text{Domain}\, X' \cdot \mathfrak{M}_{f_0}^{\mathbf{M}}] = \mathfrak{N}'$; clearly $\mathfrak{N}' \subset \mathfrak{M}_{f_0}^{\mathbf{M}}$ and $X' \,\eta\, \mathbf{M}'$, $\mathfrak{M}_{f_0}^{\mathbf{M}} \,\eta\, \mathbf{M}'$ imply $\mathfrak{N}' \,\eta\, \mathbf{M}'$. It is clear that $X' P_{\mathfrak{N}'}$ is a linear, closed operator with an everywhere dense domain, and $\eta \mathbf{M}'$. Our above relation proves

$$\mathfrak{M}_{g_0}^{\mathbf{M}} \subset [\text{Range}(X' P_{\mathfrak{N}'})].$$

On the other hand

$$\begin{aligned}
[\text{Range}(X' P_{\mathfrak{N}'})^*] &= \mathfrak{H} - (f; X' P_{\mathfrak{N}'} f = 0) \\
&\subset \mathfrak{H} - (f; P_{\mathfrak{N}'} f = 0) \\
&= \mathfrak{H} - (\mathfrak{H} - \mathfrak{N}') = \mathfrak{N}' \subset \mathfrak{M}_{f_0}^{\mathbf{M}}.
\end{aligned}$$

Using Lemma 6.2.1 we obtain

$$\mathfrak{M}_{g_0}^{\mathbf{M}} \subset [\text{Range}(X' P_{\mathfrak{N}'})] \sim [\text{Range}(X' P_{\mathfrak{N}'})^*] \subset \mathfrak{M}_{f_0}^{\mathbf{M}},$$

(the $\sim$ is $(\cdots \mathbf{M}')$), and so $\mathfrak{M}_{g_0}^{\mathbf{M}} \precsim \mathfrak{M}_{f_0}^{\mathbf{M}} (\cdots \mathbf{M}')$.

Lemma 9.3.2. $\mathfrak{M}_{g_0}^{\mathbf{M}'} \precsim \mathfrak{M}_{f_0}^{\mathbf{M}'} (\cdots \mathbf{M})$ implies $\mathfrak{M}_{g_0}^{\mathbf{M}} \precsim \mathfrak{M}_{f_0}^{\mathbf{M}} (\cdots \mathbf{M}')$.

Proof: $\mathfrak{M}_{g_0}^{\mathbf{M}'} \precsim \mathfrak{M}_{f_0}^{\mathbf{M}'}(\cdots\mathbf{M})$ means $\mathfrak{M}_{g_0}^{\mathbf{M}'} \sim \mathfrak{N}\ (\cdots\mathbf{M})$, $\mathfrak{N} \subset \mathfrak{M}_{f_0}^{\mathbf{M}'}$. So a partially isometric $U \in \mathbf{M}$ maps $\mathfrak{M}_{g_0}^{\mathbf{M}'}$ on $\mathfrak{N}$ while U^* maps $\mathfrak{N}$ on $\mathfrak{M}_{g_0}^{\mathbf{M}'}$.

Now $Ug_0 \in \mathfrak{N} \subset \mathfrak{M}_{f_0}^{\mathbf{M}'}$, so by Lemma 9.2.1 $Ug_0 = X'Y'f_0$, X', Y' linear, closed, and with everywhere dense domains. By Lemma 9.3.1

$$\mathfrak{M}_{f_0}^{\mathbf{M}} \succsim \mathfrak{M}_{Y'f_0}^{\mathbf{M}} \succsim \mathfrak{M}_{X'Y'f_0}^{\mathbf{M}} = \mathfrak{M}_{Ug_0}^{\mathbf{M}} \quad (\cdots\mathbf{M}').$$

But $U^*Ug_0 = g_0$ and so

$$\mathfrak{M}_{g_0}^{\mathbf{M}} = [Ag_0; A \in \mathbf{M}] = [AU^*Ug_0; A \in \mathbf{M}] \subset [BUg_0; B \in \mathbf{M}] = \mathfrak{M}_{Ug_0}^{\mathbf{M}}.$$

So $\mathfrak{M}_{g_0}^{\mathbf{M}} \subset \mathfrak{M}_{Ug_0}^{\mathbf{M}} \precsim \mathfrak{M}_{f_0}^{\mathbf{M}}\ (\cdots\mathbf{M}')$, and hence $\mathfrak{M}_{g_0}^{\mathbf{M}} \precsim \mathfrak{M}_{f_0}^{\mathbf{M}}(\cdots\mathbf{M}')$.

Lemma 9.3.3. $\mathfrak{M}_{f_0}^{\mathbf{M}'} \gtreqless \mathfrak{M}_{g_0}^{\mathbf{M}'}(\cdots\mathbf{M})$ is equivalent to $\mathfrak{M}_{f_0}^{\mathbf{M}} \gtreqless \mathfrak{M}_{g_0}^{\mathbf{M}}(\cdots\mathbf{M}')$, respectively.

Proof: $\mathfrak{M}_{f_0}^{\mathbf{M}'} \succsim \mathfrak{M}_{g_0}^{\mathbf{M}'}(\cdots\mathbf{M})$ implies $\mathfrak{M}_{f_0}^{\mathbf{M}} \succsim \mathfrak{M}_{g_0}^{\mathbf{M}}$ by Lemma 9.3.2. The converse follows by replacing $\mathbf{M}$ by $\mathbf{M}'$ (and so $\mathbf{M}'$ by $\mathbf{M}'' = \mathbf{M}$); thus $\mathfrak{M}_{f_0}^{\mathbf{M}'} \succsim \mathfrak{M}_{g_0}^{\mathbf{M}'}(\cdots\mathbf{M})$ and $\mathfrak{M}_{f_0}^{\mathbf{M}} \succsim \mathfrak{M}_{g_0}^{\mathbf{M}}(\cdots\mathbf{M}')$ are equivalent. The same follows for $\precsim$ by interchanging f_0 and g_0.

The same follows for $\sim$ because it means that $\precsim$ and $\succsim$ both hold; it follows for $\prec$ because this means that $\precsim$ holds, but not $\sim$; finally it follows for $\succ$ by interchanging again f_0 and g_0. Thus all statements of our Lemma are proved.

Chapter X: Ratio of $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'})$ and $D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}})$

10.1. The last §§ give the basis for a quantitative comparison of $\mathfrak{M}_f^{\mathbf{M}'}$ with $\mathfrak{M}_f^{\mathbf{M}}$. $\mathbf{M}, \mathbf{M}'$ will again be factors; $D_{\mathbf{M}}(\mathfrak{M})$, $D_{\mathbf{M}'}(\mathfrak{M}')$ arbitrarily normalised relative dimension functions of $\mathbf{M}, \mathbf{M}'$ respectively; Δ, Δ' their respective ranges. We define:

Definition 10.1.1. Denote the range of $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'})$ (for all $f \in \mathfrak{H}$) by Δ_0 and the range of $D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}})$ (for all $f \in \mathfrak{H}$) by Δ_0'.

Lemma 10.1.1. If we let $D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}})$ correspond to $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'})$ a one-to-one mapping of Δ_0 on Δ_0' results. Describe this mapping by $\alpha' = \phi(\alpha)$, $\alpha \in \Delta_0$, $\alpha' \in \Delta_0'$. $\phi(\alpha)$ is a monotone increasing function of α.

Proof: Lemma 9.3.3 can be formulated like this:

$$D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) \gtreqless D_{\mathbf{M}}(\mathfrak{M}_g^{\mathbf{M}'}) \quad \text{is equivalent to} \quad D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}}) \gtreqless D_{\mathbf{M}'}(\mathfrak{M}_g^{\mathbf{M}}),$$

respectively. This implies all parts of our Lemma.

We will now determine $\Delta_0, \Delta_0', \phi(\alpha)$ and the set of all pairs $\mathfrak{M}_f^{\mathbf{M}'}, \mathfrak{M}_f^{\mathbf{M}}$.

Lemma 10.1.2. If $A \in \mathbf{M}$, then $[Ag; g \in \mathfrak{M}_f^{\mathbf{M}'}] = \mathfrak{M}_{Af}^{\mathbf{M}'}$.

Proof: We have

$$\begin{aligned}[Ag; g \in \mathfrak{M}_f^{\mathbf{M}'}] &= [Ag; g \in [B'f; B' \in \mathbf{M}']] \\ &= [Ag; g \in (B'f; B' \in \mathbf{M}')] \\ &= [AB'f; B' \in \mathbf{M}'] = [B'Af; B' \in \mathbf{M}'] = \mathfrak{M}_{Af}^{\mathbf{M}'}.\end{aligned}$$

Lemma 10.1.3. The necessary and sufficient conditions for the existence of an $f \in \mathfrak{H}$ with $\mathfrak{M}_f^{\mathbf{M}'} = \mathfrak{M}$, $\mathfrak{M}_f^{\mathbf{M}} = \mathfrak{N}$ are these: $\mathfrak{M}\ \eta\ \mathbf{M}$, $\mathfrak{N}\ \eta\ \mathbf{M}'$, $D_{\mathbf{M}}(\mathfrak{M}) \in \Delta_0$, $D_{\mathbf{M}'}(\mathfrak{N}) = \phi(D_{\mathbf{M}}(\mathfrak{M}))$.

Proof: The necessity being obvious, let us consider the sufficiency. Our assumptions imply the existence of a g with $D_{\mathbf{M}}(\mathfrak{M}_g^{\mathbf{M}'}) = D_{\mathbf{M}}(\mathfrak{M})$, $D_{\mathbf{M}'}(\mathfrak{M}_g^{\mathbf{M}}) = D_{\mathbf{M}'}(\mathfrak{N})$. So $\mathfrak{M}_g^{\mathbf{M}'} \sim \mathfrak{M}\ (\cdots \mathbf{M})$, $\mathfrak{M}_g^{\mathbf{M}} \sim \mathfrak{N}\ (\cdots \mathbf{M}')$. Let $U \in \mathbf{M}$ be partially isometric with the initial and final sets $\mathfrak{M}_g^{\mathbf{M}'}, \mathfrak{M}$ and $U' \in \mathbf{M}'$ similarly with $\mathfrak{M}_g^{\mathbf{M}}, \mathfrak{N}$.

Consider now $\mathfrak{M}_{U'g}^{\mathbf{M}'}$. It is equal to $[A'U'g; A' \in \mathbf{M}']$, thus certainly a subset of $[B'g; B' \in \mathbf{M}'] = \mathfrak{M}_g^{\mathbf{M}'}$. But on the other hand for every $B' \in \mathbf{M}'$ we have for $A' = B'U'^* \in \mathbf{M}'$, $A'U'g = B'U'^*U'g = B'P_{\mathfrak{M}_g^{\mathbf{M}}}g = B'g$. So our set is equal to $\mathfrak{M}_g^{\mathbf{M}'}$. Now Lemma 10.1.2 gives

$$\mathfrak{M}_{UU'g}^{\mathbf{M}'} = [Uh; h \in \mathfrak{M}_{U'g}^{\mathbf{M}'}] = [Uh; h \in \mathfrak{M}_g^{\mathbf{M}'}] = \mathfrak{M}.$$

Similarly $\mathfrak{M}_{U'Ug}^{\mathbf{M}} = \mathfrak{N}$. Thus $f = UU'g = U'Ug$ meets all our requirements.

Lemma 10.1.4. $\alpha_1, \alpha_2, \ldots \in \Delta_0$, $\sum_{i=1}^\infty \alpha_i \in \Delta$, $\sum_{i=1}^\infty \phi(\alpha_i) \in \Delta'$ imply

$$\sum_{i=1}^\infty \alpha_i \in \Delta_0, \qquad \sum_{i=1}^\infty \phi(\alpha_i) = \phi\left(\sum_{i=1}^\infty \alpha_i\right).$$

Proof: We wish to construct a sequence of mutually orthogonal

$$\mathfrak{M}_1, \mathfrak{M}_2, \ldots\ \eta\ \mathbf{M}$$

so that $D_{\mathbf{M}}(\mathfrak{M}_i) = \alpha_i$. We proceed by induction. Assume that $\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_{i-1}$ have already been constructed, fulfilling these requirements as well as

$$D_{\mathbf{M}}\big(\mathfrak{H} - [\mathfrak{M}_1, \ldots, \mathfrak{M}_{i-1}]\big) \geq \sum_{j=i}^\infty \alpha_j,$$

where $i = 1, 2, \ldots$. (For $i = 1$, this is the case, as $D_{\mathbf{M}}(\mathfrak{H}) \geq \sum_{j=1}^{\infty} \alpha_j$, because $\sum_{j=1}^{\infty} \alpha_j \in \Delta$.) We then construct $\mathfrak{M}_i$ as follows: If $\alpha_i = \infty$ then $\sum_{j=i}^{\infty} \alpha_j = \infty$ and $\mathfrak{H} - [\mathfrak{M}_1, \ldots, \mathfrak{M}_{i-1}]$ is infinite. Apply Lemma 7.2.3 to it, and put $\mathfrak{M}_i = \mathfrak{N}$ for the resulting $\mathfrak{N}$ which obviously meets all requirements. If α_i is finite, choose an $\mathfrak{M}'_i \;\eta\; \mathbf{M}$ with $D_{\mathbf{M}}(\mathfrak{M}'_i) = \alpha_i$. Then

$$\mathfrak{M}'_i \precsim \mathfrak{H} - [\mathfrak{M}_1, \ldots, \mathfrak{M}_{i-1}], \qquad \mathfrak{M}'_i \sim \mathfrak{M}_i \subset \mathfrak{H} - [\mathfrak{M}_1, \ldots, \mathfrak{M}_{i-1}],$$

and use this $\mathfrak{M}_i$. Then

$$\begin{aligned} D_{\mathbf{M}}(\mathfrak{M}_i) &= \alpha_i, \\ D_{\mathbf{M}}\big(\mathfrak{H} - [\mathfrak{M}_1, \ldots, \mathfrak{M}_i]\big) &= D_{\mathbf{M}}\big(\mathfrak{H} - [\mathfrak{M}_1, \ldots, \mathfrak{M}_{i-1}]\big) - D_{\mathbf{M}}(\mathfrak{M}_i) \\ &\geq \sum_{j=i}^{\infty} \alpha_j - \alpha_i = \sum_{j=i+1}^{\infty} \alpha_j, \end{aligned}$$

and again all conditions are fulfilled.

Choose similarly a sequence $\mathfrak{N}_1, \mathfrak{N}_2, \ldots \;\eta\; \mathbf{M}'$ so that $D_{\mathbf{M}'}(\mathfrak{N}_i) = \phi(\alpha_i)$. (Now $\sum_{i=1}^{\infty} \phi(\alpha_i) \in \Delta'$ must be used.) By Lemma 10.1.3 an $f_i \in \mathfrak{H}$ with

$$\mathfrak{M}_{f_i}^{\mathbf{M}'} = \mathfrak{M}_i, \qquad \mathfrak{M}_{f_i}^{\mathbf{M}} = \mathfrak{N}_i$$

exists. As we can replace f_i by any $\alpha'_i f_i$, we may assume $\|f_i\| \leq 1/i$.

As we see, the $\mathfrak{M}_{f_i}^{\mathbf{M}'}$, $i = 1, 2, \ldots$, are mutually orthogonal and so are the $\mathfrak{M}_{f_i}^{\mathbf{M}}$, $i = 1, 2, \ldots$. So the f_i are mutually orthogonal, and as $\sum_{i=1}^{\infty} \|f_i\|^2 \leq \sum_{i=1}^{\infty} 1/i^2$ is finite, we can form $f = \sum_{i=1}^{\infty} f_i$ (cf. (16, p. 67)).

As $f_i \in \mathfrak{M}_{f_i}^{\mathbf{M}'}$, therefore $f \in [\mathfrak{M}_{f_i}^{\mathbf{M}'}; i = 1, 2, \ldots]$ and considering the mutual orthogonality of the $\mathfrak{M}_{f_i}^{\mathbf{M}'}$, $i = 1, 2, \ldots$, we have $f_i = E_{f_i}^{\mathbf{M}'} f$. Similarly $f \in [\mathfrak{M}_{f_i}^{\mathbf{M}}; i = 1, 2, \ldots]$ and $f_i = E_{f_i}^{\mathbf{M}} f$.

Now

$$\begin{aligned} \mathfrak{M}_f^{\mathbf{M}'} = [A'f; A' \in \mathbf{M}'] &= \left[\sum_{i=1}^{\infty} A' f_i; A' \in \mathbf{M}'\right] \\ &\subset [[A' f_i; A' \in \mathbf{M}']; i = 1, 2, \ldots] = [\mathfrak{M}_{f_i}^{\mathbf{M}'}; i = 1, 2, \ldots] \end{aligned}$$

and

$$\mathfrak{M}_f^{\mathbf{M}'} = [A'f; A' \in \mathbf{M}'] \supset [B' E_{f_i}^{\mathbf{M}'} f; B' \in \mathbf{M}'] = [B' f_i; B' \in \mathbf{M}'] = \mathfrak{M}_{f_i}^{\mathbf{M}'}.$$

Thus

$$\mathfrak{M}_f^{\mathbf{M}'} = [\mathfrak{M}_{f_i}^{\mathbf{M}'}; i = 1, 2, \ldots],$$

similarly $\mathfrak{M}_f^{\mathbf{M}} = [\mathfrak{M}_{f_i}^{\mathbf{M}}; i = 1, 2, \ldots]$. So we have

$$D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) = \sum_{i=1}^{\infty} D_{\mathbf{M}}(\mathfrak{M}_{f_i}^{\mathbf{M}'}) = \sum_{i=1}^{\infty} \alpha_i,$$

$$D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}}) = \sum_{i=1}^{\infty} D_{\mathbf{M}'}(\mathfrak{M}_{f_i}^{\mathbf{M}}) = \sum_{i=1}^{\infty} \phi(\alpha_i).$$

Therefore $\sum_{i=1}^{\infty} \alpha_i \in \Delta_0$ and $\sum_{i=1}^{\infty} \phi(\alpha_i) = \phi(\sum_{i=1}^{\infty} \alpha_i)$.

Lemma 10.1.5. $0 \in \Delta_0$; an $\alpha \in \Delta_0$ with $\alpha > 0$ exists; if $\alpha \in \Delta$, $\alpha \leq \beta \in \Delta_0$ then $\alpha \in \Delta_0$.

Proof: For $f = 0$, $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) = D_{\mathbf{M}}((0)) = 0$; for $f \neq 0$, $\mathfrak{M}_f^{\mathbf{M}'} \neq (0)$, $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) > 0$. $\alpha \leq \beta \in \Delta_0$, $\alpha \in \Delta$ implies that an f with $D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'}) = \beta$ and an $\mathfrak{N}\ \eta\ \mathbf{M}$ with $D_{\mathbf{M}}(\mathfrak{N}) = \alpha$ exist. So $\mathfrak{N} \precsim \mathfrak{M}_f^{\mathbf{M}'}$, $\mathfrak{N} \sim \mathfrak{N}' \subset \mathfrak{M}_f^{\mathbf{M}'}$, and by Lemma 10.1.2

$$\mathfrak{M}_{P_{\mathfrak{N}'} f}^{\mathbf{M}'} = [P_{\mathfrak{N}'} g; g \in \mathfrak{M}_f^{\mathbf{M}'}] = \mathfrak{N}' \sim \mathfrak{N},$$

so $D_{\mathbf{M}}(\mathfrak{M}_{P_{\mathfrak{N}'} f}^{\mathbf{M}'}) = D_{\mathbf{M}}(\mathfrak{N}) = \alpha$.

10.2. Proof of $D_{\mathbf{M}'}(\mathfrak{M}_f^{\mathbf{M}}) = C D_{\mathbf{M}}(\mathfrak{M}_f^{\mathbf{M}'})$ and of $\Delta_0 = \Delta$ or $\Delta_0' = \Delta'$. Theorem X With the help of Lemmas 10.1.4, 10.1.5 the discussion of Δ_0, Δ_0' and $\phi(\alpha)$ can now be completed. We will use Theorem VIII, and discuss the three cases (I), (II), (III) for $\mathbf{M}$ separately.

Assume first case (I) for $\mathbf{M}$. Then we have case (I) for $\mathbf{M}'$ too, by Lemmas 3.2.3, 3.2.4, and 8.6.1. Use $D_{\mathbf{M}}(\mathfrak{M})$, $D_{\mathbf{M}'}(\mathfrak{M})$ in their standard normalisations. By Lemma 10.1.5 $0 \in \Delta_0$; and an $\alpha \in \Delta_0$ with $\alpha > 0$, so $\alpha \geq 1$ exists, which implies, as $1 \in \Delta$, that $1 \in \Delta_0$. Similarly $0, 1 \in \Delta_0'$. As $\phi(\alpha)$ is monotone increasing (by Lemma 10.1.1) and as 0 is the smallest element of both Δ_0 and Δ_0', so $\phi(0) = 0$, and as 1 is the smallest element $\neq 0$ of both Δ_0 and Δ_0', so $\phi(1) = 1$.

Denote the case of $\mathbf{M}$ more precisely by $\mathrm{I}_{m'}$, where $m' = 1, 2, \ldots$ or ∞; similarly $\mathbf{M}'$'s by $\mathrm{I}_{n'}$, where $n' = 1, 2, \ldots$ or ∞. If $p = 0, 1, 2, \ldots, \min(m', n')$, then apply Lemma 10.1.4 with $\alpha_1 = \cdots = \alpha_p = 1$, $\alpha_{p+1} = \alpha_{p+2} = \cdots = 0$ if p is finite, and $\alpha_1 = \alpha_2 = \cdots = 1$ if p is infinite. Then $p \in \Delta_0$, $\phi(p) = p$ obtains.

If Δ_0 had further elements, we would have

$$q \in \Delta_0, \qquad q \neq 0, 1, 2, \ldots, \min(m', n').$$

But $q \in \Delta$, $q = 0, 1, 2, \ldots, m'$, therefore $\min(m', n') < q \leq m'$. Thus $\phi(\min(m', n')) < \phi(q)$ (by Lemma 10.1.1), but $\phi(\min(m', n')) = \min(m', n')$ and $\phi(q) \in \Delta_0' \subset \Delta'$, so $\min(m', n') < \phi(q) \leq n'$. So $\min(m', n') < m'$ and $< n'$, which is impossible. Thus Δ_0 is precisely the set $0, 1, 2, \ldots, \min(m', n')$ and in it $\phi(\alpha) = \alpha$; therefore Δ_0' is the same set. We see that we have either $\Delta_0 = \Delta$, or $\Delta_0' = \Delta'$.

Assume next case (III) for **M**. Δ has precisely two elements: $0, \infty$, so that $\Delta_0 = \Delta$ by Lemma 10.1.5. So $\Delta_0 = (0, \infty)$, $\Delta_0' = (\phi(0), \phi(\infty)) = (0, \phi(\infty))$. Lemma 10.1.5 applied to $\mathbf{M}'$ excludes the existence of an $\alpha' \in \Delta'$ with $0 < \alpha' < \phi(\infty)$. This excludes case (II) for $\mathbf{M}'$. Case (I) for $\mathbf{M}'$ is ruled out as above (interchange $\mathbf{M}, \mathbf{M}'$), so we have case (III) for $\mathbf{M}'$ too. Now we have for the same reasons as in **M**, $\Delta_0' = \Delta' = (0, \infty)$, $\phi(\infty) = \infty$. So we have again $\phi(\alpha) = \alpha$, but now $\Delta_0 = \Delta$ and $\Delta_0' = \Delta'$.

Assume finally case (II) for **M**. The above arguments exclude cases (I) and (III) for $\mathbf{M}'$ (interchange $\mathbf{M}, \mathbf{M}'$!), so we have case (II) for $\mathbf{M}'$ too. We have four possibilities: **M** in a case ($\mathrm{II}_{m'}$) and $\mathbf{M}'$ in a case ($\mathrm{II}_{n'}$) with $m', n' = 1, \infty$.

Consider a pair $\alpha \in \Delta_0$, $\alpha' \in \Delta_0'$ with $\alpha' = \phi(\alpha)$ and a $p = 1, 2, \ldots$. Then $(\alpha/p) \in \Delta_0$, $(\alpha'/p) \in \Delta_0'$. Assume $(\alpha'/p) > \phi(\alpha/p)$. Obviously $p(\alpha/p) = \alpha \in \Delta_0$. Further $p\phi(\alpha/p) < \alpha' \in \Delta_0' \subset \Delta'$; as we have case (II) for $\mathbf{M}'$, this implies $p\phi(\alpha/p) \in \Delta'$. Thus Lemma 10.1.4 applies for $\alpha_1 = \cdots = \alpha_p = (\alpha/p)$, $\alpha_{p+1} = \alpha_{p+2} = \cdots = 0$, giving $p\phi(\alpha/p) = \phi(\alpha) = \alpha'$, $(\alpha'/p) = \phi(\alpha/p)$. This contradicts our assumption $(\alpha'/p) > \phi(\alpha/p)$. Interchanging of $\mathbf{M}, \mathbf{M}'$ excludes $(\alpha'/p) < \phi(\alpha/p)$ too. So we have proved $\alpha'/p = \phi(\alpha/p)$. In other words:

$$\phi(\alpha/p) = (1/p)\phi(\alpha).$$

Take now a $q = 0, 1, \ldots, p$. We have $(q/p)\alpha \leq \alpha \in \Delta_0 \subset \Delta$, so (owing to case (II) for **M** and for $\mathbf{M}'$) $(q/p)\alpha \in \Delta$, $q\phi(\alpha/p) \in \Delta'$. Therefore Lemma 10.1.4 applies for $\alpha_1 = \cdots = \alpha_q = \alpha/p$, $\alpha_{q+1} = \alpha_{q+2} = \cdots = 0$, giving $(q/p)\alpha \in \Delta_0$ and $\phi((q/p)\alpha) = q\phi(\alpha/p) = (q/p)\phi(\alpha)$. In other words: If $0 \leq \beta \leq \alpha$ then $\phi(\beta) = (\beta/\alpha)\phi(\alpha)$ if β/α is rational.

By Lemma 10.1.5 $0 \leq \beta \leq \alpha$ implies $\beta \in \Delta_0$ so $\phi(\beta)$ is defined in all this interval. As it is monotone increasing (Lemma 10.1.1), we have $\phi(\beta) = (\beta/\alpha)\phi(\alpha)$ for all these β, that is $\phi(\alpha)/\alpha = \phi(\beta)/\beta$ (assuming $\alpha, \beta \neq 0$).

Assume $\alpha, \beta \in \Delta_0$ and $\neq 0$. If $\alpha \geq \beta$ then $0 \leq \beta \leq \alpha$ and we know that $\phi(\alpha)/\alpha = \phi(\beta)/\beta$. By symmetry this equation holds for $\alpha \leq \beta$ too, that is, it holds always. So $\phi(\alpha)/\alpha$ is constant, say C. C is finite and positive by its nature. We have $\phi(\alpha) = C\alpha$ if $\alpha \in \Delta_0$, $\alpha \neq 0$ and of course for $\alpha = 0$ too.

Let the l.u.b. of Δ_0 be m'', and that of Δ_0', n''. We have $0 < m'' \leq m'$, $0 < n'' \leq n'$, and of course $n'' = Cm''$. m'' belongs to Δ_0: Choose $\alpha_1, \alpha_2, \ldots$ with $\alpha_i \geq 0$, $\sum_1^i \alpha_i < m''$, $\sum_1^\infty \alpha_i = m''$ (for instance $\alpha_i = m''/2^i$ if m'' is finite, and $\alpha_i = 1$ if m'' is infinite), and apply Lemma 10.1.4. Similarly n'' belongs to Δ_0'.

Assume now $m'' < m'$, $n'' < n'$. Then m'', n'' are both finite, and a $\vartheta > 1, < 2$ with $\vartheta m'' < m'$, $\vartheta n'' < n'$ exists. Then $(\vartheta/2)m'' < m'$ and so $(\vartheta/2)m'' \in \Delta_0$ while $2(\vartheta/2)m'' = \vartheta m'' \leq m'$ is in Δ, $2\phi((\vartheta/2)m'') = 2C(\vartheta/2)m'' = \vartheta n'' \leq n'$ is in Δ'. So Lemma 10.1.4 applies for $\alpha_1 = \alpha_2 = (\vartheta/2)m''$, $\alpha_3 = \alpha_4 = \cdots = 0$ giving $\vartheta m'' \in \Delta_0$. This is impossible, as $\vartheta m'' > m''$.

So we have $m'' = m'$ and $n'' \leq n'$ or $m'' \leq m'$ and $n'' = n'$, and besides $n'' = Cm''$. Let us now consider the four possible combinations of $m', n' = 1, \infty$.

If $m' = n' = 1$ then $C \geq 1$ necessitates $n'' = n' = 1$, $m'' = (1/C)n' = 1/C \leq 1$ so $\Delta_0' = \Delta'$, while $C \leq 1$ implies $m'' = m' = 1$, $n'' = Cm'' = C \leq 1$ so $\Delta_0 = \Delta$. As both Δ, Δ' (that is $D_{\mathbf{M}}(\mathfrak{M})$, $D_{\mathbf{M}'}(\mathfrak{M})$) are normalised, C has an invariant meaning. If $m' = 1$, $n' = \infty$ then Δ' is not normalised. So we can make $C = 1$. Now obviously $m'' = m' = 1$, $n'' = m'' = 1 < \infty$. So $\Delta_0 = \Delta$. If $m' = \infty$, $n' = 1$ then Δ is not normalised. So we can make $C = 1$ again. Now obviously $n'' = n' = 1$, $m'' = n'' = 1 < \infty$. So $\Delta_0' = \Delta'$. If finally $m' = n' = \infty$ then both Δ and Δ' are not normalised. So we can make $C = 1$ again, and then $m'' = n'' = \infty$ and thus $\Delta_0 = \Delta$, $\Delta_0' = \Delta'$.

All these results together with those of Lemmas 10.1.1 and 10.1.3 give the

Theorem X. In a factorisation $\mathbf{M}, \mathbf{M}'$ the factors $\mathbf{M}, \mathbf{M}'$ must both belong to class (I), or both to class (II), or both to class (III).

The function $\phi(\alpha)$ defined in Lemma 10.1.1 always has the form $\phi(\alpha) = C\alpha$ where C is a finite, positive constant. In case (I), using the standard normalisations of Δ, Δ' necessarily $C = 1$; in case (II), if both $\mathbf{M}, \mathbf{M}'$ are in case (II_1) and using the standard normalisations of Δ, Δ' the value of C is uniquely defined; in case (II), if at least one of $\mathbf{M}, \mathbf{M}'$ is in case (II_∞) and if we normalise the corresponding Δ or Δ' suitably, then we can make $C = 1$; in case (III) the value of C is arbitrary (as $\alpha = 0, \infty$) so we can take $C = 1$.

Always either $\Delta_0 = \Delta$ or $\Delta_0' = \Delta'$; details are given in the discussion above.

An f with $\mathfrak{M}_f^{\mathbf{M}'} = \mathfrak{M}$, $\mathfrak{M}_f^{\mathbf{M}} = \mathfrak{N}$ exists if and only if $\mathfrak{M}\ \eta\ \mathbf{M}$, $\mathfrak{N}\ \eta\ \mathbf{M}'$ and if $D_{\mathbf{M}}(\mathfrak{M}) \in \Delta_0$, $D_{\mathbf{M}'}(\mathfrak{N}) \in \Delta_0'$, $D_{\mathbf{M}'}(\mathfrak{N}) = C D_{\mathbf{M}}(\mathfrak{M})$. (Owing to the last condition each one of the two preceding conditions implies the other.)

This Theorem brings us nearer to the answers of Problems 4 and 7: It excludes certain combinations of cases for $\mathbf{M}, \mathbf{M}'$ and it shows that $\mathbf{M}, \mathbf{M}'$ possess a further invariant, if $\mathbf{M}, \mathbf{M}'$ are both of class (II_1), the number C. This latter statement is of course only then of value, if such $\mathbf{M}, \mathbf{M}'$ both of class (II_1), and with various values of C, do really exist. We will see that this is the case, and besides get further information on all these points in §§13.2 and 13.3.

Chapter XI: Composition and Decomposition of Factors

11.1. A natural question to ask is this:

Problem 8. Under what conditions can two given $\mathbf{M}, \mathbf{N}$ belong both to one factorisation $\mathbf{M}_1, \ldots, \mathbf{M}_n$?

The following Lemma gives a necessary and sufficient condition but thereby raises a new problem.

Lemma 11.1.1. $\mathbf{M}, \mathbf{N}$ belong both to one (suitably chosen) factorisation if and only if (i) $\mathbf{M}, \mathbf{N}$ are factors, (ii) $\mathbf{M}, \mathbf{N}$ commute (that is: $\mathbf{M} \subset \mathbf{N}'$), (iii) $\mathrm{R}(\mathbf{M}, \mathbf{N})$ is a factor.

Proof: If $\mathbf{M}, \mathbf{N}$ belong to a factorisation $\mathbf{M}_1, \ldots, \mathbf{M}_n$, then $\mathbf{M} = \mathbf{M}_i$, $\mathbf{N} = \mathbf{M}_j$, $i \neq j$. Thus (i) is fulfilled by Lemma 3.1.1; (ii) by definition; and (iii) by Lemma 3.1.1 again, as we may make $i = 1$, $j = 2$ by a permutation of $\mathbf{M}_1, \ldots, \mathbf{M}_n$ and then remember (cf. the remark at the end of §3.1) that $\mathrm{R}(\mathbf{M}_1, \mathbf{M}_2)$, $\mathrm{R}(\mathbf{M}_3, \ldots, \mathbf{M}_n)$ too is a factorisation if $n > 3$ (for $n = 2$, $\mathrm{R}(\mathbf{M}_1, \mathbf{M}_2) = \mathbf{B}$ is trivially a factor). Thus our condition is necessary.

Assume now that (i)–(iii) are fulfilled. $(\mathrm{R}(\mathbf{M}, \mathbf{N}))' = \mathbf{M}'\mathbf{N}'$ commutes with $\mathbf{M}$ and $\mathbf{N}$; $\mathbf{M}, \mathbf{N}$ commute with each other; and

$$\mathrm{R}(\mathbf{M}, \mathbf{N}, (\mathrm{R}(\mathbf{M}, \mathbf{N}))') = \mathrm{R}(\mathrm{R}(\mathbf{M}, \mathbf{N}), (\mathrm{R}(\mathbf{M}, \mathbf{N}))') = \mathbf{B}$$

as $\mathrm{R}(\mathbf{M}, \mathbf{N})$ is a factor. So $\mathbf{M}, \mathbf{N}, (\mathrm{R}(\mathbf{M}, \mathbf{N}))'$ is a factorisation and it contains $\mathbf{M}, \mathbf{N}$. Thus our condition is sufficient.

The new problem therefore is:

Problem 9. If $\mathbf{M}, \mathbf{N}$ are two factors which commute, under what conditions will then $\mathrm{R}(\mathbf{M}, \mathbf{N})$ be a factor? How does the class of $\mathrm{R}(\mathbf{M}, \mathbf{N})$ (in the sense of Theorem VIII) connect with the classes of $\mathbf{M}$ and $\mathbf{N}$?

We will only succeed in giving a partial answer. This answer could be obtained by using somewhat less formal machinery than we are going to use. But it seems reasonable to make use of it, because it makes the algebraic side of these questions clearer.

We define:

Definition 11.1.1. Let $\mathbf{M}, \mathbf{P}$ be rings which contain 1. If $\mathbf{M} \subset \mathbf{P}$ we define $\mathbf{M}'_{\mathbf{P}} = \mathbf{P} \cdot \mathbf{M}'$ ($\mathbf{M}'_{\mathbf{P}}$ too is a ring which contains 1 and $\subset \mathbf{P}$).

This notion shares many formal properties with $\mathbf{M}'$ with which it coincides for $\mathbf{P} = \mathbf{B}$. As the decisive property of $\mathbf{M}'$ is $\mathbf{M}'' = \mathbf{M}$ (cf. (18, p. 397)), it is reasonable to define:

Definition 11.1.2. $\mathbf{P}$ is normal, if $\mathbf{M} \subset \mathbf{P}$ implies $\mathbf{M}''_{\mathbf{P}} = \mathbf{M}$.

Lemma 11.1.2. If $\mathbf{P}$ is normal, then it is a factor; and every factorisation $\mathbf{Q}, \mathbf{P}'$ is a coupled one.

Proof: Put $\mathbf{M} = (\alpha 1)$. Then $\mathbf{M} \subset \mathbf{P}$ so $\mathbf{M}''_{\mathbf{PP}} = \mathbf{M}$; now $(\alpha 1)'_{\mathbf{P}} = \mathbf{P}(\alpha 1)' = \mathbf{PB} = \mathbf{P}$, $\mathbf{P}'_{\mathbf{P}} = \mathbf{P} \cdot \mathbf{P}'$. So we have $\mathbf{P} \cdot \mathbf{P}' = (\alpha 1)$. So $\mathbf{P}$ is a factor. That $\mathbf{Q}, \mathbf{P}'$ is a factorisation, means $\mathbf{Q} \subset \mathbf{P}'' = \mathbf{P}$ and $\mathrm{R}(\mathbf{Q}, \mathbf{P}') = \mathbf{B}$, that is $(\mathrm{R}(\mathbf{Q}, \mathbf{P}'))' = \mathbf{B}'$, $\mathbf{Q}'_{\mathbf{P}} = \mathbf{Q}'\mathbf{P} = (\mathrm{R}(\mathbf{Q}, \mathbf{P}'))' = \mathbf{B}' = (\alpha 1)$. So $\mathbf{Q} = \mathbf{Q}''_{\mathbf{PP}} = (\alpha 1)'_{\mathbf{P}} = \mathbf{P}$.

Our notion of normalcy could easily be expressed in terms of the conditions which characterise the "B-lattices" of G. Birkhoff (cf. (1, p. 445)) but we will not enter on this aspect of the subject at present.

The connection with Problem 9 is established by the following Lemma:

Lemma 11.1.3. The factors $\mathbf{M}, \mathbf{N}$ are commutative and $\mathrm{R}(\mathbf{M}, \mathbf{N})$ is a factor too (that is: the conditions of Lemma 11.1.1 are fulfilled), if a $\mathbf{P}$ with $\mathbf{M} \subset \mathbf{P}$, $\mathbf{N} \subset \mathbf{P}'$ exists, so that $\mathbf{P}, \mathbf{P}'$ are both normal.

This is in particular the case, if $\mathbf{M}, \mathbf{N}$ commute, and besides $\mathbf{M}, \mathbf{M}'$ or $\mathbf{N}, \mathbf{N}'$ are both normal.

Proof: $\mathbf{M} \subset \mathbf{P} \subset \mathbf{N}'$ shows that $\mathbf{M}, \mathbf{N}$ commute. We have

$$(\mathrm{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}}))'_{\mathbf{P}} = \mathbf{P} \cdot (\mathrm{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}}))' = \mathbf{PM}'(\mathbf{M}'_{\mathbf{P}})' = \mathbf{M}'\mathbf{M}''_{\mathbf{PP}}$$

and as $\mathbf{M} \subset \mathbf{P}$, $\mathbf{P}$ normal, this is $\mathbf{M}'\mathbf{M} = (\alpha 1)$. Therefore, considering $\mathrm{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}}) \subset \mathbf{P}$, $\mathbf{P}$ normal, we have

$$\mathrm{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}}) = (\mathrm{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}}))''_{\mathbf{PP}} = (\alpha 1)'_{\mathbf{P}} = \mathbf{P}.$$

Similarly $\mathrm{R}(\mathbf{N}, \mathbf{N}'_{\mathbf{P}'}) = \mathbf{P}'$. Therefore

$$\begin{aligned}\mathrm{R}(\mathrm{R}(\mathbf{M}, \mathbf{N}), \mathrm{R}(\mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'})) &= \mathrm{R}(\mathbf{M}, \mathbf{N}, \mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'}) \\ &= \mathrm{R}(\mathrm{R}(\mathbf{M}, \mathbf{M}'_{\mathbf{P}}), \mathrm{R}(\mathbf{N}, \mathbf{N}'_{\mathbf{P}'})) \\ &= \mathrm{R}(\mathbf{P}, \mathbf{P}') = \mathbf{B}. \qquad (\mathbf{P} \text{ is a factor!})\end{aligned}$$

Now $\mathbf{M}$ commutes with $\mathbf{M}'_{\mathbf{P}}$ because $\mathbf{M}'_{\mathbf{P}} \subset \mathbf{M}'$, and with $\mathbf{N}'_{\mathbf{P}'}$ because $\mathbf{M} \subset \mathbf{P}$, $\mathbf{N}'_{\mathbf{P}'} \subset \mathbf{P}'$. So $\mathbf{M}$ commutes with $\mathrm{R}(\mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'})$. Similarly $\mathbf{N}$ commutes with $\mathrm{R}(\mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'})$. Therefore $\mathrm{R}(\mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'})$ commutes with $\mathrm{R}(\mathbf{M}, \mathbf{N})$. Thus $\mathrm{R}(\mathbf{M}, \mathbf{N})$, $\mathrm{R}(\mathbf{M}'_{\mathbf{P}}, \mathbf{N}'_{\mathbf{P}'})$ is a factorisation, and so (by Lemma 3.1.1) $\mathrm{R}(\mathbf{M}, \mathbf{N})$ is a factor.

The second half of our Lemma obtains by putting $\mathbf{P} = \mathbf{M}$ respectively $\mathbf{P} = \mathbf{N}'$.

11.2. We derive some properties of normalcy.

Lemma 11.2.1. The operation $\mathbf{M}'_{\mathbf{P}}$ (for $\mathbf{M} \subset \mathbf{P}$) and the normalcy of $\mathbf{P}$ are invariants under (A^*, AB)-ring-isomorphisms of $\mathbf{P}$.

Proof: The first statement is obvious, the second follows immediately from the first one.

Lemma 11.2.2. If $\mathbf{P}$ is a factor of class (I), then $\mathbf{P}, \mathbf{P}'$ are both normal.

Proof: As $\mathbf{P}'$ is of class (I) too (by Lemmas 3.2.4 and 8.6.1, or by Theorem X), it suffices to consider $\mathbf{P}$ alone. Now Lemma 8.6.1 and Theorem II imply that $\mathbf{P}$ is algebraically (even fully) ring-isomorphic to the $\mathbf{B}$ of some space $\mathfrak{H}$. Therefore Lemma 11.2.1 permits us to assume $\mathbf{P} = \mathbf{B}$.

But then $\mathbf{M}'_{\mathbf{P}}$ becomes $\mathbf{M}'_{\mathbf{B}} = \mathbf{M}'$ and as $\mathbf{M}'' = \mathbf{M}$ (cf. (18, p. 397)) therefore $\mathbf{P} = \mathbf{B}$ is normal.

Note that Lemmas 11.1.2 and 11.2.2 give together the statement of Lemma 3.2.4 again: Every direct factorisation $\mathbf{M}, \mathbf{N}$ is coupled (let $\mathbf{M}, \mathbf{N}$ correspond to $\mathbf{Q}, \mathbf{P}'$ respectively).

As we will find non-coupled factorisations of class (II) (cf. §13.4), Lemma 11.1.2 implies that non-normal factors of class (II) do exist. We did not succeed however in showing that no factor of class (II) is normal. Nevertheless the only non-normal $\mathbf{P}$'s we know are of class (II).

Thus the following question is only partially answered:

Problem 10. Which factors are normal?

The second half of Problem 9 remains to be discussed: Determining the class of $\mathrm{R}(\mathbf{M}, \mathbf{N})$ in terms of the classes of $\mathbf{M}$ and $\mathbf{N}$. If $\mathbf{M}$ or $\mathbf{N}$ belongs to case (I) (by symmetry we may assume that $\mathbf{N}$ does), then the results of the next two §§ permit us to solve this problem. In all other cases our information is incomplete.

11.3. We will now consider an operation which is in a certain sense inverse to the "composition" R($\mathbf{M}, \mathbf{N}$). It is based on the following Lemma:

Lemma 11.3.1. Let $\mathbf{M}$ be a ring which contains 1, E a projection $\in \mathbf{M}$. If $A' \in \mathbf{M}'$ then form $EA' = A'E = A'_E$. Then these A'_E are identical with those (bounded) operators $\bar{A}$ for which $E\bar{A} = \bar{A}E = \bar{A}$ and which commute with all EBE, $B \in \mathbf{M}$.

Proof: The necessity is clear: If $A' \in \mathbf{M}'$ then A' commutes with $E \in \mathbf{M}$ so we can define

$$EA' = A'E = A'_E; \qquad EA'_E = EEA' = EA' = A'_E, \qquad A'_E E = A'EE = A'E = A'_E,$$

and if $B \in \mathbf{M}$ then

$$EBEA'_E = EBA'_E = EBEA' = A'EBE = A'_E BE = A'_E EBE.$$

Let us now consider the sufficiency. Assume therefore $E\bar{A} = \bar{A}E = \bar{A}$ and that $\bar{A}$ commutes with all EBE, $B \in \mathbf{M}$.

As $\bar{A}$ is bounded, there exists an $\alpha > 0$ so that $\alpha^2 \cdot 1 - \bar{A}^*\bar{A}$ is (semi-) definite, and then there exists a unique (semi-) definite $\bar{C}$ with $\bar{C}^2 = \alpha^2 \cdot 1 - \bar{A}^*\bar{A}$. (That is: $\bar{C} = (\alpha^2 \cdot 1 - \bar{A}^*\bar{A})^{1/2}$, cf. (21, p. 303).) The EBE, $B \in \mathbf{M}$ commute with $\bar{A}$, therefore with $\bar{A}^*$, $\bar{A}^*\bar{A}$ and $\bar{C}$ too. In particular $E = EEE$ does so.

Assume now $B_1, \ldots, B_i \in \mathbf{M}$, $f_1, \ldots, f_i$ arbitrary. We then form:

$$\begin{aligned}
&\left\| \sum_{j=1}^{i} B_j \bar{C} E f_j \right\|^2 + \left\| \sum_{j=1}^{i} B_j \bar{A} E f_j \right\|^2 \\
&= \left(\sum_{j=1}^{i} B_j \bar{C} E f_j, \sum_{k=1}^{i} B_k \bar{C} E f_k \right) + \left(\sum_{j=1}^{i} B_j \bar{A} E f_j, \sum_{k=1}^{i} B_k \bar{A} E f_k \right) \\
&= \left(\sum_{j,k=1}^{i} \{ E\bar{C}B_k^* B_j \bar{C} E + E\bar{A}^* B_k^* B_j \bar{A} E \} f_j, f_k \right) \\
&= \left(\sum_{j,k=1}^{i} \{ \bar{C}EB_k^* B_j E\bar{C} + \bar{A}^* EB_k^* B_j E\bar{A} \} f_j, f_k \right) \\
&= \left(\sum_{j,k=1}^{i} \{ EB_k^* B_j E(\bar{C}^2 + \bar{A}^*\bar{A}) \} f_j, f_k \right)
\end{aligned}$$

$$= \left(\sum_{j,k=1}^{i} \{EB_k^* B_j E(\alpha^2 1)\} f_j, f_k \right)$$

$$= \alpha^2 \left(\sum_{j=1}^{i} B_j E f_j, \sum_{k=1}^{i} B_k E f_k \right) = \alpha^2 \left\| \sum_{j=1}^{i} B_j E f_j \right\|^2,$$

and therefore

$$\left\| \sum_{j=1}^{i} B_j \bar{A} E f_j \right\| \leq \alpha \left\| \sum_{j=1}^{i} B_j E f_j \right\|.$$

Thus $\sum_{j=1}^{i} B_j E f_j = 0$ implies $\sum_{j=1}^{i} B_j \bar{A} E f_j = 0$ and so, owing to the linearity,

$$\bar{\bar{A}} \left(\sum_{j=1}^{i} B_j E f_j \right) = \sum_{j=1}^{i} B_j \bar{A} E f_j$$

($i = 0, 1, 2, \ldots$; $B_j \in \mathbf{M}$, f_j arbitrary for $j = 1, \ldots, i$) defines a one-valued linear operator $\bar{\bar{A}}$. We can now write $\|\bar{\bar{A}} f\| \leq \alpha \|f\|$ so that $\bar{\bar{A}}$ is continuous. Therefore we can form the closure of $\bar{\bar{A}}$, $\tilde{\bar{A}}$. This will be one-valued, again fulfilling $\|\tilde{\bar{A}} f\| \leq \alpha \|f\|$ and Domain $\tilde{\bar{A}} = [\text{Domain } \bar{\bar{A}}]$. (All this results by direct application of the criterium of (21, p. 300).)

If $B \in \mathbf{M}$ then

$$\bar{\bar{A}} \left(B \sum_{j=1}^{i} B_j E f_j \right) = \bar{\bar{A}} \left(\sum_{j=1}^{i} B B_j E f_j \right) = \sum_{j=1}^{i} (B B_j) \bar{A} E f_j$$

$$= B \left(\sum_{j=1}^{i} B_j \bar{A} E f_j \right) = B \left(\bar{\bar{A}} \left(\sum_{j=1}^{i} B_j E f_j \right) \right),$$

so $\bar{\bar{A}} \, \eta \, \mathbf{M}'$ (cf. Lemma 4.2.2). Therefore $\tilde{\bar{A}} \, \eta \, \mathbf{M}'$ and $\mathfrak{M}_0 = \text{Domain } \tilde{\bar{A}} \, \eta \, \mathbf{M}'$. Thus $F_0 = P_{\mathfrak{M}_0} \in \mathbf{M}'$. Note that Domain $\tilde{\bar{A}} \supset \text{Domain } \bar{\bar{A}} \supset \mathfrak{M}$, if $E = P_{\mathfrak{M}}$ (put $i = 1$, $B_1 = 1$), so $F_0 \geq E$.

Our definition of F_0 shows, that $\tilde{\bar{A}} F_0$ is everywhere defined. It is bounded too, and as $\tilde{\bar{A}} \, \eta \, \mathbf{M}'$, $F_0 \in \mathbf{M}'$, therefore $\tilde{\bar{A}} F_0 \, \eta \, \mathbf{M}'$. These facts imply together $\tilde{\bar{A}} F_0 \in \mathbf{M}'$. Put $A' = \tilde{\bar{A}} F_0$.

Now $\widetilde{\bar{A}}Ef = \bar{A}Ef$ (put $i = 1$, $B_1 = 1$ in the definition), and so $\widetilde{\bar{A}}E = \bar{A}E = \bar{A}$. Therefore $A'E = \widetilde{\bar{A}}F_0E = \widetilde{\bar{A}}E = \bar{A}$. In other words: $EA' = A'E = A'_E = \bar{A}$, completing the proof.

We can now carry out the constructions which are necessary for our "decomposition."

Definition 11.3.1. Let $\mathbf{M}$ be a ring which contains 1 and $\mathfrak{M}$ a closed linear set $\neq (0)$. Consider those operators $A \in \mathbf{M}$ which are reduced by $\mathfrak{M}$ and form their parts in $\mathfrak{M}$, $A_{(\mathfrak{M})}$ (cf. (16, p. 78)). These are bounded operators in $\mathfrak{M}$. Denote the set of all these $A_{(\mathfrak{M})}$ ($A \in \mathbf{M}$ and reduced by $\mathfrak{M}$) by $\mathbf{M}_{(\mathfrak{M})}$.

Lemma 11.3.2. Let $\mathbf{M}$ be a ring which contains 1 and $\mathfrak{M}$ a closed linear set $\neq (0)$, $\mathfrak{M}\ \eta\ \mathbf{M}$, $E = P_{\mathfrak{M}}$. Then $(\mathbf{M}_{(\mathfrak{M})})' = \mathbf{M}'_{(\mathfrak{M})}$. (These are rings of operators in the space $\mathfrak{M}$.)

Proof: An operator A_0 in $\mathfrak{M}$ is at the same time an operator in $\mathfrak{H}$ but its domain is $\subset \mathfrak{M}$. If A_0 is bounded, then A_0E is everywhere defined in $\mathfrak{H}$ and so (being bounded) $\in \mathbf{B}$. As its range is $\subset \mathfrak{M}$, $E(A_0E) = A_0E$. Obviously $(A_0E)E = A_0E$ so A_0E commutes with $E = P_{\mathfrak{M}}$ and is reduced by $\mathfrak{M}$. Its parts in $\mathfrak{M}$, $\mathfrak{H} - \mathfrak{M}$ are $A_0, 0$ respectively.

Thus A_0, B_0 (in $\mathfrak{M}$) commute if and only if A_0E, B_0E (in $\mathfrak{H}$) commute. $A_0 \in (\mathbf{M}_{(\mathfrak{M})})'$ means therefore, that A_0E commutes with every B_0E, $B_0 \in \mathbf{M}_{(\mathfrak{M})}$.

The statement on B_0 means $B_0 = B_{(\mathfrak{M})}$, $B \in \mathbf{M}$ but then

$$EBE = EB = BE = B_0E.$$

(Note, that B_0E differs in general from $EB_0 = B_0$, their domains being $\mathfrak{H}$ respectively $\mathfrak{M}$.) So A_0E must commute with every EBE where $B \in \mathbf{M}$, B commutes with E. Replacing B by EBE (which commutes with E and leaves EBE unaltered) shows that any $B \in \mathbf{M}$ will do. Now Lemma 11.3.1 applies to A_0E: $A_0E = EA' = A'E$ for some $A' \in \mathbf{M}'$. This means, $A_0 = (A_0E)_{(\mathfrak{M})} = A'_{(\mathfrak{M})}$, $A_0 \in \mathbf{M}'_{(\mathfrak{M})}$.

Conversely: If $A_0 \in \mathbf{M}'_{(\mathfrak{M})}$, $B_0 \in \mathbf{M}_{(\mathfrak{M})}$ then $A_0 = A'_{(\mathfrak{M})}$, $B_0 = B_{(\mathfrak{M})}$ where $A' \in \mathbf{M}'$, $B \in \mathbf{M}$ so A', B commute, and A_0, B_0 too. So $A_0 \in \mathbf{M}'_{(\mathfrak{M})}$ implies $A_0 \in (\mathbf{M}_{(\mathfrak{M})})'$. So we have completely proved $(\mathbf{M}_{(\mathfrak{M})})' = \mathbf{M}'_{(\mathfrak{M})}$.

Lemma 11.3.3. Let $\mathbf{M}, \mathfrak{M}, E$ be as above. Consider the correspondence $X \longleftrightarrow X_{(\mathfrak{M})}$. This is a one-to-one mapping and even algebraic ring-isomorphism of the following rings on each other:

(i) At any rate of the ring of all $A \in \mathbf{M}$ with $EA = AE = A$ on all $\mathbf{M}_{(\mathfrak{M})}$.

(ii) If $\mathbf{M}$ is a factor, of all $\mathbf{M}'$ on all $\mathbf{M}'_{(\mathfrak{M})}$.

Proof: The invariance of $\alpha A, A^*, A+B, AB$ is obvious, only the one to one character must therefore be proved.

Ad (i): If $A_0 \in \mathbf{M}_{(\mathfrak{M})}$, form the A_0E from the proof of Lemma 11.3.2. If $A_0 = A_{(\mathfrak{M})}$, $A \in \mathbf{M}$, then as we saw, $A_0E = EAE$, so $A_0E \in \mathbf{M}$ too; and besides $A_0 = (A_0E)_{(\mathfrak{M})}$, $E(A_0E) = (A_0E)E = A_0E$. Furthermore A_0 and A_0E determine each other uniquely. This completes the proof.

Ad (ii): Every $A' \in \mathbf{M}'$ commutes with $E = P_{\mathfrak{M}} \in \mathbf{M}$ so it is reduced by $\mathfrak{M}$ and $A'_{(\mathfrak{M})}$ can be formed. $A'_{(\mathfrak{M})} = B'_{(\mathfrak{M})}$ means that $A'f = B'f$ for all $f \in \mathfrak{M}$, that is for all $f = Eg$ so $A'E = B'E$. By the corollary to Theorem III this implies $A' = B'$ as $E \neq 0$. This completes the proof.

Lemma 11.3.4. Let $\mathbf{M}, \mathfrak{M}, E$ be as above. If $\mathbf{M}$ is a factor then $\mathbf{M}_{(\mathfrak{M})}$ is one too.

Proof: Apply Lemma 11.3.2:

$$\mathbf{M}_{(\mathfrak{M})} \cdot (\mathbf{M}_{(\mathfrak{M})})' = \mathbf{M}_{(\mathfrak{M})} \cdot \mathbf{M}'_{(\mathfrak{M})} = (\mathbf{M} \cdot \mathbf{M}')_{(\mathfrak{M})} = (\alpha 1)_{(\mathfrak{M})} = (\alpha 1_{(\mathfrak{M})})$$

and $1_{(\mathfrak{M})}$ is the unity operator in $\mathfrak{M}$.

Lemma 11.3.5. Let $\mathbf{M}$ be a factor, $\mathfrak{M}, E$ as above. Then we have:

(i) If F runs over all projections $F \in \mathbf{M}$, $F \leq E$ (cf. (16, p. 76)) then $F_{(\mathfrak{M})}$ runs over all projections $\in \mathbf{M}_{(\mathfrak{M})}$. $F_{(\mathfrak{M})} \sim G_{(\mathfrak{M})}(\cdots \mathbf{M}_{(\mathfrak{M})})$, $(F, G \in \mathbf{M}, \leq E)$, is equivalent to $F \sim G\ (\cdots \mathbf{M})$.

(ii) If F' runs over all projections $F' \in \mathbf{M}'$ then $F'_{(\mathfrak{M})}$ runs over all projections $\in \mathbf{M}'_{(\mathfrak{M})}$. $F'_{(\mathfrak{M})} \sim G'_{(\mathfrak{M})}(\cdots \mathbf{M}'_{(\mathfrak{M})})$, $(F', G' \in \mathbf{M}')$, is equivalent to $F' \sim G'\ (\cdots \mathbf{M}')$.

Proof: Ad (i): The first part is a restatement of Lemma 11.3.3, (i) as $EF = FE = F$ means $F \leq E$ (cf. (16, p. 76)). As to the second part, note that $F \sim G\ (\cdots \mathbf{M})$ means $F = U^*U$, $G = UU^*$ for some $U \in \mathbf{M}$ (cf. Lemma 4.3.1 and Definition 6.1.1) and $F_{(\mathfrak{M})} \sim G_{(\mathfrak{M})}(\cdots \mathbf{M}_{(\mathfrak{M})})$ means similarly

$$F_{(\mathfrak{M})} = (U_{(\mathfrak{M})})^* U_{(\mathfrak{M})}, \qquad G_{(\mathfrak{M})} = U_{(\mathfrak{M})}(U_{(\mathfrak{M})})^*$$

(cf. as above) for some $U \in \mathbf{M}$, $EU = UE = U$ (cf. Lemma 11.3.3, (i)). The latter equation means again $F = U^*U$, $G = UU^*$ (by Lemma 11.3.3, (i), owing to the algebraic ring-isomorphism), so that all we need to prove is that the U of the first case fulfill automatically $EU = UE = U$. As the initial and final sets of that U have the projections $F, G \leq E$ they are both $\subset \mathfrak{M}$ and this implies $EU = UE = U$.

Ad (ii): This follows immediately from Lemma 8.5.1 as $\mathbf{M}'$ and $\mathbf{M}'_{(\mathfrak{M})}$ are algebraically ring-isomorphic.

Lemma 11.3.6. Let $\mathbf{M}$ be a factor, $\mathfrak{M}, E$ as above. Let $D_{\mathbf{M}}(F)$, $D_{\mathbf{M}'}(F')$ ($F \in \mathbf{M}$ respectively $F' \in \mathbf{M}'$) be relative dimension functions in $\mathbf{M}$ respectively $\mathbf{M}'$. Then

$$D_{\mathbf{M}}^{(\mathfrak{M})}(F_{(\mathfrak{M})}) = D_{\mathbf{M}}(F) \quad \text{and} \quad D_{\mathbf{M}'}^{(\mathfrak{M})}(F'_{(\mathfrak{M})}) = D_{\mathbf{M}'}(F')$$

are relative dimension functions in $\mathbf{M}_{(\mathfrak{M})}$ respectively $\mathbf{M}'_{(\mathfrak{M})}$.
Proof: Owing to Definition 9.1.1 this follows immediately from Lemma 11.3.5.

Lemma 11.3.7. Let $\mathbf{M}$ be a factor, $\mathfrak{M}, E$ as above. Let

$$D_{\mathbf{M}}(F), \quad D_{\mathbf{M}'}(F'), \quad D_{\mathbf{M}}^{(\mathfrak{M})}(F), \quad D_{\mathbf{M}'}^{(\mathfrak{M})}(F')$$

be the relative dimension functions of Lemma 11.3.6 and $\Delta, \Delta', \Delta_{(\mathfrak{M})}, \Delta'_{(\mathfrak{M})}$ their respective ranges. Then $\Delta_{(\mathfrak{M})}$ is the set of all $\alpha \in \Delta$, $\alpha \leq D_{\mathbf{M}}(E)$ while $\Delta'_{(\mathfrak{M})} = \Delta'$.
Proof: $\Delta_{(\mathfrak{M})}$ is by Lemma 11.3.5, (i) and Lemma 11.3.6, the set of all $D_{\mathbf{M}}(F)$, $F \leq E$, that is the set of all $D_{\mathbf{M}}(\mathfrak{N})$, $\mathfrak{N} \subset \mathfrak{M}$. Replacing these $\mathfrak{N}$ by all $\mathfrak{N} \sim \mathfrak{N}' \subset \mathfrak{M}$, that is by all $\mathfrak{N} \precsim \mathfrak{M}$, obviously does not affect the set of their $D_{\mathbf{M}}(\mathfrak{N})$. In other words: we have all $D_{\mathbf{M}}(\mathfrak{N})$ with $D_{\mathbf{M}}(\mathfrak{N}) \leq D_{\mathbf{M}}(\mathfrak{M})$, that is all $\alpha \in \Delta$ with $\alpha \leq D_{\mathbf{M}}(\mathfrak{M}) = D_{\mathbf{M}}(E)$.
$\Delta'_{(\mathfrak{M})} = \Delta'$ follows immediately from Lemma 11.3.5, (ii), and Lemma 11.3.6.

11.4. The treatment of §11.3 was asymmetric, insofar as the $\mathfrak{M}$ occurring in it was assumed to be $\eta\mathbf{M}$. We will now obtain symmetric results by considering simultaneously an $\mathfrak{M}\ \eta\ \mathbf{M}$ and an $\mathfrak{M}'\ \eta\ \mathbf{M}'$ and applying the preceding Lemmas twice.

Lemma 11.4.1. Let $\mathbf{M}$ be a factor; $\mathfrak{M}, \mathfrak{M}'$ closed linear sets both $\neq (0)$; $\mathfrak{M}\ \eta\ \mathbf{M}$, $\mathfrak{M}'\ \eta\ \mathbf{M}'$, $E = P_{\mathfrak{M}}$, $E' = P_{\mathfrak{M}'}$. Perform the operations of Definition 11.3.1 for the closed linear set $\mathfrak{M} \cdot \mathfrak{M}'$. The correspondence $X \longleftrightarrow X_{(\mathfrak{M}\cdot\mathfrak{M}')}$ is then a one-to-one mapping and even an algebraic ring-isomorphism of the following rings on each other:

(i) Of the ring of all $A \in \mathbf{M}$ with $EA = AE = A$ on all $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$.

(ii) Of the ring of all $A' \in \mathbf{M}'$ with $E'A' = A'E' = A'$ on all $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$.

Besides we have

(iii) $(\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')})' = \mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$, $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$ being a factor.

(These are rings of operators in the space $\mathfrak{M} \cdot \mathfrak{M}' \neq (0)$.)

Proof: As $E \in \mathbf{M}$, $E' \in \mathbf{M}'$ therefore E, E' commute, and so $EE' = P_{\mathfrak{M}\cdot\mathfrak{M}'}$. The corollary to Theorem III gives, owing to $E, E' \neq 0$ that $EE' \neq 0$, $\mathfrak{M} \cdot \mathfrak{M}' \neq (0)$. As $E' \in \mathbf{M}'$ therefore $E'_{(\mathfrak{M})} \in \mathbf{M}'_{(\mathfrak{M})}$ but $E'_{(\mathfrak{M})} = EE' = P_{\mathfrak{M}\cdot\mathfrak{M}'}$, so $\mathfrak{M} \cdot \mathfrak{M}'\ \eta\ \mathbf{M}_{(\mathfrak{M})}$.

Now (i) results by applying first Lemma 11.3.3 (i) to $\mathbf{M}, \mathbf{M}', \mathfrak{M}$ and then Lemma 11.3.3 (ii) to $\mathbf{M}'_{(\mathfrak{M})}, \mathbf{M}_{(\mathfrak{M})}, \mathfrak{M} \cdot \mathfrak{M}'$. (ii) follows from (i) by interchanging $\mathbf{M}, \mathfrak{M}$ with $\mathbf{M}', \mathfrak{M}'$. (iii) follows by applying Lemmas 11.3.2 and 11.3.4 first to $\mathbf{M}, \mathbf{M}', \mathfrak{M}$ and then to $\mathbf{M}'_{(\mathfrak{M})}, \mathbf{M}_{(\mathfrak{M})}, \mathfrak{M} \cdot \mathfrak{M}'$.

Lemma 11.4.2. Let $\mathbf{M}, \mathfrak{M}, \mathfrak{M}', E, E'$ be as above. Then we have:

(i) If F runs over all projections $F \in \mathbf{M}$, $F \leq E$ then $F_{(\mathfrak{M}\cdot\mathfrak{M}')}$ runs over all projections $\in \mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$. If $D_{\mathbf{M}}(F)$ is a relative dimension function in $\mathbf{M}$ then

$$D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')}) = D_{\mathbf{M}}(F)$$

is one in $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$. Denote their ranges by Δ respectively $\Delta_{(\mathfrak{M}\cdot\mathfrak{M}')}$, then $\Delta_{(\mathfrak{M}\cdot\mathfrak{M}')}$ is the set of all $\alpha \in \Delta$, $\alpha \leq D_{\mathbf{M}}(E)$.

(ii) If F' runs over all projections $F' \in \mathbf{M}'$, $F' \leq E'$, then $F'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ runs over all projections $\in \mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$. If $D_{\mathbf{M}'}(F')$ is a relative dimension function in $\mathbf{M}'$, then

$$D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F'_{(\mathfrak{M}\cdot\mathfrak{M}')}) = D_{\mathbf{M}'}(F')$$

is one in $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$. Denote their ranges by Δ' respectively $\Delta'_{(\mathfrak{M}\cdot\mathfrak{M}')}$, then $\Delta'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ is the set of all $\alpha' \in \Delta'$, $\alpha' \leq D_{\mathbf{M}'}(E')$.

Proof: (i) results by applying Lemmas 11.3.5 (i) respectively (ii), 11.3.6 and 11.3.7 first to $\mathbf{M}, \mathbf{M}', \mathfrak{M}$ and then to $\mathbf{M}'_{(\mathfrak{M})}, \mathbf{M}_{(\mathfrak{M})}, \mathfrak{M} \cdot \mathfrak{M}'$. (ii) follows from (i) by interchanging $\mathbf{M}, \mathfrak{M}$ with $\mathbf{M}', \mathfrak{M}'$.

We see that the coupled factorisation $\mathbf{M}, \mathbf{M}'$ in $\mathfrak{H}$ generates a coupled factorisation $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}, \mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ in $\mathfrak{M} \cdot \mathfrak{M}' \neq (0)$. Lemma 11.4.2 gives us the means to determine the classes of the latter factors in terms of those of the former ones.

Lemma 11.4.3. Let $\mathbf{M}, \mathbf{M}', \mathfrak{M}, \mathfrak{M}', E, E'$ be as above. Then $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}, \mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ belong to the same ones of the classes (I), (II), (III), as $\mathbf{M}, \mathbf{M}'$ (cf. Theorem X). In particular:

(i) If $\mathbf{M}, \mathbf{M}'$ are in class (I), and if $D_{\mathbf{M}}(F)$, $D_{\mathbf{M}'}(F')$ are given in their standard normalisations, that the same is true for $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$, $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F'_{(\mathfrak{M}\cdot\mathfrak{M}')})$. We have class ($\mathrm{I}_n$) or ($\mathrm{I}_\infty$) for $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$ (respectively $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$) corresponding to whether $\mathfrak{M}$ (respectively $\mathfrak{M}'$) is finite-n-dimensional or infinite-dimensional.

(ii) If $\mathbf{M}, \mathbf{M}'$ are in class (II), then $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$ (respectively $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$) are in class ($\mathrm{II}_1$) or ($\mathrm{II}_\infty$) corresponding to whether $\mathfrak{M}$ (respectively $\mathfrak{M}'$) is finite or infinite. So if $\mathfrak{M}, \mathfrak{M}'$ are both finite, the C of Theorem X (referred to the standard normalisations) has an invariant meaning. If we then vary $\mathfrak{M}, \mathfrak{M}'$ then C will be proportional to $D_{\mathbf{M}}(\mathfrak{M})/D_{\mathbf{M}'}(\mathfrak{M}')$.

(iii) If $\mathbf{M}, \mathbf{M}'$ are in class (III), no further comment is necessary.

Proof: Let first $\mathbf{M}, \mathbf{M}'$ be in class (I). Let $D_{\mathbf{M}}(F)$, $D_{\mathbf{M}'}(F')$ be in standard normalisation. Put $D_{\mathbf{M}}(E) = m$, $D_{\mathbf{M}'}(E') = n$ then $m, n > 0$, that is $m, n = 1, 2, \ldots, \infty$. So the ranges of $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$ and $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F'_{(\mathfrak{M}\cdot\mathfrak{M}')})$ are $0, 1, \ldots, m$ respectively $0, 1, \ldots, n$. Thus $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$, $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ belong to class (I), and (i) is true.

Let second, $\mathbf{M}, \mathbf{M}'$ be in class (II). Choose $D_{\mathbf{M}}(F)$ and then $D_{\mathbf{M}'}(F')$ in any normalisations. Put $D_{\mathbf{M}}(E) = \alpha_0$, $D_{\mathbf{M}'}(E') = \alpha'_0$, then $\alpha_0, \alpha'_0 > 0$ and the ranges of $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$ and $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F'_{(\mathfrak{M}\cdot\mathfrak{M}')})$ consist of all α with $0 \leq \alpha \leq \alpha_0$ respectively of all α' with $0 \leq \alpha' \leq \alpha'_0$. Thus $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$, $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ belong to class (II), and all parts of (ii), except those referring to C are proved.

Concerning C (cf. Theorem X) observe this: $\mathfrak{M}_f^{\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}}$ for $f \in \mathfrak{M}\cdot\mathfrak{M}'$ is

$$[A_{(\mathfrak{M}\cdot\mathfrak{M}')}f; A \in \mathbf{M}],$$

that is $[AEE'f; A \in \mathbf{M}, A$ commutes with $E]$. Then we may write as well $EAEE'f$ and replacing A by EAE (this is $\in \mathbf{M}$, commutes with E, and leaves $EAEE'f$ unaltered), we can admit all $A \in \mathbf{M}$. Now $EAEE'f = EAf$ as $f \in \mathfrak{M}\cdot\mathfrak{M}'$ so we have

$$[EAf; A \in \mathbf{M}] = [Eg; g \in \mathfrak{M}_f^{\mathbf{M}}] = [EE_f^{\mathbf{M}}h; h \in \mathfrak{H}].$$

But $E \in \mathbf{M}$, $E_f^{\mathbf{M}} \in \mathbf{M}'$ commute, so $E \cdot E_f^{\mathbf{M}} = P_{\mathfrak{M}\cdot\mathfrak{M}_f^{\mathbf{M}}}$; therefore

$$\mathfrak{M}_f^{\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}} = \mathfrak{M}\cdot\mathfrak{M}_f^{\mathbf{M}}, \qquad E_f^{\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}} = (E_f^{\mathbf{M}})_{(\mathfrak{M}\cdot\mathfrak{M}')}.$$

Besides $f \in \mathfrak{M}' \,\eta\, \mathbf{M}'$ implies $\mathfrak{M}_f^{\mathbf{M}} \subset \mathfrak{M}'$, $\mathfrak{M}\cdot\mathfrak{M}_f^{\mathbf{M}} \subset \mathfrak{M}\cdot\mathfrak{M}'$; therefore we may even write

$$E_f^{\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}} = (E_f^{\mathbf{M}})_{(\mathfrak{M}\cdot\mathfrak{M}')}.$$

Similarly

$$E_f^{(\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')})'} = (E_f^{\mathbf{M}'})_{(\mathfrak{M}\cdot\mathfrak{M}')}.$$

This proves that $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$, $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F'_{(\mathfrak{M}\cdot\mathfrak{M}')})$ have the same C as $D_{\mathbf{M}}(F)$, $D_{\mathbf{M}'}(F')$. If $\mathfrak{M}, \mathfrak{M}'$ are finite, we must pass to the standard normalisations, that is divide by

$$D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(1_{(\mathfrak{M}\cdot\mathfrak{M}')}) = D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(E_{(\mathfrak{M}\cdot\mathfrak{M}')}) = D_{\mathbf{M}}(E) = D_{\mathbf{M}}(\mathfrak{M}),$$
$$D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(1_{(\mathfrak{M}\cdot\mathfrak{M}')}) = D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(E'_{(\mathfrak{M}\cdot\mathfrak{M}')}) = D_{\mathbf{M}'}(E') = D_{\mathbf{M}'}(\mathfrak{M}').$$

This multiplies C by $D_{\mathbf{M}}(\mathfrak{M})/D_{\mathbf{M}'}(\mathfrak{M}')$, proving the rest of (ii).

Let finally $\mathbf{M}, \mathbf{M}'$ be in class (III). Then $D_{\mathbf{M}}(F)$, $D_{\mathbf{M}'}(F')$ have both the range $0, \infty$. The ranges of $D_{\mathbf{M}}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F_{(\mathfrak{M}\cdot\mathfrak{M}')})$, $D_{\mathbf{M}'}^{(\mathfrak{M}\cdot\mathfrak{M}')}(F'_{(\mathfrak{M}\cdot\mathfrak{M}')})$ must therefore be 0 alone or $0, \infty$. The former is impossible (because $\mathfrak{M}\cdot\mathfrak{M}' \neq (0)$), so we have $0, \infty$ again, that is, $\mathbf{M}_{(\mathfrak{M}\cdot\mathfrak{M}')}$, $\mathbf{M}'_{(\mathfrak{M}\cdot\mathfrak{M}')}$ are in class (III). (iii) is void.

Lemma 11.4.3 shows that if case (II) occurs at all, then combinations $\mathbf{M}, \mathbf{M}'$ of two cases (II_1) with an arbitrarily prescribed C (cf. Theorem X) exist. In fact: Choose $\mathbf{M}$ of class (II), then $\mathbf{M}'$ is of class (II) too. Choose $D_{\mathbf{M}}(\mathfrak{M})$, $D_{\mathbf{M}'}(\mathfrak{N})$ in such a normalisation that $D_{\mathbf{M}}(\mathfrak{H}) \geq 1$, $D_{\mathbf{M}'}(\mathfrak{H}) \geq 1$. Then $\mathfrak{M}\ \eta\ \mathbf{M}$, $\mathfrak{M}'\ \eta\ \mathbf{M}'$ can be chosen with arbitrarily prescribed $D_{\mathbf{M}}(\mathfrak{M}) = \alpha_0$, $D_{\mathbf{M}'}(\mathfrak{M}') = \alpha_0'$ if only $0 \leq \alpha_0, \alpha_0' \leq 1$. Thus α_0/α_0' can be prescribed arbitrarily, and with it C. Thus the invariant C exists effectively, as discussed at the end of §10.2, if case (II) exists at all. That this is the case will be seen in Chapter XIII.

11.5. We now return to the question formulated at the end of §11.2. We will determine the class of $\mathrm{R}(\mathbf{M}, \mathbf{N})$ if $\mathbf{M}, \mathbf{N}$ are two factors which commute, $\mathbf{N}$ being of class (I).

Lemma 11.5.1. Let $\mathbf{M}, \mathbf{N}$ be two factors which commute, form the factor $\mathrm{R}(\mathbf{M}, \mathbf{N})$. Let φ be minimal with respect to $\mathbf{N}$ (cf. Definition 5.1.3). Then $\mathfrak{M}_\varphi^{\mathbf{N}'}\ \eta\ \mathbf{N}$ and so $\eta \mathrm{R}(\mathbf{M}, \mathbf{N})$, and

$$X \longleftrightarrow X_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$$

is an algebraic ring-isomorphism of $\mathbf{M}$ and $(\mathrm{R}(\mathbf{M}, \mathbf{N}))_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$.

Proof: $\mathfrak{M}_\varphi^{\mathbf{N}'} \eta \mathbf{N}$ is obvious; as $\mathbf{N} \subset \mathrm{R}(\mathbf{M}, \mathbf{N})$, $\mathbf{N}' \supset (\mathrm{R}(\mathbf{M}, \mathbf{N}))'$ it implies $\mathfrak{M}_\varphi^{\mathbf{N}'} \eta \mathrm{R}(\mathbf{M}, \mathbf{N})$. So $E_\varphi^{\mathbf{N}'} \in \mathbf{N}$ and $\mathrm{R}(\mathbf{M}, \mathbf{N})$; thus it commutes with every $X \in \mathbf{M}$.

$X \longleftrightarrow X_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$ is an algebraic ring-isomorphism if it is a one to one mapping; besides it is a one to one mapping of all $Y \in \mathrm{R}(\mathbf{M}, \mathbf{N})$ with $E_\varphi^{\mathbf{N}'} Y = Y E_\varphi^{\mathbf{N}'} = Y$ on $(\mathrm{R}(\mathbf{M}, \mathbf{N}))_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$ (by Lemma 11.3.3, (i)). So we must only prove this: The correspondence

$$X_{(\mathfrak{M}_\varphi^{\mathbf{N}'})} = Y_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$$

generates a one to one mapping of all $X \in \mathbf{M}$ on all $Y \in \mathrm{R}(\mathbf{M}, \mathbf{N})$ with $E_\varphi^{\mathbf{N}'} Y = Y E_\varphi^{\mathbf{N}'} = Y$. If $X \in \mathbf{M}$ is given, then $Y = E_\varphi^{\mathbf{N}'} X$ has the above properties, and clearly $Y_{(\mathfrak{M}_\varphi^{\mathbf{N}'})} = X_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$. Besides if X is given, our correspondence determines Y uniquely, because such a Y is uniquely determined by its $Y_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$, just because the correspondence $Y \longleftrightarrow Y_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$, $Y \in \mathrm{R}(\mathbf{M},\mathbf{N})$, $E_\varphi^{\mathbf{N}'}Y = YE_\varphi^{\mathbf{N}'} = Y$ is one-to-one (by Lemma 11.3.3, (i)). So $X_{(\mathfrak{M}_\varphi^{\mathbf{N}'})} = Y_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$ is a one-to-one mapping of all $X \in \mathbf{M}$ on a part of the $Y \in \mathrm{R}(\mathbf{M},\mathbf{N})$, $E_\varphi^{\mathbf{N}'}Y = YE_\varphi^{\mathbf{N}'} = Y$. All we must prove now is that all these Y correspond to some X, that is, that they all have the form $Y = E_\varphi^{\mathbf{N}'}X$, $X \in \mathbf{M}$.

We may as well prove: If $Y \in \mathrm{R}(\mathbf{M},\mathbf{N})$ then $E_\varphi^{\mathbf{N}'}YE_\varphi^{\mathbf{N}'} = E_\varphi^{\mathbf{N}'}X$ for some $X \in \mathbf{M}$. Let $\mathbf{P}$ be the set of those Y_0 for which this holds for all $Y = AY_0B$, $A, B \in \mathbf{N}$. That is: For which these $E_\varphi^{\mathbf{N}'}AY_0BE_\varphi^{\mathbf{N}'} \in \mathbf{M}_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$, cf. Lemma 11.3.3 (i). (Observe $E_\varphi^{\mathbf{N}'} \in \mathbf{N} \subset \mathbf{M}'$.) Clearly $\mathbf{P}$ is weakly closed (along with $\mathbf{M}_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$) and $Y_1, Y_2 \in \mathbf{P}$ imply $\alpha Y_1, Y_1^*, Y_1 + Y_2 \in \mathbf{P}$ and $Y_1Y_2 \in \mathbf{P}$, the last one because

$$\begin{aligned}
&\sum_{p=1}^{\infty} E_\varphi^{\mathbf{N}'}AY_1U_{1,p}E_\varphi^{\mathbf{N}'} \cdot E_\varphi^{\mathbf{N}'}U_{p,1}Y_2BE_\varphi^{\mathbf{N}'} \\
&\qquad = \sum_{p=1}^{\infty} E_\varphi^{\mathbf{N}'}AY_1U_{1,p}P_{[f_{1,1},f_{1,2},\ldots]}U_{p,1}Y_2BE_\varphi^{\mathbf{N}'} \\
&\qquad = \sum_{p=1}^{\infty} E_\varphi^{\mathbf{N}'}AY_1P_{[f_{p,1},f_{p,2},\ldots]}Y_2BE_\varphi^{\mathbf{N}'} = E_\varphi^{\mathbf{N}'}AY_1Y_2BE_\varphi^{\mathbf{N}'}.
\end{aligned}$$

(Choose the $U_{m,p}$, $f_{m,p}$ by Lemma 5.3.6 for $\mathbf{N}$ with $f_{1,1} = \varphi$.) So $\mathbf{P}$ is a ring.

If $Y_0 = XZ$, $X \in \mathbf{M}$, $Z \in \mathbf{N}$ then for $A, B \in \mathbf{N}$

$$E_\varphi^{\mathbf{N}'}AXZBE_\varphi^{\mathbf{N}'} = E_\varphi^{\mathbf{N}'}AZBE_\varphi^{\mathbf{N}'} \cdot X.$$

As $E_\varphi^{\mathbf{N}'}$ is minimal with respect to $\mathbf{N}$, the last part of the proof of Lemma 5.1.3 applies:

$$E_\varphi^{\mathbf{N}'}AZBE_\varphi^{\mathbf{N}'} = \alpha E_\varphi^{\mathbf{N}'}$$

(replace there $\mathbf{M}, E = E_f^{\mathbf{M}'}, B$ by $\mathbf{N}, E_\varphi^{\mathbf{N}'}, E_\varphi^{\mathbf{N}'}AZBE_\varphi^{\mathbf{N}'}$). Thus the above expression is $= \alpha E_\varphi^{\mathbf{N}'}X \in \mathbf{M}_{(\mathfrak{M}_\varphi^{\mathbf{N}'})}$. So $Y_0 = XZ \in \mathbf{P}$. This implies $\mathbf{P} \supset \mathbf{M}$ and $\mathbf{N}$, and so $\mathbf{P} \supset \mathrm{R}(\mathbf{M},\mathbf{N})$. Application to $Y_0 \in \mathrm{R}(\mathbf{M},\mathbf{N})$, $A = B = 1$ completes the proof.

Lemma 11.5.2. Let $\mathbf{M}, \mathbf{N}$ be two factors which commute, $\mathbf{N}$ being of class (I). Then $\mathrm{R}(\mathbf{M},\mathbf{N})$ belongs to the same class (I), (II), (III) as $\mathbf{M}$. In particular:

(i) If $\mathbf{M}$ is in a class (I_m), $m = 1, 2, \ldots, \infty$, and $\mathbf{N}$ is in a class (I_n), $n = 1, 2, \ldots, \infty$ then $\mathrm{R}(\mathbf{M},\mathbf{N})$ is in class (I_{mn}).

(ii) If $\mathbf{M}$ is in class (II_m), $m = 1, \infty$ and $\mathbf{N}$ is in class (I_n), $n = 1, 2, \ldots, \infty$ then $\mathrm{R}(\mathbf{M}, \mathbf{N})$ is in class (II_p), where $p = 1$ if m, n are both finite, and $p = \infty$ if m or n is infinite.

(iii) If $\mathbf{M}$ is in class (III), no further comment is necessary.

Proof: As $\mathbf{N}$ is of class (I), minimal φ's (with respect to $\mathbf{N}$) exist by Theorem IV. Choose one, and put $\mathfrak{M} = \mathfrak{M}_\varphi^{\mathbf{N}'}$, then $\mathbf{M}$ is algebraically ring-isomorphic to $(\mathrm{R}(\mathbf{M}, \mathbf{N}))_{(\mathfrak{M})}$ by Lemma 11.5.1. Thus it has the same class by Theorem IX. Now Lemma 11.4.3 shows (replacing there $\mathbf{M}, \mathfrak{M}$ by $\mathrm{R}(\mathbf{M}, \mathbf{N}), \mathfrak{M}$), that $\mathbf{M}$ and $\mathrm{R}(\mathbf{M}, \mathbf{N})$ belong to the same class (I), (II), (III).

Let us now consider (i)–(iii).

If 1 is infinite with respect to $\mathbf{M}$ then it is obviously so for $\mathrm{R}(\mathbf{M}, \mathbf{N})$ too; the same being true for $\mathbf{N}$ and $\mathrm{R}(\mathbf{M}, \mathbf{N})$. This takes care of (i) and of (ii) if $m = \infty$ or $n = \infty$, (iii) is void. So we need only to consider (i) and (ii) for m, n finite.

Construct for $\mathbf{N}$ the normalised orthogonal system $\varphi_1, \varphi_2, \ldots$ from Lemma 5.3.4, where all φ_i are minimal, $\mathfrak{M}_{\varphi_1}^{\mathbf{N}'}, \mathfrak{M}_{\varphi_2}^{\mathbf{N}'}, \ldots$ are mutually orthogonal, and $[\mathfrak{M}_{\varphi_1}^{\mathbf{N}'}, \mathfrak{M}_{\varphi_2}^{\mathbf{N}'}, \ldots] = \mathfrak{H}$. We know by Lemmas 5.3.5–5.3.8 that as $\mathbf{N}$ is in the case (I_n), the number of these φ_i's is n: $\varphi_1, \ldots, \varphi_n$ (n is finite!).

Lemma 11.5.1 applies to every φ_i. So $(\mathrm{R}(\mathbf{M}, \mathbf{N}))_{(\mathfrak{M}_{\varphi_i}^{\mathbf{N}'})}$ is in the same case as $\mathbf{M}$, (I_m) resp. (II_1) (for (i) resp. (ii)). By this, $\mathfrak{M}_{\varphi_i}^{\mathbf{N}'}$ has the (relative) dimension m in the standard normalisation of $(\mathrm{R}(\mathbf{M}, \mathbf{N}))_{(\mathfrak{M}_{\varphi_i}^{\mathbf{N}'})}$ resp. a finite relative dimension in any normalisation of $(\mathrm{R}(\mathbf{M}, \mathbf{N}))_{(\mathfrak{M}_{\varphi_i}^{\mathbf{N}'})}$. The same is true by Lemma 11.4.3, (i) resp. (ii), for $\mathfrak{M}_{\varphi_i}^{\mathbf{N}'}$ and $\mathrm{R}(\mathbf{M}, \mathbf{N})$. As the $\mathfrak{M}_{\varphi_i}^{\mathbf{N}'}$ are mutually orthogonal and

$$[\mathfrak{M}_{\varphi_1}^{\mathbf{N}'}, \ldots, \mathfrak{M}_{\varphi_n}^{\mathbf{N}'}] = \mathfrak{H},$$

the additivity of the relative dimension implies that it is mn (standard normalisation) resp. finite (any normalisation) for $\mathfrak{H}$ in $\mathrm{R}(\mathbf{M}, \mathbf{N})$. Thus we have case (I_{mn}) resp. (II_1) for $\mathrm{R}(\mathbf{M}, \mathbf{N})$.

This completes the proof.

Lemma 11.5.2 shows that if case (II) exists at all, then (II_∞) exists too. ((II_1) has been discussed in §11.4.) It suffices to take an $\mathbf{M}_0$ of case (II) in an $\mathfrak{H}_1$, any $\mathbf{N}_0$ of case (I_∞) in an $\mathfrak{H}_2$ (for instance: $\mathfrak{H}_2$ a Hilbert space, $\mathbf{N}_0 = \mathbf{B}_2$), and then form $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, $\mathbf{M} = \mathbf{M}_0^{(1)}$, $\mathbf{N} = \mathbf{N}_0^{(2)}$. Then $\mathrm{R}(\mathbf{M}, \mathbf{N})$ will certainly be of class (II_∞). But all this will be settled exhaustively in Chapter XIII.

Part IV: Examples of case (II)

Chapter XII: Construction of $\mathbf{M}, \mathbf{M}'$

12.1. The considerations at the end of §8.6 have established the existence of all cases (I_n), (I_∞) and of all their combinations for factors $\mathbf{M}$ resp. coupled factorisation $\mathbf{M}, \mathbf{M}'$. We are going to do now the same for the cases (II_1), (II_∞). We will thus answer a great part of the questions raised by Problem 3 and Problem 4.

Our first objective is to construct certain coupled factorisations $\mathbf{M}, \mathbf{M}'$ which belong to the classes (II_1), (II_1) (the C of Theorem X being = 1) or (II_∞), (II_∞). These factorisations will contain many arbitrary parameters, and therefore represent a rather wide variety of examples. In fact, further constructions based on them will answer Problem 1 in the negative.

Definition 12.1.1. We will consider groups $\mathfrak{G}$ of the following kind:

(i) $\mathfrak{G}$ is a group, that is a composition rule ab, an inverse a^{-1} and a unity 1 defined in $\mathfrak{G}$, with the properties

$$(ab)c = a(bc), \qquad aa^{-1} = a^{-1}a = 1, \qquad a1 = 1a = a.$$

(So ab is associative but not necessarily commutative).

(ii) $\mathfrak{G}$ is finite or countably infinite.

Definition 12.1.2. We will consider spaces S of the following kind: A Lebesgue outer measure is defined in S, that is, for every subset T of S a real number $\mu^*(T) \geq 0, \leq \infty$ is given, with the following properties:

(i) $T_1 \subset T_2$ implies $\mu^*(T_1) \leq \mu^*(T_2)$.

(ii) For every (finite or infinite) sequence $T_1, T_2, \ldots$

$$\mu^*(T_1 \dotplus T_2 \dotplus \cdots) \leq \mu^*(T_1) + \mu^*(T_2) + \cdots .$$

We now define measurability following Carathéodory (cf. (4, p. 246)):

(∗) A subset T of S is *measurable*, if and only if, for every subset T' of S,

$$\mu^*(T') = \mu^*(T \cdot T') + \mu^*(T' - T \cdot T').$$

We then write $\mu(T)$ for $\mu^*(T)$ and call it the *measure* of T.

We now continue the enumeration of our postulates.

(iii) If $\mu^*(T) < \alpha$ then a measurable T_0 with $T_0 \supset T$, $\mu(T_0) < \alpha$ exists.

(iv) A (finite or infinite) sequence $T^{(1)}, T^{(2)}, \ldots$ of measurable sets with finite measure exists, with the following property: If $x, y \in S$, and if $x \in T^{(i)}$ is equivalent to $y \in T^{(i)}$ (for all $i = 1, 2, \ldots$), then $x = y$.

Observe the following consequences of Definition 12.1.2: $\mu^*(T) \geq 0, \leq \infty$ and conditions (i)–(ii) correspond to Carathéodory's conditions (I)–(III), without (IV), (cf. (4, pp. 238–239)). This omission however is natural, because (IV) makes explicit use of the notion of distance, whereas we did not require that a distance (or any sort of a topology) be defined in S. (iii) is, (remembering (i)) equivalent to Carathéodory's condition (V), (cf. (4, p. 258)), therefore our outer measure is a regular measure function (cf. loc. cit.) if we disregard the topological condition (IV). Closer inspection of Carathéodory's deductions shows that in consequence, all those results on outer measure, measurability, and measure loc. cit. remain true which are not explicitly topological in their statements. This applies to the considerations on pp. 246–250 loc. cit. (cf. the remark on p. 257 there). Thus 0 and S are measurable; if T', T'' are measurable, then $T' - T'T''$ is too; if $T', T'', \ldots$ are measurable, then $T' \dotplus T'' \dotplus \cdots$ and $T' \cdot T'' \cdots$ are too. Besides, the well known convergence theorems on measure are valid.

As we see, the chief difference between the characteristics of the situation in S and those of the one which exists loc. cit., is this: The sets which are known to be measurable, and which serve as a basis for all constructions of measurable functions (cf. below Definition 12.1.3) are there the Borel-sets (a topological notion) while they are now the sets $T^{(1)}, T^{(2)}, \ldots$ from (iv).

In fact, this is essentially the meaning of (iv), which is the only one of our postulates without an analogue on Carathéodory's list: It serves to replace the (absent or ignored) topology in S, and in particular the "topological separability" of S.

Let us mention the following obvious examples of spaces S:

Example a). Let S be a (finite or infinite) sequence $S = (x_1, x_2, \ldots)$. Let a corresponding sequence of real numbers $\bar{\alpha}_n \geq 0, < \infty$ be given. Define

$$\mu^*(T) = \sum_{x_n \in T} \bar{\alpha}_n .$$

Then (i), (ii) are obviously fulfilled; (iii) is fulfilled because all sets $T \subset S$ are measurable; (iv) is fulfilled, because we may choose for $T^{(1)}, T^{(2)}, \ldots$ the set of all one-element $T \subset S$.

Example b). Let S be any measurable subset of a finite dimensional Euclidean space. Let $\mu^*(T)$ be the common Lebesgue measure. Then (i)–(iii) are obviously fulfilled; (iv) is fulfilled because we may choose for $T^{(1)}, T^{(2)}, \ldots$ the sequence $Ss_1, Ss_2, \ldots$, where $s_1, s_2, \ldots$ are all spheres with rational coordinates of the center and a rational radius.

We now introduce Lebesgue integration in the customary way:

Definition 12.1.3. A complex valued function $f(x)$, defined for all $x \in S$ is called *measurable* if the sets $(x; \Re(f(x)) < \alpha)$, $(x; \Im(f(x)) < \alpha)$ ($\Re$ = real part, $\Im$ = imaginary part) are measurable for every real α. The notions of summability and of the (Lebesgue) integral $\int_S f(x)\,dx$ of a measurable function $f(x)$ are then defined in any of the several equivalent customary ways. (The procedure of Carathéodory, (4, pp. 418–427), which coincides with Lebesgue's "geometrical" definition, (10, pp. 116–120), would necessitate introducing first the notion of outer measure in a product space. Lebesgue's "analytical" definition, (10, pp. 112–116), however, is directly applicable; another very convenient definition is due to Bochner, (2, pp. 265–267)).

We now form three functional spaces:

Definition 12.1.4. Let $\mathfrak{H}_{\mathfrak{G}}$ be the set of all complex valued functions $f(a)$, defined for all $a \in \mathfrak{G}$ with a finite $\sum_{a\in\mathfrak{G}} |f(a)|^2$. Define

$$(f, g)_{\mathfrak{G}} = \sum_{a\in\mathfrak{G}} f(a)\overline{g(a)}.$$

Let $\mathfrak{H}_S$ be the set of all complex valued functions $f(x)$, defined for all $x \in S$ which are measurable in x, and with a finite $\int_S |f(x)|^2\,dx$. Consider f, g as identical if $f(x) = g(x)$ except for an x-set of measure 0. Define

$$(f, g)_S = \int_S f(x)\overline{g(x)}\,dx.$$

Let $\mathfrak{H}_{S\mathfrak{G}}$ be the set of all complex valued functions $F(x, a)$ defined for all $x \in S$, $a \in \mathfrak{G}$, which are measurable in x (for every fixed $a \in \mathfrak{G}$) and with a finite $\sum_{a\in\mathfrak{G}} \int_S |F(x, a)|^2\,dx$. Consider F, G as identical, if $F(x, a) = G(x, a)$ except for an x-set of measure 0 (depending on a). Define

$$(F, G)_{S\mathfrak{G}} = \sum_{a\in\mathfrak{G}} \int_S F(x, a)\overline{G(x, a)}\,dx.$$

$\alpha\cdot$ and $+$ are defined in the obvious way in all three spaces $\mathfrak{H}_{\mathfrak{G}}$, $\mathfrak{H}_S$, $\mathfrak{H}_{S\mathfrak{G}}$.

By Definition 12.1.2 (iv), no two points $x \neq y$ can have the property

$$x, y \notin T^{(1)} \dotplus T^{(2)} \dotplus \cdots .$$

$S - (T^{(1)} \dotplus T^{(2)} \dotplus \cdots)$ is measurable; and if its measure is finite, we can add it to the sequence $T^{(1)}, T^{(2)}, \ldots$ thus forcing $T^{(1)} \dotplus T^{(2)} \dotplus \cdots = S$. So the only case where this is not possible is when

$$S - (T^{(1)} \dotplus T^{(2)} \dotplus \cdots) = (x_0), \qquad \mu((x_0)) = \infty.$$

Then we must have $f(x_0) = 0$ and $F(x_0, a) = 0$ for all $f \in \mathfrak{H}_S$ resp. $F \in \mathfrak{H}_{S\mathfrak{G}}$, so we may omit x_0 from S. (If $x \in S$, $x \neq x_0$ then $x \in T^{(i)}$ for some $i = 1, 2, \ldots$, and so $\mu^*((x)) \leq \mu(T^{(i)}) < \infty$. Therefore Definition 12.1.5, cf. below, excludes $x_0 a \neq x_0$ that is, we have $x_0 a = x_0$. So the omission of x_0 does not affect the operation xa either). In what follows we can thus assume

$$T^{(1)} \dotplus T^{(2)} \dotplus \cdots = S.$$

If $\mu(S) = 0$, then $\mathfrak{H}_S$ and $\mathfrak{H}_{S\mathfrak{G}}$ would consist of 0 only. We therefore assume explicitly that $\mu(S) > 0$.

Lemma 12.1.1. *All three spaces $\mathfrak{H}_{\mathfrak{G}}$, $\mathfrak{H}_S$, $\mathfrak{H}_{S\mathfrak{G}}$ fulfill the postulates A, B, C, E of (16, pp. 64–66); thus they are finite dimensional Euclidean or Hilbert spaces (and $\neq 0$).*

Using the complete normalised orthogonal set of all functions

$$\varphi_{a_0}(a) = \begin{cases} 1 & \text{for } a = a_0, \\ 0 & \text{for } a \neq a_0, \end{cases} \qquad a_0 \in \mathfrak{G} \text{ in } \mathfrak{H}_{\mathfrak{G}},$$

and setting up the correspondence

$$F(x, a) \sim \langle F(x, \bar{a}_1), F(x, \bar{a}_2), \ldots \rangle$$

($\bar{a}_1, \bar{a}_2, \ldots$ is some enumeration of $\mathfrak{G}$, so that the above a_0 runs over $\bar{a}_1, \bar{a}_2, \ldots$), $\mathfrak{H}_{S\mathfrak{G}}$ is isomorphic with $\mathfrak{H}_{\mathfrak{G}} \otimes \mathfrak{H}_S$ in the sense of Definition 2.4.1 and Lemma 2.4.1.

We will use in what follows the more symmetric notation

$$F(x, a) \sim \langle F(x, a_0); a_0 \in \mathfrak{G} \rangle.$$

Proof: If we use an enumeration $\bar{a}_1, \bar{a}_2, \ldots$ of $\mathfrak{G}$, and put $f(\bar{a}_n) = x_n$, $n = 1, 2, \ldots$ then $f(a) \sim (x_1, x_2, \ldots)$ and our definition coincides with the original definition of finite dimensional Euclidean or of Hilbert space (cf. (16, p. 69)). Concerning $\mathfrak{H}_S$ it suffices to observe that the considerations of (16, pp. 108–111), on A, B, E apply immediately to it; while those on C apply, if the system of neighborhoods occurring there is replaced by $T^{(1)}, T^{(2)}, \ldots$.

As $\mu(S) = \mu(T^{(1)} \dotplus T^{(2)} \dotplus \cdots) > 0$ some $\mu(T^{(i)}) > 0$. At the same time it is $< \infty$. Then

$$f(x) = \begin{cases} 1, & x \in T^{(i)}, \\ 0, & x \notin T^{(i)} \end{cases}$$

belongs to $\mathfrak{H}_S$ and is $\neq 0$. So $\mathfrak{H}_S \neq (0)$. The last part of our Lemma results immediately by comparing our definitions with Definition 2.4.1 and Lemma 2.4.1. As $\mathfrak{H}_{S\mathfrak{G}}$ is isomorphic with $\mathfrak{H}_{\mathfrak{G}} \otimes \mathfrak{H}_S$ it must fulfill A, B, C, E, too. As $\mathfrak{H}_S \neq (0)$, therefore $\mathfrak{H}_{S\mathfrak{G}} \neq (0)$.

Heretofore the connection between S and $\mathfrak{G}$ was merely formal, due to our forming the space $\mathfrak{H}_{S\mathfrak{G}} = \mathfrak{H}_{\mathfrak{G}} \otimes \mathfrak{H}_S$. An intrinsic connection is established by the following assumptions:

Definition 12.1.5. $\mathfrak{G}$ is an m-group in S, if for every $a \in \mathfrak{G}$ there is defined a one to one mapping $x \rightleftarrows xa$ of S on itself, with the following properties:

(i) If $T \subset S$ and T_a is the $x \rightleftarrows xa$ image of T then $\mu^*(T) = \mu^*(T_a)$. (This implies that the measurability of T is equivalent to that of T_a.)

(ii) $(xa)b = x(ab)$. (This implies that $x \cdot 1 = x$ and that $x \rightleftarrows xa^{-1}$ is inverse to $x \rightleftarrows xa$.)

(iii) If $a \neq 1$ then $xa = x$ holds only for x-sets of measure 0.

$\mathfrak{G}$ is *ergodic* in S, if besides (i)–(iii) we have further:

(iv) If a measurable $T \subset S$ differs from each T_a, $a \in \mathfrak{G}$ only by a set of measure 0 (but depending on a, this means $\mu((T \dotplus T_a) - (T \cdot T_a)) = 0$) then either $\mu(T) = 0$ or $\mu(S - T) = 0$.

Observe that (iv) differs from the customary definitions of ergodicity in two ways: First $\mathfrak{G}$ is not necessarily the simple translation group, not even necessarily Abelian, second (which is essential if $\mu(S) = \infty$) we did not restrict the validity of (iv) to T's with a finite $\mu(T)$ only.

12.2. We introduce and discuss first some operators in $\mathfrak{H}_S$.

Lemma 12.2.1. *Let $\mathfrak{G}$ be an m-group in S. Define the operators:*

(i) $U_a f(x) = f(xa)$ *for any* $a \in \mathfrak{G}$.

(ii) $L_{\varphi(x)} f(x) = \varphi(x) f(x)$ *for any bounded and measurable complex valued function*

$\varphi(x)$ defined for all $x \in S$. These operators are bounded, U_a is even unitary.

Proof: We have

$$\int_S |U_a(f(x))|^2\,dx = \int_S |f(xa)|^2\,dx = \int_S |f(x)|^2\,dx$$

so U_a is bounded, and $\|U_a f\| = \|f\|$. Now clearly $U_{a^{-1}}U_a = U_aU_{a^{-1}} = 1$ so U_a has an inverse, thus it is unitary (cf. (16, p. 71)).

If $|\varphi(x)| \leq C$ for all $x \in S$, then

$$\begin{aligned}\int_S |L_{\varphi(x)}f(x)|^2\,dx &= \int_S |\varphi(x)|^2|f(x)|^2\,dx \\ &\leq C^2\int_S |f(x)|^2\,dx, \qquad \|L_{\varphi(x)}f\| \leq C\|f\|.\end{aligned}$$

So $L_{\varphi(x)}$ too is bounded.

Lemma 12.2.2. *Let $\mathfrak{G}$ be an m-group in S. Let L be the set of all operators $L_{\varphi(x)}$ from Lemma 12.2.1, (ii). Then L is a ring and $L = L'$.*

Proof: As L' is necessarily a ring, it suffices to prove $L = L'$. As

$$(L_{\varphi(x)})^* = L_{\overline{\varphi(x)}}, \qquad L_{\varphi(x)}L_{\psi(x)} = L_{\varphi(x)\psi(x)}$$

therefore every $L_{\psi(x)}$ commutes with every $L_{\varphi(x)}$ and $(L_{\varphi(x)})^*$ thus $L_{\psi(x)} \in L'$ and so $L \subset L'$. So we must only prove $L' \subset L$.

Assume therefore $A \in L'$. Then A commutes with every $L_{\varphi(x)}$, that is $AL_{\varphi(x)}f(x) = L_{\varphi(x)}Af(x)$. That is:

$$A(\varphi(x)f(x)) = \varphi(x)Af(x). \tag{$*$}$$

The assumptions are: $\varphi(x)$, $f(x)$ are measurable, $\varphi(x)$ is bounded, $\int_S |f(x)|^2\,dx$ is finite, the equation holds with the exception of an x-set of measure 0. Such exceptions, by the way, are admitted in all the equations we are going to derive.

Let $T^{(1)}, T^{(2)}, \ldots$ be the sets from Definition 12.1.2 (iv); assume (following the remark before Lemma 12.1.1), that $T^{(1)} \dotplus T^{(2)} \dotplus \cdots = S$. Define

$$e_i(x) = \begin{cases} 1, & x \in T^{(1)} \dotplus \cdots \dotplus T^{(i)}, \\ 0, & x \notin T^{(1)} \dotplus \cdots \dotplus T^{(i)}. \end{cases}$$

Apply now ($*$) to $f = e_i$. Then $A(e_i(x)\varphi(x)) = (Ae_i(x))\varphi(x)$ obtains. For $i \geq j$, $e_i(x)e_j(x) = e_j(x)$, so $\varphi = e_j$ gives $Ae_j(x) = (Ae_i(x))e_j(x)$. Thus if $x \in T^{(1)} \dotplus \cdots \dotplus T^{(j)}$,

$Ae_j(x) = Ae_i(x)$; in other words: for $x \in T^{(1)} \dotplus \cdots \dotplus T^{(j)}$ all $Ae_i(x)$, $i \geq j$, agree. Thus for every x all $Ae_i(x)$ with a sufficiently great i have the same value. Call this value $\psi(x)$ as $\psi(x) = \lim_{i\to\infty} Ae_i(x)$ this is a measurable function. Now we have $Ae_i(x) = \psi(x)e_i(x)$. And our original equation becomes: $Ae_i(x)\varphi(x) = \psi(x)e_i(x)\varphi(x)$. That is: $Af(x) = \psi(x)f(x)$ where $f(x) = e_i(x)\varphi(x)$. Thus this equation is proved if $f(x)$ is bounded and 0 for all $x \notin T^{(1)} \dotplus \cdots \dotplus T^{(i)}$ for some $i = 1, 2, \ldots$.

A is bounded, say $\|Af\| \leq C\|f\|$; therefore

$$\int_S |\psi(x)|^2 |f(x)|^2 \, dx \leq C^2 \int_S |f(x)|^2 \, dx$$

for these f's. Now put

$$f(x) = \begin{cases} 1, & |\psi(x)| \geq C+1, \quad x \in T^{(1)} \dotplus \cdots \dotplus T^{(i)}, \\ 0, & \text{otherwise.} \end{cases}$$

Denote the set of all x with $|\psi(x)| \geq C+1$ by T_0. Then the above inequality gives:

$$(C+1)^2 \mu\big(T_0(T^{(1)} \dotplus \cdots \dotplus T^{(i)})\big) \leq C^2 \mu\big(T_0(T^{(1)} \dotplus \cdots \dotplus T^{(i)})\big),$$

$$\mu\big(T_0(T^{(1)} \dotplus \cdots \dotplus T^{(i)})\big) = 0$$

and passing to the limit as $i \to \infty$, $\mu(T_0) = 0$. So we have everywhere $|\psi(x)| \leq C+1$ (as T_0 does not matter), that is: $\psi(x)$ is bounded. We can therefore write: $Af = L_\psi f$ if $f(x)$ is as described above.

Now these f form an everywhere dense set in $\mathfrak{H}_S$. If any $f \in \mathfrak{H}_S$ is given, define

$$f_i(x) = \begin{cases} f(x), & |f(x)| < i, \quad x \in T^{(1)} \dotplus \cdots \dotplus T^{(i)}, \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \ldots$; then every f_i meets these requirements, and the f_i converge for $i \to \infty$ to f (in $\mathfrak{H}_S$). As A, L_ψ are both bounded, we have therefore $Af = L_\psi f$ for every f; that is $A = L_\psi \in L$. So we have established $L' \subset L$ too, thus completing the proof.

Lemma 12.2.3. *Let $\mathfrak{G}$ be an m-group in S. The equation $L_{\varphi(x)} = L_{\psi(x)}U_a$ holds if and only if either $a = 1$, $\varphi(x) = \psi(x)$ (except for an x-set of measure 0) or $a \neq 1$, $\varphi(x) = \psi(x) = 0$ (except for an x-set of measure 0).*

Proof: The sufficiency of these conditions is obvious, so we need only to prove their necessity. Assume therefore $L_{\varphi(x)} = L_{\psi(x)}U_a$. So we have

$$\varphi(x)f(x) = \psi(x)f(xa)$$

whenever $\int_S |f(x)|^2\,dx$ is finite. This equation, and similarly all those we are going to derive, holds with the exception of an x-set of measure 0. Use the e_i from the proof of Lemma 12.2.2, then we have: $e_i(x)\varphi(x) = e_i(xa)\psi(x)$, that is $\varphi(x) = \psi(x)$ if x and $xa \in T^{(1)} \dotplus \cdots \dotplus T^{(i)}$. Summing over all $i = 1, 2, \ldots$ we obtain $\varphi(x) = \psi(x)$ for $x \in S$. This settles the case $a = 1$. Assume now $a \neq 1$. Substituting $\varphi(x) = \psi(x)$ in our original equation gives $\varphi(x)f(x) = \varphi(x)f(xa)$ and so

$$\int_S |\varphi(x)|^2 |f(x) - f(xa)|^2\,dx = 0.$$

Define now

$$e_i'(x) = \begin{cases} 1, & x \in T^{(i)}, \\ 0, & x \notin T^{(i)}, \end{cases}$$

then $e_i' \in \mathfrak{H}_S$ and we can put $f = e_i'$. Then $|f(x) - f(xa)|^2 = 1$ for

$$x \in (T^{(i)} \dotplus T^{(i)}_{a^{-1}}) - T^{(i)}T^{(i)}_{a^{-1}}$$

and so our integral formula becomes

$$\int_{S_i} |\varphi(x)|^2\,dx = 0, \qquad S_i = (T^{(i)} \dotplus T^{(i)}_{a^{-1}}) - T^{(i)}T^{(i)}_{a^{-1}}.$$

Put now

$$S' = \sum_{i=1}^{\infty} \big((T^{(i)} \dotplus T^{(i)}_{a^{-1}}) - T^{(i)}T^{(i)}_{a^{-1}}\big).$$

As the integrand above is $|\varphi(x)|^2 \geq 0$, adding of the above equations over all $i = 1, 2, \ldots$ gives

$$\int_{S'} |\varphi(x)|^2\,dx = 0.$$

Now $x \in S - S'$ means $x \notin (T^{(i)} \dotplus T^{(i)}_{a^{-1}}) - T^{(i)}T^{(i)}_{a^{-1}}$ for all $i = 1, 2, \ldots$; thus $x \in$ or $\notin$ both $T^{(i)}$ and $T^{(i)}_{a^{-1}}$, that is both x and xa are $\in T^{(i)}$ or $\notin T^{(i)}$. By Definition 12.1.2, (iv), this implies $x = xa$. As $a \neq 1$, Definition 12.1.5, (iii) now necessitates $\mu(S - S') = 0$. Thus $\int_S |\varphi(x)|^2\, dx = 0$ too, and therefore $\varphi(x) = 0$ (except for an x-set of measure 0). This completes the proof.

Lemma 12.2.4. *Let $\mathfrak{G}$ be ergodic in S. Let L be as in Lemma 12.2.2 and let U be the set of all U_a, $a \in \mathfrak{G}$. Then $L \cdot U' = (\alpha 1)$.*

Proof: $\alpha 1$ belongs obviously to U', and as $\alpha 1 = L_\alpha$, $\alpha 1 \in L$ so $\alpha 1 \in L{\cdot}U'$, $L{\cdot}U' \supset (\alpha 1)$. So we need only prove $L \cdot U' \subset (\alpha 1)$. Assume therefore $A \in L \cdot U'$. Then $A = L_{\varphi(x)}$ and as $A \in U'$ therefore $U_a A = A U_a$, $A = U_a^{-1} A U_a$, $L_{\varphi(x)} = L_{\varphi(xa^{-1})}$ for every $a \in \mathfrak{G}$. By Lemma 12.2.3 this means $\varphi(x) = \varphi(xa^{-1})$ except for an x-set of measure 0 (depending on $a \in \mathfrak{G}$).

Put $T_\alpha = (x; \mathfrak{R}(\varphi(x)) < \alpha)$. The above relation proves that $x \in T_\alpha$ and $xa^{-1} \in T_\alpha$ are equivalent, except for an x-set of measure 0; so

$$\mu((T_\alpha \dotplus T_{\alpha a}) - (T_\alpha T_{\alpha a})) = 0.$$

Therefore Definition 12.1.5, (iv) (the ergodicity) implies

$$\mu(T_\alpha) = 0 \quad \text{or} \quad \mu(S - T_\alpha) = 0.$$

Denote the set of all (real) α's with $\mu(T_\alpha) = 0$ by N. If $\alpha \leq \beta$ then $T_\alpha \subset T_\beta$ so $\beta \in N$ implies $\alpha \in N$. So if α_0 is the least upper bound of N, $-\infty \leq \alpha_0 \leq \infty$, then $\alpha < \alpha_0$ implies $\alpha \in N$, that is $\mu(T_\alpha) = 0$, and $\alpha > \alpha_0$ implies $\alpha \notin N$, that is $\mu(S - T_\alpha) = 0$.

So for every rational $\alpha < \alpha_0$, $\mu((x; \mathfrak{R}(\varphi(x)) < \alpha)) = 0$ and adding over them gives $\mu((x; \mathfrak{R}(\varphi(x)) < \alpha_0)) = 0$; for every rational $\alpha > \alpha_0$, $\mu((x; \mathfrak{R}(\varphi(x)) \geq \alpha)) = 0$ and adding over them gives: $\mu((x; \mathfrak{R}(\varphi(x)) > \alpha_0)) = 0$. Adding again:

$$\mu((x; \mathfrak{R}(\varphi(x)) \neq \alpha_0)) = 0.$$

Similarly we obtain: $\mu((x; \mathfrak{I}(\varphi(x)) \neq \beta_0)) = 0$ for a suitable β_0, and if we put $\alpha_1 = \alpha_0 + i\beta_0$ then adding these two last equations gives:

$$\mu((x; \varphi(x) \neq \alpha_1)) = 0.$$

So we have $\varphi(x) = \alpha_1$ except for an x-set of measure 0; and so $A = L_{\varphi(x)} = \alpha_1 1$, $A \in (\alpha 1)$. Thus $L \cdot U' \subset (\alpha 1)$, completing the proof.

12.3. We pass now to constructions in $\mathfrak{H}_{S\mathfrak{G}}$.

Lemma 12.3.1. *Let $\mathfrak{G}$ be an m-group in S. Define the operators*

(i) $\overline{U}_{a_0}F(x,a) = F(xa_0, aa_0)$,
(ii) $\overline{V}_{a_0}F(x,a) = F(x, a_0^{-1}a)$,
(iii) $\overline{W}F(x,a) = F(xa^{-1}, a^{-1})$,
(iv) $\overline{L}_{\varphi(x)}F(x,a) = \varphi(x)F(x,a)$,
(v) $\overline{M}_{\varphi(x)}F(x,a) = \varphi(xa^{-1})F(x,a)$,

for any $a_0 \in \mathfrak{G}$ and any bounded and measurable complex valued $\varphi(x)$ defined for all $x \in S$. These operators are bounded, $\overline{U}_{a_0}$, $\overline{V}_{a_0}$, $\overline{W}$ are even unitary. Furthermore $\overline{W} = \overline{W}^{-1}$ and

$$\overline{WU}_{a_0}\overline{W} = \overline{V}_{a_0}; \qquad \overline{WL}_{\varphi(x)}\overline{W} = \overline{M}_{\varphi(x)};$$

in other words: $F \mapsto \overline{W}F$ is an involutory spatial isomorphism of $\mathfrak{H}_{S\mathfrak{G}}$ which interchanges $\overline{U}_{a_0}, \overline{L}_{\varphi(x)}$ with $\overline{V}_{a_0}, \overline{M}_{\varphi(x)}$.

Proof: The boundedness and unitarity are proved in the same way as in the proof of Lemma 12.2.1. $\overline{W}^2 = 1$ results from direct computation, and implies $\overline{W} = \overline{W}^{-1}$; the two last equations result from direct computation.

Lemma 12.3.2. *Let $\mathfrak{G}$ be an m-group in S. Let I be the set of all $\overline{U}_{a_0}$ and all $\overline{L}_{\varphi(x)}$ and J the set of all $\overline{V}_{a_0}$ and all $\overline{M}_{\varphi(x)}$ (cf. Lemma 12.3.1). Form for each bounded operator $\overline{A}$ in $\mathfrak{H}_{S\mathfrak{G}}$ ($A \in \mathbf{B}_{S\mathfrak{G}}$) the decomposition from Definition 2.4.2,*

$$\overline{A} \sim \langle A_{a,b} \rangle_{a,b \in \mathfrak{G}}$$

where every $A_{a,b} \in \mathbf{B}_S$. (Cf. the decomposition $\mathfrak{H}_{S\mathfrak{G}} = \mathfrak{H}_{\mathfrak{G}} \otimes \mathfrak{H}_S$ from Lemma 12.1.1. As described there, we replace the indices $t, s = 1, 2, \ldots$ by indices $a, b \in \mathfrak{G}$ as already the complete normalised orthogonal set φ_{a_0}, $a_0 \in \mathfrak{G}$ in $\mathfrak{H}_{\mathfrak{G}}$ has been indexed in this way.)

Then $\overline{A} \in I'$ if and only if $A_{a,b}$ has the form

$$A_{a,b} = L_{\chi_{ab^{-1}}(xb^{-1})}$$

and $\overline{A} \in J'$ if and only if $A_{a,b}$ has the form

$$A_{a,b} = L_{\chi_{a^{-1}b}(x)} U_{b^{-1}a}.$$

Here $\chi_c(x)$ must be a bounded and measurable function of x for every $c \in \mathfrak{G}$. (We do not determine for which systems $\chi_c(x)$, $c \in \mathfrak{G}$, a bounded $\overline{A}$, to which the given $\chi_c(x)$ corresponds, actually does exist.)

Proof: Remember the definition of the $A_{a,b}$ (Definition 2.4.2, with our present notations):

$$\overline{A}\langle f(x)\varphi_{a'}(a); a \in \mathfrak{G}\rangle = \langle A_{a',b} f(x); b \in \mathfrak{G}\rangle;$$

that is (if we replace a', a by a, b so that now $x \in S$ and $b \in \mathfrak{G}$ are the variables, and $a \in \mathfrak{G}$ is a parameter):

$$\overline{A} f(x)\varphi_a(b) = A_{a,b} f(x).$$

Then the definitions of $\overline{U}_{a_0}$, $\overline{L}_{\varphi(x)}$ and $\overline{W}$ give: If $\overline{A} = \langle A_{a,b}\rangle_{a,b\in\mathfrak{G}}$, then

$$\begin{aligned}
\overline{U}_{a_0}^{-1}\overline{A}\overline{U}_{a_0} &= \left\langle U_{a_0}^{-1} A_{aa_0^{-1},ba_0^{-1}} U_{a_0}\right\rangle_{a,b\in\mathfrak{G}},\\
\overline{W}\overline{A}\overline{W} &= \left\langle U_b^{-1} A_{a^{-1},b^{-1}} U_a\right\rangle_{a,b\in\mathfrak{G}},\\
\overline{L}_{\varphi(x)}\overline{A} &= \langle L_{\varphi(x)} A_{a,b}\rangle_{a,b\in\mathfrak{G}},\\
\overline{A}\overline{L}_{\varphi(x)} &= \langle A_{a,b} L_{\varphi(x)}\rangle_{a,b\in\mathfrak{G}}.
\end{aligned}$$

Now $\overline{A} \in I'$ means that $\overline{A}$ commutes with all $\overline{L}_{\varphi(x)}$, $((L_{\varphi(x)})^* = L_{\overline{\varphi(x)}})$ and all $\overline{U}_{a_0}$, $((\overline{U}_{a_0})^* = (\overline{U}_{a_0})^{-1} = \overline{U}_{a_0^{-1}})$. This means $A_{a,b} \in L'$ that is $A_{a,b} \in L$ (by Lemma 12.2.2), and

$$U_{a_0}^{-1} A_{aa_0^{-1},ba_0^{-1}} U_{a_0} = A_{a,b}.$$

The first statement means $A_{a,b} = L_{\omega_{a,b}(x)}$ where $\omega_{a,b}(x)$ is a bounded and measurable function of x for every choice of $a, b \in \mathfrak{G}$. Now

$$\begin{aligned}
U_{a_0}^{-1} A_{aa_0^{-1},ba_0^{-1}} U_{a_0} &= U_{a_0}^{-1} L_{\omega_{aa_0^{-1},ba_0^{-1}}(x)} U_{a_0}\\
&= L_{\omega_{aa_0^{-1},ba_0^{-1}}(xa_0^{-1})}
\end{aligned}$$

so the remaining requirement is:

$$\omega_{aa_0^{-1},ba_0^{-1}}(xa_0^{-1}) = \omega_{a,b}(x).$$

This equation, as the following ones, is valid with the exception of x-sets of measure 0.

Put $a_0 = b$ and $\chi_c(x) = \omega_{c,1}(x)$ then $\omega_{a,b}(x) = \chi_{ab^{-1}}(xb^{-1})$ obtains. Conversely: If this equation holds for a system of bounded measurable functions $\chi_c(x)$ then the $\omega_{a,b}(x)$ are such too, and

$$\omega_{aa_0^{-1},ba_0^{-1}}(xa_0^{-1}) = \omega_{a,b}(x).$$

So the characterisation of the $\overline{A} \in I'$ is given by $A_{a,b} = L_{\chi_{ab^{-1}}(xb^{-1})}$. This proves our statement concerning I'.

As the spatial isomorphism $F \mapsto \overline{WF}$ of $\mathfrak{H}_{S\mathfrak{G}}$ carries I into J it carries I' into J'. So $\overline{B} \in J'$ means $\overline{B} = \overline{WAW}$ for some $\overline{A} \in I'$. The above result concerning the $\overline{A} \in I'$, together with our formula $\overline{WAW}$ gives therefore:

$$\begin{aligned} B_{a,b} &= U_b^{-1} A_{a^{-1},b^{-1}} U_a = U_b^{-1} L_{\chi_{a^{-1}b}(xb)} U_a \\ &= L_{\chi_{a^{-1}b}(x)} U_b^{-1} U_a = L_{\chi_{a^{-1}b}(x)} U_{b^{-1}a}. \end{aligned}$$

This proves our statement concerning J'.

Lemma 12.3.3. *Let $\mathfrak{G}$ be an m-group in S, and I, J as above. Then $R(I) = J'$, $R(J) = I'$.*

Proof: For $\overline{A} = \overline{V}_{a_0}$, $A_{a,b} = \delta_{a_0ab^{-1}} \cdot 1$ (define $\delta_c = 1$ for $c = 1$, $= 0$ for $c \neq 1$), and for $\overline{A} = \overline{M}_{\varphi(x)}$, $A_{a,b} = \delta_{ab^{-1}} L_{\varphi(xb^{-1})}$. So $\overline{V}_{a_0}$ and $\overline{M}_{\varphi(x)} \in I'$ (with $\chi_c(x) = \delta_{a_0c}$ resp. $\delta_c\varphi(x)$), that is $J \subset I'$. As I' is a ring, this implies $R(J) \subset I'$.

If $\overline{A} \in I'$, $\overline{B} \in J'$ then we have

$$A_{a,b} = L_{\chi_{ab^{-1}}(xb^{-1})}, \qquad B_{a,b} = L_{\xi_{a^{-1}b}(x)} U_{b^{-1}a}.$$

Now put $\overline{C} = \overline{AB}$, $\overline{D} = \overline{BA}$ then the formula (iv) of Lemma 2.4.3 gives:

$$\begin{aligned} C_{a,b} &= \sum_c A_{c,b} B_{a,c} \\ &= \sum_c L_{\chi_{cb^{-1}}(xb^{-1})} L_{\xi_{a^{-1}c}(x)} U_{c^{-1}a} \\ &= \sum_c L_{\chi_{ac^{-1}b^{-1}}(xb^{-1}) \xi_{c^{-1}}(x)} U_c, \end{aligned}$$

(we replaced the summation index c by ac^{-1}) and

$$\begin{aligned} D_{a,b} &= \sum_c B_{c,b} A_{a,c} \\ &= \sum_c L_{\xi_{c^{-1}b}(x)} U_{b^{-1}c} L_{\chi_{ac^{-1}}(xc^{-1})} \\ &= \sum_c L_{\xi_{c^{-1}b}(x) \chi_{ac^{-1}}(xb^{-1})} U_{b^{-1}c} \\ &= \sum_c L_{\chi_{ac^{-1}b^{-1}}(xb^{-1}) \xi_{c^{-1}}(x)} U_c, \end{aligned}$$

(we replaced the summation index c by bc, remember that the order of summation in $\sum_c$ does not matter by Lemma 2.4.3 (iv)). So we have $\overline{C} = \overline{D}$, $\overline{AB} = \overline{BA}$. As

$\overline{B} \in J'$ implies $\overline{B}^* \in J'$, this means $\overline{A} \in J''$; as $\overline{A}$ was an arbitrary element of I' we have $I' \subset J''$. As $1 \in J$, therefore $J'' = R(J)$ (cf. (18, p. 397)), so $I' \subset R(J)$.

Thus we have proved $R(J) = I'$. The spatial isomorphism $F \mapsto \overline{W}F$ of $\mathfrak{H}_{S\mathfrak{G}}$ interchanges I with J, therefore we have proved $R(I) = J'$ too. Thus the proof is complete.

Lemma 12.3.4. *Let $\mathfrak{G}$ be an m-group in S. Put $\mathbf{M} = R(I) = J'$ then $\mathbf{M}' = R(J) = I'$. If $\mathfrak{G}$ is ergodic in S, then $\mathbf{M}$ is a factor.*

Proof: Clearly $\mathbf{M}' = (R(I))' = I'$. If $\overline{A} \in \mathbf{M} \cdot \mathbf{M}'$ then we have by Lemma 12.3.2

$$A_{a,b} = L_{\chi_{ab^{-1}}(xb^{-1})} = L_{\xi_{a^{-1}b}(x)} U_{b^{-1}a}.$$

So if $a \neq b$ that is $b^{-1}a \neq 1$ then Lemma 12.2.3 gives $\chi_{ab^{-1}}(xb^{-1}) = \xi_{a^{-1}b}(x) = 0$. That is: $\chi_c(x) = \xi_c(x) = 0$ if $c \neq 1$. For $a = b$, that is, $b^{-1}a = 1$ similarly $\chi_1(xb^{-1}) = \xi_1(x)$ obtains. This gives for $b = 1$, $\chi_1(x) = \xi_1(x)$ and then in general $\chi_1(xb^{-1}) = \chi_1(x)$. In other words:

$$U_b^{-1} L_{\chi_1(x)} U_b = L_{\chi_1(x)}.$$

So we found the following conditions: $\chi_c(x) = \xi_c(x)$ for all $c \in \mathfrak{G}$, if $c \neq 1$ then even $= 0$; if $c = 1$ then $L_{\chi_1(x)} \in L \cdot U'$.

Now if $\mathfrak{G}$ is ergodic in S, then Lemma 12.2.4 gives $L \cdot U' = (\alpha 1)$ so we have $L_{\chi_1(x)} = \alpha 1$, $\chi_1(x) = \alpha$. This means $\chi_c(x) = \xi_c(x) = \alpha\delta_c$, that is $\overline{A} = \alpha 1$. Thus we have proved $\mathbf{M} \cdot \mathbf{M}' = (\alpha 1)$ that is: $\mathbf{M}$ is a factor.

12.4. From now on we assume that $\mathfrak{G}$ is ergodic in S.

Lemma 12.4.1. *Write for an $\overline{A} \in \mathbf{M}$ or $\in \mathbf{M}'$ that*

$$\overline{A} \cong [[\chi_c(x)]]_{x \in S, c \in \mathfrak{G}}$$

if $A \sim \langle A_{a,b} \rangle_{a,b \in \mathfrak{G}}$ and

$$A_{a,b} = L_{\chi_{ab^{-1}}(x)} U_{b^{-1}a} \quad \text{resp.} \quad A_{a,b} = L_{\chi_{ab^{-1}}(xb^{-1})}$$

(cf. Lemma 12.3.2). If now

$$\overline{A} \cong [[\chi_c(x)]], \qquad \overline{B} \cong [[\xi_c(x)]], \qquad \overline{C} \cong [[\eta_c(x)]]$$

then the following rules of computation hold:

(i) *If $\overline{C} = \alpha\overline{A}$ then $\eta_c(x) = \alpha\chi_c(x)$.*

(ii) *If* $\overline{C} = \overline{A}^*$ *then* $\eta_c(x) = \overline{\chi_{c^{-1}}(xc^{-1})}$.

(iii) *If* $\overline{C} = \overline{A} + \overline{B}$ *then* $\eta_c(x) = \chi_c(x) + \xi_c(x)$.

(iv) *If* $\overline{C} = \overline{AB}$ *then*

$$\eta_c(x) = \sum_a \chi_a(x)\xi_{ca^{-1}}(xa^{-1}).$$

The $\sum_a$ *converges "en mesure" (cf. the precise description of this notion at the end of the proof), irrespective of the order in which the* $a \in \mathfrak{G}$ *are gone through.*

Proof: Consider first $\overline{A} \in \mathbf{M}'$. (i), (iii) follow immediately from Lemma 2.4.3, (i), (iii); (ii) follows from Lemma 2.4.3, (ii), by the following consideration:

$$C_{a,b} = L_{\eta_{ab^{-1}}(xb^{-1})}, \qquad C_{a,b} = A^*_{b,a} = L_{\overline{\chi_{ba^{-1}}(xa^{-1})}},$$

$$\eta_{ab^{-1}}(xb^{-1}) = \overline{\chi_{ba^{-1}}(xa^{-1})}, \qquad \eta_c(x) = \overline{\chi_{c^{-1}}(xc^{-1})}.$$

Finally (iv) results from Lemma 2.4.3, (iv):

$$\begin{aligned} C_{a,b} &= L_{\eta_{ab^{-1}}(xb^{-1})}, \\ C_{a,b} &= \sum_c A_{c,b}B_{a,c} \\ &= \sum_c L_{\chi_{cb^{-1}}(xb^{-1})}L_{\xi_{ac^{-1}}(xc^{-1})} \\ &= \sum_c L_{\chi_{cb^{-1}}(xb^{-1})\xi_{ac^{-1}}(xc^{-1})}, \end{aligned}$$

and putting $b = 1$, and writing c, a for a, c,

$$L_{\eta_c(x)} = \sum_a L_{\chi_a(x)\xi_{ca^{-1}}(xa^{-1})}$$

is strongly convergent, irrespective of the order in which the $a \in \mathfrak{G}$ are gone through.

Let now $T \subset S$ be a measurable set of finite measure, put

$$e_T(x) = \begin{cases} 1, & x \in T, \\ 0, & x \notin T \end{cases}$$

so that $e_T \in \mathfrak{H}_S$. Then if $\bar{a}_1, \bar{a}_2, \dots$ is an enumeration of $\mathfrak{G}$ we have

$$\left\| L_{\eta_c(x)}e_T - \sum_{j=1}^{i} L_{\chi_{\bar{a}_j}(x)\xi_{c\bar{a}_j^{-1}}(x\bar{a}_j^{-1})}e_T \right\| \longrightarrow 0 \quad \text{for } i \to \infty$$

so

$$\int_S \left| \eta_c(x) e_T(x) - \sum_{j=1}^{i} \chi_{\bar{a}_j}(x) \xi_{c\bar{a}_j^{-1}}(x\bar{a}_j^{-1}) e_T(x) \right|^2 dx \longrightarrow 0,$$

$$\int_T \left| \eta_c(x) - \sum_{j=1}^{i} \chi_{\bar{a}_j}(x) \xi_{c\bar{a}_j^{-1}}(x\bar{a}_j^{-1}) \right|^2 dx \longrightarrow 0.$$

This implies for every $\epsilon > 0$ that

$$\mu\left(\left(x; \left| \eta_c(x) - \sum_{j=1}^{i} \chi_{\bar{a}_j}(x) \xi_{c\bar{a}_j^{-1}}(x\bar{a}_j^{-1}) \right| \geq \epsilon \right) \cdot T \right) \longrightarrow 0$$

for $i \to \infty$. That is: $\sum_{j=1}^{\infty} \chi_{\bar{a}_j}(x) \xi_{c\bar{a}_j^{-1}}(x\bar{a}_j^{-1})$ converges "en mesure" to $\eta_c(x)$. As $\bar{a}_1, \bar{a}_2, \ldots$ was an arbitrary enumeration of $\mathfrak{G}$ this proves (iv). (Of course, if $\mathfrak{G}$ is finite, the convergence considerations are unnecessary, the sum $\sum_a$ being simply equal to $\eta_c(x)$.)

Consider now $\overline{A} \in \mathbf{M}$. The spatial isomorphism $F \mapsto \overline{W}F$ in $\mathfrak{H}_{S\mathfrak{G}}$ carries I into J and therefore $\mathbf{M}'$ into $\mathbf{M}$. So we have $\overline{A} = \overline{W}\overline{B}\overline{W}$, $\overline{B} \in \mathbf{M}'$. This $\overline{B} \cong [[\chi_c(x)]]_{x \in S, c \in \mathfrak{G}}$. Now the computation at the end of the proof of Lemma 12.3.2 shows that then $\overline{A} \cong [[\chi_c(x)]]_{x \in S, c \in \mathfrak{G}}$ with the same $\chi_c(x)$. Therefore, the rules of computation established in $\mathbf{M}'$ carry over immediately to $\mathbf{M}$.

After these preparations we are in the position to take the decisive step:

Definition 12.4.1. Assume $\overline{A} \in \mathbf{M}$ or $\mathbf{M}'$. Form the system of functions $\chi_c(x)$ with $\overline{A} \cong [[\chi_c(x)]]_{x \in S, c \in \mathfrak{G}}$ (cf. Lemma 12.4.1). Consider the two following cases:

(α) $\mu(S)$ is finite. Then, as $\chi_1(x)$ is bounded and measurable, $\int_S \chi_1(x)\, dx$ exists, and it is a finite complex number.

(β) $\chi_1(x) \geq 0$ for all $x \in S$. Then $\int_S \chi_1(x)\, dx$ exists again, and it is a real number $\geq 0, \leq \infty$.

In both cases define

$$t(\overline{A}) = \int_S \chi_1(x)\, dx.$$

In what follows immediately, we will really make use of case (β) only; case (α) is needed for later applications (§15.4).

Lemma 12.4.2. *Assume $\overline{A}, \overline{B} \in \mathbf{M}$ or $\mathbf{M}'$. Then we have:*

(i) $t(\alpha\overline{A}) = \overline{\alpha t(\overline{A})}$.

(ii) $t(\overline{A}^*) = \overline{t(\overline{A})}$.

(iii) $t(\overline{A} + \overline{B}) = t(\overline{A}) + t(\overline{B})$.

These equations are to be so understood, that whenever case (α) or (β) holds for the right sides, it holds for the left sides too (for (i) with case (β) however, $\alpha \geq 0$ is needed), and both are equal. Finally we have (automatically with case (β) on both sides):

(iv) $t(\overline{A}^*\overline{A}) = t(\overline{A}\overline{A}^*) \geq 0$.

Proof: (i)–(iii) follow directly from Lemma 12.4.1, (i)–(iii) resp. In order to prove (iv), put

$$\overline{A}^*\overline{A} \cong [[\omega_c(x)]]_{x\in S, c\in\mathfrak{G}}, \qquad \overline{A}\overline{A}^* \cong [[\nu_c(x)]]_{x\in S, c\in\mathfrak{G}}.$$

As $\overline{A} \cong [[\chi_c(x)]]_{x\in S, c\in\mathfrak{G}}$, we have

$$\overline{A}^* \cong [[\overline{\chi_{c^{-1}}(xc^{-1})}]]_{x\in S, c\in\mathfrak{G}}$$

and

$$\begin{aligned} \omega_c(x) &= \sum_a \chi_{ca^{-1}}(xa^{-1})\overline{\chi_{a^{-1}}(xa^{-1})} \\ &= \sum_a \chi_{ca}(xa)\overline{\chi_a(xa)}, \\ \nu_c(x) &= \sum_a \chi_a(x)\overline{\chi_{ac^{-1}}(xc^{-1})}. \end{aligned}$$

(Use Lemma 12.4.1, (ii) and (iv). The sums converge “en mesure,” irrespectively of the order.) Thus

$$\omega_1(x) = \sum_a |\chi_a(xa)|^2, \qquad \nu_1(x) = \sum_a |\chi_a(x)|^2.$$

We have case (β) for both, and

$$\begin{aligned} t(\overline{A}^*\overline{A}) &= \int_S \omega_1(x)\,dx = \sum_a \int_S |\chi_a(xa)|^2\,dx \\ &= \sum_a \int_S |\chi_a(x)|^2\,dx, \\ t(\overline{A}\overline{A}^*) &= \int_S \nu_1(x)\,dx = \sum_a \int_S |\chi_a(x)|^2\,dx, \end{aligned}$$

proving our statements.

Lemma 12.4.3. *If $\overline{E}$ runs over all projections in $\mathbf{M}$ or in $\mathbf{M}'$ then $t(\overline{E})$ is always defined (case (β) holds), and it is a relative dimension function for $\mathbf{M}$ resp. $\mathbf{M}'$. In this normalisation the C of Theorem X is $= 1$.*

If $T \subset S$ is measurable, define

$$e_T(x) = \begin{cases} 1, & x \in T, \\ 0, & x \notin T. \end{cases}$$

Then an $\overline{E} \cong [[\delta_c e_T(x)]]_{x \in S, c \in \mathfrak{G}}$ exists, in $\mathbf{M}$ as well as in $\mathbf{M}'$, it is a projection, and $t(\overline{E}) = \mu(T)$.

Proof: As $\overline{E}^* \overline{E} = \overline{E}^2 = \overline{E}$ for every projection $\overline{E}$, therefore Lemma 12.4.2 (iv) secures case (β) for $\overline{E}$ and $t(\overline{E}) \geq 0, \leq \infty$. If $T \subset S$ is measurable, then

$$\begin{aligned} \overline{L}_{e_T(x)} &= \langle \delta_{ab^{-1}} L_{e_T(x)} \rangle_{a,b \in \mathfrak{G}} \\ &= \langle L_{\delta_{ab^{-1}} e_T(x)} U_{b^{-1}a} \rangle_{a,b \in \mathfrak{G}} \cong [[\delta_c e_T(x)]]_{x \in S, c \in \mathfrak{G}} \end{aligned}$$

in $\mathbf{M}$ and

$$\begin{aligned} \overline{M}_{e_T(x)} &= \langle \delta_{ab^{-1}} L_{e_T(xa^{-1})} \rangle_{a,b \in \mathfrak{G}} \\ &= \langle L_{\delta_{ab^{-1}} e_T(xb^{-1})} \rangle_{a,b \in \mathfrak{G}} \cong [[\delta_c e_T(x)]]_{x \in S, c \in \mathfrak{G}} \end{aligned}$$

in $\mathbf{M}'$. So the $\overline{E}$'s required at the end of our Lemma exist; clearly $\overline{E}^* = \overline{E}$, $\overline{E}^2 = \overline{E}$ so that $\overline{E}$ is a projection; and

$$t(\overline{E}) = \int_S e_T(x)\, dx = \int_T 1\, dx = \mu(T).$$

Thus the last statement of our Lemma is true.

As we observed in the proof of Lemma 12.1.1, there exists at least one element of the sequence $T^{(1)}, T^{(2)}, \ldots$ from Definition 12.1.2, (iv), for which $\mu(T^{(i)}) > 0, < \infty$. So we have for its $\overline{E}$ (cf. above) $t(\overline{E}) > 0, < \infty$.

Now Lemma 8.3.5 implies that $t(\overline{E})$ is a relative dimension function for $\mathbf{M}$ resp. $\mathbf{M}'$ if we can only establish for it the conditions (ii), (iii) in Definition 8.2.1. (iii) follows immediately from Lemma 12.4.2, (iii), because under those assumptions $P_{[\mathfrak{M},\mathfrak{N}]} = P_{\mathfrak{M}} + P_{\mathfrak{N}}$. (ii) follows from Lemma 12.4.2, (iv), because $\overline{E} \sim \overline{F}$ $(\cdots \mathbf{M})$ resp.

$(\cdots \mathbf{M}')$ implies the existence of a U ($\in \mathbf{M}$ resp. $\in \mathbf{M}'$) with $U^*U = \overline{E}$, $UU^* = \overline{F}$ (cf. Lemma 4.3.1).

Our last task is to determine the C of Theorem X. If $\overline{E} \in \mathbf{M}$, $\overline{E} \cong [[\chi_c(x)]]_{x\in S, c\in\mathfrak{G}}$ then we saw at the end of the proof of Lemma 12.4.1 that $\overline{WEW} \in \mathbf{M}'$, $\overline{WEW} \cong [[\chi_c(x)]]_{x\in S, c\in\mathfrak{G}}$. Thus $t(\overline{E}) = t(\overline{WEW})$. Now, owing to the character of the spatial isomorphism $F \mapsto \overline{W}F$ it carries every $\mathfrak{M}_F^{\mathbf{M}'}$ into the corresponding $\mathfrak{M}_{\overline{W}F}^{\mathbf{M}}$ therefore

$$\overline{W}\overline{E}_F^{\mathbf{M}'}\overline{W} = \overline{E}_{\overline{W}F}^{\mathbf{M}}.$$

This proves $t(\overline{E}_F^{\mathbf{M}'}) = t(\overline{E}_{\overline{W}F}^{\mathbf{M}})$ and thus, if $\overline{W}F = \pm F$, $t(\overline{E}_F^{\mathbf{M}'}) = t(\overline{E}_F^{\mathbf{M}})$.

If besides $F \neq 0$, then $\overline{E}_F^{\mathbf{M}'} \neq 0$ and (as $t(\overline{E})$ is a relative weight function for $\mathbf{M}$), $t(\overline{E}_F^{\mathbf{M}'}) > 0$. So the above relations imply, considering $t(\overline{E}_F^{\mathbf{M}}) = Ct(\overline{E}_F^{\mathbf{M}'})$, that $C = 1$. So we need only to find an $F \neq 0$ with $\overline{W}F = \pm F$. Choose any $G \neq 0$; as $\overline{W}^2 = 1$ we have $\overline{W}(G \pm \overline{W}G) = \pm(G \pm \overline{W}G)$ and as

$$G = \frac{1}{2}\big((G + \overline{W}G) + (G - \overline{W}G)\big)$$

therefore both $G \pm \overline{W}G$ cannot be $= 0$. So either $F = G + \overline{W}G$ or $F = G - \overline{W}G$ meets our requirements.

Thus all parts of our Lemma are proved.

We restate the results obtained thus far:

Theorem XI. *Let $\mathfrak{G}$ be a finite or countably infinite group (Definition 12.1.1), S a space with a Lebesgue outer measure (Definitions 12.1.2 and 12.1.3). Let $\mathfrak{H}_{\mathfrak{G}}$, $\mathfrak{H}_S$, $\mathfrak{H}_{S\mathfrak{G}}$ be the spaces derived from them (Definition 12.1.4). Assume that $\mathfrak{G}$ is ergodic in S (Definition 12.1.5).*

Form the operators $\overline{U}_{a_0}$, $\overline{V}_{a_0}$, $\overline{W}$, $\overline{L}_{\varphi(x)}$, $\overline{M}_{\varphi(x)}$ in $\mathfrak{H}_{S\mathfrak{G}}$ (Lemma 12.3.1) and with their help the rings $\mathbf{M}, \mathbf{M}'$ (Lemma 12.3.4).

$F \mapsto \overline{W}F$ is an involutory isomorphism of $\mathfrak{H}_{S\mathfrak{G}}$ which interchanges $\mathbf{M}$ with $\mathbf{M}'$.

Certain systems of bounded and measurable x-functions $\chi_c(x)$, $c \in \mathfrak{G}$ are in a one-to-one correspondence with all $\overline{A} \in \mathbf{M}$ and at the same time with all $\overline{A} \in \mathbf{M}'$. We denote both correspondences by

$$\overline{A} \cong [[\chi_c(x)]]_{x\in S, c\in\mathfrak{G}}$$

(Lemma 12.4.1). Using these correspondences, the computation rules in $\mathbf{M}$ as well as those in $\mathbf{M}'$ can be expressed in terms of the $\chi_c(x)$; we get in both cases the same rules (Lemma 12.4.1, (i)–(iv)).

$\mathbf{M}, \mathbf{M}'$ are coupled factors. Relative dimension functions $D_{\mathbf{M}}(\overline{E})$, $D_{\mathbf{M}'}(\overline{E})$ for $\mathbf{M}$ resp. $\mathbf{M}'$ ($\overline{E}$ a projection $\in \mathbf{M}$ resp. $\in \mathbf{M}'$) can be defined as follows: Write

$$\overline{E} \cong [[\chi_c(x)]]_{x \in S, c \in \mathfrak{G}}$$

then

$$D_{\mathbf{M}}(\overline{E}) \quad \text{resp.} \quad D_{\mathbf{M}'}(\overline{E}) = \int_S \chi_1(x)\, dx.$$

(We have $\chi_1(x) \geq 0$ for all $x \in S$, therefore the integral on the right hand is always defined; it represents a real number ≥ 0, $\leq \infty$).

In this normalisation the C of Theorem X is $= 1$.

If $T \subset S$ is measurable, and

$$e_T(x) = \begin{cases} 1, & x \in T, \\ 0, & x \notin T, \end{cases}$$

then we have for both projections $\overline{E} = \overline{L}_{e_T(x)} \in \mathbf{M}$ and $\overline{E}' = \overline{M}_{e_T(x)} \in \mathbf{M}'$ this: $D_{\mathbf{M}}(\overline{E})$ resp. $D_{\mathbf{M}'}(\overline{E}') = \mu(T)$.

Chapter XIII: Properties of $\mathbf{M}, \mathbf{M}'$

13.1. The characterisation of $\mathbf{M}, \mathbf{M}'$ by Theorem XI makes the determination of their classes easy. This determination is the object of the discussions which follow. If we wanted to use these $\mathbf{M}, \mathbf{M}'$ solely to obtain an example of case (II), then it could be shortened considerably: the examples (α)–(γ) after Lemma 13.2.1 could be obtained in a few lines, together with Lemma 13.1.2 for that special case, while Lemma 13.1.1 would be entirely unnecessary. But owing to the general interest of these $\mathbf{M}, \mathbf{M}'$ we prefer to give an exhaustive discussion.

In what follows we use throughout the notations and the assumptions of Theorem XI.

Lemma 13.1.1. *Case* (I) *holds for $\mathbf{M}, \mathbf{M}'$ if and only if a point $x \in S$ with $\mu^*((x)) > 0$ exists. If this is the case, S can be characterised, after the omission of a subset of measure* 0, *(which therefore is unessential), as follows:*

Every one-point set (x) in S ($x \in S$) is measurable, and $\mu((x))$ has a value independent of x: $\mu((x)) = \epsilon > 0, < \infty$, $\mathfrak{G}$ is simply transitive in S, that is if x_0 is any fixed element of S, then $a \rightleftarrows x_0 a$ is a one to one mapping of $\mathfrak{G}$ on S.

If $\mathfrak{G}$ *and* S *have* $n = 1, 2, \ldots, \infty$ *elements, then* $\mathbf{M}, \mathbf{M}'$ *are both in case* (I_n). *The* $D_{\mathbf{M}}(E)$, $D_{\mathbf{M}'}(E)$ *of Theorem XI go over into the standard normalisation, if both are multiplied by* $1/\epsilon$.

Proof: Assume first that case (I) holds. Then all values of $D_{\mathbf{M}}(\overline{E})$ are integer multiples of an $\epsilon_0 > 0$ and so all $\mu(T)$, $T \subset S$ and measurable, are so. Now a T with $\mu(T) > 0, < \infty$ exists (cf. the beginning of the proof of Lemma 12.4.3), therefore for one of these $T \subset S$, $\mu(T)$ assumes a minimal value. Denote it by T_0. Thus for any measurable $T \subset T_0$ we have, owing to $0 \leq \mu(T) \leq \mu(T_0)$ either $\mu(T_0) = \mu(T)$ or $\mu(T) = 0$.

Now consider the $T^{(1)}, T^{(2)}, \ldots$ of Definition 12.1.2, (iv), assuming that $T^{(1)} \dotplus T^{(2)} \dotplus \cdots = S$ (cf. the remark before Lemma 12.1.1), and that 0 is one of them (it can be added to their system without altering its character). Consider an $x \in S$. Denote those n's for which $x \in T^{(n)}$ by $m_1, m_2, \ldots$ and those for which $x \notin T^{(n)}$ by $n_1, n_2, \ldots$. Owing to Definition 12.1.2 (iv), we have

$$(x) = T^{(m_1)} \cdot T^{(m_2)} \cdots (S - T^{(n_1)}) \cdot (S - T^{(n_2)}) \cdots .$$

As $x \in T^{(1)} \dotplus T^{(2)} \dotplus \cdots$ the sequence $m_1, m_2, \ldots$ is not empty; as 0 is a $T^{(n)}$ the sequence $n_1, n_2, \ldots$ is not empty either; by repeating m or n infinitely many times we can secure that both sequences are infinite. So we see: For every $x \in S$ there exist two infinite sequences $m_1, m_2, \ldots$ and $n_1, n_2, \ldots$ so that

$$(x) = \prod_{i=1}^{\infty} T^{(m_i)} \cdot (S - T^{(n_i)}).$$

Now assume $x \in T_0$ and $\mu^*((x)) = 0$. Put

$$U_j = T_0 \cdot \prod_{i=1}^{j} T^{(m_i)} \cdot (S - T^{(n_i)}).$$

Then we have $T_0 \supset U_1 \supset U_2 \supset \cdots$, $(x) = U_1 U_2 \cdots$. So $\lim_{j\to\infty} \mu(U_j) = \mu((x)) = 0$ (all U_j are measurable along with T_0 and the $T^{(n)}$), and so a j with $\mu(U_j) < \mu(T_0)$ exists, (remember that $\mu(T_0) > 0$). By our assumptions on T_0 this implies $\mu(U_j) = 0$.

Now consider all sets U of this form:

$$U = T_0 \cdot \prod_{i=1}^{j} T^{(m_i)} \cdot (S - T^{(n_i)}),$$

$j = 1, 2, \ldots; m_1, \ldots, m_j, n_1, \ldots, n_j = 1, 2, \ldots$ for which $\mu(U) = 0$. Their set is countable: $U^{(1)}, U^{(2)}, \ldots$. So $\mu(U^{(1)} \dotplus U^{(2)} \dotplus \cdots) = 0$. Thus an $x \in T_0$ with $x \notin U^{(1)} \dotplus U^{(2)} \dotplus \cdots$ exists. By what we proved above, this excludes $\mu^*((x)) = 0$. So an $x \in S$ with $\mu^*((x)) > 0$ exists.

Assume now conversely the existence of an $x \in S$ with $\mu^*((x)) > 0$. As

$$(x) = \prod_{i=1}^{\infty} T^{(m_i)}(S - T^{(n_i)})$$

(cf. above), (x) is measurable, $\mu((x)) > 0$. Besides $(x) \subset T^{(m_1)}$, $\mu((x)) \leq \mu(T^{(m_1)}) < \infty$. Put $\epsilon = \mu((x)) > 0, < \infty$ then $\mu((xa)) = \mu((x)) = \epsilon$ for every $a \in \mathfrak{G}$. Form now $S_1 = (xa; a \in \mathfrak{G})$, S_1 is exactly invariant under every transformation $x \mapsto xb$, $b \in \mathfrak{G}$. Thus the ergodicity of $\mathfrak{G}$ in S (cf. Definition 12.1.5, (iv)) requires $\mu(S_1) = 0$ or $\mu(S - S_1) = 0$. But $\mu(S_1) \geq \mu((x)) = \epsilon > 0$ so $\mu(S - S_1) = 0$. Omit the set $S - S_1$ from S, then we have $S_1 = S$, that is $S = (xa; a \in \mathfrak{G})$. So we have for every $x_0 \in S$, $\mu((x_0)) = \epsilon$. By Definition 12.1.5 (iii), $a \neq 1$ implies $x_0 \neq x_0 a$, so $a \neq b$ implies $x_0 a \neq x_0 b$. Thus all statements of the second section of our Lemma are established for this case.

Choose an $x_0 \in S$ and define

$$e_{(x_0)}(x) = \begin{cases} 1, & x = x_0, \\ 0, & x \neq x_0. \end{cases}$$

Then $\overline{L}_{e_{(x_0)}(x)} \in \mathbf{M}$, it is a projection, and $D_{\mathbf{M}'}(\overline{L}_{e_{(x_0)}(x)}) = \mu((x_0)) = \epsilon$. We claim that $\overline{L}_{e_{(x_0)}(x)}$ is minimal in $\mathbf{M}$. This is the case, if for every F with $\overline{L}_{e_{(x_0)}(x)} F = F \neq 0$, $E_F^{\mathbf{M}'} = \overline{L}_{e_{(x_0)}(x)}$ (cf. Lemma 5.1.2, obviously $\overline{L}_{e_{(x_0)}(x)} \neq 0$). Now $\overline{L}_{e_{(x_0)}(x)} F = F$ means $e_{x_0}(x) F(x, a) = F(x, a)$ that is $F(x, a) = 0$ if $x \neq x_0$. We must show: One such F, say F_0, gives if it is $\neq 0$ by application of operators $\overline{A} \in \mathbf{M}'$ all others. As $F_0 \neq 0$ but $F_0(x, a) = 0$ for $x \neq x_0$ an $a_0 \in \mathfrak{G}$ with $F_0(x_0, a_0) \neq 0$ exists. Now

$$\frac{1}{\overline{\overline{F_0(x_0, a_0)}}} \overline{M}_{e_{(x_0 a_1^{-1})}(x)} \overline{V}_{a_1 a_0^{-1}} \in \mathbf{M}'$$

and it transforms $F_0(x, a)$ into

$$F'_{a_1}(x, a) = \begin{cases} 1, & x = x_0, a = a_1, \\ 0, & \text{otherwise} \end{cases}$$

and all desired $F(x,a)$ are linear aggregates of these.

Thus $\overline{L}_{e_{(x_0)}(x)}$ is minimal in $\mathbf{M}$ and therefore case (I) holds for $\mathbf{M}, \mathbf{M}'$ (cf. Theorem IV). So the necessary and sufficient character of our condition is established.

As the second section of our Lemma has already been verified, all we need to consider is the last one. As $\overline{L}_{e_{(x_0)}(x)}$ is minimal in $\mathbf{M}$ and $D_{\mathbf{M}'}(\overline{L}_{e_{(x_0)}(x)}) = \epsilon$ the standard normalisation of $D_{\mathbf{M}'}(\overline{E}')$ obtains by multiplying it by $1/\epsilon$. As the spatial isomorphism $F \mapsto \overline{W}F$ interchanges $\mathbf{M}, D_{\mathbf{M}}(\overline{E})$ with $\mathbf{M}', D_{\mathbf{M}'}(\overline{E}')$ the same is true for $D_{\mathbf{M}}(\overline{E})$.

Finally, if S (that is $\mathfrak{G}$) has $n = 1, 2, \ldots, \infty$ elements, say $x_1, \ldots, x_n$ (if $n = \infty$ omit x_n), then

$$\begin{aligned} D_{\mathbf{M}'}(1) = D_{\mathbf{M}'}(\overline{L}_1) &= D_{\mathbf{M}'}(\overline{L}_{e_{(x_1)}(x)+\cdots+e_{(x_n)}(x)}) \\ &= D_{\mathbf{M}'}(\overline{L}_{e_{(x_1)}(x)} + \cdots + \overline{L}_{e_{(x_n)}(x)}) \\ &= D_{\mathbf{M}'}(\overline{L}_{e_{(x_1)}(x)}) + \cdots + D_{\mathbf{M}'}(\overline{L}_{e_{(x_n)}(x)}) = n\epsilon. \end{aligned}$$

In the standard normalisation this gives n, so we have case (I_n) for $\mathbf{M}'$. The spatial isomorphism gives the same for $\mathbf{M}$.

Lemma 13.1.2. *Case* (II) *holds for* $\mathbf{M}, \mathbf{M}'$ *if and only if there is for every* $x \in S$, $\mu^*((x)) = 0$. *If* $\mu(S)$ *is finite, then case* (II_1) *holds for both* $\mathbf{M}, \mathbf{M}'$ *and the* $D_{\mathbf{M}}(\overline{E})$, $D_{\mathbf{M}'}(\overline{E}')$ *of Theorem XI go over into the standard normalisation, if both are multiplied by* $1/\mu(S)$. *If* $\mu(S)$ *is infinite, then case* (II_∞) *holds for both* $\mathbf{M}, \mathbf{M}'$.

Proof: As an $E' \in \mathbf{M}'$ with $D_{\mathbf{M}'}(\overline{E}') > 0, < \infty$ exists (cf. the beginning of the proof of Lemma 12.4.3), case (III) cannot hold. So either case (I) or case (II) holds, and therefore Lemma 13.1.1 implies that our present condition is necessary and sufficient for case (II).

As

$$D_{\mathbf{M}}(1) = D_{\mathbf{M}}(\overline{M}_1) = D_{\mathbf{M}}(\overline{M}_{e_S(x)}) = \mu(S)$$

and

$$D_{\mathbf{M}'}(1) = D_{\mathbf{M}'}(\overline{L}_1) = D_{\mathbf{M}'}(\overline{L}_{e_S(x)}) = \mu(S),$$

the other statements of our Lemma follow immediately.

The two preceding Lemmas characterise $\mathbf{M}, \mathbf{M}'$ completely. We see (using the terminology of Theorem VIII): The $\mathbf{M}, \mathbf{M}'$ as constructed in Theorem XI are always of the same class; this class is finite or infinite if $\mu(S)$ is finite resp. infinite; it is discrete or continuous if the measure $\mu(T)$, $T \subset S$ is discrete resp. continuous (that is, if an $x \in S$ with $\mu^*((x)) > 0$ does resp. does not exist); it is never purely infinite. And the difference between the discrete and the continuous cases corresponds to that one between effective transitivity of $\mathfrak{G}$ in S, and ergodicity without such transitivity.

13.2. Examples of case (I), based on Lemma 13.1.1, are so obvious that we need not be concerned with them. (Owing to the one-to-one correspondence between S and $\mathfrak{G}$ we may even put, without any loss of generality, $S = \mathfrak{G}$. Then we can make $\epsilon = 1$ and then $\mu^*(T)$, $T \subset S$ becomes $\mu^*(T) =$ (number of elements in T).) We wish however to give effective examples of case (II), based on Lemma 13.1.2. These examples will be of a very special type, as we will only consider Abelian groups $\mathfrak{G}$, and even those highly specialised.

Lemma 13.2.1. *Let S be either the set of all real numbers, S_∞, or the set of all real numbers ≥ 0, < 1, which can be (and in what follows will be) looked at as the set of all real numbers* (mod 1)*, S_1. Use the common Lebesgue measure in S. Let $\mathfrak{G}$ be a module of real numbers, that is a finite or countably infinite set with these properties: $0 \in \mathfrak{G}$, $a, b \in \mathfrak{G}$ imply $-a \in \mathfrak{G}$, $a + b \in \mathfrak{G}$. $\mathfrak{G}$ is a group, if we define unit $= 0$, $a^{-1} = -a$, $ab = a + b$.*

$\mathfrak{G}$ is an m-group in S, if we define $xa = x + a$. For $S = S_1$ we insist that $1 \in \mathfrak{G}$.

Excepting $\mathfrak{G} = 0$ and $\mathfrak{G} = (\dots, -2a_0, -a_0, 0, a_0, 2a_0, \dots)$ (for a fixed $a_0 > 0$; if $S = S_1$, then necessarily $a_0 = 1/n$, $n = 1, 2, \dots$), $\mathfrak{G}$ is always ergodic in S.

Proof: $\mathfrak{G}$ satisfies obviously Definition 12.1.1. S satisfies Definition 12.1.2 because it is a special case of Example b) given after it; it is easy to verify Definition 12.1.5, (i)–(iii), for $\mathfrak{G}$ and S, that is the m-group character. It remains for us to decide about Definition 12.1.5 (iv), that is ergodicity.

Every $T \subset S$ is invariant under $\mathfrak{G} = (0)$ and the set

$$T = (x;\ pa_0 \leq x < (p + \tfrac{1}{2})a_0, \quad p = 0, \pm 1, \pm 2, \dots) \subset S$$

(for $S = S_1$ take it (mod 1)) is invariant under

$$\mathfrak{G} = (\dots, -2a_0, -a_0, 0, a_0, 2a_0, \dots).$$

Thus these $\mathfrak{G}$ are not ergodic.

Assume now that $\mathfrak{G}$ is not of this form. If $S = S_\infty$ and some $a_0 \neq 0$ is in $\mathfrak{G}$ we can map S_∞ and $\mathfrak{G}$ by the one-to-one transformation $x \longleftrightarrow (1/a_0)x$, $a \longleftrightarrow (1/a_0)a$. This is an isomorphism for everything that is important now and it carries a_0 into 1. For this reason we may assume $1 \in \mathfrak{G}$ originally. If $1 \in \mathfrak{G}$, then all sets T which are invariant under $\mathfrak{G}$ can be looked at (mod 1); then we can do this for $S = S_\infty$, that is we obtain $S = S_1$. So we need to discuss $S = S_1$ only.

Now $\mu(S) = \mu(S_1) = 1$ is finite, so every

$$e_T(x) = \begin{cases} 1, & x \in T, \\ 0, & x \notin T \end{cases}$$

is $e_T \in \mathfrak{H}_S$. Ergodicity means: $U_a(e_T) = e_T$ for all $a \in \mathfrak{G}$ means $e_T = 0$ or 1 or more generally: $f \in \mathfrak{H}_S, U_a f = f$ for all $a \in \mathfrak{G}$ means $f =$ constant.

The $\varphi_n(x) = e^{2\pi i n x}$, $n = 0, \pm 1, \pm 2, \ldots$ form a complete normalised orthogonal set in $\mathfrak{H}_S$, $U_a \varphi_n = e^{2\pi i n a} \varphi_n$. Put

$$f = \sum_{n=-\infty}^{\infty} \alpha_n \varphi_n, \qquad \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \text{ finite;}$$

this orthogonal expansion in the space $\mathfrak{H}_S$ is numerically the "in the mean" convergent Fourier expansion of f, then

$$U_a f = \sum_{n=-\infty}^{\infty} e^{2\pi i n a} \alpha_n \varphi_n.$$

$U_a f = f$ means $\alpha_n = e^{2\pi i n a} \alpha_n$, that is $\alpha_n = 0$, except if na is an integer for every $a \in \mathfrak{G}$. If this happens for any $n \neq 0$ then it is the case for $|n| > 0$ too, so we may assume $n = 1, 2, \ldots$.

Denote the set of all na, $a \in \mathfrak{G}$ by $\mathfrak{G}'$. $\mathfrak{G}'$ then consists of integers. $0 \in \mathfrak{G}'$; $p, q \in \mathfrak{G}'$ imply $-p, p + q \in \mathfrak{G}'$. Thus $\mathfrak{G}'$ consists of all multiples of a fixed $q = 1, 2, \ldots$. As $1 \in \mathfrak{G}$, $n \in \mathfrak{G}'$, q is a divisor of n, say $n = qm$. Thus

$$\mathfrak{G} = \left(\ldots, -\frac{2q}{n}, -\frac{q}{n}, 0, \frac{q}{n}, \frac{2q}{n}, \ldots\right) = \left(\ldots, -\frac{2}{m}, -\frac{1}{m}, 0, \frac{1}{m}, \frac{2}{m}, \ldots\right),$$

contradicting our assumption concerning $\mathfrak{G}$.

So $\alpha_n = 0$ whenever $n \neq 0$ and therefore $f = \alpha_0 \varphi_0 = \alpha_0 =$ constant. This completes the proof.

Now as $\mu(S_\infty) = \infty$ and $\mu(S_1) = 1$ we see by Lemma 13.1.2 that every ergodic $\mathfrak{G}$ gives an example of $\mathbf{M}, \mathbf{M}'$ in class (II_∞) resp. (II_1). Some simple $\mathfrak{G}'$ which are obviously ergodic by Lemma 13.2.1 are these:

(α) $\mathfrak{G}$ is the set $\mathfrak{G}_\theta$ of all $m + n\theta$, $m, n = 0, \pm 1, \pm 2, \dots$ where θ is any given irrational number.

(β) $\mathfrak{G}$ is the set $\mathfrak{G}_{\text{rat.}}$ of all rational numbers.

(γ) $\mathfrak{G}$ is the set $\mathfrak{G}_{\text{rat.},p}$ of all rational numbers of the form

$$\frac{m}{p^n}, \qquad m = 0, \pm 1, \pm 2, \dots, \qquad n = 0, 1, 2, \dots$$

where p is any given number $2, 3, \dots$ (not necessarily a prime!).

13.3. The above examples (with $S = S_\infty$) show that factorisations $\mathbf{M}, \mathbf{M}'$ of the classes $(\mathrm{II}_\infty), (\mathrm{II}_\infty)$ exist. Then Lemma 11.4.3, (ii) and the remarks after this Lemma show that the combinations of classes $(\mathrm{II}_1), (\mathrm{II}_\infty)$; $(\mathrm{II}_\infty), (\mathrm{II}_1)$; $(\mathrm{II}_1), (\mathrm{II}_1)$ exist too, and that in the last case the C of Theorem X can be prescribed arbitrarily. That any combination of cases $(\mathrm{I}_m), (\mathrm{I}_n)$, $m, n = 1, 2, \dots$, exists, has been remarked at the end of §8.6. All other combinations, except $(\mathrm{III}_\infty), (\mathrm{III}_\infty)$, have been excluded by Theorem X. By Theorem X again $(\mathrm{III}_\infty), (\mathrm{III}_\infty)$ exist if and only if factors in case (III) exist at all. So we have:

Theorem XII. *Factorisations* $\mathbf{M}, \mathbf{M}'$ *belonging to the following classes do exist:* $(\mathrm{I}_m), (\mathrm{I}_n)$, $m, n = 1, 2, \dots, \infty$; $(\mathrm{II}_m), (\mathrm{II}_n)$, $m, n = 1, \infty$, *and in case* $m, n = 1$ *even the* C *of Theorem X can be prescribed arbitrarily.* $(\mathrm{III}_\infty), (\mathrm{III}_\infty)$ *exists if and only if factors in case* (III) *exist, a problem as yet unsolved. All other combinations of classes do not exist.*

This answers Problem 2, negatively (considering Lemma 8.6.1), and answers Problem 3, Problem 4 as completely as it is possible at the present state of our knowledge.

13.4. We can use our $\mathbf{M}, \mathbf{M}'$ to construct a factorisation in which the factors $\mathbf{M}$ and $\mathbf{N}$ are not coupled ($\mathbf{M} \neq \mathbf{N}'$), and thus answer Problem 1 negatively. This leads to the partial answer to Problem 10 discussed in §11.2.

In what follows we use throughout the notations and the assumptions of Theorem XI.

Lemma 13.4.1. *Let* $\mathfrak{G}^0$ *be a proper subgroup of* $\mathfrak{G}$ *which is ergodic in* S. *Every* $\overline{A} \in \mathbf{M}'$ *is* $\overline{A} \cong [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$, *consider those for which* $\chi_c(x) = 0$ *whenever* $c \notin \mathfrak{G}^0$. *Denote their set by* $\mathbf{N}$. *Then* $\mathbf{N}$ *is a ring,* $\mathbf{N} \subsetneq \mathbf{M}'$, $\mathbf{M}' \cdot \mathbf{N}' = (\alpha 1)$.

Proof: $\mathbf{N} \subset \mathbf{M}'$ is obvious. As $\overline{V}_{a_0} \in \mathbf{M}'$,

$$\overline{V}_{a_0} \cong [[\delta_{c a_0^{-1}}]]_{x \in S,\, c \in \mathfrak{G}},$$

therefore $\overline{V}_{a_0} \in \mathbf{N}$ if and only if $a_0 \in \mathfrak{G}^0$, so $\mathfrak{G}^0 \subsetneq \mathfrak{G}$ implies $\mathbf{N} \subsetneq \mathbf{M}'$.

The computation rules of Lemma 12.4.1 show that $A, B \in \mathbf{N}$ imply $\alpha A, A^*, A + B, AB \in \mathbf{N}$. So we must only prove that $\mathbf{N}$ is weakly closed in order to show that it is a ring. As $\mathbf{M}'$ is weakly closed (being a ring) even relative weak closure of $\mathbf{N}$ in $\mathbf{M}'$ suffices.

Now an $\overline{A} \in \mathbf{M}'$ belongs to $\mathbf{N}$ if and only if $a_0 \notin \mathfrak{G}^0$ implies $\chi_{a_0}(x) = 0$ for $\overline{A} \cong [[\chi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$; that is if $ab^{-1} \notin \mathfrak{G}^0$ implies $A_{a,b} = 0$ for $\overline{A} \sim \langle A_{a,b} \rangle_{a,b \in \mathfrak{G}}$. So we need only to prove: If two $a, b \in \mathfrak{G}$ are given, then the set of all $\overline{A} \sim \langle A_{a',b'} \rangle_{a',b' \in \mathfrak{G}}$ with $A_{a,b} = 0$ is weakly closed. Now

$$(A_{a,b} f, g) = (\overline{A} f^{[a]}, g^{[b]}),$$

where $h^{[c]}(x, d) = \delta_{c,d} h(x)$ ($h \in \mathfrak{H}_S$, $h^{[c]} \in \mathfrak{H}_{S\mathfrak{G}}$) and so our condition is $(\overline{A} f^{[a]}, g^{[b]}) = 0$ for every pair $f, g \in \mathfrak{H}_S$. This describes obviously a weakly closed set. Thus $\mathbf{N}$ is a ring and $\mathbf{N} \subsetneqq \mathbf{M}'$.

Assume now $\overline{A} \in \mathbf{M}' \cdot \mathbf{N}'$. As $\overline{M}_{\chi(x)} \in \mathbf{M}'$,

$$\overline{M}_{\chi(x)} \cong [[\delta_c \chi(x)]]_{x \in S,\, c \in \mathfrak{G}}$$

and as $c \notin \mathfrak{G}^0$ implies $c \neq 1$, $\delta_c = 0$ so we have $\overline{M}_{\chi(x)} \in \mathbf{N}$. So $\overline{A}$ must commute with every $\overline{M}_{\chi(x)}$ and (cf. above) with every $\overline{V}_{a_0}$, $a_0 \in \mathfrak{G}^0$. Now $\overline{A} \in \mathbf{M}'$, $\overline{A} \cong [[\xi_c(x)]]_{x \in S,\, c \in \mathfrak{G}}$ so our requirements are (by Lemma 2.4.3, (iv))

$$\xi_c(x)\chi(xc^{-1}) = \xi_c(x)\chi(x), \qquad \xi_{a_0^{-1} c a_0}(x a_0) = \xi_c(x).$$

The first equation reads

$$L_{\xi_c(x)}\chi = L_{\xi_c(x)} U_{c^{-1}} \chi$$

for all bounded $\chi \in \mathfrak{H}_S$. Thus it is for all

$$\chi(x) = e_T(x) = \begin{cases} 1, & x \in T, \\ 0, & x \notin T, \end{cases} \qquad T \subset S, \quad \mu(T) \text{ finite};$$

and for their linear aggregates and their condensation points; that is for all $\chi \in \mathfrak{H}_S$ (cf. the second section of the proof of Lemma 12.1.1). So $L_{\xi_c(x)} = L_{\xi_c(x)} U_{c^{-1}}$ which implies by Lemma 12.2.3, that $\xi_c(x) = 0$ if $c \neq 1$ (and so $c^{-1} \neq 1$).

The second equation reads:

$$U_{a_0}^{-1} L_{\xi_1(x)} U_{a_0} = L_{\xi_1(x)}.$$

Denote the set of all U_{a_0}, $a_0 \in \mathfrak{G}^0$ by $\mathfrak{U}^0$. Then we have $L_{\xi_1(x)} \in \mathfrak{L} \cdot (\mathfrak{U}^0)'$. As $\mathfrak{G}^0$ is ergodic in S, Lemma 12.2.4 applies: $L_{\xi_1(x)} = \alpha 1 = L_\alpha$, $\xi_1(x) = \alpha$. So we have $\xi_c(x) = \delta_c \alpha$, $\overline{A} = \alpha 1$. This completes the proof of $\mathbf{M}' \cdot \mathbf{N}' = (\alpha 1)$.

Lemma 13.4.2. *Let* $\mathfrak{G}^0, \mathbf{N}$ *be as above. Then* $\mathbf{M}, \mathbf{N}$ *is a factorisation, but* $\mathbf{M}, \mathbf{N}$ *are not coupled factors.*

Proof: As $\mathbf{N} \subset \mathbf{M}'$ and

$$\mathrm{R}(\mathbf{M}, \mathbf{N}) = (\mathrm{R}(\mathbf{M}, \mathbf{N}))'' = (\mathbf{M}' \cdot \mathbf{N}')' = (\alpha 1)' = \mathbf{B},$$

therefore $\mathbf{M}, \mathbf{N}$ is a factorisation. As $\mathbf{N} \subsetneqq \mathbf{M}'$ they are not coupled.

It is easy to give examples of groups $\mathfrak{G}, \mathfrak{G}^0$ as required by Lemma 13.4.1 and Lemma 13.4.2: We use the examples (α)–(γ) at the end of §13.1. So $\mathfrak{G} = \mathfrak{G}_\theta$, $\mathfrak{G}^0 = \mathfrak{G}_{k\theta}$, θ irrational, $k = 2, 3, \ldots$, will do; or $\mathfrak{G} = \mathfrak{G}_{\text{rat.}}$, $\mathfrak{G}^0 = \mathfrak{G}^0_{\text{rat.},q}$, $q = 2, 3, \ldots$ or $\mathfrak{G} = \mathfrak{G}_{\text{rat.},p}$, $\mathfrak{G}^0 = \mathfrak{G}_{\text{rat.},q}$, $p, q = 2, 3, \ldots$, q such a divisor of p no power of which is divisible by p (that is: q does not contain all prime factors of p, for instance $p = 6$, $q = 2$).

Thus Lemma 13.4.2 implies that the $\mathbf{M}, \mathbf{M}'$ of Theorem XI are not normal for $\mathfrak{G} = \mathfrak{G}_\theta, \mathfrak{G}_{\text{rat.}}, \mathfrak{G}_{\text{rat.},p}$ if p is not the power of a prime. (For $\mathbf{M}'$ this results directly, for $\mathbf{M}$ it follows then from the spatial isomorphism $F \mapsto \overline{W F}$ in $\mathfrak{H}_{S\mathfrak{G}}$ which interchanges $\mathbf{M}$ with $\mathbf{M}'$.) So we have now the examples of not normal factors of class (II) we wanted. The question whether all factors of class (II) are not normal, remains nevertheless open; we can even not decide whether $\mathbf{M}, \mathbf{M}'$ for $\mathfrak{G} = \mathfrak{G}_{\text{rat.},p}$, p power of a prime, are normal. (Cf. note at end.)

This is the extent to which we can answer Problem 10.

Part V: The finite cases

Chapter XIV: The Generalised additivity of $D_{\mathbf{M}}(\mathfrak{M})$

14.1. The object of the chapters which follow is the investigation of the finite cases (cf. Theorem VIII), that is, of all factors $\mathbf{M}$ which belong to cases (I_n), $n = 1, 2, \ldots$, or (II_1). The outstanding fact is that they all behave very similarly. This is remarkable, because an $\mathbf{M}$ in a case (I_n) is ring-isomorphic to the ring of all (bounded) operators of an n-dimensional Euclidean space and therefore completely elementary; while an **M** in case (II_1) lies in a Hilbert space $\mathfrak{H}$, unbounded operators can be $\eta\mathbf{M}$, operators in **M** can have continuous spectra, etc.

All our results will be valid for all finite cases, that is, both for (I_n) and (II_1). The really interesting application is of course (II_1), while most of our results are void or trivial or well known facts from linear geometry, when applied to (I_n). Our reason for including the (I_n) too in all our statements is only the desire to stress the analogy between (I_n) and (II_1), and to put every result concerning (II_1) at once in the right light by showing what it would mean for (I_n).

Our notations will be these: $\mathfrak{H}$ is a space; **M** a factor in $\mathfrak{H}$ belonging to a finite case, (I_n), $n = 1, 2, \ldots$, or (II_1); $D_{\mathbf{M}}(\mathfrak{M})$ a relative dimension function for **M** with the normalisation $D_{\mathbf{M}}(\mathfrak{H}) = 1$. For ($\mathrm{I}_n$) this is the standard normalisation multiplied by $1/n$; for (II_1) it is the standard normalisation itself. In the next § (and only there) however these assumptions will not be used, the class of **M** and the normalisation of $D_{\mathbf{M}}(\mathfrak{H})$ being arbitrary.

14.2. We discuss the behavior of $D_{\mathbf{M}}(\mathfrak{M})$ for such $\mathfrak{M}$'s which need not to be orthogonal, nor comparable ($\mathfrak{M} \subset \mathfrak{N}$), thus generalising the statements of Definition 8.2.1.

Lemma 14.2.1. *Assume $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$. Then*

$$D_{\mathbf{M}}([\mathfrak{M}, \mathfrak{N}] - \mathfrak{N}) \leq D_{\mathbf{M}}(\mathfrak{M}).$$

Proof: This follows immediately from Lemma 7.3.4.

Lemma 14.2.2. *Assume $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$ and $D_{\mathbf{M}}(\mathfrak{M}) > D_{\mathbf{M}}(\mathfrak{N})$. Then $\mathfrak{M} \cdot (\mathfrak{H} - \mathfrak{N}) \neq (0)$.*

Proof: We have, using Lemma 14.2.1,

$$\begin{aligned} D_{\mathbf{M}}([\mathfrak{N}, \mathfrak{H} - \mathfrak{M}] \cdot \mathfrak{M}) &= D_{\mathbf{M}}([\mathfrak{N}, \mathfrak{H} - \mathfrak{M}] - (\mathfrak{H} - \mathfrak{M})) \\ &\leq D_{\mathbf{M}}(\mathfrak{N}) < D_{\mathbf{M}}(\mathfrak{M}), \end{aligned}$$

so $[\mathfrak{N}, \mathfrak{H} - \mathfrak{M}] \cdot \mathfrak{M} \neq \mathfrak{M}$. But $[\mathfrak{N}, \mathfrak{H} - \mathfrak{M}] \cdot \mathfrak{M} \subset \mathfrak{M}$, therefore $\mathfrak{M} - [\mathfrak{N}, \mathfrak{H} - \mathfrak{M}] \cdot \mathfrak{M} \neq (0)$. Now

$$\begin{aligned} \mathfrak{M} - [\mathfrak{N}, \mathfrak{H} - \mathfrak{M}] \cdot \mathfrak{M} &= \mathfrak{M} \cdot (\mathfrak{H} - [\mathfrak{N}, \mathfrak{H} - \mathfrak{M}]) \\ &= \mathfrak{M}(\mathfrak{H} - \mathfrak{N})(\mathfrak{H} - (\mathfrak{H} - \mathfrak{M})) \\ &= \mathfrak{M} \cdot (\mathfrak{H} - \mathfrak{N}) \cdot \mathfrak{M} = \mathfrak{M} \cdot (\mathfrak{H} - \mathfrak{N}). \end{aligned}$$

So we proved $\mathfrak{M} \cdot (\mathfrak{H} - \mathfrak{N}) \neq (0)$.

Lemma 14.2.3. *Assume $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$. Then*

$$D_{\mathbf{M}}([\mathfrak{M}, \mathfrak{N}]) \leq D_{\mathbf{M}}(\mathfrak{M}) + D_{\mathbf{M}}(\mathfrak{N}),$$

and equality holds if $\mathfrak{M} \cdot \mathfrak{N} = (0)$.

Proof: Put $\mathfrak{M}' = [\mathfrak{M}, \mathfrak{N}] - \mathfrak{N}$. Then $\mathfrak{M}' \,\eta\, \mathbf{M}$, $\mathfrak{M}', \mathfrak{N}$ are orthogonal, $[\mathfrak{M}', \mathfrak{N}] = [\mathfrak{M}, \mathfrak{N}]$, and so Definition 8.2.1(iii) applies. Thus

$$\begin{aligned} D_{\mathbf{M}}([\mathfrak{M}, \mathfrak{N}]) = D_{\mathbf{M}}([\mathfrak{M}', \mathfrak{N}]) &= D_{\mathbf{M}}(\mathfrak{M}') + D_{\mathbf{M}}(\mathfrak{N}) \\ &\le D_{\mathbf{M}}(\mathfrak{M}) + D_{\mathbf{M}}(\mathfrak{N}), \end{aligned}$$

considering $D_{\mathbf{M}}(\mathfrak{M}') \le D_{\mathbf{M}}(\mathfrak{M})$ by Lemma 14.2.1.

We see that we have equality if $D_{\mathbf{M}}(\mathfrak{M}') = D_{\mathbf{M}}(\mathfrak{M})$. If this is not the case, then $D_{\mathbf{M}}(\mathfrak{M}') < D_{\mathbf{M}}(\mathfrak{M})$, and so Lemma 14.2.2 applies: $\mathfrak{M} \cdot (\mathfrak{H} - \mathfrak{M}') \ne (0)$. Now $f \in \mathfrak{M} \cdot (\mathfrak{H} - \mathfrak{M}')$ means $f \in \mathfrak{M}$ and the orthogonality of f to $\mathfrak{M}' = [\mathfrak{M}, \mathfrak{N}] - \mathfrak{N}$. But as $f \in \mathfrak{M} \subset [\mathfrak{M}, \mathfrak{N}]$, the second statement means

$$f \in [\mathfrak{M}, \mathfrak{N}] - ([\mathfrak{M}, \mathfrak{N}] - \mathfrak{N}) = \mathfrak{N}.$$

So we have $f \in \mathfrak{M} \cdot \mathfrak{N}$, and therefore the above result states $\mathfrak{M} \cdot \mathfrak{N} \ne (0)$. This completes the proof.

Lemma 14.2.4. *Assume* $\mathfrak{M}, \mathfrak{N} \,\eta\, \mathbf{M}$. *Then*

$$D_{\mathbf{M}}([\mathfrak{M}, \mathfrak{N}]) + D_{\mathbf{M}}(\mathfrak{M} \cdot \mathfrak{N}) = D_{\mathbf{M}}(\mathfrak{M}) + D_{\mathbf{M}}(\mathfrak{N}).$$

Proof: Put $\mathfrak{M}_1 = \mathfrak{M} - \mathfrak{M} \cdot \mathfrak{N}$. Then $\mathfrak{M}_1, \mathfrak{M} \cdot \mathfrak{N}$ are orthogonal, $[\mathfrak{M}_1, \mathfrak{M} \cdot \mathfrak{N}] = \mathfrak{M}$, so by Definition 8.2.1(iii),

$$D_{\mathbf{M}}(\mathfrak{M}) = D_{\mathbf{M}}(\mathfrak{M}_1) + D_{\mathbf{M}}(\mathfrak{M} \cdot \mathfrak{N}).$$

Furthermore $[\mathfrak{M}_1, \mathfrak{M} \cdot \mathfrak{N}, \mathfrak{N}] = [\mathfrak{M}, \mathfrak{N}]$, and if $f \in \mathfrak{M}_1 \cdot \mathfrak{N}$, then $f \in \mathfrak{M}_1 \cdot \mathfrak{M} \cdot \mathfrak{N}$, but $\mathfrak{M}_1$ is orthogonal to $\mathfrak{M} \cdot \mathfrak{N}$, so $f = 0$, which proves $\mathfrak{M}_1 \cdot \mathfrak{N} = (0)$. So Lemma 14.2.3 applies:

$$D_{\mathbf{M}}([\mathfrak{M}, \mathfrak{N}]) = D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{N}]) = D_{\mathbf{M}}(\mathfrak{M}_1) + D_{\mathbf{M}}(\mathfrak{N}).$$

Adding $D_{\mathbf{M}}(\mathfrak{M} \cdot \mathfrak{N})$ to both sides gives, considering our previous equation, the desired formula.

Lemma 14.2.5. *Let* $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ *be a (finite or infinite) sequence, all* $\mathfrak{M}_i \,\eta\, \mathbf{M}$. *Then*

$$D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \ldots]) \le \sum_i D_{\mathbf{M}}(\mathfrak{M}_i).$$

The equality holds if and only if one of these two conditions is satisfied:

(i) $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \ldots])$ *is infinite.*

(ii) $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \ldots])$ *is finite, and* $[\mathfrak{M}_1, \ldots, \mathfrak{M}_{i-1}] \cdot \mathfrak{M}_i = (0)$ *for* $i = 2, 3, \ldots$.
In case (ii) there is even

$$[\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \ldots] \cdot [\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \ldots] = (0)$$

if $m_1, m_2, \ldots$ *and* $n_1, n_2, \ldots$ *are any two sequences without common elements.*
Proof: We can assume that the sequence $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ is infinite, as we could otherwise insert infinitely many terms = (0).

Lemma 8.3.3 can be applied to

$$\mathfrak{M}_1 \subset [\mathfrak{M}_1, \mathfrak{M}_2] \subset [\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3] \subset \cdots$$

and it gives

$$D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \ldots]) = \lim_{i \to \infty} D_{\mathbf{M}}([\mathfrak{M}_1, \ldots, \mathfrak{M}_i]).$$

Now Lemma 14.2.3 can be applied to $[\mathfrak{M}_1, \ldots, \mathfrak{M}_{i-1}], \mathfrak{M}_i$, and it gives

$$D_{\mathbf{M}}([\mathfrak{M}_1, \ldots, \mathfrak{M}_i]) \leq D_{\mathbf{M}}([\mathfrak{M}_1, \ldots, \mathfrak{M}_{i-1}]) + D_{\mathbf{M}}(\mathfrak{M}_i),$$

with an equality if (ii) holds; therefore

$$D_{\mathbf{M}}([\mathfrak{M}_1, \ldots, \mathfrak{M}_i]) \leq \sum_{j=1}^{i} D_{\mathbf{M}}(\mathfrak{M}_j),$$

again with an equality if (ii) holds. So we have proved the relation with $\leq$ in general, and the sufficiency of (ii) for the equality. (i) too implies equality, because if $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \ldots])$ is infinite, then $\sum_i D_{\mathbf{M}}(\mathfrak{M}_i)$ is infinite too.

The necessity of (i) or (ii) means that if

$$D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \ldots]) = \sum_i D_{\mathbf{M}}(\mathfrak{M}_i)$$

and is finite, then (ii) holds. We prove that in this case

$$[\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \ldots] \cdot [\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \ldots] = (0)$$

if $m_1, m_2, \ldots$ and $n_1, n_2, \ldots$ have no common elements. This proves (ii) (put $m_1 = 1, \ldots, m_{i-1} = i - 1$ and $n_1 = i$), and at the same time it justifies the last statement of our Lemma too. As we can add to the sequence $n_1, n_2, \ldots$ all

$i = 1, 2, \ldots$ which differ from $m_1, m_2, \ldots$ and $n_1, n_2, \ldots$, we may assume that $n_1, n_2, \ldots$ is the complementary set to $m_1, m_2, \ldots$ in $1, 2, \ldots$.

Now we have

$$D_{\mathbf{M}}([\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \ldots]) \leq \sum_i D_{\mathbf{M}}(\mathfrak{M}_{m_i}), \qquad D_{\mathbf{M}}([\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \ldots]) \leq \sum_i D_{\mathbf{M}}(\mathfrak{M}_{n_i}),$$

and so by Lemma 14.2.4

$$\begin{aligned}
&D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \ldots]) + D_{\mathbf{M}}([\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \ldots] \cdot [\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \ldots]) \\
&= D_{\mathbf{M}}([[\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \ldots], [\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \ldots]]) + D_{\mathbf{M}}([\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \ldots] \cdot [\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \ldots]) \\
&= D_{\mathbf{M}}([\mathfrak{M}_{m_1}, \mathfrak{M}_{m_2}, \ldots]) + D_{\mathbf{M}}([\mathfrak{M}_{n_1}, \mathfrak{M}_{n_2}, \ldots]) \\
&\leq \sum_i D_{\mathbf{M}}(\mathfrak{M}_{m_i}) + \sum_i D_{\mathbf{M}}(\mathfrak{M}_{n_i}) = \sum_i D_{\mathbf{M}}(\mathfrak{M}_i) = D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \ldots]).
\end{aligned}$$

As $D_{\mathbf{M}}([\mathfrak{M}_1, \mathfrak{M}_2, \ldots])$ is finite by assumption, this implies that the dimension of the intersection is ≤ 0, so it is 0, and therefore the intersection is (0). This completes the proof.

Chapter XV: The Relative Trace

15.1. From now on we make the assumption of §14.1: $\mathbf{M}$ is a factor in a finite case $((\mathrm{I}_n)$, $n = 1, 2, \ldots$, or $(\mathrm{II}_1))$, and $D_{\mathbf{M}}(\mathfrak{M})$ is normalised by $D_{\mathbf{M}}(\mathfrak{H}) = 1$.

We define an analogue of the trace of common (finite dimensional) matrix theory, more precisely: We define a notion which shares most essential properties of the trace, and coincides with it in the discrete cases (I_n), $n = 1, 2, \ldots$. The importance of this notion lies of course in its validity in the continuous case (II_1).

Definition 15.1.1. Let $A \in \mathbf{M}$ be Hermitian. Form its resolution of unity $E(\lambda)$, $-\infty < \lambda < \infty$. (Cf. (16, p. 92). As A is bounded this coincides with Hilbert's spectral form. The description which we will use is discussed in (18, pp. 389–390, footnote 42, and p. 418).) It is characterised and uniquely determined, as discussed loc. cit., by the following properties:

(α) $E(\lambda)$ is a projection, defined for all $-\infty < \lambda < \infty$.
(β) $\lambda \leq \mu$ implies $E(\lambda)$ part of $E(\mu)$.
(γ) $\lambda \geq \lambda_0$, $\lambda \to \lambda_0$ implies $E(\lambda) \to E(\lambda_0)$ in the strong topology.
(δ) There exists a c so that $E(\lambda) = 0$ if $\lambda \leq -c$, and $E(\lambda) = 1$ if $\lambda \geq c$.

(ϵ) For all $f, g \in \mathfrak{H}$,

$$(Af, g) = \int_{-\infty}^{\infty} \lambda \, d(E(\lambda)f, g).$$

(This is a numerical Stieltjes integral, certainly convergent, because we can replace $\int_{-\infty}^{\infty}$ by $\int_{-c}^{c}$ owing to (δ); and $(E(\lambda)f, g)$ is of bounded variation owing to (β), cf. (16, p. 93).)

This equation may be written symbolically as

$$A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda). \tag{*}$$

As $1 \in \mathbf{M}$, therefore $A \in \mathbf{M}$ has the consequence that all $E(\lambda) \in \mathbf{M}$ (cf. (18, p. 390)). Therefore all $D_{\mathbf{M}}(E(\lambda))$ are defined, and we can form the numerical Stieltjes integral

$$T_{\mathbf{M}}(A) = \int_{-\infty}^{\infty} \lambda \, dD_{\mathbf{M}}(E(\lambda)). \tag{**}$$

(This integral is convergent, because we can replace $\int_{-\infty}^{\infty}$ by $\int_{-c}^{c}$ owing to (δ), as $D_{\mathbf{M}}(E(\lambda)) = 0$ resp. $= 1$ if $\lambda < -c$ resp. $\lambda \geq c$, and $D_{\mathbf{M}}(E(\lambda))$ is monotone owing to (β).)

We call this $T_{\mathbf{M}}(A)$, which is thus defined for all $A \in \mathbf{M}$, the relative trace of A. It is clear how the symbolic equation (*) suggests the definition (**). We shall see in §15.2, after Lemma 15.2.2, that in the discrete cases (I_n), $n = 1, 2, \ldots$, our $T_{\mathbf{M}}(A)$ is $1/n$ times the trace of A taken in the usual sense for matrices of degree n.

15.2. An alternative definition of the trace can be obtained by using an extension of the "Minimax principle" (cf. (5, pp. 26–29)). This extension is as follows:

Lemma 15.2.1. *Let $A \in \mathbf{M}$ be Hermitian, and $E(\lambda)$, $-\infty < \lambda < \infty$, its resolution of unity. Form the quantity*

$$e(\alpha) = \underset{\substack{\mathfrak{M} \eta \mathbf{M} \\ D_{\mathbf{M}}(\mathfrak{M}) \geq \alpha}}{\text{g.l.b.}} \left\{ \underset{\substack{f \in \mathfrak{M} \\ \|f\| = 1}}{\text{l.u.b.}} (Af, f) \right\} \tag{‡}$$

for every $\alpha > 0, \leq 1$. Then at the same time

$$e(\alpha) = \underset{[D_{\mathbf{M}}(E(\lambda)) \geq \alpha]}{\text{g.l.b.}} \lambda. \tag{‡‡}$$

Proof: Consider the set of all λ with $D_{\mathbf{M}}(E(\lambda)) \geq \alpha$. As it contains $\lambda = c$ it is not empty; as it does not contain $\lambda = -c$ it is not the set of all real numbers; it contains with λ every $\lambda' > \lambda$ by Definition 15.1.1 (β); it contains its g.l.b. λ_0 by Definition 15.1.1 (γ). Thus it is the set of all $\lambda \geq \lambda_0$. So we must prove $e(\alpha) = \lambda_0$.

Define $\mathfrak{M}_0$ by $P_{\mathfrak{M}_0} = E(\lambda_0)$, $\mathfrak{M}_0 \,\eta\, \mathbf{M}$ because $E(\lambda_0) \in \mathbf{M}$, and $D_{\mathbf{M}}(\mathfrak{M}_0) = D_{\mathbf{M}}(E(\lambda_0)) \geq \alpha$. For $f \in \mathfrak{M}_0$ we have $E(\lambda_0)f = f$, and so $E(\lambda)f = f$ if $\lambda \geq \lambda_0$. Thus

$$\begin{aligned}
(Af, f) &= \int_{-\infty}^{\infty} \lambda \, d(E(\lambda)f, f) = \int_{-c}^{\lambda_0} \lambda \, d(E(\lambda)f, f) \\
&\leq \int_{-c}^{\lambda_0} \lambda_0 \, d(E(\lambda)f, f) \\
&= \lambda_0\big((E(\lambda_0)f, f) - (E(-c)f, f)\big) = \lambda_0(f, f) = \lambda_0\|f\|^2,
\end{aligned}$$

and so $\text{l.u.b.}_{[f \in \mathfrak{M}_0,\ \|f\|=1]}(Af, f) \leq \lambda_0$. As $\mathfrak{M}_0 \,\eta\, \mathbf{M}$, $D_{\mathbf{M}}(\mathfrak{M}_0) \geq \alpha$, this proves $e(\alpha) \leq \lambda_0$.

Assume now $e(\alpha) < \lambda_0$. Then by (‡) a $\mathfrak{M} \,\eta\, \mathbf{M}$, $D_{\mathbf{M}}(\mathfrak{M}) \geq \alpha$, with

$$\underset{\substack{f \in \mathfrak{M} \\ \|f\|=1}}{\text{l.u.b.}}(Af, f) < \lambda_0$$

must exist. Choose a

$$\lambda' > \underset{\substack{f \in \mathfrak{M} \\ \|f\|=1}}{\text{l.u.b.}}(Af, f), \qquad \lambda' < \lambda_0.$$

By the definition of λ_0, the relation $\lambda' < \lambda_0$ has the consequence $D_{\mathbf{M}}(E(\lambda')) < \alpha$. Define $\mathfrak{M}'$ by $P_{\mathfrak{M}'} = E(\lambda')$; then $\mathfrak{M}' \,\eta\, \mathbf{M}$ because $E(\lambda') \in \mathbf{M}$. We have

$$D_{\mathbf{M}}(\mathfrak{M}') = D_{\mathbf{M}}(E(\lambda')) < \alpha \leq D_{\mathbf{M}}(\mathfrak{M}),$$

so by Lemma 14.2.2 $\mathfrak{M} \cdot (\mathfrak{H} - \mathfrak{M}') \neq (0)$. Therefore an $f_0 \in \mathfrak{M} \cdot (\mathfrak{H} - \mathfrak{M}')$, $f_0 \neq 0$, exists.

Now as $f_0 \in \mathfrak{H} - \mathfrak{M}'$, we have $E(\lambda')f_0 = 0$, and so $E(\lambda)f_0 = 0$ if $\lambda \leq \lambda'$. Thus

$$\begin{aligned}
(Af_0, f_0) &= \int_{-\infty}^{\infty} \lambda \, d(E(\lambda)f_0, f_0) = \int_{\lambda'}^{c} \lambda \, d(E(\lambda)f_0, f_0) \\
&\geq \int_{\lambda'}^{c} \lambda' \, d(E(\lambda)f_0, f_0) \\
&= \lambda'\big((E(c)f_0, f_0) - (E(\lambda')f_0, f_0)\big) = \lambda'(f_0, f_0) = \lambda'\|f_0\|^2.
\end{aligned}$$

Multiplying f_0 with $1/\|f_0\|$, we can obtain $\|f_0\| = 1$. But at the same time $f_0 \in \mathfrak{M}$ too, therefore we have

$$\underset{\substack{f \in \mathfrak{M} \\ \|f\|=1}}{\text{l.u.b.}} (Af, f) \geq \lambda',$$

contradicting our original assumptions on λ'. Thus there must be $e(\alpha) = \lambda_0$, and the proof is completed.

Lemma 15.2.2. *Let $A \in \mathbf{M}$ be Hermitian, and $e(\alpha)$ as in Lemma 15.2.1. Then*

$$T_{\mathbf{M}}(A) = \int_0^1 e(\alpha)\, d\alpha.$$

Proof: We have $T_{\mathbf{M}}(A) = \int_{-\infty}^{\infty} \lambda\, d(D_{\mathbf{M}}(E(\lambda)))$. It suffices to remember the definition of the Riemann–Stieltjes integral in order to see that this coincides with the Riemann integral

$$\int_0^1 \left(\underset{[D_{\mathbf{M}}(E(\lambda)) \geq \alpha]}{\text{g.l.b.}} \lambda \right) d\alpha \qquad \text{(cf. (19, p. 198)).}$$

But this is, by formula (‡‡) of Lemma 15.2.1, equal to $\int_0^1 e(\alpha)\, d\alpha$.

Observe that in the case (I_n), $D_{\mathbf{M}}(\mathfrak{M})$ assumes no other values than $0, 1/n, 2/n, \ldots, (n-1)/n, 1$; therefore $e(\alpha)$ is constant in each interval $(p-1)/n < \alpha \leq p/n$, $p = 1, \ldots, n$. Thus Lemma 15.2.2 gives

$$T_{\mathbf{M}}(A) = \int_0^1 e(\alpha)\, d\alpha = \frac{1}{n} \sum_{p=1}^{n} e\left(\frac{p}{n}\right),$$

and one verifies easily that $e(p/n)$ is the proper value No. p (from below) of A. Thus $T_{\mathbf{M}}(A)$ is $1/n$ times the trace of A. A more direct proof is given in §15.4, after Theorem XIII. In the case (II_1), $\int_0^1 e(\alpha)\, d\alpha$ becomes an integral with a really continuous domain. Owing to the analogy with the case (I_n), it seems natural to introduce the following terminology:

Definition 15.2.1. Let $A \in \mathbf{M}$ be Hermitian, and $e(\alpha)$ as in Lemma 15.2.1. Then we call $e(\alpha)$ the proper value No. α (from below) of A, on the continuous scale $0 < \alpha \leq 1$.

The immediate application of our new way to express $T_{\mathbf{M}}(A)$ is this:

Lemma 15.2.3. *Let $A, B \in \mathbf{M}$ be Hermitian, and $A - B$ (semi-)definite. Then*

$$T_{\mathbf{M}}(A) \geq T_{\mathbf{M}}(B).$$

Proof: Form the $e(\alpha)$ of Lemma 15.2.1 for A and for B, denoting it by $e(\alpha)$ resp. $\eta(\alpha)$. As the definiteness of $A - B$ means $((A-B)f, f) \geq 0$, $(Af, f) \geq (Bf, f)$ for every f, therefore (‡) of Lemma 15.2.1 gives $e(\alpha) \geq \eta(\alpha)$. Now Lemma 15.2.2 gives $T_{\mathbf{M}}(A) \geq T_{\mathbf{M}}(B)$.

Note that $T_{\mathbf{M}}(A) \geq 0$ for definite A is obvious, and that this would imply directly our Lemma, if we knew that $T_{\mathbf{M}}(A - B) = T_{\mathbf{M}}(A) - T_{\mathbf{M}}(B)$. But this relation is only established at present if A, B commute (cf. Lemma 15.3.4 and the remarks after it), and this makes the above simplification impracticable.

15.3. We derive now the main formal properties of $T_{\mathbf{M}}(A)$.

Lemma 15.3.1. *Let $E \in \mathbf{M}$ be a projection. Then $T_{\mathbf{M}}(E) = D_{\mathbf{M}}(E)$.*

Proof:

$$E(\lambda) = \begin{cases} E, & \lambda \geq 1, \\ 0, & \lambda < 1 \end{cases}$$

is E's resolution of unity, as conditions (α)–(ϵ) from Definition 15.1.1 are easily verified for it. So

$$D_{\mathbf{M}}(E(\lambda)) = \begin{cases} D_{\mathbf{M}}(E), & \lambda \geq 1, \\ 0, & \lambda < 1, \end{cases}$$

and so

$$T_{\mathbf{M}}(E) = \int_{-\infty}^{\infty} \lambda \, dD_{\mathbf{M}}(E(\lambda)) = 1 \cdot D_{\mathbf{M}}(E) = D_{\mathbf{M}}(E).$$

Lemma 15.3.2. *Let $A \in \mathbf{M}$ be Hermitian, and a, b two real numbers. Then*

$$T_{\mathbf{M}}(aA + b1) = aT_{\mathbf{M}}(A) + b.$$

Proof: We first prove $T_{\mathbf{M}}(-A) = -T_{\mathbf{M}}(A)$, because this will permit us to restrict ourselves afterwards to the case $a \geq 0$.

Let $E(\lambda)$ be A's resolution of unity. Then $1 - E(-\lambda)$ fulfills for $-A$ all conditions (α)–(ϵ) of Definition 15.1.1 except (γ). The function

$$E_1(\lambda) = \lim_{\substack{\lambda' > \lambda \\ \lambda' \to \lambda}} \left(1 - E(-\lambda')\right)$$

fulfills, as a simple argument shows, all of them. So $E_1(\lambda)$ is $-A$'s resolution of unity. Now $E_1(\lambda) \neq 1 - E(-\lambda)$ only in the points of discontinuity of $E(-\lambda)$, that is in the points $-\lambda$ where λ runs over all point proper values of A. This is a countable set. A fortiori $D_{\mathbf{M}}(E_1(\lambda)) \neq D_{\mathbf{M}}(1 - E(-\lambda))$ holds only in a countable set. Therefore we can, by the definition of Riemann–Stieltjes integrals, replace $d(D_{\mathbf{M}}(E_1(\lambda)))$ by $d(D_{\mathbf{M}}(1 - E(-\lambda)))$ in every Riemann–Stieltjes integral in which it occurs. Thus

$$\begin{aligned} T_{\mathbf{M}}(-A) &= \int_{-\infty}^{\infty} \lambda \, dD_{\mathbf{M}}(E_1(\lambda)) = \int_{-\infty}^{\infty} \lambda \, dD_{\mathbf{M}}(1 - E(-\lambda)) \\ &= \int_{-\infty}^{\infty} (-\lambda) \, dD_{\mathbf{M}}(E(-\lambda)) = \int_{\infty}^{-\infty} \lambda \, dD_{\mathbf{M}}(E(\lambda)) \\ &= -\int_{-\infty}^{\infty} \lambda \, dD_{\mathbf{M}}(E(\lambda)) = -T_{\mathbf{M}}(A). \end{aligned}$$

Assume now $a \geq 0$. Denote the $e(\alpha)$ of A and of $aA + b1$ by $e(\alpha)$ resp. $\eta(\alpha)$. For $\|f\| = 1$,

$$((aA + b1)f, f) = a(Af, f) + b(f, f) = a(Af, f) + b,$$

and as $a \geq 0$, this formula can be used when forming g.l.b.'s and l.u.b.'s. Therefore (‡) of Lemma 15.2.1 gives $\eta(\alpha) = ae(\alpha) + b$, and then Lemma 15.2.2 gives $T_{\mathbf{M}}(aA + b1) = aT_{\mathbf{M}}(A) + b$. This completes the proof.

Lemma 15.3.3. *$T_{\mathbf{M}}(A)$ is a continuous function of A if the uniform topology for operators (cf. (18, p. 384)) is used.*

Proof: We will show: If $\|(A - B)f\| \leq \epsilon\|f\|$ for every $f \in \mathfrak{H}$, then

$$|T_{\mathbf{M}}(A) - T_{\mathbf{M}}(B)| \leq \epsilon.$$

Under the above assumption we have

$$|((A - B)f, f)| \leq \|(A - B)f\| \, \|f\| \leq \epsilon\|f\|^2,$$

$$(Af, f) \leq (Bf, f) + \epsilon\|f\|^2 = ((B + \epsilon 1)f, f).$$

So $(B + \epsilon 1) - A$ is definite, and by Lemma 15.2.3, $T_{\mathbf{M}}(B + \epsilon 1) \geq T_{\mathbf{M}}(A)$. But by Lemma 15.3.2, $T_{\mathbf{M}}(B + \epsilon 1) = T_{\mathbf{M}}(B) + \epsilon$, so we have $T_{\mathbf{M}}(A) - T_{\mathbf{M}}(B) \leq \epsilon$. Interchanging A, B (the assumption was symmetric in A, B), we obtain $T_{\mathbf{M}}(A) - T_{\mathbf{M}}(B) \geq -\epsilon$. So $|T_{\mathbf{M}}(A) - T_{\mathbf{M}}(B)| \leq \epsilon$, completing the proof.

We will prove elsewhere considerably more about the continuity of $T_{\mathbf{M}}(A)$. (Cf. note at the end of this paper.)

Lemma 15.3.4. *Let $A, B \in \mathbf{M}$ be Hermitian, and commute with each other. Then*

$$T_{\mathbf{M}}(A+B) = T_{\mathbf{M}}(A) + T_{\mathbf{M}}(B).$$

Proof: Let $E(\lambda)$ and $F(\lambda)$ be the resolutions of unity belonging to A, resp. B. By (18, pp. 390–391, in particular footnote 43), A, B are limits of uniformly convergent sequences of linear aggregates of the $E(\lambda)$, resp. the $F(\lambda)$. Therefore it suffices, by Lemma 15.3.3, to prove

$$T_{\mathbf{M}}\left(\sum_{i=1}^{m} a_i E(\lambda_i) + \sum_{j=1}^{n} b_j F(\mu_j)\right) = T_{\mathbf{M}}\left(\sum_{i=1}^{m} a_i E(\lambda_i)\right) + T_{\mathbf{M}}\left(\sum_{j=1}^{n} b_j F(\mu_j)\right).$$

Observe that if A, B commute, all $E(\lambda), F(\mu)$ commute with each other (cf. (16, p. 115)).

So it suffices to prove this:

$$T_{\mathbf{M}}\left(\sum_{\rho=1}^{p}(c_\rho + d_\rho)E_\rho\right) = T_{\mathbf{M}}\left(\sum_{\rho=1}^{p} c_\rho E_\rho\right) + T_{\mathbf{M}}\left(\sum_{\rho=1}^{p} d_\rho E_\rho\right)$$

if $E_1, \ldots, E_p$ are commutative projections. (Substitute $p = m+n$, $E(\lambda_1), \ldots, E(\lambda_m)$, $F(\mu_1), \ldots, F(\mu_n)$, $a_1, \ldots, a_m, 0, \ldots, 0$, and $0, \ldots, 0, b_1, \ldots, b_n$ for $p, E_1, \ldots, E_p$, $c_1, \ldots, c_p, d_1, \ldots, d_p$.)

This would follow immediately from

$$T_{\mathbf{M}}\left(\sum_{\rho=1}^{p} c_\rho E_\rho\right) = \sum_{\rho=1}^{p} c_\rho D_{\mathbf{M}}(E_\rho) \tag{\#}$$

(for all choices of $c_1, \ldots, c_p$, provided that $E_1, \ldots, E_p$ commute).

Consider the 2^p terms arising when computing

$$(E_1 + (1-E_1)) \cdots (E_p + (1-E_p)).$$

They are products of commutative projections $\in \mathbf{M}$, thus projections $\in \mathbf{M}$; their sum is 1, and the sum of those which arise from the term E_ρ in the factor $E_\rho + (1-E_\rho)$ is E_ρ. Denote them by $E'_1, \ldots, E'_q$, $q = 2^p$; as $E'_1 + \cdots + E'_q = 1$, they are mutually orthogonal. Write

$$E_\rho = E'_{\tau_{\rho,1}} + \cdots + E'_{\tau_{\rho,s_\rho}}.$$

We claim: If (#) holds for $E'_1, \dots, E'_q$ (and all $c'_1, \dots, c'_q$), then it holds for $E_1, \dots, E_p$ too. Indeed:

$$
\begin{aligned}
T_{\mathbf{M}}\left(\sum_{\rho=1}^{p} c_\rho E_\rho\right) &= T_{\mathbf{M}}\left(\sum_{\rho=1}^{p} c_\rho \sum_{\sigma=1}^{s_\rho} E'_{\tau_{\rho,\sigma}}\right) \\
&= T_{\mathbf{M}}\left(\sum_{\tau=1}^{q}\left(\sum_{\substack{\rho,\sigma \\ \tau_{\rho,\sigma}=\tau}} c_\rho\right) E'_\tau\right) \\
&= \sum_{\tau=1}^{q}\left(\sum_{\substack{\rho,\sigma \\ \tau_{\rho,\sigma}=\tau}} c_\rho\right) D_{\mathbf{M}}(E'_\tau) \\
&= \sum_{\rho=1}^{p} c_\rho \left(\sum_{\sigma=1}^{s_\rho} D_{\mathbf{M}}(E'_{\tau_{\rho,\sigma}})\right) = \sum_{\rho=1}^{p} c_\rho D_{\mathbf{M}}(E_\rho).
\end{aligned}
$$

In other words: We may assume in (#) that $E_1, \dots, E_p$ are mutually orthogonal, and $E_1 + \cdots + E_p = 1$.

If now two c_ρ's in (#) are equal, say $c_\rho = c_\sigma$ for some $\rho \neq \sigma$, then E_ρ, E_σ coalesce to $E_\rho + E_\sigma$ in both sides of (#). So we may assume that all c_ρ's are different. As a permutation does not matter, we may even assume that $c_1 < c_2 < \cdots < c_p$.

Define now

$$
G(\lambda) = \begin{cases} 0, & \lambda < c_1, \\ E_1 + \cdots + E_\rho, & c_\rho \leq \lambda < c_{\rho+1}, \quad \rho = 1, \dots, p-1, \\ 1, & \lambda \geq c_p. \end{cases}
$$

One verifies immediately the conditions (α)–(ϵ) of Definition 15.1.1 for these $G(\lambda)$ and $\sum_{\rho=1}^{p} c_\rho E_\rho$; therefore it is the resolution of unity which belongs to $\sum_{\rho=1}^{p} c_\rho E_\rho$.

Now Definition 15.1.1 gives

$$
\begin{aligned}
T_{\mathbf{M}}\left(\sum_{\rho=1}^{p} c_\rho E_\rho\right) &= \int_{-\infty}^{\infty} \lambda\, dD_{\mathbf{M}}(G(\lambda)) \\
&= \sum_{\rho=1}^{p} c_\rho \big(D_{\mathbf{M}}(E_1 + \cdots + E_\rho) - D_{\mathbf{M}}(E_1 + \cdots + E_{\rho-1})\big) \\
&= \sum_{\rho=1}^{p} c_\rho D_{\mathbf{M}}(E_\rho).
\end{aligned}
$$

This completes the proof.

Observe that we should expect from the analogy of the trace of finite dimensional matrices, that the additivity of $T_{\mathbf{M}}(A)$ expressed in Lemma 15.3.4 holds even without assuming the commutativity of A, B. We will come back to this question at the end of §15.4, Problem 11.

15.4. The relation $\mathfrak{M} \sim \mathfrak{N}$ can be expressed in a new way, owing to $D_{\mathbf{M}}(\mathfrak{H}) = 1$.

Lemma 15.4.1. *$\mathfrak{M} \sim \mathfrak{N}$ $(\eta\mathbf{M})$ is equivalent to $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$ and the existence of a unitary $V \in \mathbf{M}$ which maps $\mathfrak{M}$ on $\mathfrak{N}$.*

Proof: If $\mathfrak{M}\ \eta\ \mathbf{M}$ and such a $V \in \mathbf{M}$ exists, then $U = VP_{\mathfrak{M}}$ is $\in \mathbf{M}$, partially isometric, with the initial and final sets $\mathfrak{M}$, resp. $\mathfrak{N}$, so $\mathfrak{M} \sim \mathfrak{N}$ $(\eta\mathbf{M})$.

Conversely: If $\mathfrak{M} \sim \mathfrak{N}$ $(\eta\mathbf{M})$, then

$$
\begin{aligned}
D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}) &= D_{\mathbf{M}}(\mathfrak{H}) - D_{\mathbf{M}}(\mathfrak{M}) \\
&= D_{\mathbf{M}}(\mathfrak{H}) - D_{\mathbf{M}}(\mathfrak{N}) = D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{N})
\end{aligned}
$$

(this step is possible because $D_{\mathbf{M}}(\mathfrak{H}) = 1$ is finite), and so $\mathfrak{H} - \mathfrak{M} \sim \mathfrak{H} - \mathfrak{N}$. Let U_1, U_2 be $\in \mathbf{M}$, partially isometric, and have the respective initial sets $\mathfrak{M}, \mathfrak{H} - \mathfrak{M}$ and the respective final sets $\mathfrak{N}, \mathfrak{H} - \mathfrak{N}$. One verifies easily that $V = U_1 + U_2 \in \mathbf{M}$ is unitary, and that it maps $\mathfrak{M}$ on $\mathfrak{N}$. $\mathfrak{M}, \mathfrak{N}\ \eta\ \mathbf{M}$ is obvious.

This proof was based on the finiteness of $D_{\mathbf{M}}(\mathfrak{H})$, that is, of $\mathfrak{H}$ (with respect to $\mathbf{M}$). In fact the finiteness of $\mathfrak{H}$ is necessary and sufficient for the Lemma itself: Because if $\mathfrak{H}$ is infinite, then $\mathfrak{H} \sim \mathfrak{M}$ $(\eta\mathbf{M})$ for some $\mathfrak{M} \subsetneq \mathfrak{H}$, and no unitary U maps $\mathfrak{H}$ on an $\mathfrak{M} \neq \mathfrak{H}$.

We now state the essential invariance property of $T_{\mathbf{M}}(A)$:

Lemma 15.4.2. *Let $A \in \mathbf{M}$ be Hermitian and $U \in \mathbf{M}$ be unitary. Then*

$$T_{\mathbf{M}}(U^{-1}AU) = T_{\mathbf{M}}(A).$$

Proof: If A has the resolution of unity $E(\lambda)$, then $U^{-1}AU$ has clearly $U^{-1}E(\lambda)U$. Put $E(\lambda) = P_{\mathfrak{M}(\lambda)}$, $U^{-1}E(\lambda)U = P_{\mathfrak{N}(\lambda)}$; then $\mathfrak{N}(\lambda)$ is $\mathfrak{M}(\lambda)$'s image by U^{-1}. So $\mathfrak{M}(\lambda) \sim \mathfrak{N}(\lambda)$ $(\eta\mathbf{M})$, $D_{\mathbf{M}}(\mathfrak{M}(\lambda)) = D_{\mathbf{M}}(\mathfrak{N}(\lambda))$, $D_{\mathbf{M}}(E(\lambda)) = D_{\mathbf{M}}(U^{-1}E(\lambda)U)$. Now Definition 15.1.1 gives $T_{\mathbf{M}}(A) = T_{\mathbf{M}}(U^{-1}AU)$.

We are now able to characterise the relative trace by inner properties:

Lemma 15.4.3. *Assume that a (real and finite) numerical function $T'(A)$ is defined for Hermitian $A \in \mathbf{M}$, and that it has the following properties:*

(i) $T'(1) = 1$.
(ii) $T'(aA) = aT'(A)$ *if a is real.*
(iii) $T'(A + B) = T'(A) + T'(B)$ *if A, B commute.*
(iv) $T'(A) \geq 0$ *if A is (semi-)definite.*
(v) $T'(U^{-1}AU) = T'(A)$ *if $U \in \mathbf{M}$ is unitary.*

This is the case if and only if $T'(A)$ is relative trace: $T'(A) = T_{\mathbf{M}}(A)$ for Hermitian $A \in \mathbf{M}$.

Proof: $T_{\mathbf{M}}(A)$ has the properties (i)–(v) by Lemma 15.3.2, Lemma 15.3.4, Lemma 15.2.3, and Lemma 15.4.2. Therefore only the converse problem remains: to determine $T'(A)$ if it fulfills (i)–(v). Assume therefore that such a $T'(A)$ is given.

If E is a projection $\in \mathbf{M}$, then, as E is definite, we have $T'(E) \geq 0$ by (iv). If $E \sim F$ $(\eta\mathbf{M})$, then by Lemma 15.4.1 a unitary $U \in \mathbf{M}$ with $F = U^{-1}EU$ exists, and so by condition (v) $T'(E) = T'(F)$. If E, F are orthogonal, that is $EF = FE = 0$, then they commute, and therefore $T'(E + F) = T'(E) + T'(F)$. So the conditions (ii), (iii) of Definition 8.2.1 are fulfilled, and as $T'(1) = 1 > 0, < \infty$, therefore Lemma 8.3.5 applies: $T'(E)$ is a relative dimension function (with respect to $\mathbf{M}$). So $T'(E) = cD_{\mathbf{M}}(E)$ with a suitable constant c; but as $T'(1) = D_{\mathbf{M}}(1) = 1$ by (i), therefore $c = 1$, that is: $T'(E) = D_{\mathbf{M}}(E)$.

Thus if $E_1, \ldots, E_p$ are mutually orthogonal projections, $E_1 + \cdots + E_p = 1$, and $c_1 < c_2 < \cdots < c_p$, then we have: If $\rho \neq \sigma$, then $E_\rho E_\sigma = E_\sigma E_\rho = 0$, so E_ρ, E_σ commute, and so by (ii), (iii)

$$T'\left(\sum_{\rho=1}^{p} c_\rho E_\rho\right) = \sum_{\rho=1}^{p} c_\rho T'(E_\rho) = \sum_{\rho=1}^{p} c_\rho D_{\mathbf{M}}(E_\rho),$$

and this, by the considerations at the end of the proof of Lemma 15.3.4, is

$T_{\mathbf{M}}(\sum_{\rho=1}^{p} c_\rho E_\rho)$. So the desired equation $T'(A) = T_{\mathbf{M}}(A)$ holds for all A's of the above form $\sum_{\rho=1}^{p} c_\rho E_\rho$.

(i)–(iii) imply in general $T'(aA + b1) = aT'(A) + b$. (iii), (iv) imply $T'(A) \geq T'(B)$ if $A - B$ is (semi-)definite and if A, B commute. Thus the argument of the proof of Lemma 15.3.3 applies to $T'(A)$ too, if we restrict ourselves to some commutative set of operators: Then $T'(A)$ is a continuous function of A, if the uniform topology for operators is used. We know from Lemma 15.3.3 that this holds for $T_{\mathbf{M}}(A)$ too.

Now form the operators mentioned in (18, pp. 390–391, in particular footnote 43): They were denoted by

$$A^{K\epsilon} = \sum_{\nu=1}^{n} \lambda_\nu \big(E(\lambda_\nu) - E(\lambda_{\nu-1})\big).$$

As we pointed out there, they are uniformly convergent with the limit A; and as all $E(\lambda)$ commute with each other and with A, the same is true for the $A^{K\epsilon}$. Thus $T'(A) = T_{\mathbf{M}}(A)$ follows, if we can prove $T'(A^{K\epsilon}) = T_{\mathbf{M}}(A^{K\epsilon})$. But this equation holds, because $A^{K\epsilon}$ has the above discussed form. Put $p = n$, $c_\nu = \lambda_\nu$, $E_\nu = E(\lambda_\nu) - E(\lambda_{\nu-1})$. Thus the proof is completed.

We restate the results obtained thus far:

Theorem XIII. *Let $\mathbf{M}$ be a factor of a finite class ((I_n), or (II_1)). Then there exists one and only one (real and finite) numerical function $T'(A)$, defined for all Hermitian $A \in \mathbf{M}$, which has the following properties:*

(i) $T'(1) = 1$.
(ii) $T'(aA) = aT'(A)$ *if a is real.*
(iii) $T'(A + B) = T'(A) + T'(B)$ *if A, B commute.*
(iv) $T'(A) \geq 0$ *if A is (semi-)definite.*
(v) $T'(U^{-1}AU) = T'(A)$ *if $U \in \mathbf{M}$ is unitary.*

This unique $T'(A)$ is the relative trace of A, $T_{\mathbf{M}}(A)$. Further essential properties of this $T'(A)$ are given in Lemma 15.2.3, Lemma 15.3.1, and Lemma 15.3.3. Equivalent definitions are contained in Definition 15.1.1 and in Lemma 15.2.2.

In particular we have (this follows from Lemma 15.3.1, so from (i)–(v)):

(vi) For a projection $E \in \mathbf{M}$, $T'(E) = 0$ implies $E = 0$.

In case (I_n) we obtain by this unicity theorem an easy determination of $T_{\mathbf{M}}(A)$: $\mathbf{M}$ in case (I_n) means $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, $\mathbf{M} = \mathbf{B}_1^{(1)}$, $\mathfrak{H}_1$ has dimension n. Now Lemma

2.4.6 shows that, using the symbolism

$$A^0 \sim \langle A_{t,s} \rangle_{t,s=1,\dots,n}$$

of Definition 2.4.2, the $A^0 \in \mathbf{M}$ are characterised by

$$A^0 \sim \langle a_{t,s} 1 \rangle_{t,s=1,\dots,n},$$

and thus correspond to the numerical matrices $\{a_{t,s}\}_{t,s=1,\dots,n}$. Now

$$T'(A^0) = \frac{1}{n} \sum_{t=1}^{n} a_{t,t}$$

fulfills (i)–(v), as one verifies easily; therefore

$$T_{\mathbf{M}}(A^0) = \frac{1}{n} \sum_{t=1}^{n} a_{t,t}.$$

The examples of case (II_1) given in Theorem XI, considering Lemma 13.1.2, can be discussed too. If $\bar{A} \in \mathbf{M}$ or $\bar{A} \in \mathbf{M}'$, write $\bar{A} \simeq [\chi_c(x)]_{x \in S, c \in \mathfrak{G}}$, and form

$$t(\bar{A}) = \int_S \chi_1(x)\, dx$$

(cf. loc. cit. and Lemma 12.4.2). Then $T'(\bar{A}) = (1/\mu(S)) t(\bar{A})$ fulfills (i)–(iii), as one verifies easily with the help of Lemma 12.4.2(i)–(iii).

As to (iv) observe, that every definite operator $\bar{A}$ has the form $\bar{B}^2$, where $\bar{B} = \alpha(\bar{A})$, $\alpha(x) = |x|^{1/2}$. Thus $\bar{B}$ belongs to our ring ($\mathbf{M}$, resp. $\mathbf{M}'$) and is Hermitian. A fortiori $\bar{A} = \bar{B}^* \bar{B}$, and now $t(\bar{A}) = t(\bar{B}^* \bar{B}) \geq 0$, $T'(\bar{A}) \geq 0$, follow from Lemma 12.4.2(iv).

(v) is proved as follows: As

$$A = \tfrac{1}{2}(A + A^*) + i\, \tfrac{1}{2i}(A - A^*),$$

it suffices to consider Hermitian operators A; and as

$$A = \left(\tfrac{1}{2}(A+1)\right)^2 - \left(\tfrac{1}{2}(A-1)\right)^2,$$

we may even assume $A = B^2$, B Hermitian. So we must prove

$$T'(U^{-1} B^2 U) = T'(B^2),$$

that is $t(U^{-1}B^2U) = t(B^2)$. This coincides with Lemma 12.4.2(iv), if we put there $A = BU$.

In all these cases, (iii), that is $T_{\mathbf{M}}(A + B) = T_{\mathbf{M}}(A) + T_{\mathbf{M}}(B)$, holds even without the restriction that A, B commute. This raises the following problem:

Problem 11. *Is $T_{\mathbf{M}}(A + B) = T_{\mathbf{M}}(A) + T_{\mathbf{M}}(B)$ always true, without further restrictions on A, B?*

The answer is certainly yes in case (I_n), and for all known examples of case (II_1). (Cf. also note at the end of this paper.)

15.5. The statements of Theorem XIII concerning the unique determination of $T'(A)$ by the properties enumerated there can be formulated and discussed, even if $\mathbf{M}$ is not assumed to be a factor, but only a ring with $1 \in \mathbf{M}$.

So if $\mathbf{M}$ is a ring with $1 \in \mathbf{M}$, we may ask: When do the conditions (i)–(vi) of Theorem XIII have a unique solution? This is the case if $\mathbf{M}$ is a factor and in a finite case. Let us now consider the converse problem. Assume that $\mathbf{M}$ is a ring with $1 \in \mathbf{M}$, and that (i)–(vi) have a unique solution, say $T^0(A)$.

Consider $\mathbf{M} \cdot \mathbf{M}'$. This is a ring, and therefore it is the ring generated by all projections $E \in \mathbf{M} \cdot \mathbf{M}'$ (cf. (18, p. 392)). Consider a projection $E \in \mathbf{M} \cdot \mathbf{M}'$. If $A \in \mathbf{M}$, then A and E commute, as $E \in \mathbf{M}'$, and so $AE = AEE = EAE$. Thus if A is Hermitian, resp. (semi-)definite, AE is too. Besides $A, E \in \mathbf{M}$, so $AE \in \mathbf{M}$. Now take two constants $a_1, a_2 > 0$, and define

$$T'(A) = a_1 T^0(A) + a_2 T^0(AE).$$

Then $T'(A)$ behaves with respect to the conditions (i)–(vi) of Theorem XIII as follows: (i) holds if $a_1 + a_2 T^0(E) = 1$. (ii) holds at any rate. (iii) holds, because $E \in \mathbf{M}'$ commutes with the $A, B \in \mathbf{M}$, so if these A, B commute, then AE, BE do too. (iv) holds, because AE is (semi-)definite along with A. (v) holds, because $E \in \mathbf{M}'$ commutes with the $U \in \mathbf{M}$, so $(U^{-1}AU)E = U^{-1}(AE)U$. (vi) holds, because if F is a projection, therefore (semi-)definite, then FE is it too; thus $T^0(F) \geq 0$, $T^0(FE) \geq 0$, and $T'(F) = 0$ implies $T^0(F) = 0$, $F = 0$.

Under these conditions our assumption necessitates $T'(A) = T^0(A)$ for each Hermitian $A \in \mathbf{M}$. So in particular $T'(E) = T^0(E)$, but the definition gives $T'(E) = (a_1 + a_2)T^0(E)$. Therefore $(a_1 + a_2 - 1)T^0(E) = 0$. Thus we have proved:

$$a_1, a_2 > 0, \quad a_1 + a_2 T^0(E) = 1 \quad \text{must imply} \quad (a_1 + a_2 - 1)T^0(E) = 0.$$

This is clearly not the case if $T^0(E) \neq 0, 1$; therefore we have either $T^0(E) = 0$, $E = 0$ (by (vi)), or $T^0(E) = 1$, $T^0(1 - E) = 1 - T^0(E) = 0$, $1 - E = 0$, $E = 1$ (by (i), (iii), (vi)).

Thus $\mathbf{M} \cdot \mathbf{M}' = \mathbf{R}(0, 1) = (\alpha 1)$, and so $\mathbf{M}$ is a factor.

Now the argument made at the beginning of the proof of Lemma 15.4.3 shows, remembering Lemma 8.3.5, that $T^0(E)$ is a relative dimension function with respect to $\mathbf{M}$ (E runs over all projections $E \in \mathbf{M}$); $T^0(1) = 1$ (by (i)) being finite, 1 is finite with respect to $\mathbf{M}$, and so is $\mathfrak{H}$. Therefore $\mathbf{M}$ is in one of the finite cases: (I_n), $n = 1, 2, \ldots$, or (II_1).

Consequently we have proved this:

Theorem XIV. *Let* $\mathbf{M}$ *be a ring with* $1 \in \mathbf{M}$. *The conditions (i)–(vi) of Theorem XIII admit a unique solution if and only if* $\mathbf{M}$ *is a factor, and belongs to one of the finite cases:* (I_n), $n = 1, 2, \ldots$, *or* (II_1).

Note that while (i)–(v) imply (vi) if $\mathbf{M}$ is a factor, they may not imply it if $\mathbf{M}$ is merely a ring with $1 \in \mathbf{M}$, and therefore the unicity of their solution may not guarantee the factor character of $\mathbf{M}$. This is the reason why (i)–(v) were sufficient in Theorem XIII, but (i)–(vi) were needed in Theorem XIV. In fact there exist rings $\mathbf{M}$ with $1 \in \mathbf{M}$ for which (i)–(v) have a unique solution, and which nevertheless are not factors. Easy considerations, which we will not detail here, show in fact that the ring $\mathbf{M} = (P_{[\varphi]})'$ ($\mathfrak{H}$ any space of ∞ dimensions, $\varphi \in \mathfrak{H}$, $\|\varphi\| = 1$) is such; the unique solution of (i)–(v) being $T^0(A) = (A\varphi, \varphi)$, and $\mathbf{M}$ is not a factor, because $P_{[\varphi]} \in \mathbf{M} \cdot \mathbf{M}'$.

We have not attempted in this chapter to give anything like a complete theory of the trace. We wanted to give only a brief summary of the most characteristic features.

Many further problems could be handled with our present methods. For instance, Lemma 15.3.3 could be strengthened so as to apply even to the strong topology of operators (cf. (18, p. 381)); connections could be established between $T_{\mathbf{M}}(A)$ and the (Af, f), etc. We propose to come back to this subject, as well as to Problem 11, in subsequent papers.

Chapter XVI: Unbounded Operators

16.1. $\mathbf{M}$ is assumed throughout this chapter too to be a factor in a finite case ((I_n), $n = 1, 2, \ldots$, or (II_1)), and that $D_{\mathbf{M}}(\mathfrak{M})$ is normalised by $D_{\mathbf{M}}(\mathfrak{H}) = 1$.

The following Lemma is a consequence of Lemma 6.2.1, valid in the finite cases only:

Lemma 16.1.1. *If X is a linear, closed operator with an everywhere dense domain, and $X \,\eta\, \mathbf{M}$, then $(f; Xf = 0)$ and $(f; X^*f = 0)$ are linear closed sets $\eta\mathbf{M}$, and*

$$D_{\mathbf{M}}((f; Xf = 0)) = D_{\mathbf{M}}((f; X^*f = 0)).$$

Proof: $X^*f = 0$ means $(X^*f, g) = (f, Xg) = 0$ for all $g \in \text{Domain}\, X$, that is, that f is orthogonal to $\text{Range}\, X$. So $(f; X^*f = 0) = \mathfrak{H} - [\text{Range}\, X]$, therefore $(f; X^*f = 0) \,\eta\, \mathbf{M}$, and

$$D_{\mathbf{M}}((f; X^*f = 0)) = D_{\mathbf{M}}(\mathfrak{H}) - D_{\mathbf{M}}([\text{Range}\, X]) = 1 - D_{\mathbf{M}}([\text{Range}\, X]).$$

Replacing X by X^*, we see that $(f; Xf = 0) \,\eta\, \mathbf{M}$, and

$$D_{\mathbf{M}}((f; Xf = 0)) = 1 - D_{\mathbf{M}}([\text{Range}\, X^*]).$$

Now Lemma 6.2.1 states

$$D_{\mathbf{M}}([\text{Range}\, X]) = D_{\mathbf{M}}([\text{Range}\, X^*]),$$

therefore

$$D_{\mathbf{M}}((f; Xf = 0)) = D_{\mathbf{M}}((f; X^*f = 0)).$$

In the cases (I_n), $n = 1, 2, \ldots$, this is the statement that the number of linearly independent solutions of a linear equation $Xf = 0$ coincides with the one for the adjoint equation $X^*f = 0$. We see now, that owing to our notion of relative dimension, it is true in the case (II_1) too. Observe, that these are the only cases where it holds: In any infinite case $\mathfrak{H}$ is infinite, so $\mathfrak{H} \sim \mathfrak{M} \subsetneq \mathfrak{H}$, and if $U \in \mathbf{M}$ is the partially isometric operator which maps $\mathfrak{H}$ on $\mathfrak{M}$, then

$$(f; Uf = 0) = (0), \qquad (f; U^*f = 0) = \mathfrak{H} - \mathfrak{M} \neq (0).$$

16.2. The key to the deductions which follow is contained in this definition:

Definition 16.2.1. Let $\mathfrak{A}$ be an arbitrary linear set in $\mathfrak{H}$ ($\mathfrak{A}$ is not necessarily closed, and not necessarily $\eta\mathbf{M}$). If a sequence $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ of linear closed sets exists which has the following properties:

(i) $\mathfrak{M}_i \,\eta\, \mathbf{M}$.
(ii) $\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \cdots \subset \mathfrak{A}$.
(iii) $[\mathfrak{M}_1, \mathfrak{M}_2, \ldots] = \mathfrak{H}$.

then $\mathfrak{A}$ is called essentially dense.

We prove:

Lemma 16.2.1. *Every essentially dense $\mathfrak{A}$ is everywhere dense too.*

Proof: As all $\mathfrak{M}_i \subset \mathfrak{A}$, and as $\mathfrak{A}$ is a linear set,

$$\mathfrak{M}_1, \mathfrak{M}_2, \ldots \subset \mathfrak{A} \subset \mathfrak{H}, \qquad \mathfrak{H} = [\mathfrak{M}_1, \mathfrak{M}_2, \ldots] \subset [\mathfrak{A}] \subset \mathfrak{H},$$

$[\mathfrak{A}] = \mathfrak{H}$. As $\mathfrak{A}$ is linear, $[\mathfrak{A}]$ consists of its condensation points only, therefore $\mathfrak{A}$ is everywhere dense.

Lemma 16.2.2. *If $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ is a (finite or infinite) sequence of essentially dense sets, then $\mathfrak{A}_1 \cdot \mathfrak{A}_2 \cdots$ is essentially dense too.*

Proof: We can assume that the sequence $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ is infinite, as we could otherwise insert infinitely many terms $\mathfrak{A}_i = \mathfrak{H}$.

Form the sequences from Definition 16.2.1: $\mathfrak{M}_{i,1}, \mathfrak{M}_{i,2}, \ldots$ for $\mathfrak{A}_i$, $i = 1, 2, \ldots$. By Lemma 8.3.3,

$$\lim_{p \to \infty} D_{\mathbf{M}}(\mathfrak{M}_{i,p}) = D_{\mathbf{M}}([\mathfrak{M}_{i,1}, \mathfrak{M}_{i,2}, \ldots]) = D_{\mathbf{M}}(\mathfrak{H}) = 1,$$

and so we can form a sequence $\rho_{i,1} < \rho_{i,2} < \cdots$, with

$$D_{\mathbf{M}}(\mathfrak{M}_{i,\rho_{i,\nu}}) > 1 - \frac{1}{2^{i+\nu}}, \qquad D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_{i,\rho_{i,\nu}}) = D_{\mathbf{M}}(\mathfrak{H}) - D_{\mathbf{M}}(\mathfrak{M}_{i,\rho_{i,\nu}}) < \frac{1}{2^{i+\nu}}.$$

Form $\overline{\mathfrak{M}}_\nu = \mathfrak{M}_{1,\rho_{1,\nu}} \cdot \mathfrak{M}_{2,\rho_{2,\nu}} \cdots$. Clearly $\overline{\mathfrak{M}}_\nu$ is a closed linear set, and $\overline{\mathfrak{M}}_\nu \,\eta\, \mathbf{M}$. As $\rho_{i,\nu} < \rho_{i,\nu+1}$, therefore $\mathfrak{M}_{i,\rho_{i,\nu}} \subset \mathfrak{M}_{i,\rho_{i,\nu+1}}$, and so $\overline{\mathfrak{M}}_\nu \subset \overline{\mathfrak{M}}_{\nu+1}$; as $\mathfrak{M}_{i,\rho_{i,\nu}} \subset \mathfrak{A}_i$, so $\overline{\mathfrak{M}}_\nu \subset \mathfrak{A}_1 \cdot \mathfrak{A}_2 \cdots$. Finally we have

$$\begin{aligned} D_{\mathbf{M}}(\mathfrak{H} - \overline{\mathfrak{M}}_\nu) &= D_{\mathbf{M}}([\mathfrak{H} - \mathfrak{M}_{1,\rho_{1,\nu}}, \mathfrak{H} - \mathfrak{M}_{2,\rho_{2,\nu}}, \ldots]) \\ &\leq \sum_{i=1}^{\infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_{i,\rho_{i,\nu}}) < \sum_{i=1}^{\infty} \frac{1}{2^{i+\nu}} = \frac{1}{2^\nu} \end{aligned}$$

by Lemma 14.2.5, and so

$$D_{\mathbf{M}}(\mathfrak{H} - [\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \ldots]) \leq D_{\mathbf{M}}(\mathfrak{H} - \overline{\mathfrak{M}}_\nu) < \frac{1}{2^\nu}.$$

This holds for every $\nu = 1, 2, \ldots$, therefore

$$D_{\mathbf{M}}(\mathfrak{H} - [\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \ldots]) = 0, \quad \mathfrak{H} - [\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \ldots] = (0), \quad [\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \ldots] = \mathfrak{H}.$$

Thus the sequence $\overline{\mathfrak{M}}_1, \overline{\mathfrak{M}}_2, \ldots$ fulfills all conditions (i)–(iii) of Definition 16.2.1 for the linear set $\mathfrak{A}_1 \cdot \mathfrak{A}_2 \cdots$; therefore $\mathfrak{A}_1 \cdot \mathfrak{A}_2 \cdots$ is essentially dense.

Lemma 16.2.3. *Let $\mathfrak{A}$ be an essentially dense set, and X a linear closed operator with an everywhere dense domain, and $X \,\eta\, \mathbf{M}$. Then the set*

$$(f; f \in \text{Domain}\, X,\ Xf \in \mathfrak{A})$$

is essentially dense.

Proof: Let W, B be the canonical decomposition of X: B self adjoint and definite, W partially isometric, $W \in \mathbf{M}$, $B \,\eta\, \mathbf{M}$, and $X = WB$ (cf. Definition 4.4.1 and Lemma 4.4.1).

Let $E(\lambda)$, $-\infty < \lambda < \infty$, be the resolution of unity which belongs to B. This means, as B is not necessarily bounded, that $E(\lambda)$ fulfills the conditions (α)–(γ) of Definition 15.1.1 unaltered, but instead of (δ)–(ϵ) these modifications:

(δ') If $\lambda \to -\infty$, resp. $+\infty$, then $E(\lambda) \to 0$, resp. 1, in the sense of strong operator convergence.

(ϵ') $f \in \text{Domain}\, B$ if and only if $\int_{-\infty}^{\infty} \lambda^2 \, d\|E(\lambda)f\|^2$ is finite, and then

$$(Bf, g) = \int_{-\infty}^{\infty} \lambda \, d(E(\lambda)f, g) \qquad \text{(cf. (15, p. 92))}.$$

The (semi-)definiteness of B means that $E(\lambda) = 0$ for $\lambda < 0$ (cf. (21, p. 302)). Put $E(\lambda) = P_{\mathfrak{M}(\lambda)}$.

As $B \,\eta\, \mathbf{M}$, B is invariant under every unitary transformation $U' \in \mathbf{M}'$, and so is $E(\lambda)$; thus $E(\lambda) \in \mathbf{M}$, $\mathfrak{M}(\lambda) \,\eta\, \mathbf{M}$, and $E(\lambda)$ and $\mathfrak{M}(\lambda)$ obviously reduce B. If $f \in \mathfrak{M}(\lambda)$, then $E(\lambda)f = f$, so

$$E(\lambda')f = \begin{cases} f, & \lambda' \geq \lambda, \\ 0, & \lambda' < 0. \end{cases}$$

Therefore (δ') shows that Bf is defined, and (ϵ') that $0 \leq (Bf, f) \leq \lambda \|f\|^2$. Thus the part of B in $\mathfrak{M}(\lambda)$ is bounded and everywhere defined (in $\mathfrak{M}(\lambda)$), or: $BE(\lambda)$ is bounded and everywhere defined (in $\mathfrak{H}$).

We know that $\mathfrak{M}(1) \subset \mathfrak{M}(2) \subset \cdots$, and as $\lim_{i\to\infty} E(i) = 1$, we have $[\mathfrak{M}(1), \mathfrak{M}(2), \ldots] = \mathfrak{H}$. Thus by Lemma 8.3.3,

$$\lim_{i\to\infty} D_{\mathbf{M}}(\mathfrak{M}(i)) = D_{\mathbf{M}}(\mathfrak{H}) = 1,$$

$$\lim_{i\to\infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}(i)) = \lim_{i\to\infty}\left(D_{\mathbf{M}}(\mathfrak{H}) - D_{\mathbf{M}}(\mathfrak{M}(i))\right) = 0.$$

Choose next the sequence $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ from Definition 16.2.1 for $\mathfrak{A}$. Then $\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \cdots \subset \mathfrak{A}$, $[\mathfrak{M}_1, \mathfrak{M}_2, \ldots] = \mathfrak{H}$, therefore as above,

$$\lim_{i\to\infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_i) = 0.$$

Form now the set

$$\mathfrak{M}^{(i)} = (f; f \in \mathfrak{M}(i),\ WBE(i)f \in \mathfrak{M}_i).$$

As $BE(i)$ is everywhere defined and bounded, $\mathfrak{M}^{(i)}$ is linear and closed. As $f \in \mathfrak{M}(i)$ implies $WBE(i)f = WBf = Xf$, we have

$$\mathfrak{M}^{(i)} \subset (f; f \in \text{Domain}\, X,\ Xf \in \mathfrak{M}_i) \subset (f; f \in \text{Domain}\, X,\ Xf \in \mathfrak{A}).$$

Finally $f \in \mathfrak{M}^{(i)}$ implies $f \in \mathfrak{M}(i+1)$, and $E(i+1)f = f$, and so $\mathfrak{M}^{(i)} \subset \mathfrak{M}^{(i+1)}$. Thus we have

$$\mathfrak{M}^{(1)} \subset \mathfrak{M}^{(2)} \subset \cdots \subset (f; f \in \text{Domain}\, X,\ Xf \in \mathfrak{A}).$$

The definition of $\mathfrak{M}^{(i)}$ makes it obvious that it is invariant under all unitary transformations $U' \in \mathbf{M}'$, therefore $\mathfrak{M}^{(i)}\ \eta\ \mathbf{M}$.

Now

$$\begin{aligned}\mathfrak{M}^{(i)} &= (f; f \in \mathfrak{M}(i),\ WBE(i)f \in \mathfrak{M}_i)\\ &= \mathfrak{M}(i) \cdot (f; WBE(i)f \in \mathfrak{M}_i)\\ &= \mathfrak{M}(i) \cdot (f; P_{\mathfrak{H}-\mathfrak{M}_i} WBE(i)f = 0).\end{aligned}$$

Now by Lemma 16.1.1,

$$\begin{aligned}D_{\mathbf{M}}((f; P_{\mathfrak{H}-\mathfrak{M}_i} WBE(i)f = 0)) &= D_{\mathbf{M}}((f; (P_{\mathfrak{H}-\mathfrak{M}_i} WBE(i))^* f = 0))\\ &= D_{\mathbf{M}}((f; (BE(i))^* W^* P_{\mathfrak{H}-\mathfrak{M}_i} f = 0)),\end{aligned}$$

and

$$(f; (BE(i))^* W^* P_{\mathfrak{H}-\mathfrak{M}_i} f = 0) \supset (f; P_{\mathfrak{H}-\mathfrak{M}_i} f = 0) = \mathfrak{M}_i.$$

So

$$D_{\mathbf{M}}((f; P_{\mathfrak{H}-\mathfrak{M}_i} WBE(i)f = 0)) \geq D_{\mathbf{M}}(\mathfrak{M}_i),$$

$$D_{\mathbf{M}}(\mathfrak{H} - (f; P_{\mathfrak{H}-\mathfrak{M}_i} WBE(i)f = 0)) \leq D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_i).$$

Now we obtain, using Lemma 14.2.3, or Lemma 14.2.5, this:

$$\begin{aligned} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}^{(i)}) &= D_{\mathbf{M}}([\mathfrak{H} - \mathfrak{M}(i), \mathfrak{H} - (f; P_{\mathfrak{H}-\mathfrak{M}_i} WBE(i) f = 0)]) \\ &\leq D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}(i)) + D_{\mathbf{M}}(\mathfrak{H} - (f; P_{\mathfrak{H}-\mathfrak{M}_i} WBE(i) f = 0)) \\ &\leq D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}(i)) + D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_i). \end{aligned}$$

Thus

$$\lim_{i\to\infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}(i)) = 0, \qquad \lim_{i\to\infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}_i) = 0$$

give

$$\lim_{i\to\infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}^{(i)}) = 0.$$

By Lemma 8.3.4,

$$\begin{aligned} D_{\mathbf{M}}(\mathfrak{H} - [\mathfrak{M}^{(1)}, \mathfrak{M}^{(2)}, \ldots]) &= D_{\mathbf{M}}((\mathfrak{H} - \mathfrak{M}^{(1)}) \cdot (\mathfrak{H} - \mathfrak{M}^{(2)}) \cdot (\mathfrak{H} - \mathfrak{M}^{(3)}) \cdots) \\ &= \lim_{i\to\infty} D_{\mathbf{M}}(\mathfrak{H} - \mathfrak{M}^{(i)}) = 0, \end{aligned}$$

$$\mathfrak{H} - [\mathfrak{M}^{(1)}, \mathfrak{M}^{(2)}, \ldots] = (0), \qquad [\mathfrak{M}^{(1)}, \mathfrak{M}^{(2)}, \ldots] = \mathfrak{H}.$$

Thus $\mathfrak{M}^{(1)}, \mathfrak{M}^{(2)}, \ldots$ fulfill the conditions (i)–(iii) of Definition 16.2.1 for $(f; f \in \text{Domain}\, X,\ Xf \in \mathfrak{A})$. Therefore this set is essentially dense, and the proof is completed.

Putting $\mathfrak{A} = \mathfrak{H}$, we see that Domain X itself is essentially dense. This, of course, could have been easily proved directly too.

The considerations of this § are special cases of a generalisation of the notion of relative dimensionality (which was only defined for linear closed sets $\eta\mathbf{M}$ so far) to all linear sets. A theory very similar to that of Lebesgue's interior measure results. We do not go into the details of this subject here, but it will be dealt with in subsequent papers.

16.3. The Lemmas 16.2.1–16.2.3 have an interesting application to unbounded operators.

Definition 16.3.1. Let $x_1, y_1, x_2, y_2, \ldots$ be a (finite or infinite) sequence of symbolic variables. By a (non-commutative) monomial we mean an expression of the form $z_1 \cdots z_n$, where $n = 0, 1, 2, \ldots$, and each z_ν is some x_i or y_i. If $n = 0$, we use the symbol 1. The order in which the factors $z_1, \ldots, z_n$ occur is essential; the number n is the degree of the monomial. By a (non-commutative) polynomial we mean a finite linear aggregate of non-commutative monomials, that is, an expression of the form

$$p(x_1, y_1, \ldots) = \sum_{\rho=1}^{p} a_\rho z_1^{(\rho)} \cdots z_{n_\rho}^{(\rho)},$$

where $p = 0, 1, 2, \ldots$; $a_1, \ldots, a_p$ are complex numbers, and each $z_\nu^{(\rho)}$ is some x_i or y_i. If $p = 0$, we use the symbol 0. Monomial terms of the form $0 \cdot z_1 \cdots z_n$ cannot be omitted, but two terms $a \cdot z_1 \cdots z_n$ and $b \cdot z_1 \cdots z_n$ can be contracted to one, and the order of the additive terms does not matter. We consider monomials as special cases of polynomials. If

$$p(x_1, y_1, \ldots) = \sum_{\rho=1}^{p} a_\rho z_1^{(\rho)} \cdots z_{n_\rho}^{(\rho)}$$

is a polynomial, then ${}^{(r)}p(x_1, y_1, \ldots)$, the reduced polynomial of $p(x_1, y_1, \ldots)$, obtains from it by omitting all terms with $a_\rho = 0$ (that is, of the form $0 \cdot z_1 \cdots z_n$).

If $p(x_1, y_1, x_2, y_2, \ldots)$, $q(x_1, y_1, x_2, y_2, \ldots)$ are two polynomials, then

$$ap(x_1, y_1, x_2, y_2, \ldots), \qquad p(x_1, y_1, x_2, y_2, \ldots) + q(x_1, y_1, x_2, y_2, \ldots),$$

$$p(x_1, y_1, x_2, y_2, \ldots) \cdot q(x_1, y_1, x_2, y_2, \ldots)$$

are those which result by carrying out the indicated operations term by term, but without reducing (if the process ${}^{(r)}$ is not explicitly indicated).

The symbol $^+$ is defined as follows, in successive steps:

$$x_i^+ = y_i, \qquad y_i^+ = x_i; \qquad (z_1 \cdots z_n)^+ = z_n^+ \cdots z_1^+;$$

$$\left(\sum_{\rho=1}^{p} a_\rho z_1^{(\rho)} \cdots z_{n_\rho}^{(\rho)} \right)^+ = \sum_{\rho=1}^{p} \overline{a_\rho} \left(z_1^{(\rho)} \cdots z_{n_\rho}^{(\rho)} \right)^+.$$

Obviously

$$\left({}^{(r)}p(x_1, y_1, \ldots)\right)^+ = {}^{(r)}\left(p(x_1, y_1, x_2, y_2, \ldots)^+\right).$$

Definition 16.3.2. Let $x_1, y_1, x_2, y_2, \ldots$ be a (finite or infinite) sequence of symbolic variables; $X_1, X_2, \ldots$ a corresponding sequence of linear, closed operators, each one having an everywhere dense domain, so that to every pair of variables x_i, y_i one operator X_i corresponds. If

$$p(x_1, y_1, x_2, y_2, \ldots) = \sum_{\rho=1}^{p} a_\rho z_1^{(\rho)} \cdots z_{n_\rho}^{(\rho)}$$

is a polynomial, we mean by

$$p(X_1, X_1^*, X_2, X_2^*, \ldots)$$

the operator which arises if we replace in

$$\sum_{\rho=1}^{p} a_\rho z_1^{(\rho)} \cdots z_{n_\rho}^{(\rho)}$$

every x_i by X_i, and every y_i by X_i^*. Here the domain and the values of $p(X_1, X_1^*, X_2, X_2^*, \ldots)$ are to be determined with the help of the following rules (cf. (18, p. 404)):

$0f$ and $1f$ are everywhere defined and have the values 0, resp. f. $(aX)f$ is defined if and only if Xf is defined (even for $a = 0$), and its value is $a(Xf)$. $(X + Y)f$ is defined if and only if both Xf and Yf are defined, and its value is $Xf + Yf$. $(XY)f$ is defined if and only if both Yf and $X(Yf)$ are defined, and its value is $X(Yf)$.

Variables x_i, y_i, resp. the corresponding operators X_i, X_i^*, which do not occur in the expression

$$p(x_1, y_1, x_2, y_2, \ldots) = \sum_{\rho=1}^{p} a_\rho z_1^{(\rho)} \cdots z_{n_\rho}^{(\rho)}$$

can be omitted in the expression $p(x_1, y_1, x_2, y_2, \ldots)$, resp. $p(X_1, X_1^*, \ldots)$. (So we can write $p(x_1)$ for $p(x_1, y_1) = x_1$, but not for $p(x_1, y_1) = x_1 + 0 \cdot y_1$.) $p(X_1, X_1^*, \ldots)$ and ${}^{(r)}p(X_1, X_1^*, \ldots)$ are not identical in general. Thus if $p(x_1) = 0 \cdot x_1$, ${}^{(r)}p(x_1) = 0$, then $p(X_1) = 0 \cdot X_1$ and ${}^{(r)}p(X_1) = 0$, and these differ, because they have different domains, if X_1 is not everywhere defined. In this particular case, however, we can obtain equality by forming the closure [ ] of all operators concerned (cf. (16, pp. 70–71), where a ~ is used to denote closure): As X_1 has an everywhere dense domain, therefore $[\,0 \cdot X_1\,] = 0$, and so $[\,p(X_1)\,] = [\,{}^{(r)}p(X_1)\,]$. But in general this procedure too breaks down: Define

$$p(x_1, x_2) = 0 \cdot x_1 + 0 \cdot x_2, \qquad {}^{(r)}p(x_1, x_2) = 0,$$

and let X_1, X_2 be two operators for which Domain $X_1 \cdot$ Domain $X_2 = (0)$ (cf. (17, p. 230)). Then $p(X_1, X_2)$ has the domain (0), and there of course the value 0. Thus the same is true for $[\, p(X_1, X_2)\,]$, while ${}^{(r)}p(X_1, X_2) = 0$, $[\,{}^{(r)}p(X_1, X_2)\,] = 0$. Thus $[\, p(X_1, X_2)\,]$ has the domain (0), while $[\,{}^{(r)}p(X_1, X_2)\,]$ has the domain $\mathfrak{H}$, and so they are different.

Another "pathological" possibility is that $(X_1X_2)^*$ may not be $X_2^*X_1^*$. In fact it can happen that X_1X_2 has no closure at all, and therefore no adjoint with an everywhere dense domain (cf. (21, pp. 296 and 300); our statement means that X_1X_2 is not a **-operator, cf. loc. cit.); or again $(X_1X_2)^*$ may exist and be a proper extension of $X_2^*X_1^*$. We will not go presently into the details of this matter.

For $X_1, X_2, \ldots \eta\, \mathbf{M}$, where $\mathbf{M}$ is a factor in one of the finite cases ((I_n), $n = 1, 2, \ldots$, or (II_1)), as we now always assume, nothing of this sort happens, and the algebra of the $[\, p(X_1, X_1^*, X_2, X_2^*, \ldots)\,]$ works perfectly smoothly. This is the main result of the considerations which follow. For the cases (I_n), $n = 1, 2, \ldots$, these things are obvious (then even the [ ] is superfluous, because every linear operator is closed), and the essential content of our results is that it is the same way in case (II_1). So while the usually considered case (I_∞) (including $\mathbf{M} = \mathbf{B}$ for a Hilbert space $\mathfrak{H}$) is highly pathological in this respect, (II_1) turns out to be the appropriate generalisation of the (I_n).

16.4. We prove first two Lemmas which have considerable interest of their own. In particular the first one proves that the "spectral problem" (the existence of a "resolution of unity" for every linear, closed Hermitian operator $X \,\eta\, \mathbf{M}$) has always a satisfactory solution. (Cf. (16, p. 92); observe the contrast with (I_∞).)

Lemma 16.4.1. *Every linear, closed, Hermitian operator $X \,\eta\, \mathbf{M}$ is maximal and self adjoint (cf. (16, p. 88)).*

Proof: Let U be the Cayley transform of X, $\mathfrak{E}$, $\mathfrak{F}$ the connected linear, closed sets (cf. (16, p. 80)). Then $U, \mathfrak{E}, \mathfrak{F}, E = P_{\mathfrak{E}}$ are invariant under every unitary transformation $U' \in \mathbf{M}'$ along with X, so they are all $\eta\mathbf{M}$. Thus $UE \,\eta\, \mathbf{M}$, and it is obviously partially isometric, with the initial and final sets $\mathfrak{E}$, resp. $\mathfrak{F}$, so UE is bounded, and therefore $UE \in \mathbf{M}$. Similarly $E \in \mathbf{M}$.

We know that $[\mathrm{Range}(U - 1)] = \mathfrak{H}$, and as Uf is only defined if $f \in \mathfrak{E}$, we may as well write $[\mathrm{Range}(UE - E)] = \mathfrak{H}$. Now $UE - E = (UE - E)E$, $(UE - E)^* = E(UE - E)^*$, and so $[\text{Range}(UE-E)^*] \subset [\text{Range}\, E] = \mathfrak{E}$. So by Lemma 6.2.1,

$$\begin{aligned} D_{\mathbf{M}}(\mathfrak{E}) &\geq D_{\mathbf{M}}([\text{Range}(UE-E)^*]) = D_{\mathbf{M}}([\text{Range}(UE-E)]) \\ &= D_{\mathbf{M}}(\mathfrak{H}) = 1. \end{aligned}$$

Considering the properties of the partially isometric operator UE, we have further $D_{\mathbf{M}}(\mathfrak{E}) = D_{\mathbf{M}}(\mathfrak{F})$. Therefore $D_{\mathbf{M}}(\mathfrak{H}-\mathfrak{E}) = D_{\mathbf{M}}(\mathfrak{H}-\mathfrak{F}) = 0$, that is 0, and so $\mathfrak{H}-\mathfrak{E} = \mathfrak{H}-\mathfrak{F} = (0)$. So X is maximal and self adjoint by (16, p. 88).

Lemma 16.4.2. *Let X, Y be linear, closed operators, with everywhere dense domains, $X, Y \,\eta\, \mathbf{M}$. If Y is an extension of X (that is: whenever Xf is defined, Yf is defined too, and $Xf = Yf$; cf. (15, p. 70)), then $X = Y$.*

Proof: Let W, B be the canonical decomposition of Y: B self adjoint and (semi-)definite, W partially isometric, $W \,\eta\, \mathbf{M}$, $B \,\eta\, \mathbf{M}$, and $Y = WB$ (cf. Definition 4.4.1 and Lemma 4.4.1). At the same time $B = W^*Y$ (cf. (21, p. 306)), so $Y = WW^*Y$, and as Y is a continuation of X, so $X = WW^*X$ too.

$W^*Y = B$ is self adjoint, so it is Hermitian, and so W^*X is too; besides it is linear, closed, and has an everywhere dense domain along with X. As $W^*X \,\eta\, \mathbf{M}$, therefore Lemma 16.4.1 applies: W^*X is maximal. But W^*Y is a Hermitian continuation of W^*X, so $W^*X = W^*Y$, $WW^*X = WW^*Y$, $X = Y$. Thus the proof is completed.

We pass now to the discussion of the (non-commutative) polynomials of operators.

Lemma 16.4.3. *Let $X_1, X_2, \ldots$ be a (finite or infinite) sequence of linear, closed operators, each one having an everywhere dense domain. Assume that all $X_i \,\eta\, \mathbf{M}$. Let $\mathfrak{D}\mathfrak{P}$ be the common part of the domains of the operators $p(X_1, X_1^*, X_2, X_2^*, \ldots)$, where $p(x_1, y_1, x_2, y_2, \ldots)$ runs over all (non-commutative) polynomials of $x_1, y_1, x_2, y_2, \ldots$. Then $\mathfrak{D}\mathfrak{P}$ is essentially dense and everywhere dense.*

Proof: It suffices, by Lemma 16.2.1, to prove the essential density. It suffices furthermore, obviously, to let $p(x_1, y_1, x_2, y_2, \ldots)$ run over all monomials only. These are countable in number, say $p^{(\sigma)}(x_1, y_1, \ldots)$, $\sigma = 1, 2, \ldots$; therefore

$$\mathfrak{D}\mathfrak{P} = \prod_{\sigma=1}^{\infty} \text{Domain}\, p^{(\sigma)}(x_1, y_1, \ldots).$$

By Lemma 16.2.2, we need only to prove that every Domain $p(X_1, X_1^*, \ldots)$ is essentially dense. That is: That for every monomial $p(x_1, y_1, \ldots) = z_1 \cdots z_n$ (cf. Definition 16.3.1), Domain $p(X_1, X_1^*, \ldots)$ is essentially dense.

We prove this by induction on the degree n. For $n = 0$, the operator in question is 1, its domain is $\mathfrak{H}$, and so our statement is obvious. Assume now that it is already established for $n-1$, $n = 1, 2, \ldots$, and let us consider n. Let $p(x_1, y_1, \ldots) = z_1 \cdots z_n$ be a monomial of degree n. Put $q(x_1, y_1, \ldots) = z_1 \cdots z_{n-1}$, and write $p(X_1, X_1^*, \ldots) = A$, $q(X_1, X_1^*, \ldots) = B$. z_n is a variable x_i or y_i; put correspondingly $Y = X_i$, resp. $Y = X_i^*$. Now $A = BY$, so

$$\text{Domain}\, A = (f; f \in \text{Domain}\, Y,\ Yf \in \text{Domain}\, B).$$

Domain B is essentially dense by our induction assumption (the degree of $q(x_1, y_1, \ldots) = z_1 \cdots z_{n-1}$ is $n-1$); Y is linear, closed, and has an everywhere dense domain (for X_i this was assumed, for X_i^* it follows from (21, p. 301)). Therefore Lemma 16.2.3 applies, and Domain A is essentially dense too. Thus the proof is completed.

Lemma 16.4.4. *Let $X_1, X_2, \ldots$ be a (finite or infinite) sequence of linear, closed operators, each one having an everywhere dense domain. Assume that all $X_i\ \eta\ \mathbf{M}$. Let $p(x_1, y_1, x_2, y_2, \ldots)$, $q(x_1, y_1, x_2, y_2, \ldots)$, $r(x_1, y_1, x_2, y_2, \ldots)$ be (non-commutative) polynomials of the symbolic variables x_i, y_i, corresponding to X_i, $i = 1, 2, \ldots$. Then we have:*

(i) $[\,p(X_1, X_1^*, X_2, X_2^*, \ldots)\,]$ *can be formed; it is linear, closed, has an everywhere dense domain, and it is $\eta\mathbf{M}$ too.*

(ii) If ${}^{(r)}p(x_1, y_1, x_2, y_2, \ldots) = {}^{(r)}q(x_1, y_1, x_2, y_2, \ldots)$, *then*

$$[\,p(X_1, X_1^*, X_2, X_2^*, \ldots)\,] = [\,q(X_1, X_1^*, X_2, X_2^*, \ldots)\,];$$

that is, $[\,p(X_1, X_1^*, X_2, X_2^*, \ldots)\,]$ *depends on* ${}^{(r)}p(x_1, y_1, x_2, y_2, \ldots)$ *only.*

(iii) If $p(x_1, y_1, x_2, y_2, \ldots)^+ = q(x_1, y_1, x_2, y_2, \ldots)$, *then*

$$[\,p(X_1, X_1^*, X_2, X_2^*, \ldots)\,]^* = [\,q(X_1, X_1^*, X_2, X_2^*, \ldots)\,].$$

(iv) If $a p(x_1, y_1, x_2, y_2, \ldots) = q(x_1, y_1, x_2, y_2, \ldots)$, *then*

$$[\,a[\,p(X_1, X_1^*, X_2, X_2^*, \ldots)\,]\,] = [\,q(X_1, X_1^*, X_2, X_2^*, \ldots)\,].$$

(If $a \neq 0$, then the first [ ] is obviously unnecessary.)

(v) If $p(x_1, y_1, x_2, y_2, \ldots) + q(x_1, y_1, x_2, y_2, \ldots) = r(x_1, y_1, x_2, y_2, \ldots)$, *then*

$$[\,[\,p(X_1, X_1^*, X_2, X_2^*, \ldots)\,] + [\,q(X_1, X_1^*, X_2, X_2^*, \ldots)\,]\,] = [\,r(X_1, X_1^*, X_2, X_2^*, \ldots)\,].$$

(vi) If $p(x_1, y_1, x_2, y_2, \ldots) \cdot q(x_1, y_1, x_2, y_2, \ldots) = r(x_1, y_1, x_2, y_2, \ldots)$, then

$$[\,[\,p(X_1, X_1^*, X_2, X_2^*, \ldots)\,] \cdot [\,q(X_1, X_1^*, X_2, X_2^*, \ldots)\,]\,] = [\,r(X_1, X_1^*, X_2, X_2^*, \ldots)\,].$$

Proof: Ad (i): $p(X_1, X_1^*, \ldots)$ and $p(X_1, X_1^*, \ldots)^+$ have everywhere dense domains by Lemma 16.4.3. One verifies immediately that

$$(p(X_1, X_1^*, \ldots) f, g) = (f, p(X_1, X_1^*, \ldots)^+ g)$$

for all $f \in \text{Domain}\, p(X_1, X_1^*, \ldots)$, $g \in \text{Domain}\, p(X_1, X_1^*, \ldots)^+$, and so $p(X_1, X_1^*, \ldots)^*$ is an extension of $p(X_1, X_1^*, \ldots)^+$. Therefore $p(X_1, X_1^*, \ldots)^*$ has an everywhere dense domain, and so we can form $[\,p(X_1, X_1^*, \ldots)\,]$ (cf. (21, p. 300)). $[\,p(X_1, X_1^*, \ldots)\,]$ is invariant under every unitary transformation $U' \in \mathbf{M}'$, along with $X_1, X_2, \ldots$, so it is $\eta\mathbf{M}$.

Ad (ii):

$${}^{(r)}p(X_1, X_1^*, X_2, X_2^*, \ldots)$$

is obviously an extension of

$$p(X_1, X_1^*, X_2, X_2^*, \ldots);$$

therefore $[\,{}^{(r)}p(X_1, X_1^*, \ldots)\,]$ is one of $[\,p(X_1, X_1^*, \ldots)\,]$. By (i), both operators are linear, closed, have everywhere dense domains, and are $\eta\mathbf{M}$. So by Lemma 16.4.2,

$$[\,{}^{(r)}p(X_1, X_1^*, \ldots)\,] = [\,p(X_1, X_1^*, X_2, X_2^*, \ldots)\,].$$

Applying this to both $p(x_1, y_1, x_2, y_2, \ldots)$ and $q(x_1, y_1, x_2, y_2, \ldots)$ gives the statement of (ii).

Ad (iii): As we saw in the proof of (i), $p(X_1, X_1^*, \ldots)^*$ is an extension of $p(X_1, X_1^*, \ldots)^+$, that is, of $q(X_1, X_1^*, \ldots)$. The first operator is obviously equal to $[\,p(X_1, X_1^*, \ldots)\,]^*$, and as it is linear, closed, it must even be an extension of $[\,q(X_1, X_1^*, \ldots)\,]$. As these are both linear, closed operators with everywhere dense domains, Lemma 16.4.2 applies again: They must be equal.

Ad (iv)–(vi): All these statements are proved in the same way; therefore it suffices to consider one of them, say (vi):

Clearly

$$[\,p(X_1, X_1^*, X_2, X_2^*, \ldots)\,] \cdot [\,q(X_1, X_1^*, X_2, X_2^*, \ldots)\,]$$

is an extension of

$$p(X_1, X_1^*, X_2, X_2^*, \ldots) \cdot q(X_1, X_1^*, X_2, X_2^*, \ldots) = r(X_1, X_1^*, X_2, X_2^*, \ldots).$$

The first mentioned operator is linear closed, and therefore it is an extension of $[\,r(X_1, X_1^*, X_2, X_2^*, \ldots)\,]$, too. Now we can infer

$$[\,[\,p(X_1, X_1^*, X_2, X_2^*, \ldots)\,] \cdot [\,q(X_1, X_1^*, X_2, X_2^*, \ldots)\,]\,] = [\,r(X_1, X_1^*, X_2, X_2^*, \ldots)\,]$$

from Lemma 16.4.2, completing the proof.

Of course (iv) for $a = 0$ is trivial, even without the first [ ].

We restate the results of this §.

Theorem XV. *Let* **M** *be a factor of a finite class* ((I_n), $n = 1, 2, \ldots$, *or* (II_1)). *Denote by* $U(\mathbf{M})$ *the set of all linear, closed operators with an everywhere dense domain, which are* $\eta\mathbf{M}$. *Then for* $X, Y \in U(\mathbf{M})$ *the operators* $[\,aX\,]$ *(coinciding with* aX *for* $a \neq 0$*),* $[\,X + Y\,]$, $[\,XY\,]$ *can be formed, and are* $\in U(\mathbf{M})$ *again. These operations satisfy all customary laws of matrix algebra:*

$$[\,X + Y\,] = [\,Y + X\,]; \qquad [\,[\,X + Y\,] + Z\,] = [\,X + [\,Y + Z\,]\,].$$

$$[\,a[\,X + Y\,]\,] = [\,a[\,X\,] + b[\,Y\,]\,]; \qquad [\,(a + b)X\,] = [\,aX + bX\,].$$

$$[\,[\,XY\,]Z\,] = [\,X[\,YZ\,]\,]; \qquad [\,[\,aX\,]Y\,] = [\,a[\,XY\,]\,]; \qquad [\,a[\,bX\,]\,] = [\,(ab)X\,].$$

$$[\,[\,X + Y\,]Z\,] = [\,[\,XZ\,] + [\,YZ\,]\,]; \qquad [\,X[\,Y + Z\,]\,] = [\,[\,XY\,] + [\,XZ\,]\,].$$

$$[\,aX\,]^* = [\,\overline{a}X^*\,]; \qquad [\,X + Y\,]^* = [\,X^* + Y^*\,]; \qquad [\,XY\,]^* = [\,Y^*X^*\,].$$

More generally: The (non-commutative) polynomials

$$[\,p(X_1, X_1^*, X_2, X_2^*, \ldots)\,]$$

(cf. Definition 16.3.1 and Definition 16.3.2) are $\in U(\mathbf{M})$ *again;*

$$[\,p(X_1, X_1^*, X_2, X_2^*, \ldots)\,]$$

depends only on the reduced form ${}^{(r)}p(x_1, y_1, x_2, y_2, \ldots)$ *of* $p(x_1, y_1, x_2, y_2, \ldots)$, *and the laws of matrix algebra hold for the*

$$[\,p(X_1, X_1^*, X_2, X_2^*, \ldots)\,]$$

(cf. Lemma 16.4.4(iii)–(vi)).

Every Hermitian $X \in U(\mathbf{M})$ *is maximal and self adjoint, and thus it has a unique resolution of unity (cf. (15, p. 92)).*

For $X, Y \in U(\mathbf{M})$, *whenever* Y *is an extension of* X *(cf. (15, p. 70)), then* $X = Y$; *that is: proper extensions do not exist in* $U(\mathbf{M})$.

(Added in proof, Jan. 29, 1936.) The authors have since succeeded in establishing the following further facts concerning the finite cases:

(i) $T_{\mathbf{M}}(A+B) = T_{\mathbf{M}}(A) + T_{\mathbf{M}}(B)$ unrestrictedly. (Cf. Problem 11.)
(ii) $T_{\mathbf{M}}(A)$ is weakly continuous in A. (Cf. Lemma 15.5.5.)
(iii) If $C = 1$, then an f exists, for which identically $T_{\mathbf{M}}(A) = (Af, f)$.
(iv) In the same case certain isomorphisms between $\mathbf{M}, \mathbf{M}'$, and $\mathfrak{H}$ exist.
(v) Several examples of Case II, thus (α)–(γ) on page 134, are spacially isomorphic. (This answers the question at the end of §13.4.) The proofs will appear in a subsequent paper.

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